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On the interactions of job creation, migration, and natural population increase within the framework of a dynamic demoeconomic model

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Abstract. Within the framework of a dynamic demoeconomic model the dominant role of job creation and migration for regional development is examined. The model is formulated by means of continuous dynamic equations for which an analytical solution is available for one region. On the basis of the solution the time paths of the model variables are specified. Numerical examples show the impact of the theoretical issues involved. The extension to two or more regions is discussed, which provides a comparison with models developed by Keyfitz and Rogers.

1 Introduction

For some time now, a new trend in regional modelling can be observed as a result of the rising interest of planners and social scientists in demographic problems. In addition, demographers have begun to discover the relevance of regional differentiation for the explanation of demographic facts and trends. The combination of economic and demographic aspects in model building has led to a new class of models, the so-called demoeconomic models.

The demoeconomic models differ from the multiregional accounting models developed by Rees, Wilson, and others. Whereas the aim of the accounting models is solving problems of estimation, accounting, and forecasting of large systems of multiregional population oriented data, the aim of the demoeconomic models is a more theoretical one. These models start with hypotheses on the interaction of demographic and economic forces and their effects on the population growth of a region as well as on the population distribution of a system of regions. The elements of the demoeconomic models—the variables and the mathematical functions—are, of course, dynamic as is the nature of the problems on which they try to provide insight. In this article as well as in the literature quoted, the dynamic functions are mostly formulated in a continuous rather than in a discrete form. This allows the use of the effective mathematical tools of calculus. However, the model presented here can also be easily formulated in a discrete form.

2 The central role of migration for economic growth and population change

The basic link between regional economic and demographic aspects is migration. The role of migration both for the economic and for the demographic development of a region or a system of regions can hardly be overestimated. In many countries the in-migration rate has the same relevance as the birth rate for population change; for the Federal Republic of Germany the annual rate of in-migration is 580000 and the annual birth rate is 590000 (average over 1975-1979). This means that the relative weight of the population increasing components 'in-migrations' and 'births' is about 1:1, even at the national level (see table 1). At the regional level this relation increases considerably, depending on the degree of the regionalisation of the country. Similar proportions exist between the components 'deaths' and 'out-migrations' (Birg, 1979, page 91; 1981). Most people who plan to migrate from one region to another need a new job in the region of destination (except when commuting is possible); therefore migration is closely related to the job creating economic forces of a region. But the economic forces in turn depend—in the long run—largely on migration, because the region's stock of human capital, which is relevant for production and innovation, is a principal consequence of the flow of skills incorporated in the migrating labour force.

Table 1. Relative weights of population increasing components.

Regional partition level	Births to in-migration ratio
national level	1:1 up to 1:1·3
79 planning regions	1:4·4
450 'Kreise' (counties)	1:5·5
communal level (8000)	more than 1:5·5

3 A demoeconomic model including migration

For any region demoeconomic interaction is comprised of four basic processes:
(1) economic change causes change in the number and quality of jobs,
(2) the number and quality of jobs influences in-migration as well as out-migration,
(3) migration determines population change,
(4) population change in turn causes, as a feedback, economic change.
The variables of these four groups interact simultaneously.

To describe the elements of the four processes in terms of dynamic equations, the complexity of real world phenomena has to be reduced considerably to solve the model analytically, that is, to achieve the desired detailed description of the time path of all model variables.

Starting with the job creation process, two groups of jobs are distinguished: the number of jobs in the basic sector, A_1, and in the nonbasic sector, A_2. The total number of jobs at the beginning of period t is the sum of A_1 and A_2. By definition:

$$A(t) = A_1(t)+A_2(t) \ . \tag{1}$$

The assumption is made that the number of jobs in the basic sector grows at the constant rate denoted by the parameter a in the following equation:

$$A_1(t) = A_1(0)\exp(at) \ . \tag{2}$$

For the number of jobs in the nonbasic sector an additive linear dependency on the number of inhabitants $P(t)$ and on the number of jobs in the basic sector is assumed:

$$A_2(t) = b_1P(t)+b_2A_1(t) \ , \qquad 0 < b_1 \ , \quad b_2 < 1 \ , \tag{3}$$

where b_1 and b_2 are constants. To specify the influence of the number of jobs on in-migration, two categories of in-migration variables are defined: the potential number of in-migrants $M_p^{in}(t)$ and the effective number of in-migrants $M_e^{in}(t)$. It is assumed that regional conditions such as high housing rents and bottlenecks in the quantitative supply of dwellings restrict the number of actual in-migrations. Furthermore it is assumed that the effect of the restrictions can be quantified by a variable $R(t)$ which grows at a constant rate, here denoted by h:

$$R(t) = R(0)\exp(ht) \ . \tag{4}$$

The actual number of in-migrants is then defined as the difference between the potential number of in-migrants and the restriction variable $R(t)$:

$$M_e^{in}(t) = M_p^{in}(t)-R(t) \ . \tag{5}$$

To assure that for any period the restriction effects are less than the potential in-migrations, the condition

$$R(t) < M_{\mathrm{p}}^{\mathrm{in}} \tag{5a}$$

is required, whereby the growth rate h must not exceed the growth rate of $M_{\mathrm{p}}^{\mathrm{in}}(t)$.

It is now assumed that there are two different groups of potential in-migrants; first, persons with a propensity to migrate into the region for noneconomic reasons, $M_{\mathrm{p},1}^{\mathrm{in}}$, and, second, persons who plan to migrate for economic (occupational) reasons, $M_{\mathrm{p},2}^{\mathrm{in}}$, so that:

$$M_{\mathrm{p}}^{\mathrm{in}} = M_{\mathrm{p},1}^{\mathrm{in}} + M_{\mathrm{p},2}^{\mathrm{in}} \ , \tag{6}$$

by definition.

For those migrants who are not economically motivated, for example students and retired people, the assumption is made that their number grows at a constant rate w:

$$M_{\mathrm{p},1}^{\mathrm{in}}(t) = M_{\mathrm{p},1}^{\mathrm{in}}(0)\exp(wt) \ . \tag{7}$$

For the economically motivated migrations it is assumed that the potential number of in-migrants is a function of the number of job opportunities $Q(t)$. The number of job opportunities is in turn a function of the stock of jobs:

$$M_{\mathrm{p},2}^{\mathrm{in}}(t) = gQ(t) \ , \qquad g > 0 \ , \tag{8}$$

where g is a constant, and

$$Q(t) = cA(t) \ , \qquad c > 0 \ , \tag{9}$$

where c is a constant. The job opportunity variable $Q(t)$ is one of the basic variables of the model. It is important to note that the number of job opportunities $Q(t)$ is not defined as the difference between the supply and the demand side of the labour market. $Q(t)$ simply quantifies the number of jobs which are vacant in period t. The vacancies may be caused by the retirement of employees, by the creation of new jobs, or by various vacancies which occur in the matching process.

Empirical investigations have shown the unemployment rate to be an inadequate variable in explaining migration flows. For the Federal Republic of Germany, model tests were as negative as were those for the United States of America [for the USA see Rogers (1968, page 80) and for the FRG Birg (1979, page 102)]. In both cases the unemployment rate was highly significant, but it did not have the sign suggested by theory. Rogers states: "Since ... the unemployment rate ... is highly significant, this result is disturbing". It would be less disturbing if it is simply considered that potential migration is not orientated on differences between the supply and the demand sides of the labour market but on the number of vacant jobs. Even in the case when the demand for jobs exceeds the supply, there is always a considerable number of jobs which become vacant because of fluctuations in the labour market which could be occupied by in-migrants. For this reason the in-migration function (8) is not based on the difference between the demand and supply variables but on the absolute number of job opportunities.

The same arguments hold with respect to the explanation of the number of out-migrants, $M^{\mathrm{out}}(t)$; this variable depends on the number of job opportunities in the region of destination. But since a model for a single region is first being considered, it is not necessary and not even possible to specify the out-migration function by analogy with the in-migration function. In section 6 two-region models will be discussed and a further consideration of the adequate specification of the migration functions made.

For the one-region model the number of out-migrations is specified as a function of the number of inhabitants. Despite its simplicity, this function has a coefficient of determination of more than 0·9 even in cross-region-analysis (Birg, 1979, page 93):

$$M^{\text{out}}(t) = kP(t)\ , \qquad 0 < k < 1\ , \tag{10}$$

where k is a constant. To complete the model a balance of change equation is added which defines the number of the inhabitants at the end of the interval [0, t] as the sum of the birth surplus (or deficit) and the net migration. For simplicity it is assumed that the rate of change due to the birth surplus is a constant factor r of the number of inhabitants, that is, $rP(t)$. The population change rate at time t then is given by:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = rP(t) + M_\mathrm{e}^{\text{in}}(t) - M^{\text{out}}(t)\ . \tag{11}$$

Integrating equation (11), the number of inhabitants at the end of the time interval $[0, t]$ is obtained:

$$P(t) = P(0) + r\int_0^t P(\tau)\mathrm{d}\tau + \int_0^t M_\mathrm{e}^{\text{in}}(\tau)\mathrm{d}\tau - \int_0^t M^{\text{out}}(\tau)\mathrm{d}\tau\ . \tag{12}$$

For empirical analysis in the framework of forecasting the following alternative form of equation (12) may be used:

$$P(t) = P_n(t) + \int_0^t M_\mathrm{e}^{\text{in}}(\tau)\mathrm{d}\tau - \int_0^t M^{\text{out}}(\tau)\mathrm{d}\tau\ . \tag{12a}$$

In equation (12a) the variable $P_n(t)$ gives the number of inhabitants due to the natural increase of the number of inhabitants $P(0)$, that is, the population living in the region *already at the starting point* of the process. If for this group a natural increase at a constant growth rate r is assumed, then

$$P_n(t) = P(0)\exp(rt)\ , \tag{13}$$

whereby, for simplicity it is further assumed that for in-migrants and out-migrants the number of births equals the number of deaths. Otherwise the two integral expressions have to be multiplied by appropriate parameters to take account of the natural increase of the two groups of migrants.

The advantage of using equation (12a) instead of equation (12) for forecasting purposes is that the variable $P_n(t)$ can be computed outside the forecasting model because it is independent of the migration variables, whereas the expression

$$r\int_0^t P(\tau)\mathrm{d}\tau$$

in equation (12) depends on the variables $M_\mathrm{e}^{\text{in}}(t)$ and $M^{\text{out}}(t)$ so that natural increase cannot be computed independently from migration. For an example of interregional population forecastings on the basis of equation (12a) see Birg (1979; 1981).

4 Derivation of the analytical solution of the model

The model described in section 3 consists of equations (1) to (12). Equation (12) is not independent of equation (11), therefore there are eleven equations with eleven variables.

Solving the model requires a reformulation of the eleven equations in such a way that each variable is a function of the time variable only. Substituting equations (2) and (3) into equation (1), and equations (7), (8), and (9) into equation (6) gives the following equations:

$$A(t) = [A_1(0)+b_2A_1(0)]\exp(at)+b_1P(t) , \quad (14)$$

$$M_{\mathrm{p}}^{\mathrm{in}}(t) = M_{\mathrm{p},1}^{\mathrm{in}}(0)\exp(wt)+gcA(t) . \quad (15)$$

Substitution of equation (14) into equation (15) gives:

$$M_{\mathrm{p}}^{\mathrm{in}}(t) = M_{\mathrm{p},1}^{\mathrm{in}}(0)\exp(wt)+gc[A_1(0)+b_2A_1(0)]\exp(at)+gcb_1P(t) . \quad (16)$$

The in-migration function (5) is obtained by the use of equations (4) and (16):

$$M_{\mathrm{e}}^{\mathrm{in}}(t) = M_{\mathrm{p},1}^{\mathrm{in}}(0)\exp(wt)+gc[A_1(0)+b_2A_1(0)]\exp(at)+gcb_1P(t)-R(0)\exp(ht) . \quad (17)$$

Substitution of the reformulated in-migration function (17) and of the out-migration function (10) into equation (11) gives the following formulation of the differential equation (11):

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = rP(t)+M_{\mathrm{p},1}^{\mathrm{in}}(0)\exp(wt)+gc[A_1(0)+b_2A_1(0)]\exp(at) +gcb_1P(t)-R(0)\exp(ht)-kP(t) . \quad (18)$$

If $V(t)$ is substituted in equation (18) for the following expression,

$$V(t) = M_{\mathrm{p},1}^{\mathrm{in}}(0)\exp(wt)+gc[A_1(0)+b_1A_1(0)]\exp(at)-R(0)\exp(ht) , \quad (19)$$

one obtains:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t}+(k-gcb_1-r)P(t) = V(t) . \quad (20)$$

Equation (20) has the same structure as the following general form,

$$y'+\phi(x)y = \psi(x) , \quad (21)$$

which is obtained from the definitions

$$P(t) = y , \qquad (k-gcb_1-r) = \phi(x) , \qquad V(t) = \psi(x) . \quad (22)$$

For equation (21) the solution is (see Bronstein and Semendjajew, 1980, page 472):

$$y = \exp\left[-\int\phi(x)\mathrm{d}x\right]\left\{\int\psi(x)\exp\left[\int\phi(x)\mathrm{d}x\right]+C\right\} . \quad (23)$$

By use of equation (23) and definitions (19) and (22), the following is the solution for equation (18):

$$P(t) = P(0)\exp(\eta+r)t+\frac{M_{\mathrm{p},1}^{\mathrm{in}}(0)}{w-(\eta+r)}[\exp(wt)-\exp(\eta+r)t] +\frac{gc[A_1(0)+b_2A_1(0)]}{a-(\eta+r)}[\exp(at)-\exp(\eta+r)t] -\frac{R(0)}{h-(\eta+r)}[\exp(ht)-\exp(\eta+r)t] , \quad (24)$$

where $\eta = gcb_1-k$.

For the second version of the model based on equation (12a) instead of equation (12) the solution is:

$$P(t) = P(0)\left[\frac{r}{r-\eta}\exp(rt) - \frac{r}{r-\eta}\exp(\eta t) + \exp(\eta t)\right] + \frac{M_{\mathrm{p},1}^{\mathrm{in}}(0)}{w-\eta}[\exp(wt) - \exp(\eta t)]$$
$$+\frac{gc[A_1(0)+b_2A_1(0)]}{a-\eta}[\exp(at) - \exp(\eta t)] - \frac{R(0)}{h-\eta}[\exp(ht) - \exp(\eta t)] . \quad (25)$$

In the case when $r = 0$ (no natural increase) the two solutions are identical.

5 Basic results and numerical illustrations

The population growth described by equations (24) and (25) has the following basic characteristics:

(a) The growth rate is not constant but time-dependent. This result can easily be derived by means of the following argument: a constant growth rate λ would imply a solution of the type

$$y(t) = y(0)\exp(\lambda t) .$$

A sum of exponential functions is not, however, an exponential function and equations (24) and (25) are sums of exponential functions. Hence the growth rate is not constant with time.

(b) The population growth rate depends not only on the natural increase parameter r and on the parameters included in the in-migration and out-migration functions, but also on every other parameter included in any of the eleven equations of the model. The parameters can be separated into two groups, namely parameters which are multiplying factors and parameters which are exponents (growth rates), as given in table 2.

(c) The most sensitive parameter constellations are given by the two cases

$$\eta > 0 \text{ and } \eta < 0 .$$

If the growth rate w of the non-job oriented in-migrations and the rate of increase a of the number of jobs in basic sectors are both at least zero then population will increase (decrease) if η is greater (less) than zero [provided that h is restricted according to condition (5a)]: $P(t)$ will increase if

$$\eta > 0 , \qquad r > 0 \text{ or } r = 0 , \qquad a, w > 0 \text{ or } a, w = 0 ;$$

Table 2. The parameters, their type (whether multiplying factor or exponent), and the equation in which they are introduced.

Parameter		Equation
Multiplying factors		
g	information-competition coefficient	(8)
c	matching coefficient	(9)
b_1	population related nonbasic coefficient	(3)
b_2	production related nonbasic coefficient	(3)
k	out-migration coefficient	(10)
Growth rates		
r	rate of natural increase	(12) (13)
a	rate of increase of jobs in basic sectors	(2)
w	rate of increase of potential in-migrations which are not job oriented	(7)
h	rate of increase of in-migration restrictions	(4)

$P(t)$ will decrease if

$$\eta < 0 , \qquad r < 0 \quad \text{or} \quad r = 0 , \qquad a, w < 0 \quad \text{or} \quad a, w = 0 .$$

Interestingly $P(t)$ will increase for $\eta > 0$ even if the rate of natural population increase (r), the rate of increase (a) of the number of jobs in basic industries, and the rate of increase (w) of the number of non-job oriented in-migrants are zero.

(d) Sensitivity analysis shows that the effect of most of those parameters which are classified as multiplying factors exceeds the effect of the parameters which are classified as growth rates if the values of the parameters are chosen realistically. The sensitivity results of table 3 illustrate the effects of parameter changes in comparison with the time dependent results of table 4 as a reference.

(e) One of the most sensitive parameters is the matching coefficient c [see equation (9)] which quantifies the number of job opportunities as a linear function of the number of jobs. In the FRG this coefficient is about 25%; in other words 25% of all employees change their jobs every year:

$$c = \frac{Q(t)}{A(t)} = 0{\cdot}25 . \tag{26}$$

Table 3. The effect of a parameter change on the values of model variables.

Parameter	Initial value	Variable							
		P	P_n	$M_c^{net,*}$ [a]	A	A_1	A_2	A/P [b]	A_1/A_2 [c]
Value of variable after a 50% increase in value of parameter at $t = 50$									
c	0·25	369	165	204	133	33	100	0·36	0·33
g	0·50	369	165	204	133	33	100	0·36	0·33
b_1	0·25	274	165	109	144	33	111	0·53	0·30
b_2	0·25	189	165	24	93	33	60	0·49	0·55
k	0·05	101	165	−64	66	33	33	0·66	1·00
r	0·01	209	212	−3	93	33	60	0·45	0·55
a	0·01	199	165	34	103	42	60	0·52	0·70
w	0·01	205	165	40	93	33	60	0·45	0·55
h	0·01	139	165	−26	76	33	43	0·55	0·77
Value of variable for initial value of parameter at $t = 50$									
–	–	176	165	11	85	33	52	0·48	0·63

[a] Sum of the direct and indirect effects of cumulated net migration [see equation (29b)].
[b] A/P is the activity rate.
[c] A_1/A_2 is the basic/nonbasic relation.

Table 4. A time series of model variables [equations (1) to (11) and (12)].

Period t	Variable											
	P	P_n	A	A_1	A_2	Q	$M_{p,1}^{in}$	$M_{p,2}^{in}$	M_p^{in}	R	M_e^{in}	M^{out}
0	100	100	50	20	30	13	4	6	10	5	5	5
5	106	105	53	21	32	13	4	7	11	5	6	5
10	113	111	56	22	34	14	4	7	11	6	6	6
15	120	116	59	23	36	15	5	7	12	6	6	6
20	127	122	62	24	38	16	5	8	13	6	7	6
25	134	128	66	26	40	16	5	8	13	6	7	7
30	142	135	69	27	42	17	5	9	14	7	7	7
35	150	142	73	28	45	18	6	9	5	7	8	7
40	158	149	77	30	47	19	6	10	16	7	8	8
45	167	157	81	31	50	20	6	10	16	8	9	8
50	176	165	85	33	52	21	7	11	17	8	9	9

If this parameter is increased to a value of 0·375 (an increase of 50%) the number of inhabitants in the period $t = 50$ is 369 instead of 176, given an initial number of 100 at $t = 0$. The elasticity $\epsilon_{P(50),c}$ of the population change at $t = 50$ with respect to a change of the parameter c is 2·20:

$$\epsilon_{P(50),c} = \frac{\Delta P(50)}{P(50)} \bigg/ \frac{\Delta c}{c} = \frac{193}{176} \bigg/ \frac{0\cdot125}{0\cdot250} = \frac{1\cdot10}{0\cdot50} = 2\cdot20 \ . \tag{27}$$

(f) The elasticity of $P(50)$ with respect to the information competition coefficient g equals that with respect to c:

$$\epsilon_{P(50),g} = \epsilon_{P(50),c} = 2\cdot20 \ , \tag{28}$$

because c and g always enter equation (25) as a product. The value of the product is changed equally if either c or g vary by a certain percentage. The parameter b_1 is included in the same product as far as this product appears in η. But in the nominator of the relation in the first term of the second row of equation (25) the product gc does not contain b_1. Therefore the elasticity of b_1 is not equal to the elasticities of g and c.
(g) For demographers the outcome may be interesting in that net migration depends on the birth surplus. This can be seen from table 3; a *rise* in the rate of natural increase r causes a *decline* in the cumulated net migrations from 11 to −3. But the influence of the labour market oriented parameters c and g on net migration is even greater; an *increase* in these parameters *increases* net migration.

Migration has two effects on population. The first is the direct impact of migration in the period $[0, t]$ on the number of inhabitants at time point t, denoted as 'cumulated net migration':

$$M_c^{\text{net}}(t) = \int_0^t M^{\text{net}}(\tau)\,\mathrm{d}\tau = \int_0^t M_e^{\text{in}}(\tau)\,\mathrm{d}\tau - \int_0^t M^{\text{out}}(\tau)\,\mathrm{d}\tau \ . \tag{29}$$

The second effect is the sum of various indirect consequences of the first effect on the number of births and deaths, denoted as $x(t)$.

The number of inhabitants in period t is the sum of these two effects plus the number of inhabitants due to natural increase of the inhabitants $P(0)$, that is, that population already living in the region in period $t = 0$ [see equation (13)]:

$$P(t) = P_n(t) + M_c^{\text{net}}(t) + \int_0^t x(\tau)\,\mathrm{d}\tau \ . \tag{29a}$$

If we denote the sum of the direct and indirect effects of migration by $M_c^{\text{net},*}$, we get from equation (29a)

$$M_c^{\text{net},*}(t) = M_c^{\text{net}}(t) + \int_0^t x(\tau)\,\mathrm{d}\tau = P(t) - P_n(t) \ . \tag{29b}$$

As the direct effect $M_c^{\text{net}}(t)$ can be computed on the basis of the solution given in equation (24), the indirect effect can be derived easily. From equation (29b)

$$\int_0^t x(\tau)\,\mathrm{d}\tau = P(t) - P_n(t) - M_c^{\text{net}}(t) \ . \tag{29c}$$

6 Extension of the model to two and more regions

Two-region models are more general than is superficially implied: if a country is divided into two subregions and if there is a model which describes the population development of the two parts, then the parts may be divided again into two parts, and so on—so that an n-region model can be derived by successively applying the principles of a two-region model.

One of the most important elements of any two-region model is the migration function. According to the type of the function used two-region models can be separated into three classes:

6.1 *The Keyfitz model*

Keyfitz (1980) developed a two-region model of the type:

$$P_1(t) = P_1(0) + \int_0^t rP_1(\tau)\,\mathrm{d}\tau - \int_0^t mP_1(\tau)\,\mathrm{d}\tau \ , \tag{30}$$

where $P_1(t)$ is the population in region 1 [with initial stock $P_1(0)$], r is the rate of natural increase, and m the rate of net migration from region 1 to region 2, which means that there is a net outflow ($m > 0$). The solution of equation (30) is

$$P_1(t) = P_1(0)\exp(r-m)t \ . \tag{31}$$

The rate of natural increase is assumed to be the same in both regions. The population function for region 2 is given by the difference between total population and that of region 1:

$$P_2(t) = [P_1(0)+P_2(0)]\exp(rt) - P_1(0)\exp(r-m)t \ . \tag{32}$$

In equation (32) the sum $P_1(0)+P_2(0)$ is the initial total population in both regions which develops at the constant rate of natural increase r.

The main characteristic of the Keyfitz model is that net migration from region 1 to region 2, denoted by $M_1^{\mathrm{net}}(t)$, is a function of the population of region 1 only:

$$M_1^{\mathrm{net}}(t) = -mP_1(t) \ , \qquad m > 0 \ . \tag{33}$$

A more realistic assumption would be to make the migration flows from region 1 to region 2 (M_{12}) and from region 2 to region 1 (M_{21}), which are included in the variable M_1^{net} by

$$M_1^{\mathrm{net}} = M_{21}(t) - M_{12}(t) = -mP_1(t) \ , \tag{33a}$$

a function of the population in the region of origin as well as of the population in the region of destination.

6.2 *The Rogers model*

Rogers (1968) developed a two-region components-of-change model which was reformulated by Ledent (1978a; 1978b; 1978c). The reformulated model contains explicit specifications of functions for the in-migration and out-migration flows.

Migration from region 1 to region 2 is a linear function of the population in the region of *origin*:

$$M_{12}(t) = m_{12}P_1(t) \ , \qquad m_{12} > 0 \ , \tag{34}$$

$$M_{21}(t) = m_{21}P_2(t) \ , \qquad m_{21} > 0 \ . \tag{35}$$

In this model it need not be assumed that the rates of natural increase r_1 and r_2 in the two regions are equal. If $r_1 \neq r_2$ the basic equations are:

$$P_1(t) = P_1(0) + \int_0^t r_1P_1(\tau)\,\mathrm{d}\tau + \int_0^t m_{21}P_2(\tau)\,\mathrm{d}\tau - \int_0^t m_{12}P_1(\tau)\,\mathrm{d}\tau \ , \tag{36}$$

$$P_2(t) = P_2(0) + \int_0^t r_2P_2(\tau)\,\mathrm{d}\tau + \int_0^t m_{12}P_1(\tau)\,\mathrm{d}\tau - \int_0^t m_{21}P_2(\tau)\,\mathrm{d}\tau \ . \tag{37}$$

The model has the following analytical solution which is discussed in detail by Ledent (1978b, page 25):

$$P_1(t) = A\exp(x_1t) - B\exp(x_2t) , \tag{38}$$

$$P_2(t) = C\exp(x_1t) - D\exp(x_2t) , \tag{39}$$

where A, B, C, D, x_1 and x_2 are constants.

From equations (34) and (35) the net migration functions are obtained as:

$$M_1^{\text{net}}(t) = M_{21}(t) - M_{12}(t) = m_{21}P_2(t) - m_{12}P_1(t) , \tag{40}$$

$$M_2^{\text{net}}(t) = -M_1^{\text{net}}(t) = m_{12}P_1(t) - m_{21}P_2(t) . \tag{41}$$

If the net migration function of the Keyfitz model [equation (33)] is compared with the corresponding equation (40) of the Rogers model it can be seen that the Rogers model is more general because the net migration function includes both population variables. But if the flow functions (34) and (35) instead of the net migration functions are examined, the underlying assumption that the migration flow between two regions is determined by the population in the region of origin, but not by that in the region of destination, is certainly open to criticism.

6.3 *An alternative model*

The specification of a more realistic migration function can be based on the following considerations:

(1) A necessary condition for any move from region i to region j is that potential migrants compare the relative advantages of the competing regions i and j. Therefore the number of comparisons made between regions i and j per time period, denoted as N_{ij}^{comp}, is an upper limit for the number of migrants:

$$M_{ij} \leqslant N_{ij}^{\text{comp}} . \tag{42}$$

(2) The number of comparisons per time period is a function of two variables, first, the number of people who compare their present state, and, second, the number of alternative opportunities in the region of destination with which the present state is compared. The first variable is assumed to be a function f of the number of people living in the region of origin, $f_i(P_i)$. The second variable is assumed to be a function of the number of compared jobs in the region of destination; as most people earn their living by working, most migrants need a job in the region of destination (except the special cases already mentioned, for example, commuters, students, and dependents). The number of compared jobs in turn is assumed to be a function q_j of the opportunity variable $Q_j(t)$ which quantifies the number of jobs becoming vacant as a result of the creation of new jobs, as a result of the matching process, or of the retirement of employees in region j. As pointed out in the description of equation (9) the variable $Q_j(t)$ is not quantified as the difference between the demand and supply of jobs. To reduce the number of endogenous variables, the variable $Q_j(t)$ can be replaced by the variable $P_j(t)$ by assuming that the variable $Q_j(t)$ is a linear function of the total number of jobs $A_j(t)$—as in equation (9)—and that $A_j(t)$ is in turn a linear function of the number of inhabitants:

$$q_j[Q_j(t)] = q_j[c_jA_j(t)] . \tag{43a}$$

In equation (43a), c_j is the matching parameter introduced in equation (9). If $A_j(t)$ is assumed to be a linear function of $P_j(t)$, the activity rate $\xi_j(t)$, defined by

$$\xi_j(t) = \frac{A_j(t)}{P_j(t)} , \qquad \xi_j > 0 , \tag{43b}$$

can be used to rewrite equation (43a) as a function of $P_j(t)$: viz

$$q_j[Q_j(t)] = q_j[c_j\xi_jP_j(t)] = q_j[\gamma_jP_j(t)] \,, \tag{43c}$$

where ξ_j is a constant, and γ_j is an abbreviation for the product $c_j\xi_j$. As γ_j depends on the matching parameter c_j, the parameter γ_j may also be denoted by the term *matching parameter* or by the term *job vacancy parameter*.

To specify the functions f_i and q_j further let it be assumed that f_i and q_j are exponential functions:

$$f_i[P_i(t)] = P_i(t)^{\beta^*}, \qquad \beta^* > 0 \,, \tag{43d}$$

$$q_j[\gamma_jP_j(t)] = [\gamma_jP_j(t)]^{\alpha^*}, \qquad \alpha^* > 0 \,, \tag{43e}$$

so that the number of comparisons can be specified by means of the function ψ:

$$N_{ij}^{\mathrm{comp}} = \psi[\gamma_jP_j(t)^{\alpha^*}, P_i(t)^{\beta^*}] \,. \tag{43f}$$

The form of ψ is clearly multiplicative because the number of comparisons is the product of the number of people making comparisons and the number of opportunities with which they compare their present state, so that

$$N_{ij}^{\mathrm{comp}} = [\gamma_jP_j(t)]^{\alpha^*}P_i(t)^{\beta^*} \,. \tag{44}$$

From equation (42):

$$M_{ij}(t) \leqslant [\gamma_jP_j(t)]^{\alpha^*}P_i(t)^{\beta^*} \,. \tag{45}$$

Let it be assumed that the numerical values of the parameters α^* and β^* are adjusted (reduced) in such a way that the equality sign in equation (45) can be used. Then the migration function $M_{ij}(t)$,

$$M_{ij}(t) = [\gamma_jP_j(t)]^{\alpha}P_i(t)^{\beta} \,, \qquad \alpha, \beta > 0 \,, \tag{46}$$

corresponding to the gravity model function,

$$M_{ij}(t) = \frac{P_j(t)^{\pi_1}P_i(t)^{\pi_2}}{(D_{ij})^{\pi_3}} \,, \tag{46a}$$

is obtained, with D_{ij} denoting the distance between regions i and j. Since in the two-region case this 'distance' can be regarded as a constant, the reduction $\alpha < \alpha^*$ and $\beta < \beta^*$ can be performed in such a way that the denominator in equation (46a) can be omitted. Then the only difference between model (46) and the gravity model is that equation (46) contains an additional parameter, namely the matching parameter γ.

On the basis of equation (46), an alternative to the two-region models of Keyfitz and Rogers can be formulated as follows:

$$P_1(t) = P_1(0) + \int_0^t r_1P_1(\tau)\,\mathrm{d}\tau + \int_0^t [\gamma_1P_1(\tau)]^{\alpha}P_2(\tau)^{\beta}\mathrm{d}\tau - \int_0^t [\gamma_2P_2(\tau)]^{\alpha}P_1(\tau)^{\beta}\mathrm{d}\tau \,, \tag{47}$$

$$P_2(t) = P_2(0) + \int_0^t r_2P_2(\tau)\,\mathrm{d}\tau + \int_0^t [\gamma_2P_2(\tau)]^{\alpha}P_1(\tau)^{\beta}\mathrm{d}\tau - \int_0^t [\gamma_1P_1(\tau)]^{\alpha}P_2(\tau)^{\beta}\mathrm{d}\tau \,. \tag{48}$$

These equations cannot be solved analytically in an easy manner. But in the case of equal natural increase ($r_1 = r_2$) it is possible to derive important characteristics of the solution, especially with regard to the two questions:

(a) Which of the two regions gains and, therefore, which one loses population?

(b) If there exists an equilibrium distribution, what is the relative population in the two regions?

If there exists an equilibrium for the population distribution, the relative growth rates of the two regions must be equal:

$$\frac{dP_1(t)}{dt}\Big/P_1(t) = \frac{dP_2(t)}{dt}\Big/P_2(t) . \tag{49}$$

Since population change is the sum of natural increase and net migration, equation (49) can be formulated as

$$\frac{r_1P_1(t)+M_1^{\text{net}}(t)}{P_1(t)} = \frac{r_2P_2(t)+M_2^{\text{net}}(t)}{P_2(t)} . \tag{50}$$

In the case of equal rates of natural increase ($r_1 = r_2$) condition (50) reduces to

$$\frac{M_1^{\text{net}}(t)}{P_1(t)} = -\frac{M_1^{\text{net}}(t)}{P_2(t)} , \tag{50a}$$

because $M_1^{\text{net}}(t) = -M_2^{\text{net}}(t)$. Condition (50a) can only be satisfied if net migration is zero. By use of equation (46) this means that

$$M_1^{\text{net}}(t) = \gamma_1^\alpha P_1(t)^\alpha P_2(t)^\beta - \gamma_2^\alpha P_2(t)^\alpha P_1(t)^\beta = 0 . \tag{50b}$$

In equation (50b) $P_2(t)$ can be replaced by use of the definition that the sum of $P_1(t)$ and $P_2(t)$ equals total population $P_0(t)$:

$$M_1^{\text{net}}(t) = \gamma_1^\alpha[P_0(t)-P_1(t)]^\alpha\{\gamma^\alpha[P_0(t)-P_1(t)]^{\beta-\alpha} - \gamma_2^\alpha P_1(t)^{\beta-\alpha}\} = 0 . \tag{50c}$$

To answer the question whether there exists a terminating stable equilibrium the conditions for which net migration in equation (50c) is zero have to be sought. The variable $M_1^{\text{net}}(t)$ is zero if the first or the second bracket in equation (50c) is zero. The first is zero if $P_1 = P_0$. The second if

$$\gamma_1^\alpha[P_0(t)-P_1(t)]^{\alpha-\beta} - \gamma_2^\alpha P_1(t)^{\beta-\alpha} = 0 . \tag{51}$$

Solving this equation gives the solution

$$P_1(t) = \lambda P_0(t) ,$$

where

$$\lambda = \frac{1}{(\gamma_2/\gamma_1)^{\alpha/(\beta-\alpha)}+1} = \frac{\gamma_1^{\alpha/(\beta-\alpha)}}{\gamma_1^{\alpha/(\beta-\alpha)}+\gamma_2^{\alpha/(\beta-\alpha)}} . \tag{52}$$

This means that as long as the ratio of population in region 1 to total population is lower (higher) than λ, there will be a positive (negative) net migration: that is, if $P_1(t) < \lambda P_0(t)$ then $M_1^{\text{net}} > 0$, and if $P_1(t) > \lambda P_0(t)$ then $M_1^{\text{net}} < 0$. This can be seen by analysing the brackets in equation (50c); the first bracket is always positive if $P_1 \neq P_0$. The second increases as $P_1(t)$ decreases, hence net migration will be positive for a small and negative for a high $P_1(t)$, becoming zero at the turning point when $P_1(t) = \lambda P_0(t)$.

The characteristics of the model may be summarised as follows:

(1) There exists a stable equilibrium population distribution in the long run (say for $t = t^*$) with the following shares in the regions:

$$\frac{P_1(t^*)}{P_0} = \lambda , \qquad \frac{P_2(t^*)}{P_0} = 1-\lambda . \tag{53}$$

The shares λ and $1-\lambda$ are, according to equation (52), a function of the parameters of the migration function only—regardless of the initial populations $P_1(0)$ and $P_2(0)$ (both nonzero).

(2) The equilibrium situation depends directly on an exponent of the relative value of the two job vacancy parameters γ_1 and γ_2:

$$\frac{P_1(t^*)}{P_2(t^*)} = \left(\frac{\gamma_1}{\gamma_2}\right)^{\alpha/(\beta-\alpha)} . \qquad (54)$$

This means that the higher the job vacancy parameter γ of a region, the higher will be its relative share of population in the equilibrium situation.

(3) In the special case $\alpha = \beta$ and $\gamma_1 = \gamma_2$ the net migration is zero and is independent of the population distribution [see equation (50b)]. This means that only under these special conditions are the population shares of the regions constant with time, so that the equilibrium distribution is equal to the initial distribution. In all other cases the population distribution will change towards an equilibrium which is determined by the relative magnitudes of the respective job vacancy parameters.

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