THE REPRESENTATION TYPE OF LOCAL ALGEBRAS

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Let k be a (commutative) field, and $k\langle X_1,\dots,X_n\rangle$ the free associative algebra in n (non-commuting) variables. Denote by M_i the ideal of $k\langle X_1,\dots,X_n\rangle$ generated by all monomials of degree i. For any k-algebra A, let ${}_A\underline{\underline{m}}$ be the category of all A-modules which are finite dimensional as k-vector spaces. If I is a twosided ideal of $k\langle X_1,\dots,X_n\rangle$, then for $A = k\langle X_1,\dots,X_n\rangle/I$, the category ${}_A\underline{\underline{m}}$ is just the category of all (finite dimensional) vector spaces endowed with n endomorphisms which satisfy the relations expressed by the elements of I.

The k-algebra A is called <u>local</u>, provided $A = k\cdot 1 + \operatorname{rad} A$, where $\operatorname{rad} A$ is the Jacobson radical of A. If A is a local k-algebra, we will consider also its completion $\overline{A} = \varprojlim A/(\operatorname{rad} A)^n$. There is a canonical ring homomorphism $A \longrightarrow \overline{A}$, and A is said to be <u>complete</u> in case this homomorphism is an isomorphism. Since obviously every object in ${}_A\underline{\underline{m}}$ is annihilated by some power $(\operatorname{rad} A)^n$, the canonical homomorphism $A \longrightarrow \overline{A}$ induces an isomorphism of the categories ${}_A\underline{\underline{m}}$ and ${}_{\overline{A}}\underline{\underline{m}}$. Thus, in order to consider the behaviour of ${}_A\underline{\underline{m}}$ for a local algebras A, we may restrict to the case where A is complete.

The k-algebra A is said to be <u>wild</u> (or to be of wild representation type) provided there is a full and exact subcategory of ${}_A\underline{\underline{m}}$ which is representation equivalent to the category ${}_{k\langle X,Y\rangle}\underline{\underline{m}}$. The reason for calling it wild, is that there seems to be no hope to expect a complete classification of the indecomposable objects in ${}_{k\langle X,Y\rangle}\underline{\underline{m}}$, since for any finitely generated k-algebra B, there is a full and exact embedding of ${}_B\underline{\underline{m}}$ into ${}_{k\langle X,Y\rangle}\underline{\underline{m}}$. On the other hand, the algebra A is said to be tame (or to be of tame representation type), if there exists a complete classification of the indecomposable objects in ${}_A\underline{\underline{m}}$, and if there are not only finitely many indecomposables.

In order to distinguish the complete local algebras according to there representation type, we have to find the smallest possible wild algebras (that is, wild algebras for which all proper residue algebras are tame or of finite representation type), and the largest possible tame algebras (that is, tame algebras which do not occur as proper residue algebras of other tame algebras).

(1.1) We will have to consider several algebras which we want to introduce now. First, we mention

(a) $k\langle X,Y,Z\rangle/M_2$,

the local algebra of dimension 4 with radical square zero. Next, we single out certain residue algebras $k\langle X,Y\rangle/I$ of $k\langle X,Y\rangle$ of dimension 5, namely those with I the twosided ideal generated by the elements

(b) $X^2,\ XY,\ Y^2X,\ Y^3$;

(b^o) $X^2,\ YX,\ XY^2,\ Y^3$;

(c) $X^2,\ XY-\alpha YX,\ Y^2X,\ Y^3$ with $\alpha \neq 0$; and

(d) $X^2-Y^2,\ YX$.

Also, we are interested in another set of local algebras $k\langle X,Y\rangle/I$, where the ideal I is generated by just two elements:

(1) YX , XY ;

(2) $YX - X^n$, XY , with $n \geq 2$;

(3) $YX - X^n$, $XY - Y^m$ $n \geq 2$, $m \geq 3$;

(4) $YX - X^2$, $XY - \alpha Y^2$ $0 \neq \alpha \neq 1$ in k ;

(5) $X^2 - (YX)^nY$, $Y^2 - (XY)^nX$ $n \geq 1$;

(6) $X^2 - (YX)^nY$, Y^2 $n \geq 1$;

(7) $X^2 - (YX)^n$, $Y^2 - (XY)^n$ $n \geq 2$;

(8) $X^2 - (YX)^n$, Y^2 $n \geq 2$;

(9) X^2 , Y^2 ;

Let us mention first which algebras are known to be tame or wild.

(1.2) The algebras (a), (b), (b^o), (c) and (d) are wild.

For (a), (b) and (b^o), this was proved by Heller and Reiner [7], for (c) this was proved by Drozd [4] and Brenner [2]. In section 3, we will deal with these algebras.

(1.3) The algebras (1) – (4) and (7) – (9) are tame. Namely, we have the following theorem:

Let A be a local algebra, and assume there are elements x_1, x_2, y_1, y_2 in rad A such that rad A = $Ax_1+Ay_1 = Ax_2+Ay_2$ and $x_1x_2 = y_1y_2 = 0$, then A is tame.

The case of the algebra (1) was proved by Gelfand and Ponomarev [6] and by Szekeres (unpublished, but see [12]). The case (9), which includes the decomposition of the modular representations of the dihedral 2-groups, was proved in [11]. An indication of the method of the proof of (1.3) will be given in the last section, we follow quite closely the ideas devellopped by Gelfand and Ponomarev in the case of algebra (1).*)

(1.4) Let k be an algebraically closed field. Let A be a complete local algebra. Then either (i) A has a residue ring of type (a) – (d), or (ii) A is a residue ring of the completion of one of the algebras (1) – (9), or (iii) char k = 2, and A is isomorphic to $k\langle X,Y\rangle/I$ with I the twosided ideal generated by

(5') $X^2 - (YX)^nY + \gamma(YX)^{n+1}$, $Y^2 - (XY)^nX + \delta(YX)^{n+1}$, or

(6') $X^2 - (YX)^nY + \gamma(YX)^{n+1}$, $Y^2 + \delta(YX)^{n+1}$,

with $(\gamma,\delta) \neq (o,o)$.

In section 2 we will prove this theorem. The first step in its proof is the classification of the local algebras $k\langle X,Y\rangle/I$ of dimension 5 given by Gabriel (unpublished). Certain partial results were obtained by Dade [3], Janusz [8] and Müller [1o], when they considered the problem to bring certain algebras (group algebras of 2-groups of maximal rank) into a normal form. Drozd [4] proved the result for commutative A .

With respect to representation theory, the case (iii) in the theorem is of no real importance. Namely, the algebras (5') and (6') – as well as (5) and (6) – are Frobenius algebras, and modulo the socle, (5') and (5), as well as (6') and (6), are isomorphic (for fixed n).

Since the only indecomposable module which is not annihilated by the socle, is the algebra itself, the representation theory of (5') is identical to that of (5), and the representation theory of (6') is the same as that of (6).

(1.5) It follows from the preceding paragraphs that the only question which remains is to determine the representation type of (5) and (6). It is an interesting fact that these are "just" the group algebras of the generalised quaternion and the semi-dihedral groups. To be more precise: If k is an algebraically closed field of characteristic 2, and G is a generalised quaternion group, then the group algebra kG is of type (5'), and if G is semi-dihedral, then kG is of type (6').

It should be noted that for all other p-groups G, the representation type of kG is known: If char k = p and G is a non-cyclic p-group, then kG is wild except in the case of a two-generator 2-group of maximal rank (Krugliak [9] and Brenner [1]), that is except in the case of dihedral, semi-dihedral, and generalised quaternion groups. Namely, in all the other cases, kG has a residue ring of type (a) or (c), and therefore is wild.

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*) At the conference in Ottawa, theorem (1.3) was formulated by the author only with an additional hypothesis: that $kx_1+ky_1 = kx_2+ky_2$; the general case was conjectured. A complete proof will appear elsewhere.

2. The classification theorem

We want to prove theorem (1.4). Thus, we assume that k is algebraically closed. Let A be a complete local algebra, and let $J = \operatorname{rad} A$. We assume that A has no residue algebra of the form (a), (b), (b^o), (c) or (d). As a consequence, $\dim_k J/J^2 \leq 2$. If $\dim_k J/J^2 \leq 1$, then A is a homomorphic image of $\varprojlim k\langle X\rangle/(X^n)$, and this is a homomorphic image of the completion of the algebra (1). Thus, we may assume $\dim_k J/J^2 = 2$. Often we will denote by N a (suitable) k-subspace of A with $J = N \oplus J^2$.

(2.1) We may assume $\dim_k J^2/J^3 = 2$.

First, we show that for $\dim_k J^2/J^3 \geq 3$, there is a homomorphic image of one of the forms (a) – (d). This is obvious for dimension 4. We may assume $J^3 = 0$, and let $\dim_k J^2 = 3$. There is a non-trivial relation

$$\alpha x^2 + \beta xy + \gamma yx + \delta y^2 = 0,$$

where x, y is a basis of N. If $\alpha = \delta = 0$, then we use as additional relation $x^2 = 0$, and get as residue algebra an algebra of the form (b), (b^o) or (c). Thus, we may suppose $\alpha = 1$. Using $x' = x+\gamma y$ instead of x, we have a relation of the form

$$x'^2 + \beta' x'y + \delta' y = 0.$$

Adding the new relation $x'y = 0$, we get as residue algebra one of the form (b) or (d).

If $\dim_k J^2/J^3 = 1$, let A be the completion of some local algebra $k\langle X,Y\rangle/I$, where I is a twosided ideal. We want to construct an ideal $I' \subseteq I$ such that

$k\langle X,Y\rangle/I'$ again has no residue algebra of the form (a) – (d), but with $\dim_k J'^2/J'^3 = 2$, where $J' = \operatorname{rad} k\langle X,Y\rangle/I'$. It is fairly easy to see that $I+M_3$ contains elements x_2x_1 and y_2y_1 , where both x_1, y_1 as well as x_2, y_2 is a basis of a fixed N with $J = N \oplus J^2$. If x_2x_1+f and y_2y_1+g belong to I (with f, g in M_3), then let I' be generated by x_2x_1+f and y_2y_1+g .

(2.2) <u>There are elements</u> a, b <u>in</u> $J\setminus J^2$ <u>such that</u> ab <u>belongs to</u> J^3.

Again, we may assume $J^3 = 0$. Now A can be written in the form $k \oplus N \oplus N\otimes N/U$, where U is a subspace of $N\otimes N$ of dimension 2, and where the multiplication is given by the tensor product $\otimes$. Let x, y be a basis of N . We may assume that U intersects both $N\otimes x$ and $N\otimes y$ trivially, thus U is the graph of an isomorphism $N\otimes x \longrightarrow N\otimes y$, and therefore there is an automorphism $\varphi: N \longrightarrow N$ with $U = \{a\otimes x + \varphi(a)\otimes y \mid a \in N\}$. Let a be an eigenvector of φ with eigenvalue α . Then $0 \neq a\otimes(x+\alpha y)$ belongs to U.

(2.3) <u>There are elements</u> x_1, x_2, y_1, y_2 <u>with</u> $N = kx_1+ky_1 = kx_2+ky_2$ <u>and</u> x_2x_1, y_2y_1 in J.

Again, we may assume $J^3 = 0$. First, assume there is $x \in J\setminus J^2$ with $x^2 = 0$. Let x, y be a basis of N. There is another non-trivial relation

$$\alpha xy + \beta yx + \gamma y^2 = 0.$$

Now $\gamma \neq 0$, since otherwise we have one of the cases (b), (b^o) or (c). Thus, we may suppose $\gamma = 1$, and then

$$(y + \alpha x)(y + \beta x) = 0,$$

and we take $x_1 = x_2 = x$ and $y_2 = y + \alpha x$, $y_1 = y + \beta x$.

Next, assume $x^2 \neq 0$ for all x in $J \setminus J^2$. By (2.2), there is now a basis x, y of N with $yx = 0$. As before, we consider another non-trivial relation, say

$$\alpha x^2 + \beta y^2 + \gamma xy = 0.$$

Again, $\gamma \neq 0$, since otherwise we are dealing with the algebra (d), thus assume $\gamma = 1$. Then

$$(x + \beta y)(\alpha x + y) = 0,$$

which shows that we may take $x_2 = y$, $x_1 = x$, $y_2 = x + \beta x$ and $y_1 = \alpha x + y$.

(2.4) A/J^3 <u>is residue algebra of one of the algebras (1) – (9)</u>.

Proof: Assume first, one of the elements x_2, y_2, say x_2, is linearly independent both from x_1 and from y_1. Using a suitable multiple of x_1 for x and of y_1 for y, we may assume $x_2 = y - x$. If y_2 is also linearly independent both from x and y, then a multiple of y_2 is of the form $x - \alpha y$, with $\alpha \neq 0, 1$. Thus, A/J^3 is of the form (4). If y_2 is a multiple of x, then we have case (2) with $n=2$, if y_2 is a multiple of y, then we have case (8) with $n = 2$. In case both x_2 and y_2 are linearly dependent of x_1 or y_1, we get the cases (1) and (9).

(2.5) It remains to be shown: If, for $p \geq 3$, A/J^p is a residue ring of one of the algebras (1)–(9), then the same is true for A/J^{p+1}. Obviously, we may assume $J^{p+1} = 0$. As a by-product of our calculations, we also will determine a basis of the algebras (1)–(9).

<u>Case (1)</u>. There are elements X, Y in rad A with YX and XY in $\mathrm{rad}^p A$. Now $\mathrm{rad}^p A$ is generated by X^p and Y^p, thus there are elements $\alpha, \beta, \gamma, \delta$ in k with

$$YX + \alpha X^p + \beta Y^p = 0 \quad \text{and} \quad XY + \gamma X^p + \delta Y^p = 0.$$

If we replace X by $X' = X + \beta Y^{p-1}$ and Y by $Y' = Y + \gamma X^{p-1}$, the new relations are

$$Y'X' + \alpha X'^p = 0 \quad \text{and} \quad X'Y' + \delta Y'^p = 0.$$

We show how to get rid of α and δ. If $\alpha = \delta = 0$, we are again in case (1). If $\alpha \neq 0$, and $\delta = 0$, we replace X' by $X'' = \sqrt[p-1]{-\alpha}\, X'$, and are in case (2). If $\alpha = 0$ and $\delta \neq 0$, then we interchange X' and Y', and are in the previous situation. Finally, assume $\alpha \neq 0 \neq \delta$. Consider $X' = \lambda X''$ and $Y' = \mu Y''$ where λ, μ are elements of k which we want to determine now, in order to have X'' and Y'' satisfying the relations (3). The old relations become

$$\mu\lambda Y''X'' + \alpha\lambda^p X''^p = 0 \quad \text{and} \quad \lambda\mu X''Y'' + \delta\mu^p Y''^p = 0.$$

This means that we have to find λ, μ such that

$$\alpha\lambda^{p-1}\mu^{-1} = -1 \quad \text{and} \quad \delta\mu^{p-1}\lambda^{-1} = -1,$$

in order to have

$$Y''X'' - X''^p = 0 \quad \text{and} \quad X''Y'' - Y''^p = 0.$$

Of course it is easy to write down λ and μ explicitly, and X'' and Y'' are generators of rad A, since X' and Y' had this property. Such a change of X' and Y' will be called a scalar transformation in the later part of the proof, and usually will be left to the reader.

Case (2). We can assume $n < p$. Now the elements XY and $YX-X^n$ both belong to J^p, therefore $X^{n+1} = XYX = 0$, and J^p is generated by the single element Y^p. Assume there is a relation

$$YX - X^n + \alpha Y^p = 0,$$

then we replace X by $X' = X + \beta Y^{p-1}$, and get that A is either residue ring of an algebra of type (2) or of one of type (3); in the latter case we use an obvious scalar transformation.

Case (3). We consider the case $n \leq m = p-1$, and we want to prove that $J^p = 0$. This then implies that the algebra of type (3) has dimension $n+m+1$. By assumption, there are elements X, Y in J with $YX - X^n$ and $XY - Y^m$ in J^p. As in case (2), J^p is generated by Y^p, but

$$Y^p = YXY = X^nY = X^{n-1}Y^m = X^{n-2}Y^{2m-1} = 0,$$

since $n+2m-3 \geq p+1$.

Case (4). We assume $J^4 = 0$ and show $J^3 = 0$. There are equalities

$$X^3 = XYX = \alpha Y^2X = \alpha YX^2 = \alpha X^3, \quad \text{and}$$
$$X^2Y = \alpha XY^2 = \alpha^2 Y^3 = \alpha YXY = \alpha X^2 Y.$$

Since $\alpha \neq 1$, the monomials X^3, XYX and X^2Y are zero. Since $\alpha \neq 0$, also all the other monomials vanish.

Case (9). Assume A/J^p is a residue ring of the algebra of type (9). We distinguish two cases. First, let p be even, $p = 2q$. Then J^p is generated by the two elements $(YX)^q$ and $(XY)^q$, thus there are relations

$$X^2 + \alpha(YX)^q + \beta(XY)^q = 0, \quad Y^2 + \gamma(YX)^q + \delta(XY)^q.$$

If we replace X by $X' = X + \beta(YX)^{q-1}Y$ and $Y' = Y + \gamma(XY)^{q-1}X$, then the relations in X' and Y' (after some scalar transformation) have the form (7), (8) or (9). If p is odd, say $p = 2q+1$, then J^p is generated by the elements $(XY)^qX$ and $(YX)^qY$, and we have relations

$$X^2 + \alpha(XY)^qX + \beta(YX)^qY = 0, \; Y^2 + \gamma(XY)^qX + \delta(YX)^qY = 0.$$

This time, we replace X by $X+\alpha(XY)^q$ and Y by $Y+\delta(YX)^q$, and, again after some scalar transformation, the newe relations are of the form (5), (6) or (9).

<u>Case (8)</u> Now, let $p = 2n+1$, and assume A/J^p is generated by two elements X and Y which satisfy the relations $X^2 - (YX)^n = 0$ and $Y^2 = 0$. Now J^p is generated by the elements $(XY)^nX$ and $(YX)^nY$, but

(+) $(XY)^nX = X^3 = X(XY)^n = X^2Y(XY)^{n-1} = (XY)^nY(XY)^{n-1} = 0$,

therefore J^p is generated by the single element $(YX)^nY$. There are relations

$$X^2 - (YX)^n + \alpha(YX)^nY = 0, \quad Y^2 + \beta(YX)^nY = 0.$$

We replace X by $X' = X - \alpha YX + \alpha XY - \alpha^2 YXY$, and Y by $Y' = Y + \beta(YX)^n$. Then we get

$$X'^2 - (Y'X')^n = 0 \quad \text{and} \quad Y'^2 = 0.$$

To see the first, we note that

$$X'^2 = X^2 + \alpha X^2Y = X^2 + \alpha(YX)^nY ,$$

where the first equality stems from the fact that all the other summands cancel each other, and the second follows from the fact that $X^2 - (YX)^n$ belongs to J^p. Thus, X' and Y' satisfy relations of the form (8).

Next, let $p = 2n+2$, and A/J^p be of type (8). Then, as we have seen above, $(XY)^nX$ belongs to J^p. But then $J^p = 0$, and therefore the algebra of type (8) has dimension $4n+2$.

<u>Case (7)</u>. We assume $p = 2n+1$. We want to show that $J^p = 0$ in case A/J^p is residue algebra of the algebra (7). Using the calculation (+) of the previous case, we see that $(XY)^nX = 0$. Similarly, we have now also $(YX)^nY = 0$. This proves the assertion. As a consequence, we see that the algebra of type (7) has dimension $4n+1$.

<u>Cases (5),(6)</u>. Finally, we have to consider the situation where A/J^p is residue algebra of an algebra of type (5) or (6). We first look at the case $p = 2n+2$. Since $X^2-(YX)^nY$ belongs to J^p, it follows that

$$(YX)^{n+1} = X^3 = (XY)^{n+1}.$$

Thus, if $J^p \neq 0$, then A is a Frobenius algebra, with socle generated by the element $(YX)^{n+1}$. This shows that A is of the form (5') of (6'). But if the characteristic of k is different of 2, then it is easy to bring (5') into the form (5),and (6') into the form (6).

If $p = 2m+3$, we know from the previous consideration that $(YX)^{n+1}-(XY)^{n+1}$ belongs to J^p, and therefore

$$(XY)^{n+1}X = (YX)^{n+1}X = (YX)^nYX^2 = (YX)^nY(YX)^nY = 0,$$

and then also $(YX)^{n+1}Y = 0$. As a consequence, the algebras of type (5), (5'), (6), (6') all are of dimension $4n+4$.

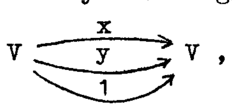
3. The wild algebras

In order to show that a given algebra A is wild, we will use the following procedure. We will start with a category $\underline{\underline{w}}$ which we know is wild, with a full subcategory $\underline{\underline{u}}$ of ${}_A\underline{\underline{m}}$, and with functors

$$U: \underline{\underline{w}} \longrightarrow \underline{\underline{u}}, \quad \text{and} \quad P: \underline{\underline{u}} \longrightarrow \underline{\underline{w}},$$

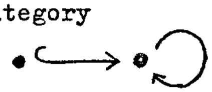
such that the composition PU is the identity functor on $\underline{\underline{w}}$. Then, obviously, $\underline{\underline{w}}$ is representation equivalent to the full subcategory of ${}_A\underline{\underline{m}}$ of all modules which are images under U.

(3.1) The algebra $A = k\langle X,Y,Z\rangle/M_2$ is wild. Following Heller and Reiner [7], we embed the category $\underline{\underline{w}} = {}_{k\langle x,y\rangle}\underline{\underline{m}}$ into ${}_A\underline{\underline{m}}$. Let $\underline{\underline{u}}$ be the full subcategory of ${}_A\underline{\underline{m}}$ consisting of all ${}_AM$ with $Z^{-1}0 = ZM$ (that is, all A-modules which are free when considered as $K\langle Z\rangle/(Z^2)$-modules). The functor U associates with ${}_{k\langle x,y\rangle}V$ the module ${}_AM$ given by the diagram



thus, as vectorspace, ${}_kM = V\oplus V$, and X operates on $V\oplus V$ by $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$, and so on. Conversely, given ${}_AM$ in $\underline{\underline{u}}$, then $P({}_AM)$ is the vector space ZM together with the two endomorphisms $x = XZ^{-1}$ and $y = YZ^{-1}$. Note that, for example, XZ^{-1} is well-defined, since $XZ^{-1}0 = XZM = 0$ according to the condition $Z^{-1}0 = ZM$, and that its image lies in ZM, using again the same condition.

(3.2) <u>The algebra</u> $A = K\langle X,Y\rangle/(X^2,YX,XY^2,Y^3)$ <u>is wild</u>. Again, we follow Heller–Reiner [7]. As $\underline{\underline{w}}$, we use the category

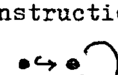


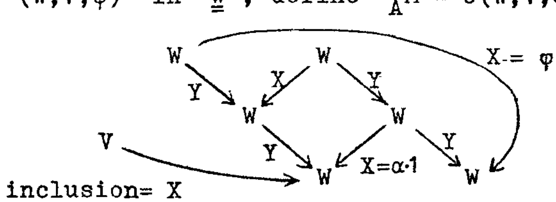
thus, an object of $\underline{\underline{w}}$ is given by a tripel (W,V,φ) with W a vector space, $V \subseteq W$ a subspace, and φ an endomorphism of W . Let $\underline{\underline{u}}$ be the full subcategory of all ${}_AM$ in ${}_A\underline{\underline{m}}$ with $XY^{-1}0 = 0$ and $Y^{-1}0 \subseteq YM$.
For (W,V,φ) in $\underline{\underline{w}}$, define ${}_AM = U(W,V,\varphi)$ by the diagram

$$V \xrightarrow{\;Y\;} W \underset{Y=1}{\overset{X=\varphi}{\rightrightarrows}} W ,$$

thus $M = V \oplus W \oplus W$, and X and Y operate on M as indicated. Conversely, for ${}_AM$ in $\underline{\underline{u}}$, let $P({}_AM) = (Y^{-1}0, Y^2M, XY^{-1})$. Obviously, Y^2M is a subspace of $Y^{-1}0$, and XY^{-1} is well-defined, since we assume $XY^{-1}0 = 0$. Also, the image of XY^{-1} lies in $Y^{-1}0$, since $YX = 0$.

(3.3) The algebra $A = k\langle X,Y\rangle/(X^2,XY,Y^2X,Y^3)$ is wild, since it is just the opposite algebra to the previously discussed one.

(3.4) <u>The algebra</u> $A = k\langle X,Y\rangle/(X^2,XY-\alpha YX,Y^2X,Y^3)$ <u>is wild</u>. We may assume $\alpha \neq 0$, and give a construction due to Drozd [4]. Again, $\underline{\underline{w}}$ is the category  .
Let $\underline{\underline{u}}$ be the full subcategory of all ${}_AM$ in ${}_A\underline{\underline{m}}$ with $YXY^{-2}XY^{-2}0 = 0$ and $YXM \subseteq Y^2M$, $XY^{-2}YXM \subseteq Y^2M$. For (W,V,φ) in $\underline{\underline{w}}$, define ${}_AM = U(W,V,\varphi)$ by the diagram



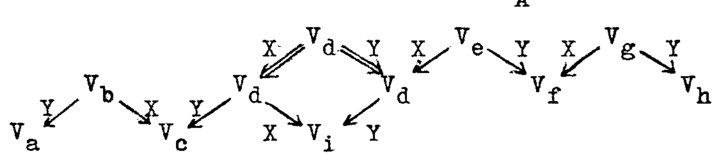
Thus, ${}_A M$ is the direct sum of six copies of W and one copy of V, and X and Y operate on it as indicated (where all but three maps are identity maps, one is given by φ, one is multiplication by α and one is the inclusion $V \subseteq W$). It remains to define P. Given ${}_A M$ in $\underline{\underline{u}}$, let $P({}_A M) = (YXM,\ XY^{-1}0\ \ YXM,\ YXY^{-2}XY^{-2})$. By the assumptions on $\underline{\underline{u}}$, $YXY^{-2}XY^{-2}$ is really an endomorphism of YXM, and it is easy to check that PU is the identity on $\underline{\underline{w}}$.

(3.5) <u>The algebra</u> $A=k\langle X,Y\rangle/(XY,X^2-Y^2)$ <u>is wild</u>. (Note that the ideal (XY,X^2-Y^2) contains M_3.)

We start with the category $\underline{\underline{w}}$ with objects V given as

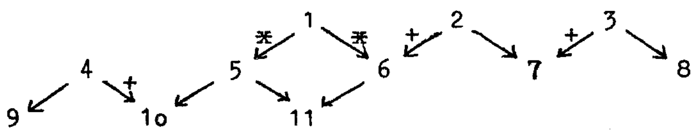
$$\begin{array}{ccccccccccccccc} V_a & \twoheadleftarrow & V_b & \hookrightarrow & V_c & \twoheadleftarrow & V_d & \hookleftarrow & V_e & \twoheadrightarrow & V_f & \hookleftarrow & V_g & \twoheadrightarrow & V_h \\ & & & & & & \downarrow & & & & & & & & \\ & & & & & & V_i & & & & & & & & \end{array},$$

that is, we consider the category of representations of the corresponding quiver such that the maps are monomorphisms or epimorphisms as indicated. This is a well-known wild category. The functor $U\colon \underline{\underline{w}} \longrightarrow {}_A\underline{\underline{m}}$ maps the representation V onto the A-module ${}_A M$ given as



where (besides two identity maps) all maps are the ones given by V.

We define a functor $P\colon {}_A\underline{\underline{m}} \longrightarrow \underline{\underline{w}}'$, where $\underline{\underline{w}}'$ is the category of <u>all</u> representations of the quiver



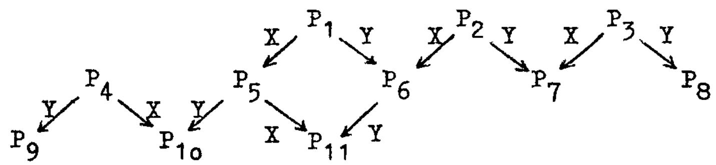
for which the square is commutative. The category $\underline{\underline{w}}$ is (equivalent to) the full subcategory of $\underline{\underline{w}}'$ of all representations for which the maps with $*$ are isomorphisms, those with + are monomorphisms, and the remaining ones are epimorphisms. We will use as $\underline{\underline{u}}$ the full subcategory of all modules ${}_AM$ in ${}_A\underline{\underline{m}}$ with $P({}_AM)$ in $\underline{\underline{w}}$.

In order to define P, we note that there is a chain of subfunctors F_i $(o \leq i \leq 11)$ of the forget functor F_o from the category ${}_A\underline{\underline{m}}$ into the category of k-vector spaces, namely

$$F_o({}_AM) = M,$$
$$F_1({}_AM) = X^{-1}YM,$$
$$F_2({}_AM) = X^{-1}YX^{-1}YM,$$
$$F_3({}_AM) = X^{-1}YXM + YM + XM,$$
$$F_4({}_AM) = YM + XM,$$
$$F_5({}_AM) = YM,$$
$$F_6({}_AM) = YX^{-1}YM,$$
$$F_7({}_AM) = YX^{-1}YX^{-1}YM,$$
$$F_8({}_AM) = YX^{-1}YXM + YXM,$$
$$F_9({}_AM) = YXM + X^2M,$$
$$F_{1o}({}_AM) = X^2M,$$
$$F_{11}({}_AM) = 0.$$

Most of the inclusions $F_{i-1} \subseteq F_i$ are trivial, otherwise we use the relations $XM \subseteq X^{-1}YM$, $YM \subseteq X^{-1}0$ and $XM \subseteq X^{-1}YX^{-1}0$.

The functor $P: {}_A\underline{\underline{m}} \longrightarrow \underline{\underline{w}}'$ is now defined componentwise by $P_i = F_i/F_{i-1}$, and those natural transformations $P_i \longrightarrow P_j$ which we need, are the ones induced by multiplication by X or Y, respectively:



Again, in order to show that these maps are defined, we need only the relation $XY = 0$. Of course, the square is commutative, since we assume $X^2 = Y^2$.

It is easy to check that the composition PU is the identity functor on $\underline{\underline{w}}$.

4. Tame algebras

We want to give some indications about the proof of theorem (1.3). In order to show that a given algebra A is tame, it is reasonable to do two things: first to write down a list of certain indecomposable modules, and then to prove that every object of $_A\underline{\underline{m}}$ can be decomposed as a direct sum of copies of these modules. In our case, the decomposition will be achieved by using several functors and natural transformations.

We will start with an index set $\underline{W}$ on which a function $\underline{W} \longrightarrow \mathbb{N}$ is defined which associates to every D in $\underline{W}$ a natural number $|D| \geq 1$, the "length" of D. To every D in $\underline{W}$ we will define either one indecomposable module $M(D)$, or a whole set of indecomposable modules $M(D,\varphi)$ indexed by the set of (equivalence classes of) indecomposable automorphisms of k-vector spaces (thus, if k is algebraically closed, we may take as index set the set of Jordan matrices).

Then, we will consider the forget functor $_A\underline{\underline{m}} \longrightarrow {}_k\underline{\underline{m}}$, which associates to every A-module the underlying vector space. For every D in $\underline{W}$, we will construct $2\cdot|D|$ subfunctors of it, denoted by $F(D,i)^+$ and $F(D,i)^-$, where $1 \leq i \leq |D|$, such that $F(D,i)^- \leq F(D,i)^+$. We will denote by $F(D,i)$ the quotient functor $F(D,i)^+/F(D,i)^-$. Then, we will construct natural transformations

$$F(D,i) \longrightarrow F(D,i+1) \quad \text{or} \quad F(D,i) \longleftarrow F(D,i+1),$$

for $1 \leq i < |D|$, and for certain elements D in $\underline{W}$ also for $i = |D|$, calculating modulo $|D|$.

In this way, we will determine for every ${}_A M$ in ${}_A\underline{\underline{m}}$ a submodule of "type D" (that is, one which is a direct sum of copies either of M(D), or of some of the $M(D,\varphi)$.), such that ${}_A M$ is the direct sum of these submodules.

Obviously, the index set $\underline{W}$ will depend on the particular algebra A . The method will be easier to visualise, if we use a specific example. We have chosen the case of the algebra (4), that is $k\langle X,Y\rangle/(YX-X^2,XY-\alpha Y^2)$ with $\alpha \neq 0,1$, since, on the one hand, the algebra is rather small, and, on the other hand, the behaviour of the remaining algebras is somewhat intermediate between that of the algebra (4) and of the well-known cases (1) and (9).

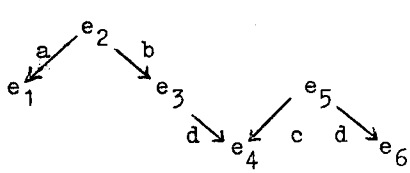
Thus, let $A = k\langle X,Y\rangle/(YX-X^2,XY-\alpha Y^2)$, and $\alpha \neq 0,1$. We will denote the elements X, Y, $X-\alpha Y$ and $X-Y$ by a,b,c,d , in order to point out that these are just four elements of kX+kY which are pairwise linearly independent. The set $\underline{W}$ will be the disjoint union of two subsets $\underline{W}_1$ and $\underline{W}_2$. Now, $\underline{W}_1$ is the set of all finite words in the letters a,b^{-1},c,d^{-1} (including the empty word 1), subject to the following rules: after c or b^{-1} follows either a or d^{-1}, after a follows only b^{-1}, and after d^{-1} follows only c . Thus, an example is the word

$$D = ab^{-1}d^{-1}cd^{-1},$$

and its length is defined to be 6 (= number of letters +1). If D and E are words, and DE is also a word, then we call DE the product of D and E . Of course, D^2

stands for DD, and so on. We call a word D non-periodic provided D^2 is also a word, and D is not of the form $D = E^n$ for some word E and $n>1$. Two non-periodic words D and E are called equivalent, provided one is a cyclic permutation of the other, and $\underline{W}_2$ will be the set of all equivalence classes of non-periodic words. Note that for elements of $\underline{W}_2$ the length is defined different: it is the precise number of letters of the corresponding word. (The word D above does not give rise to an element of $\underline{W}_2$, since D^2 is not an admissible word. An example of an element of $\underline{W}_2$ is the set of cyclic permutations of $ab^{-1}d^{-1}cd^{-1}c$.)

Next, we show how to define for D in $\underline{W}_1$ a module M(D). Namely, let M(D) be a $|D|$-dimensional vector space with base vectors $e_1,\dots,e_{|D|}$, such that X and Y operate on the base vectors according to the word D. Thus, for $D = ab^{-1}d^{-1}cd^{-1}$, we have the following schema



which means that $ae_2 = e_1$, $be_2 = e_3$, de_3 $(= (a-b)e_3) = e_4$, and so on. Note that in all but the terminal points e_1 and e_6 the action of a and b is uniquely defined. By definition this is true for e_2 . It is obvious for e_5, since the elements c and d are linearly independent. Since e_3 is image under b, we must have $ce_3 = 0$, thus also on e_3 the multiplication by two linearly independent elements (namely c and d) is given. Also,

e_4 is image both under a and b, thus we must have $ce_4 = de_4 = 0$. For the terminal points, we make the following convention. If, as in our case, also cD is a word, then we let $ce_1 = 0$, if aD is a word, we let $ae_1 = 0$, and if D starts with c, then we let $ae_1 = be_1 = 0$. Consequently, we define in our case also $ae_6 = be_6 = 0$.

In a similar way, we define for a word D in $\underline{W}_2$ and an automorphism φ of a vector space V, the module $M(D,\varphi)$. Namely, we take as underlying vector space the direct sum of $|D|$ copies of V, and define again the action of X and Y according to the word D, where all arrows but the last are taken as the identity map between the corresponding copies (as induced by the element of $kX+kY$ which correspond to the letter), and where the last letter gives just the map φ between the last and the first copy of V.

In order to define the subfunctors of the forget functor ${}_A\underline{\underline{m}} \longrightarrow {}_k\underline{\underline{m}}$ which are of interest to us, we note that the forget functor has two canonical filtrations, given by the equations $da = 0$ and $cb = 0$. Consider first the equation $da = 0$. We form finite and infinite words in the letters a and d^{-1}, and denote by $\underline{F}_a$ the set of all finite words together with those infinite words which are of the form $DE^{\infty} = DEEE\cdots$, where D and E are finite words. For every word D in $\underline{F}_a$, there are two obvious functors ${}_A\underline{\underline{m}} \longrightarrow {}_k\underline{\underline{m}}$, one defined by $M \longmapsto D(0_M)$, the other by

$M \longmapsto D(M)$. Here, we use the definition $E^{\infty}(0_M) = \bigcup E^n(0_M)$, and $E^{\infty}(M) = \bigcap E^n(M)$. It is easy to see that the set of all such functors is linearly ordered by inclusion, and we call this set the a-filtration. In a similar way, the equation $cb = 0$ gives rise to a set $\underline{F}_b$ of finite and infinite words in the letters b, c^{-1}, and then to the b-filtration.

If $F_2 \leq F_1$ are two subfunctors of the forget functor, we call $[\frac{F_1}{F_2}]$ an intervall. The intersection of the two intervalls $[\frac{F_1}{F_2}]$ and $[\frac{G_1}{G_2}]$ is defined to be the intervall

$$[\tfrac{F_1}{F_2}] \cap [\tfrac{G_1}{G_2}] = [\frac{F_1 \cap G_1}{(F_1 \cap G_2)+(F_2 \cap G_1)}]$$

For any word D in $\underline{W}$, the functors $F(D,i)^+$ and $F(D,i)^-$ are defined by intersecting suitable intervalls of the a-filtration with those of the b-filtration. We indicate the choice of the intervalls in the case of the word $D = ab^{-1}d^{-1}cd^{-1}$:

$$[\frac{F(D,1)^+}{F(D,1)^-}] = [\frac{ad^{-1}(d^{-1}a)\ M}{ad^{-1}(d^{-1}a)\ 0}] \cap [\frac{c^{-1}0}{bM}]$$

$$[\frac{F(D,2)^+}{F(D,2)^-}] = [\frac{d^{-1}(d^{-1}a)\ M}{d^{-1}(d^{-1}a)\ 0}] \cap [\frac{c^{-2}0}{c^{-1}bM}]$$

$$[\frac{F(D,3)^+}{F(D,3)^-}] = [\frac{(d^{-1}a)\ M}{(d^{-1}a)\ 0}] \cap [\frac{bc^{-2}0}{bc^{-1}bM}]$$

$$[\frac{F(D,4)^+}{F(D,4)^-}] = [\frac{(ad^{-1})\ M}{(ad^{-1})\ 0}] \cap [\frac{b^2c^{-2}0}{b^2c^{-1}bM}]$$

$$[\frac{F(D,5)^+}{F(D,5)^-}] = [\frac{(d^{-1}a)\ M}{(d^{-1}a)\ 0}] \cap [\frac{c^{-1}b^2c^{-2}0}{c^{-1}b^2c^{-1}bM}]$$

$$[\frac{F(D,6)^+}{F(D,6)^-}] = [\frac{(ad^{-1})\ M}{(ad^{-1})\ 0}] \cap [\frac{bc^{-1}b^2c^{-2}0}{bc^{-1}b^2c^{-1}bM}]$$

We now use the multiplication maps in order to define natural transformations between the quotient functors $F(D,i)$. Again, we use the word D as guide line. In our case, for example, we want to have the following transformations:

$$\begin{array}{cccccc} & F(D,2) & & & & \\ {}^{a}\swarrow & & \searrow^{b} & & & \\ F(D,1) & & F(D,3) & & F(D,5) & \\ & & {}^{d}\searrow & \swarrow^{c} & & \searrow^{d} \\ & & & F(D,4) & & F(D,6), \end{array}$$

where the letter indicates the multiplying element. Of course, it has to be checked that the multiplication maps are well-defined and act as indicated, and that they induce even isomorphisms of the corresponding components.

It then only remains to be shown that the intervalls $[{F(D,i)^{+} \atop F(D,i)^{-}}]$ cover the forget functor (that means, for every M and every $o \neq x \in M$, there is such an intervall with $x \in F(D,i)^{+}(M) \setminus F(D,i)^{-}(M)$.)

An outline of the background of the proof, may be found in Gabriel's paper [5] where he discusses the value of functor categories in order to determine all indecomposable objects of a given category.

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