THE STRUCTURE OF BALANCED RINGS

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The present paper provides a complete description of left balanced rings. Recall that a ring is called *left balanced* if every left R-module M is balanced, i.e. if, for any left R-module M, the natural homomorphism of R to the double centralizer of M is surjective. In [4], left balanced rings have been shown to be left artinian. Hence, by [7], a ring is left balanced if and only if it is a finite product of full matrix rings over local left balanced left artinian rings. Consequently, the study of left balanced rings is reduced to that of left artinian local rings.

Throughout the rest of the paper, R denotes a local left artinian ring, W is its radical, and $Q = R/W$ is the residue division ring. There is a unique simple left R-module ${}_RQ$, and E denotes its injective envelope (so E is the only indecomposable injective left R-module). Then the main result of the present paper is the following:

STRUCTURE THEOREM. *R is left balanced if and only if either* (a) *R is uniserial or* (b) $W^2 = 0$, *${}_RW$ is simple, W_R has length* 2, *and E has length* 3, *or* (c) $W^2 = 0$, *W_R is simple, ${}_RW$ has length* 2, *and E has length* 2.

Here, R is called *uniserial*, if all its left ideals and all its right ideals are linearly ordered by inclusion. The rings described under (b) and (c) will be called *exceptional*. As a matter of fact, it will be shown (Theorem 2.3) that a local ring R satisfies the conditions (b) if and only if its opposite R^* satisfies the condition (c). Consequently, an (arbitrary) ring is left balanced if and only if it is right balanced. As a by-product of our structural investigations in [4] and the present paper, a left artinian ring A is shown to be balanced if and only if every finitely generated left A-module is balanced (Remark 3.7).

In this paper, we are concerned only with the necessity of the conditions. The sufficiency of (a) is well known and the sufficiency of (b) and (c) was essentially proved in [5].

If $Q = R/W$ is finitely generated over its centre, then R is exceptional if and only if $W^2 = 0$ and $\partial\, {}_RW \times \partial W_R = 2$ (Theorem 5.2). This is a consequence of the fact that a division subring of a division ring which is finitely generated over its centre, is of left index 2 if and only if it is

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of right index 2. The existence of non-exceptional local rings satisfying the conditions $W^2 = 0$ and $\partial_R W \times \partial W_R = 2$ is equivalent to the existence of a division ring with an isomorphic subring of right index 2 and left index different from 2 (Theorem 5.3).

Finally, a categorical characterization of balanced rings is given in Theorem 4.1: a left artinian ring A is balanced if and only if the composition factors of each indecomposable left A-module are isomorphic, if every such A-module of length greater than 3 is uniserial, and any two indecomposable A-modules of a given length with isomorphic composition factors are isomorphic. In fact, a balanced ring has only finitely many isomorphism types of indecomposable modules (Theorem 4.2).

1. Preliminaries

Throughout the paper, the terminology and notation of [**4**] will be used as well as the notations R, W, Q, and E fixed already. In particular, if A is a ring with unity, A^* denotes its opposite. By an A-module we always understand a unital A-module; the symbols ${}_AM$ or M_A will be used to underline the fact that M is a left or a right A-module, respectively; ∂M will denote the length of M. It should be noted that homomorphisms always act on the side opposite to that of the operators; in particular, every left A-module M defines a right $\mathscr{C}$-module $M_{\mathscr{C}}$, where $\mathscr{C}$ is the centralizer of the A-module M.

We need three known results. In [**4**], we proved

THEOREM 1.1. *Let R be left balanced. Then*
either (i) *R is left uniserial,*
or (ii) *$W^2 = 0$, and $\partial_R W = 2$,*
or (iii) *$W^3 = 0$, $\partial_R(W/W^2) = 2$, and W^2 is the unique minimal left ideal.*

Conversely, in [**5**], certain local rings with $W^2 = 0$ were shown to be left balanced. There, Q was supposed to be commutative; however, this was used only to calculate the indecomposable injective modules. If we assume that E has the appropriate properties, then §§ 3, 4, and 5 of [**5**] give the following two descriptions of R-modules (as well as proving that exceptional rings are balanced).

THEOREM 1.2. *Let $W^2 = 0$, ${}_RW$ be simple, W_R be of length 2, and the injective envelope E of ${}_RQ$ be of length 3. Then*

(a) *every indecomposable left module is either simple, injective or isomorphic to ${}_RR$, and every left module is a direct sum of these indecomposable modules;*
(b) *R is left balanced.*

THEOREM 1.3. *Let* $W^2 = 0$, W_R *be simple,* ${}_RW$ *be of length* 2, *and the injective envelope* E *of* ${}_RQ$ *be of length* 2. *Then*

(a) *every indecomposable left module is monogenic and either simple, injective, or isomorphic to* ${}_RR$, *and every left module is a direct sum of these indecomposable modules;*

(b) R *is left balanced.*

In our investigations, the following two results on balanced modules will be required. We recall that an indecomposable R-module is said to *have large kernels* if, for every endomorphism φ of M, either $\operatorname{Ker}\varphi = 0$ or $\operatorname{Soc} M \subseteq \operatorname{Ker}\varphi$.

LEMMA 1.4. *Let* M *be a balanced indecomposable left* R*-module of finite length and* $m \in M$ *such that* $\operatorname{Ann}(m) = 0$. *Then*

(i) *denoting by* $\mathscr{C}$ *the centralizer of* M, $m\mathscr{C} = M$, *and*

(ii) *if, moreover,* M *has large kernels, then* $\operatorname{Soc} M \subseteq Rm$.

Proof. Let $\mathscr{W}$ be the radical of $\mathscr{C}$. Since M is of finite length, $\mathscr{W}$ is nilpotent and $M_{\mathscr{C}}$ has a non-trivial socle $\operatorname{Soc} M_{\mathscr{C}}$ and a non-trivial radical $M\mathscr{W}$ (see [1], Exercise 3, pp. 26–27). Now, $M/(m\mathscr{C}+M\mathscr{W})$ and $\operatorname{Soc} M_{\mathscr{C}}$ are semisimple right $\mathscr{C}$-modules. Moreover, the local ring $\mathscr{C}$ has only one isomorphism type of simple modules, and therefore, if we show that any $\mathscr{C}$-homomorphism Ψ of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon} M/(m\mathscr{C}+M\mathscr{W}) \longrightarrow \operatorname{Soc} M_{\mathscr{C}} \xrightarrow{\iota} M_{\mathscr{C}}$$

(where ε is the canonical epimorphism and ι the embedding) is trivial, then we have

$$M = m\mathscr{C}+M\mathscr{W}.$$

But M is balanced, so $\Psi x = rx$ for some $r \in R$ and any $x \in M$. It follows from $rm = \Psi m = 0$ and $\operatorname{Ann}(m) = 0$ that $r = 0$; therefore Ψ is trivial. The equality $M = m\mathscr{C}+M\mathscr{W}$ yields

$$M = m\mathscr{C}+(m\mathscr{C}+M\mathscr{W})\mathscr{W} = m\mathscr{C}+M\mathscr{W}^2$$

and, by induction,

$$M = m\mathscr{C}+M\mathscr{W}^n.$$

Since $\mathscr{W}$ is nilpotent, $M = m\mathscr{C}$. This proves (i).

Since M has large kernels, every element $\varphi \in \mathscr{W}$ satisfies $(\operatorname{Soc}_R M)\varphi = 0$, and therefore $\operatorname{Soc}_R M \subseteq \operatorname{Soc} M_{\mathscr{C}}$. Also, $m \notin M\mathscr{W}$; otherwise $m\mathscr{C}$ would belong to $M\mathscr{W}$, but $M\mathscr{W} \neq M$. This implies that, for any element $x \in \operatorname{Soc}_R M$, we can find a $\mathscr{C}$-homomorphism Ψ of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon'} M/M\mathscr{W} \longrightarrow \operatorname{Soc} M_{\mathscr{C}} \xrightarrow{\iota} M_{\mathscr{C}},$$

mapping m onto x. But Ψ is induced by left multiplication and so there is $r \in R$ with $rm = x$. The proof of Lemma 1.4 is complete.

The next result corresponds to Construction II of [4]; it implies that modules of equal length over a local left balanced, left artinian ring are isomorphic if they have simple socles.

LEMMA 1.5. *Let M_1 and M_2 be two faithful left R-modules such that, for $i \neq j$, every homomorphism $\varphi\colon M_i \to M_j$ satisfies* $(\operatorname{Soc} M_i)\varphi = 0$. *Then $M_1 \oplus M_2$ is not balanced.*

Proof. We represent the elements of the centralizer of M by matrices

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}, \quad \text{where } \varphi_{ij}\colon M_i \to M_j.$$

Take a non-zero element z of $\operatorname{Soc}_R R$, and define an additive homomorphism $\Psi\colon M \to M$ by

$$\Psi(m_1, m_2) = (zm_1, 0) \quad \text{for } (m_1, m_2) \text{ in } M_1 \oplus M_2.$$

Now, zm_i belongs to $\operatorname{Soc} M_i$; so, for $i \neq j$, we have

$$z(m_i \varphi_{ij}) = (zm_i)\varphi_{ij} = 0.$$

This implies that Ψ belongs to the double centralizer of $M_1 \oplus M_2$, because

$$[\Psi(m_1, m_2)] \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} = (zm_1\varphi_{11}, zm_1\varphi_{12}) = (zm_1\varphi_{11} + zm_2\varphi_{21}, 0)$$

$$= \Psi\left[(m_1, m_2)\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}\right].$$

Assuming that Ψ is induced by left multiplication by $\rho \in R$, the equation $(zm_1, 0) = (\rho m_1, \rho m_2)$ for all $m_i \in M_i$ implies $\rho = 0$, because M_2 is faithful. But $zM_1 \neq 0$, because M_1 is faithful. Lemma 1.5 follows.

2. Exceptional rings

The aim of § 2 is to make a detailed study of the 'exceptional' rings defined in the Introduction to be those R which satisfy conditions (b) or (c) of the Structure Theorem. We assume throughout § 2 that $W^2 = 0$.

The division ring $Q = R/W$ operates on W both from the left and the right and we get two vector spaces ${}_QW$ and W_Q. If w is a non-zero element of W, we define a subring S_w of R by

$$S_w = \{s \in R \mid sw \in wR\}.$$

Obviously, since $Ww = 0$, $W \subseteq S_w$. Moreover, if s is a unit belonging to S_w, then the equality $sw = wr$ (for a suitable $r \in R$) implies $s^{-1}w = wr^{-1}$, and thus $s^{-1} \in S_w$ as well. Consequently, S_w/W is a division subring of $Q = R/W$. If λ is a unit of R, then

$$S_{\lambda w} = \{s \in R \mid s\lambda w \in \lambda wR\} = \{s \in R \mid \lambda^{-1}s\lambda w \in wR\} = \lambda S_w \lambda^{-1}$$

and thus, in particular,

$$\dim_{(S_w/W)}Q = \dim_{(S_{\lambda w}/W)}Q.$$

Similarly, we define the subring T_w of R:

$$T_w = \{t \in R \mid wt \in Rw\}.$$

Again, $W \subseteq T_w$, T_w/W is a division subring of Q, and, for an arbitrary unit ρ of R,

$$\dim Q_{(T_w/W)} = \dim Q_{(T_{w\rho}/W)}.$$

PROPOSITION 2.1. *Let* $\dim_Q W = 1$ *and* $\dim W_Q = 2$. *Then the following statements are equivalent:*

(i) $\dim_{(S_w/W)}Q = 2$;

(ii) *there exist two linearly independent elements* v, w *of* W_Q *such that* $W = wR + S_w v$;

(iii) R *is exceptional.*

Proof. In order to prove that (i) implies (ii), let w be a non-zero element of W, $S = S_w$, and let $\dim_{(S/W)}Q = 2$. Thus, there exists $r \in R\backslash S$ such that $R = S + Sr + W$. Putting $v = rw$, one gets

$$W = Rw = (S + Sr + W)w = Sw + Sv = wR + Sv,$$

because $\dim_Q W = 1$ implies $Sw = wR$.

Now, to establish that (ii) implies (iii), we are going to show that

$$M = {}_R(R \oplus R)/D, \quad \text{where } D = \{(\lambda v, \lambda w) \mid \lambda \in R\},$$

is an indecomposable injective left R-module, hence a copy of E. To prove injectivity of M assume that a homomorphism $\varphi\colon {}_RW \to M$ is given; we are required to extend it to a homomorphism from ${}_RR$ to M. Obviously, φ is determined by the image of w; and, since $w\varphi \in WM$, we can find $w_1, w_2 \in W$ such that

$$w\varphi = (w_1, w_2) + D.$$

But, $w_2 = \lambda w$ and thus, for some $w_0 \in W$,

$$(w_1, w_2) + D = (w_1 - \lambda v, 0) + (\lambda v, \lambda w) + D = (w_0, 0) + D.$$

Now $w_0 \in wR + S_w v$, and therefore there are elements $r_1 \in R$, $s \in S_w$, and $r_2 \in R$ such that

$$w_0 = wr_1 + sv \quad \text{and} \quad sw = -wr_2.$$

We claim that the homomorphism

$$_R R \xrightarrow{(r_1, r_2)} {_R(R \oplus R)} \xrightarrow{\ \varepsilon\ } M,$$

where ε is the canonical epimorphism, is an extension of φ. Indeed, the element w is mapped into

$$w(r_1, r_2) + D = (w_0 - sv,\ -sw) + D = (w_0, 0) + D = w\varphi,$$

as required. Consequently, M is injective and, being of length 3, necessarily indecomposable. So E also has length 3.

To complete the proof, let us verify that (iii) implies (i). An indecomposable injective left R-module M of length 3 is necessarily an amalgam of two copies of $_R R$ over its socle. Thus,

$$M = {_R(R \oplus R)}/D \quad \text{with } D = \{(\lambda v, \lambda w) \mid \lambda \in R\}$$

for suitable v and w of W. Now, take an arbitrary $x \in W$ and consider the homomorphism $\varphi\colon {_R W} \to M$ mapping w into $(x, 0) + D$. Extend φ to a homomorphism from $_R R$ to M and lift the latter to

$$_R R \xrightarrow{(r_1, r_2)} {_R(R \oplus R)}.$$

Hence,

$$(wr_1, wr_2) - (x, 0) \in D,$$

and thus

$$(wr_1 - x, wr_2) = (\lambda v, \lambda w) \quad \text{for some } \lambda \in R.$$

Therefore, writing $S_w = S$,

$$\lambda \in S \quad \text{and} \quad x = wr_1 - \lambda v \in wR + Sv.$$

Thus

$$Rw = W = wR + Sv = Sw + Sv = (S + Sr)w$$

with a suitable $r \in R$, and hence

$$R = S + Sr + W.$$

Now, $r \notin S$; for, otherwise $W = wR + Srw = wR$ in contradiction to $\dim W_Q = 2$. As a consequence, $\dim {_{(S/W)}Q} = 2$, as required. The proof of Proposition 2.1 is complete.

PROPOSITION 2.2. *Let* $\dim {_Q W} = 2$ *and* $W_Q = 1$. *Then the following statements are equivalent:*

(i) $\dim Q_{(T_w/W)} = 2$;

(ii) *there exist two linearly independent elements* v, w *of* $_Q W$ *such that*

$$W = Rw + vT_w;$$

(iii) R *is exceptional.*

Proof. Both statements (i) and (ii) are dual to those of Proposition 2.1, and thus they are equivalent. In order to show that (ii) implies (iii), let v, w be the elements given in (ii). We are going to prove that R/Rw is an indecomposable injective R-module, hence a copy of E. To this end, let $\varphi\colon {}_RW \to R/Rw$ be a non-zero homomorphism. Since right multiplication by elements of R is transitive on W, we can evidently assume that $\operatorname{Ker}\varphi = Rw$. Thus, φ is determined by the conditions $w\varphi = 0$ and $v\varphi = w_0 + Rw$ for a suitable $w_0 \in W$. In view of the relation $W = Rw + vT_w$, we have

$$w_0 = rw + vt \quad \text{for some } r \in R \text{ and } t \in T_w.$$

Consequently, the homomorphism

$${}_RR \xrightarrow{\ t\ } {}_RR \xrightarrow{\ \varepsilon\ } R/Rw$$

maps Rw into 0, v into $w_0 - rw + Rw = w_0 + Rw$, and is thus an extension of φ to ${}_RR$. Therefore R/Rw is injective. It is clearly indecomposable, and clearly of length 2. So E also has length 2.

Finally, we verify that (iii) implies (i). Let M be an indecomposable injective left R-module of length 2; hence $M \cong R/Rw$ for some non-zero element $w \in W$. Let $v \in {}_QW$ so that v and w are linearly independent. Take an arbitrary element $x \in W$ and consider the homomorphism $\varphi\colon {}_RW \to M$ such that

$$v\varphi = x + Rw \quad \text{and} \quad w\varphi = 0.$$

Since M is injective, φ can be extended to a homomorphism from ${}_RR$ to M, and therefore lifted to ${}_RR \xrightarrow{\ r\ } {}_RR$. From here, it follows that $wr \in Rw$ and thus $r \in T_w$; moreover, $x - vr \in Rw$. Consequently,

$$W = Rw + vT_w,$$

which fact completes the proof of Proposition 2.2.

THEOREM 2.3. *A ring R is exceptional if and only if its opposite R^* is exceptional.*

Proof. This follows immediately from the fact that a ring satisfies the conditions of Proposition 2.1 (i) if and only if its opposite satisfies the conditions of Proposition 2.2 (i).

We conclude this section by remarking that the existence of exceptional rings will be shown in § 5. There we shall also consider the conditions under which a local ring R with

$$W^2 = 0 \quad \text{and} \quad \dim {}_QW \times \dim W_Q = 2$$

is exceptional. It will be shown that this question is equivalent to deciding whether certain division subrings of Q have left index equal to right index.

3. The structure theorem

First, we give a complete description of left balanced local rings R with $W^2 = 0$.

LEMMA 3.1. *Let* $W^2 = 0$, $\dim_Q W = 1$, *and* $\dim W_Q \geqslant 3$. *Let* u, v, w *be linearly independent in* W_Q. *Then*

$$M = {}_R(R \oplus R \oplus R)/D, \quad \textit{where } D = \{(\lambda u, \lambda v, \lambda w) \mid \lambda \in R\},$$

is an indecomposable left R-module with large kernels.

Proof. First, there is no homomorphism of M onto ${}_RR$. For, assuming the converse, we get a homomorphism

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} : {}_R(R \oplus R \oplus R) \to {}_RR$$

such that D is mapped into 0. Thus $ur_1 + vr_2 + wr_3 = 0$ and in view of the linear independence of u, v, w in W_Q, all $r_i \in W$; therefore the homomorphism cannot be surjective.

Now, $\partial(\operatorname{Soc} M) = 2$. For, assume $\partial(\operatorname{Soc} M) > 2$, and consider the submodule $M' = {}_R(R \oplus R \oplus W)/D$ of M. Obviously, M' is isomorphic to ${}_R(R \oplus R)$, and therefore $\partial(\operatorname{Soc} M') = 2$. By assumption, we can find an element $m \in \operatorname{Soc} M \setminus \operatorname{Soc} M'$. Since $M' \cap Rm = 0$ and

$$\partial(M' \oplus Rm) = \partial(M') + 1 = 5 = \partial(M),$$

we conclude that M' is a direct summand of M. But this contradicts the fact that M has no epimorphic image isomorphic to ${}_RR$.

Since $\operatorname{Soc} M \supseteq [\operatorname{Soc}_R(R \oplus R \oplus R)]/D$ and since $\partial(\operatorname{Soc} M) = 2$, we get

$$\operatorname{Soc} M = [\operatorname{Soc}_R(R \oplus R \oplus R)]/D = [\operatorname{Rad}_R(R \oplus R \oplus R)]/D = \operatorname{Rad} M.$$

Moreover, there is no indecomposable submodule of M of length 3. For, an indecomposable left R-module N of length 3 has a simple socle, and thus N would intersect trivially either $M_1 = R[(1,0,0)+D]$ or $M_2 = R[(0,1,0)+D]$. Assuming $N \cap M_1 = 0$, we conclude, in view of $\partial M_1 = 2$ and $\partial M = 5$, that M_1 is a direct summand of M. But, M_1 is obviously isomorphic to ${}_RR$, and this contradicts the fact that M has no epimorphic image isomorphic to ${}_RR$. Therefore $N \cap M_1 \neq 0$. Similarly $N \cap M_2 \neq 0$. Therefore no such N can exist. Also, there cannot be an indecomposable submodule of M of length 4. For, such a submodule would have a simple socle and thus contain an indecomposable submodule of length 3.

Finally, since M has no indecomposable epimorphic image of length 2 and no indecomposable submodules of length 3 or 4, the image of every proper endomorphism φ of M is semisimple and therefore $(\operatorname{Rad} M)\varphi = 0$; consequently, $(\operatorname{Soc} M)\varphi = 0$ and so M is an indecomposable module with large kernels.

LEMMA 3.2. *Let R be left balanced, $W^2 = 0$, and $\dim_Q W = 1$. Then R is uniserial or exceptional.*

Proof. Assuming that R is left balanced, we want first to show that $\dim W_Q \leqslant 2$. Otherwise, there are three linearly independent elements u, v, w in W_Q, and according to Lemma 3.1,

$$M = {}_R(R \oplus R \oplus R)/D, \quad \text{where } D = \{(\lambda u, \lambda v, \lambda w) \mid \lambda \in R\},$$

is an indecomposable left R-module with large kernels. Since, obviously, $\operatorname{Ann}[(1,0,0)+D] = 0$, it follows from Lemma 1.4 (ii) that

$$\operatorname{Soc} M \subseteq R[(1,0,0)+D].$$

Taking $(0,u,0)+D \in \operatorname{Soc} M$, we get $(r_0,0,0)+D = (0,u,0)+D$ for a suitable $r_0 \in R$, and thus $(r_0, -u, 0) \in D$. This is impossible, and therefore $\dim W_Q \leqslant 2$.

If $\dim W_Q = 1$, R is uniserial. Therefore, we assume that $\dim W_Q = 2$. In order to prove that R is exceptional it is sufficient to show, in view of Proposition 2.1, that $W = wR + S_w v$ for two linearly independent elements v and w of W_Q. Since the R-module

$$N = {}_R(R \oplus R)/D \quad \text{with } D = \{(\lambda v, \lambda w) \mid \lambda \in R\}$$

is indecomposable and since, obviously,

$$\operatorname{Ann}[(1,0)+D] = 0, \quad N = [(1,0)+D]\mathscr{C},$$

by Lemma 1.4 (i). Therefore, taking an arbitrary $r \in R$, there is $\varphi \in \mathscr{C}$ such that

$$[(1,0)+D]\varphi = (r,0)+D.$$

Lifting φ to

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} : {}_R(R \oplus R) \to {}_R(R \oplus R),$$

we find that $(\alpha_{11} - r, \alpha_{12}) \in D$ and, in particular, $\alpha_{11} - r \in W$ and $\alpha_{12} \in W$. Also, applying this homomorphism to $(v, w) \in D$, we obtain

$$(v\alpha_{11} + w\alpha_{21}, v\alpha_{12} + w\alpha_{22}) = (vr + w\alpha_{21}, w\alpha_{22}) \in D.$$

Hence,

$$vr + w\alpha_{21} = \lambda v \quad \text{and} \quad w\alpha_{22} = \lambda w \quad \text{for some } \lambda \in R.$$

Therefore $\lambda \in S_w$ and $vr \in wR + S_w v$. Consequently

$$W = vR + wR \subseteq Wr + S_w v,$$

as required.

LEMMA 3.3. *Let* $W^2 = 0$, $\dim_Q W = 2$, *and* $\dim W_Q \geqslant 2$. *Let* u, v *be linearly independent elements in* W_Q. *Then*

$$M = {}_R(R \oplus R)/D, \quad \textit{where } D = \{(\lambda u, \lambda v) \mid \lambda \in R\},$$

is an indecomposable left R*-module with large kernels.*

Proof. First, let $\varphi\colon M \to {}_RR$. Then, lifting φ to $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\colon {}_R(R+R) \to {}_RR$ mapping D to 0, we find that $ur_1 + vr_2 = 0$. In view of the linear independence of u, v in W_Q, r_1 and r_2 belong to W and hence φ is not surjective. As an immediate consequence, we have that $\partial(\operatorname{Soc} M) = 3$. For, assume that $\partial(\operatorname{Soc} M) > 3$ and consider the submodule $M' = R[(1,0)+D]$ which is obviously isomorphic to ${}_RR$. Since $\partial(\operatorname{Soc} M') = 2$, we find a submodule M'' of $\operatorname{Soc} M$ with $M' \cap M'' = 0$ and $\partial M'' = 2$. But

$$\partial(M' \oplus M'') = \partial M' + \partial M'' = 5 = \partial M$$

implies that M' is a direct summand of M. This contradicts the fact that M has no epimorphic image isomorphic to ${}_RR$. Since $\partial(\operatorname{Soc} M) = 3$,

$$\operatorname{Soc} M = [\operatorname{Soc}_R(R \oplus R)]/D = [\operatorname{Rad}_R(R \oplus R)]/D,$$

and therefore $\operatorname{Soc} M = \operatorname{Rad} M$.

Now, M contains no monogenic submodule of length 2. For, assume that X is such a submodule. Again, since neither $R_1 = R[(1,0)+D]$ nor $R_2 = R[(0,1)+D]$ is a direct summand of M, X necessarily contains the simple submodule $X' = R[(u,0)+D] = R[(0,-v)+D] = R_1 \cap R_2$. Therefore the submodule $N = X + R_1$ satisfies $\partial(N/\operatorname{Rad} N) = 2$ and $\partial N = 4$. Since $\partial(M/N) = 1$, $R_2 \cong R$, and

$$M/N = (R_2 + N)/N \cong R_2/(R_2 \cap N),$$

it follows that $WR_2 = R_2 \cap N \subseteq N$. Hence $WR_1 + WR_2 \subseteq N$ and

$$\partial(\operatorname{Soc} N) = 3,$$

so that N is a direct sum of X and two copies of Q. This implies that $\partial(N/\operatorname{Rad} N) = 3$, in contradiction to the formula $\partial(N/\operatorname{Rad} N) = 2$ proved above. So no such N exists.

Also, M contains no indecomposable submodule of length 4. Assume that Y is a submodule of M of length 4. First, let $\partial(\operatorname{Soc} Y) = 3$. Then Y contains necessarily a copy of ${}_RR$, so that Y splits. Second, let $\partial(\operatorname{Soc} Y) = 2$. Then, since $\partial(\operatorname{Soc} M) = 3$, M is isomorphic to $Y \oplus Q$, and this is obviously incompatible with the previously established formula $\operatorname{Rad} M = \operatorname{Soc} M$.

Now, it follows easily that each proper endomorphism of M has semisimple image, hence kernel containing $\operatorname{Rad} M$. Since $\operatorname{Rad} M = \operatorname{Soc} M$, M has large kernels and is indecomposable.

LEMMA 3.4. *Let R be left balanced, $W^2 = 0$, and $\dim_Q W = 2$. Then R is exceptional.*

Proof. Assume that R is left balanced. First, we shall prove that $\dim W_Q = 1$. Assume the contrary and choose $v \in W$, $v \neq 0$. Since $\dim_Q W = 2$ and $\dim W_Q \geqslant 2$, both Rv and vR are proper subgroups of W and therefore their set-theoretical union is a proper subset of W. It follows that there is an element $w \in W$ which is neither in Rv nor in vR. Consider the left R-module

$$M = {}_R(R \oplus R)/D \quad \text{with } D = \{(\lambda v, \lambda w) \mid \lambda \in R\}.$$

Since v and w are linearly independent in W_Q, M is, according to Lemma 3.3, an indecomposable module with large kernels. Now, since $\mathrm{Ann}[(0,1)+D] = 0$, we may apply Lemma 1.4 (ii) and find that $\mathrm{Soc}\, M \subseteq R[(0,1)+D]$. Therefore, taking $(w,0)+D \in \mathrm{Soc}\, M$, there is $r_0 \in R$ such that $(w, -r_0) \in D$. Thus, in particular, $w = \lambda v$ for some $\lambda \in R$, in contradiction to $w \notin Rv$.

Now, to complete the proof, we want to show that $W = Rw + vT_w$. Consider the R-module $N = R/Rw$; let $\mathscr{C}$ be its centralizer. Obviously, the rings $\mathscr{C}$ and T_w/Rw are isomorphic, and thus $(Rw+vT_w)/Rw$ is a non-zero $\mathscr{C}$-submodule of N. Since $\mathscr{C}$ is local, all simple $\mathscr{C}$-modules are isomorphic. Thus, the fact that $\mathrm{Rad}\, N_{\mathscr{C}} \neq N$ and $\mathrm{Soc}\, N_{\mathscr{C}}$ is essential in $N_{\mathscr{C}}$ (see again [**1**]) implies that there is a non-zero $\mathscr{C}$-homomorphism $\Psi: N \to N$ mapping N into $(Rw+vT_w)/Rw$. Since N is a balanced R-module, Ψ is induced by the ring multiplication:

$$\Psi n = rn \text{ for all } n \in N \text{ with a suitable non-zero } r \in R.$$

Consequently,

$$(rR + Rw)/Rw = \Psi(R/Rw) \subseteq (Rw + vT_w)/Rw,$$

and therefore $rR \subseteq Rw + vT_w$. Since $\dim W_Q = 1$, $W = Rw + vT_w$, as required.

REMARK. Let us point out briefly that an alternative proof of the first part of Lemma 3.4 can run as follows: having proved that $\dim W_Q = 1$, one can deduce from Lemma B of [**2**] that $\dim Q_{(T_w/W)} \leqslant 2$. By Proposition 2.2, the latter is equivalent to the fact that R is exceptional.

We are ready to *complete the proof of the Structure Theorem* by showing:

THEOREM 3.5. *If R is left balanced then it is either uniserial or exceptional.*

Proof. Let W be the radical of R and $Q = R/W$. If $W^2 = 0$, it follows from (1.2) that $\dim_Q W \leqslant 2$. Therefore, by Lemmas 3.2 and 3.4, if R is left balanced then it is uniserial or exceptional.

Now, Theorem (3.5) would be established if we show that the only left balanced ring R with $W^2 \neq 0$ are the uniserial rings. Assume this is not so, that R is left balanced, that $W^2 \neq 0$, and that R is not uniserial. Since the ring R/W^2 is left balanced, non-uniserial and the square of its radical is zero, it follows that R/W^2 is exceptional. Thus, by Theorem (1, 1), there are two remaining possibilities to exclude: (a) $W^3 = 0$, $\dim_Q(W/W^2) = 2$ and $\dim_Q W^2 = 1$, and (b) R is left uniserial but $\dim(W/W^2)_Q = 2$.

Case (a). Assume that $W^3 = 0$, $\dim_Q(W/W^2) = 2$ and $\dim_Q W^2 = 1$. Thus ${}_RW$ is a module of length 3 with a simple socle. But ${}_RW$ can be considered as a left R/W^2-module. Since R/W^2 is an exceptional ring with $\dim_Q(W/W^2) = 2$, the indecomposable injective left R/W^2-module is of length 2. Therefore no left R/W^2-module of length 3 has a simple socle.

Case (b). Assume that R is left uniserial and that $\dim(W/W^2)_Q = 2$. We may obviously assume $W^3 = 0$. In order to show that this case cannot occur, we shall construct two faithful non-isomorphic R-modules M_1 and M_2 with simple socles and use Lemma 1.5 to exhibit a non-balanced R-module, namely $M_1 \oplus M_2$.

First, since $S = R/W^2$ is exceptional, also the opposite of S is exceptional, according to Theorem 2.3. Observing that ${}_R(W/W^2)$ is simple, we can apply (1.3) (a) to the opposite of S and conclude that every *right* S-module is a direct sum of monogenic modules. In particular, W itself can be considered as a right S-module and $\operatorname{Soc} W_S = (W^2)_S$. This follows from the fact that $\operatorname{Soc} W_S$ as an ideal of R is two-sided and cannot be equal to W_S, because $W^2 \neq 0$. Now, since W_S possesses a semisimple quotient $(W/W^2)_S$ of length 2, there are, in view of the above decomposition, elements x and y in W such that $xR \cap yR = 0$, with $x + W^2$, $y + W^2$ linearly independent in $(W/W^2)_S$. Moreover, since x and y do not belong to $W^2 = \operatorname{Soc} W_R$, $\partial(xR) = \partial(yR) = 2$.

Now consider

$$M_1 = {}_R(R \oplus R)/D_1, \quad \text{where } D_1 = \{(\lambda x, \lambda y) \mid \lambda \in R\}.$$

Obviously, $\partial M_1 = 4$. The left R-module M_1 has no monogenic quotient of length 2. For, given a homomorphism $\varphi\colon M_1 \to {}_R(R/W^2)$, we can lift it to a homomorphism $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\colon {}_R(R \oplus R) \to {}_RR$. Since D_1 is mapped into W^2, $xr_1 + yr_2$ belongs necessarily to W^2. But $x + W^2$ and $y + W^2$ are linearly independent in $(W/W^2)_{R/W}$ and therefore both xr_1 and yr_2 belong to W^2. Consequently, both r_1 and r_2 lie in W and hence φ cannot be surjective. From here it follows easily that $\operatorname{Soc} M_1$ is simple. For, obviously, $N' = {}_R(R\oplus W)/D_1 \cong {}_RR$ and thus $\partial(\operatorname{Soc} N') = 1$; therefore, if $\partial(\operatorname{Soc} M_1) \geqslant 2$, then there exists a simple submodule $N'' \subseteq M_1$ such that $N' \cap N'' = 0$. Consequently,

$$\partial(N' \oplus N'') = \partial N' + 1 = 4 = \partial M,$$

and thus $M_1 = N' \oplus N''$. This contradicts the fact that there is no monogenic quotient of length 2.

To construct M_2, take a non-zero element $z \in yR \cap W^2$; such an element exists because $\partial(yR) = 2$. Define the left R-module

$$M_2 = {}_R(R\oplus R)/D_2, \quad \text{where } D_2 = \{(\lambda x, \lambda z) \mid \lambda \in R\}.$$

Again $\partial M_2 = 4$. Also $\operatorname{Soc} M_2$ is simple. For, if $\operatorname{Soc} M_2$ is not simple, then an argument similar to the one given above shows that ${}_R(W^2 \oplus R)/D_2 \cong {}_RR$ is a direct summand of M_2. Thus M_2 possesses an epimorphic image which is a monogenic R-module of length 3. But a homomorphism $\varphi\colon M_2 \to {}_RR$ can be lifted to a homomorphism $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\colon {}_R(R\oplus R) \to {}_RR$ mapping D_2 into 0. Therefore, $xr_1 + zr_2 = 0$. This relation shows that both r_1 and r_2 belong to W and thus the homomorphism φ cannot be surjective.

Now, M_1 and M_2 are two non-isomorphic left R-modules of length 4. This follows from the fact that M_2 (in contrast to M_1) has a monogenic quotient of length 2, namely ${}_R(R\oplus R)/(R\oplus W^2)$. Consequently, any homomorphism between M_1 and M_2 must have a non-trivial kernel. Since both $\operatorname{Soc} M_1$ and $\operatorname{Soc} M_2$ are simple, such a homomorphism $\varphi\colon M_i \to M_j$ (with $i \neq j$) satisfies $(\operatorname{Soc} M_i)\varphi = 0$. Thus, applying Lemma 1.5, we get a non-balanced left R-module. This contradiction shows that case (b) cannot occur.

The proof of Theorem (3.5), and therefore of the Structure Theorem, is now complete.

Since the concepts of a uniserial ring, as well as that of an exceptional ring, are self-dual, we have the following important result.

COROLLARY 3.6. *A ring is left balanced if and only if it is right balanced.*

From now on, we refer simply to a balanced ring.

REMARK 3.7. *A left artinian ring A is balanced if and only if every finitely generated left A-module is balanced.*

Proof. If every finitely generated left A-module is balanced, then A is a finite direct sum of full matrix rings over local rings R_i. Otherwise, it follows from [**9**] that there are non-balanced modules of length 2. Now, the property of being finitely generated is invariant under Morita equivalence, and so finitely generated R_i-modules are balanced (see, for example, [**2**]). Therefore it is sufficient to prove our theorem for local rings; and for these, the theorem holds because each non-balanced module used in the proofs in [**4**] and in the present paper is generated by at most four elements.

The assumption in Remark 3.7 that the ring A be left artinian is essential. It is well known (see [**1**]) that every finitely generated abelian group is balanced, although the ring of all integers is not balanced.

4. The module category of a balanced ring

It is well known (and was used in the proof of the Structure Theorem) that the property of being balanced is invariant under Morita equivalence. Here, we characterize explicitly the balanced rings A in terms of the module categories ${}_AM$.

THEOREM 4.1. *The left artinian ring A is balanced if and only if the category ${}_AM$ of all left A-modules has the following properties:*

(i) *the composition factors of each indecomposable left A-module are isomorphic,*

(ii) *every indecomposable left A-module with length greater than* 3 *is uniserial, and*

(iii) *any two indecomposable left A-modules of a given length and with isomorphic composition factors are isomorphic.*

Proof. First, note that A is the direct sum of full matrix rings over local rings R_i if and only if condition (i) is satisfied in ${}_AM$. Then, ${}_AM$ satisfies (ii) and (iii) if and only if, for all i, the categories ${}_{R_i}M$ satisfy these conditions. Therefore, it is sufficient to prove Theorem 4.1 for a local left artinian ring $A = R$.

The necessity of the conditions follows immediately from Theorem 3.5 together with Theorems (1.2) and (1.3).

In order to prove the sufficiency, let us first assume that all indecomposable left R-modules are uniserial. Then R is trivially left uniserial. Also, R is right uniserial. For otherwise ${}_R(R\oplus R)/D$, where $D = \{(\lambda u,\lambda v) \mid \lambda \in R\} + (W^2 \oplus W^2)$ with linearly independent elements u, v in $(W/W^2)_Q$, is a non-uniserial indecomposable left R-module.

If there is a non-uniserial indecomposable R-module X of length 3 with a simple socle, then X is necessarily injective and R (being a monogenic indecomposable R-module) is left uniserial. Consequently, ${}_RR$ can be embedded in X and therefore $W^2 = 0$. By Lemma 3.1, $\dim W_Q = 2$; for, otherwise, there would exist an indecomposable non-uniserial R-module of length 5. Hence, R is exceptional and thus balanced.

If there is a non-uniserial indecomposable R-module Y of length 3 with a non-simple socle, then $Y/\mathrm{Rad}\, Y$ is simple and thus, necessarily, $Y \cong {}_RR$. Consequently, $W^2 = 0$, $\dim {}_QW = 2$ and the indecomposable injective is uniserial (and of length 2). By Lemma 3.3, $\dim W_Q = 1$; hence, R is again exceptional and therefore balanced.

THEOREM 4.2. *A balanced ring has only finitely many isomorphism types of indecomposable modules.*

Proof. A balanced ring is left artinian, therefore the length of the uniserial left A-modules is bounded. For any simple module S, all indecomposable left A-modules with composition factors isomorphic to S are uniserial or of length 3, and any two of them are isomorphic, if their lengths are equal. This follows from Theorem 4.1. Since there is only a finite number of non-isomorphic simple A-modules, the number of isomorphism types of indecomposable left modules is finite. By duality, also the number of isomorphism types of indecomposable right modules is finite.

5. The residue division ring

Recall that R denotes a local left artinian ring, W is its radical, and $Q = R/W$ is the residue division ring. In this section we seek conditions under which such a ring, with

$$W^2 = 0 \quad \text{and} \quad \partial\, {}_RW \times \partial W_R = 2,$$

is exceptional. It will be shown that this question is equivalent to deciding whether certain division subrings of Q have left index equal to right index (cf. **[10]**).

We may restrict to the case $\partial\, {}_RW = 1$. We start with the following lemma which calculates the right index of S_w/W in Q.

LEMMA 5.1. *Let $W^2 = 0$ and $\dim {}_QW = 1$. Let w be a non-zero element of W. Then S_w/W is isomorphic to Q and*

$$\dim Q_{(S_w/W)} = \partial W_R.$$

Proof. Write $S = S_w$ and define a morphism $\alpha\colon Q \to W$ by

$$\alpha(r+W) = rw \text{ for } r \in R.$$

Furthermore, define a morphism $\beta\colon S/W \to Q$ by

$$\beta(s+W) = t+W \text{ for } s \in S \text{ and } t \in R \text{ such that } sw = wt.$$

Then α and β are well-defined bijections because $\mathrm{Ann}(w) = W$ and $Rw = W$. Clearly, α and β are additive and β is easily verified to be multiplicative. Therefore β is a ring isomorphism. Now, the pair

$(\alpha, \beta): Q \times S/W \to W \times Q$ satisfies, for any $r \in R$, $s \in S$, and $t \in R$ with $sw = wt$,

$$\alpha[(r+W)(s+W)] = \alpha(rs+W) = rsw = rwt = rw(t+W)$$
$$= \alpha(r+W)\beta(s+W),$$

which implies the equation $\dim Q_{S/W} = \dim W_Q = \partial W_R$.

If R is a local ring with

$$W^2 = 0, \quad \partial_R W = 1 \quad \text{and} \quad \partial W_R = 2,$$

then Lemma 5.1 shows that $\dim Q_{S_w/W} = 2$. But R is exceptional if and only if $\dim Q_{S_w/W} = 2$. Therefore, if Q has the property that any division subring of right index 2, which is isomorphic to Q, has also left index 2, then the above conditions imply that R is exceptional. In particular, we have the following theorem generalizing the result of [**6**].

THEOREM 5.2. *Assume that R/W is finitely generated over its centre. Then R is balanced if and only if it is uniserial or*

$$W^2 = 0 \quad \text{and} \quad \partial_R W \times \partial W_R = 2.$$

Proof. If R/W is finite-dimensional over its centre, then a division subring of $Q = R/W$ has right index 2 if and only if it has left index 2 ([**8**], p. 158). So R is exceptional if and only if $W^2 = 0$ and $\partial_R W \times \partial W_R = 2$.

The existence of a local ring R, with $W^2 = 0$ and $\partial_R W \times \partial W_R = 2$, which is not exceptional would imply that the concept of a balanced ring involves not only the structure of the lattices of left and of right ideals, but also the embeddings of certain subrings.

THEOREM 5.3. *The following assertions are equivalent.*

(i) *There exists a local ring R with $W^2 = 0$ and $\partial_R W \times \partial W_R = 2$, which is not exceptional.*

(ii) *There exists a division ring D with a subring D' of right index* 2 *and of left index $\neq 2$, such that D and D' are isomorphic.*

Proof. In order to show that (i) implies (ii), we may assume that the given local ring R satisfies $\partial_R W = 1$. Otherwise, we consider the dual ring. Let $D = Q$, and $D' = S_w/W$ for some non-zero element w of W. Then, according to Lemma 5.1, D and D' are isomorphic and $\dim D_{D'} = 2$; but because R is not exceptional, $\dim_{D'} D \neq 2$.

Conversely, let D' be a subring of the division ring D such that there exists an isomorphism $\gamma: D \to D'$. We form the ring R of all pairs (a, b) of elements of D with component-wise addition and the following multiplication

$$(a,b)(a',b') = (aa', ab' + b\gamma(a)).$$

Then the set of all elements $(0, b)$, with $b \in D$, is the radical W of R. The ring R is local, $W^2 = 0$, and $\partial_R W = 1$. Let $w = (0, 1)$. Then (a, b) belongs to S_w if and only if $a \in D'$. Therefore, $\partial W_R = \dim D_{D'} = 2$, according to Lemma 5.1. But $\dim_{(S_w/W)} Q \neq 2$, so R is not exceptional.

REMARK 5.4. P. M. Cohn has constructed in [**3**] an example of a division ring D with a division subring D' of right index 2, but of left index different from 2. Thus, the question is whether such a subring D' exists which is, in addition, isomorphic to D.

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