Normal Forms of Real Matrices with Respect to Complex Similarity

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ABSTRACT

This paper gives a complete classification of real linear transformations between two complex vector spaces in terms of matrices.

INTRODUCTION

In this paper, we shall consider the set of all real $2m\times 2n$ matrices ($m,n=1, 2, \ldots$). A real $2m\times 2n$ matrix will be called *formally complex* if every (k,l)-block ($1\leqslant k\leqslant m$, $1\leqslant l\leqslant n$) of its partition into 2×2 blocks has the form

$$\begin{pmatrix} a_{kl} & b_{kl} \\ -b_{kl} & a_{kl} \end{pmatrix}, \qquad a_{kl}, b_{kl}\in \mathbf{R}.$$

Two real $2m\times 2n$ matrices A, A' are said to be **C**-*similar* if there exist formally complex regular square ($2m\times 2m$ and $2n\times 2n$) matrices P, Q such that

$$PAQ=A'.$$

The following theorem provides a complete classification of the **C**-similarity classes; the symbols E_a ($a \in \mathbf{R}$), E_∞ and E_{ab} ($a, b \in \mathbf{R}$) denote the 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

respectively.

By the direct product of two matrices A and B, we shall understand the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If B is a $2m' \times 2n'$ zero matrix (allowing $m' = 0$ or $n' = 0$), we call the product a zero-augmentation of A. A $2m \times 2n$ matrix is said to be $\mathbf{C}$-indecomposable if it is not $\mathbf{C}$-similar to a direct product of two matrices with even number of rows and columns.

THEOREM . *Every (non-zero) real $2m \times 2n$ matrix is $\mathbf{C}$-similar to a zero-augmented product of matrices of the following types:*

(i) $2(p+1) \times 2p$ *matrices* ($p = 1, 2, \ldots$)

$$\begin{bmatrix} E_\infty & & & & \\ E_0 & E_\infty & & 0 & \\ & E_0 & E_\infty & & \\ & & \ddots & \ddots & \\ & 0 & & E_0 & E_\infty \\ & & & & E_0 \end{bmatrix},$$

(ii) *the corresponding transposed* $2p \times 2(p+1)$ *matrices* ($p = 1, 2, \ldots$)

$$\begin{bmatrix} E_\infty & E_0 & & & & \\ & E_\infty & E_0 & & 0 & \\ & & E_\infty & \ddots & & \\ & 0 & & \ddots & E_0 & \\ & & & & E_\infty & E_0 \end{bmatrix},$$

(iii) *square* $2p\times 2p$ *matrices with* $|a|\leqslant 1$ $(p=1,2,\ldots)$

$$\begin{bmatrix} E_a & E_\infty & & & \\ & E_a & E_\infty & 0 & \\ & & \ddots & \ddots & \\ & 0 & & E_a & E_\infty \\ & & & & E_a \end{bmatrix}$$

and

(iv) *square* $4p\times 4p$ *matrices with either* $b>0$ *or* $b=0$ *and* $a<0$ $(p=1,2,\ldots)$

$$\begin{bmatrix} E_1 & E_{ab} & 0 & E_\infty & & & & & & \\ E_{-1} & E_1 & 0 & 0 & & & & & & \\ & & E_1 & E_{ab} & 0 & E_\infty & & & 0 & \\ & & E_{-1} & E_1 & 0 & 0 & & & & \\ & & & & \ddots & & \ddots & & & \\ & & & & & & E_1 & E_{ab} & 0 & E_\infty \\ & 0 & & & & & E_{-1} & E_1 & 0 & 0 \\ & & & & & & & & E_1 & E_{ab} \\ & & & & & & & & E_{-1} & E_1 \end{bmatrix}.$$

These matrices are **C**-*indecomposable and, in the decomposition of a real* $2m\times 2n$ *matrix, they are determined (up to their order) uniquely.*

REMARK. Note that, in contrast to the fact that there are so many different **C**-similarity classes of indecomposable matrices, there is only a single **C**-similarity class of formally complex indecomposable matrices, namely that represented by E_1.

This result provides a typical illustration of some new general methods which can be used in problems in the classification of linear transformations of vector spaces. These methods were initiated by I. M. Gelfand and V. A. Ponomarev [5], who present in the same paper a conceptual proof of the Kronecker theorem on the classification of pairs of matrices [6]. Later, the functorial approach was systematically explored by I. N. Bernstein, I. M.

Gelfand and V. A. Ponomarev in [2]. Our proofs are based on results and methods developed in [4] and [7].

Throughout the paper, **R** and **C** stand for the fields of real and complex numbers, respectively.

1. PRELIMINARY RESULTS

Let us point out again that we always consider real matrices with even number of rows and columns. Each such $2m\times 2n$ matrix A describes an **R**-linear transformation φ of an n-dimensional **C**-vector space W with respect to the bases $\{v_1, v_1 i, v_2, v_2 i, \ldots, v_n, v_n i\}$ and $\{w_1, w_1 i, w_2, w_2 i, \ldots, w_m, w_m i\}$, where $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ are **C**-bases of V and W, respectively. Moreover, a $2m\times 2n$ matrix A' is **C**-similar to A if and only if it describes φ with respect to some other **C**-bases of V and W. In this way, the classification of **C**-similarity classes of matrices is interpreted as the classification of real linear transformations between two complex vector spaces.

In this section, we collect some information about the **C**-similarity classes of **C**-indecomposable matrices which will be used in the next section to prove the classification theorem. The proof of these statements will be given in Sec. 3 using a natural translation of our problem into the more general language of the representations of graphs. In particular, it will become apparent that the statements of this section which may seem to be rather technical become, in the frame-work of the representation theory, conceptual.

In what follows, $R=\mathbf{C}[z;\ -]$ will always denote the skew polynomial ring over **C** in one variable z with respect to complex conjugation: thus, elements of R are (formal) sums $\sum_{j=0}^{n} z^j c_j$ $(c_j\in\mathbf{C})$ with componentwise addition and distributive multiplication subject to the rule $cz=z\bar{c}$.

The first statement deals with non-square **C**-indecomposable matrices.

LEMMA A. *Up to* **C***-similarity, there are precisely one* **C***-indecomposable* $2(p+1)\times 2p$ *and one* **C***-indecomposable* $2p\times 2(p+1)$ *matrix for each* $p=1,2,\ldots$. *All other* **C***-indecomposable matrices are square matrices.*

The following assertion furnishes a reduction of square **C**-indecomposable matrices to **C**-irreducible ones. Here, a non-zero square $2m\times 2n$ matrix is called **C**-irreducible if it is not **C**-similar to a matrix of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is a square $2n\times 2n$ matrix.

LEMMA B. (a) *Every* **C**-*indecomposable square matrix is* **C**-*similar to a matrix of the form*

$$\begin{bmatrix} S & T & & & \\ & S & T & 0 & \\ & & \ddots & \ddots & \\ & 0 & & S & T \\ & & & & S \end{bmatrix}, \qquad (*)$$

where S is a **C**-*irreducible square matrix and the matrix*

$$\begin{pmatrix} S & T \\ 0 & S \end{pmatrix} \qquad (**)$$

is **C**-*indecomposable. Conversely, if S is a* **C**-*irreducible matrix and* (**) *is* **C**-*indecomposable, then also* (*) *is* **C**-*indecomposable.*

(b) *If* (**) *is not* **C**-*indecomposable, then there are formally complex matrices A and B such that* $T = -(AS + SB)$ *and therefore*

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} S & T \\ 0 & S \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix},$$

where I denotes the (formally complex) identity matrix. Moreover, we may assume that one diagonal entry in B, in a prescribed position, is zero. Also, if S and S′ are **C**-*irreducible matrices, then two* **C**-*indecomposable matrices*

$$\begin{bmatrix} S & T & & & \\ & S & T & 0 & \\ & & \ddots & \ddots & \\ & 0 & & S & T \\ & & & & S \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S' & T' & & & \\ & S' & T' & 0 & \\ & & \ddots & \ddots & \\ & 0 & & S' & T' \\ & & & & S' \end{bmatrix}$$

are **C**-*similar if and only if S and S′ are* **C**-*similar and the matrices have the same size.*

Now, we formulate a description of **C**-irreducible matrices in terms of simple R-modules.

LEMMA C. *Let* M_R *be a simple* R-*module with a* **C**-*basis*

$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_d\}$, *and let* $\mathbf{u}_l z = \sum_{k=1}^{d} \mathbf{u}_k (a_{kl} + i b_{kl})$ *with* $a_{kl}, b_{kl} \in \mathbf{R}$. *Then*

$$\begin{bmatrix} C_{11}+E_1 & C_{12} & \cdots & C_{1d} \\ C_{21} & C_{22}+E_1 & \cdots & C_{2d} \\ & & \ddots & \\ C_{d1} & C_{d2} & \cdots & C_{dd}+E_1 \end{bmatrix} \quad \text{with} \quad C_{kl} = \begin{pmatrix} a_{kl} & b_{kl} \\ b_{kl} & -a_{kl} \end{pmatrix}$$

is a **C**-*irreducible matrix; two such matrices derived from the* R-*modules* M_R *and* M'_R, *respectively, are* **C**-*similar if and only if* M_R *and* M'_R *are isomorphic. In this way, we obtain representatives of all* **C**-*similarity classes of irreducible square matrices with the exception of one class represented by the matrix*

$$E_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We conclude this introductory section with a description of the simple R-modules.

LEMMA D. *The* R-*modules*

$$R/(z-r)R \qquad \text{with real} \quad r \geqslant 0$$

and

$$R/(z^2-c)R \qquad \text{with complex} \quad c = a+ib,$$

where $b>0$ *or* $b=0$ *and* $a<0$, *are simple and pairwise non-isomorphic. Any simple* R-*module is isomorphic to one of them.*

2. PROOF OF THE THEOREM

Here, we establish our Theorem assuming Lemmas A, B, C and D.

PROPOSITION 1. *With respect to* **C**-*similarity, the matrices of the type* (i) *and* (ii) *described in the Theorem are indecomposable.*

Proof. We shall prove that the matrices A_p $(p=1,2,\dots)$ of the type (i) described in the Theorem are indecomposable; the other part of Proposition 1 follows by duality.

Now, A_p describes an **R**-linear transformation φ of a p-dimensional **C**-vector space V into a $(p+1)$-dimensional **C**-vector space W with respect to bases $\{v_1, v_1 i, v_2, v_2 i, \dots, v_p, v_p i\}$ and $\{w_1, w_1 i, w_2, w_2 i, \dots, w_{p+1}, w_{p+1} i\}$, where $\{v_1, v_2, \dots, v_p\}$ and $\{w_1, w_2, \dots, w_{p+1}\}$ are **C**-bases of V and W, respectively.

In order to prove indecomposability of A_p, we have to show that there is no non-trivial decomposition of the **C**-vector spaces $V_{\mathbf{C}} = V'_{\mathbf{C}} \oplus V''_{\mathbf{C}}$ and $W_{\mathbf{C}} = W'_{\mathbf{C}} \oplus W''_{\mathbf{C}}$ such that φ decomposes into $\varphi' : V' \to W'$ and $\varphi'' : V'' \to W''$. This is trivial for $p=1$; for, without loss of generality, $V' = V$ and the **C**-closure $\overline{\varphi(V)}$ of $\varphi(V)$ [i.e., the least **C**-vector subspace of W containing $\varphi(V)$] equals W.

We proceed by induction; let $p>1$. The **C**-interior $W^0 = \underline{\varphi(V)}$ of $\varphi(V)$ [i.e., the largest **C**-vector subspace contained in $\varphi(V)$] is the $(p-1)$-dimensional **C**-subspace generated by $(w_2, \dots, w_p)$. The **R**-subspace of V generated by $(v_1, v_2, v_2 i, \dots, v_{p-1}, v_{p-1} i, v_p i)$ [if $p=2$, by $(v_1, v_2 i)$] is $\varphi^{-1}(W^0)$. Thus $V^0 = \underline{\varphi^{-1}(W^0)}$ is the $(p-2)$-dimensional **C**-subspace generated by $(v_2, \dots, v_{p-1})$ (if $p=2$, by the empty sequence). The **R**-linear transformation φ^0 from V^0 to W^0 induced by φ is **C**-indecomposable. For $p=2$ this is so because of the **C**-dimensions of V^0 and W^0. For $p>2$ it follows by induction because the matrix of the transformation relative to the corresponding **R**-bases is A_{p-2}. Since $\varphi(V) = \varphi(V') + \varphi(V'')$, $\varphi(V') \subseteq W'$ and $\varphi(V'') \subseteq W''$, we have $\varphi(V) = [\varphi(V) \cap W'] \oplus [\varphi(V) \cap W'']$. This implies readily that $W^0 = (W^0 \cap W') \oplus (W^0 \cap W'')$ (see [3], p. 312). Since $W' \cap W'' = 0$ and φ is injective, we get $\varphi^{-1}(W^0) = [\varphi^{-1}(W^0) \cap V'] \oplus [\varphi^{-1}(W^0) \cap V'']$. As above, this implies $V^0 = (V^0 \cap V') \oplus (V^0 \cap V'')$. Since the decomposition of φ^0 must be trivial, we may assume that (say) $W^0 = W^0 \cap W'$, $W^0 \subseteq W'$. Since φ is injective, $\varphi^{-1}(W^0) \subseteq V'$. Consequently $V = \overline{\varphi^{-1}(W^0)} \subseteq V'$, $W = \overline{\varphi(V)} \subseteq W'$, i.e., $V = V'$, $W = W'$. ∎

Proposition 2. *Up to* **C**-*similarity,*

$$E_a \quad \text{with} \quad |a| \leqslant 1$$

and

$$\begin{pmatrix} E_1 & E_{ab} \\ E_{-1} & E_1 \end{pmatrix} \qquad \text{with either} \quad b>0 \quad \text{or} \quad b=0 \text{ and } a<0$$

are precisely all the square **C**-*irreducible matrices.*

Proof. The proof follows immediately from Lemmas D and C. First, let $\{\mathbf{u}\}$, where $\mathbf{u} = 1 + (z - r)R$, be a $\mathbf{C}$-basis of $R/(z-r)R$ with a real $r \geqslant 0$. Then $\mathbf{u}z = \mathbf{u}r$, and hence

$$\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$$

is $\mathbf{C}$-irreducible; this matrix can be normalized to

$$\begin{pmatrix} 1 & 0 \\ 0 & (1-r)/(1+r) \end{pmatrix},$$

i.e. to E_a with $-1 < a \leqslant 1$. Since E_{-1} is $\mathbf{C}$-irreducible by Lemma C, we get the first part of Proposition 2.

Second, let $\{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = 1 + (z^2 - c)R$ and $\mathbf{u}_2 = z + (z^2 - c)R$, be a $\mathbf{C}$-basis of $R/(z^2 - c)R$, $c = a + ib$. Then $\mathbf{u}_1 z = \mathbf{u}_2$ and $\mathbf{u}_2 z = \mathbf{u}_1 c$. Hence, according to Lemma C, $\begin{pmatrix} E_1 & E_{ab} \\ E_{-1} & E_1 \end{pmatrix}$, with the appropriate conditions, is $\mathbf{C}$-irreducible.

Finally, Lemma C asserts that, in this way, we obtain all $\mathbf{C}$-irreducible matrices. ■

PROPOSITION 3. *The matrices*

$$A = \begin{pmatrix} E_a & E_\infty \\ 0 & E_a \end{pmatrix} \qquad \text{with} \quad |a| \leqslant 1$$

and

$$B = \begin{pmatrix} E_1 & E_{ab} & 0 & E_\infty \\ E_{-1} & E_1 & 0 & 0 \\ 0 & 0 & E_1 & E_{ab} \\ 0 & 0 & E_{-1} & E_1 \end{pmatrix} \qquad \begin{matrix} \text{with either} \quad b > 0 \\ \text{or} \quad b = 0 \quad \text{and} \quad a < 0 \end{matrix}$$

are $\mathbf{C}$-indecomposable.

Proof. The matrix A defines an $\mathbf{R}$-linear mapping from $V = \mathbf{v}_1\mathbf{C} + \mathbf{v}_2\mathbf{C}$ to another two-dimensional $\mathbf{C}$-vector space $W_{\mathbf{C}}$; the elements of V will be written as column vectors

$$\mathbf{r}=\begin{bmatrix} r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}=(r_1,r_2,r_3,r_4)^T$$

with respect to the $\mathbf{R}$-basis $\{\mathbf{v}_1,\mathbf{v}_1 i,\mathbf{v}_2,\mathbf{v}_2 i\}$ of V. Observe that multiplication by the complex number i yields $\mathbf{r}\cdot i=(-r_2,r_1,-r_4,r_3)^T$. In view of Lemma B, it is sufficient to show that there is no vector $\mathbf{x}=(x_1,x_2,1,0)^T$ of V such that both $A\mathbf{x}=(x_1,ax_2,1,0)^T$ and $A(\mathbf{x}\cdot i)=(-x_2,ax_1+1,0,a)^T$ belong to a one-dimensional $\mathbf{C}$-subspace of $W_{\mathbf{C}}$. Assume that $A(\mathbf{x}\cdot i)=A(\mathbf{x})\cdot(r+si)$ for some real r and s. Then $(-x_2,ax_1+1,0,a)=(x_1r-ax_2s,ax_2r+x_1s,r,s)$. Consequently, $r=0$, $s=a$, and we get a contradiction.

In the case of the matrix B we proceed similarly. Again, we consider the matrix as an $\mathbf{R}$-linear mapping from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$. Obviously, B is $\mathbf{C}$-indecomposable if and only if the matrix

$$C=\begin{bmatrix}
0 & 0 & c & d & 0 & 0 & 0 & 0\\
0 & 1 & d & -c & 0 & 0 & 0 & q\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & 0 & 0 & c & d\\
 & & & & 0 & 1 & d & -c\\
 & 0 & & & 1 & 0 & 1 & 0\\
 & & & & 0 & -1 & 0 & 0
\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}
1 & 0 & -1 & 0 & & & & \\
 & 1 & 0 & 1 & & & 0 & \\
 & & 2 & 0 & & & & \\
 & & & 2 & & & & \\
 & & & & 1 & 0 & -1 & 0\\
 & & & & & 1 & 0 & -1\\
 & 0 & & & & & 2 & 0\\
 & & & & & & & 2
\end{bmatrix}$$

$$\times B\begin{bmatrix}
2 & 0 & 1 & 0 & & & & \\
 & 2 & 0 & 1 & & & 0 & \\
 & & 1 & 0 & & & & \\
 & & & 1 & & & & \\
 & & & & 2 & 0 & 1 & 0\\
 & & & & & 2 & 0 & 1\\
 & 0 & & & & & 1 & 0\\
 & & & & & & & 1
\end{bmatrix},$$

with $c = (a-1)/4$, $d = b/4$, $q = 1/4$, is **C**-indecomposable. Also, since $\begin{pmatrix} E_1 & E_{ab} \\ E_{-1} & E_1 \end{pmatrix}$ is **C**-irreducible, the matrix $S = \begin{pmatrix} E_\infty & E_{cd} \\ E_{-1} & E_0 \end{pmatrix}$ (which occurs in C) is **C**-irreducible. According to Lemma B, if C is **C**-decomposable, then

$$\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} = \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} C \begin{pmatrix} I & D' \\ 0 & I \end{pmatrix}$$

with formally complex 4×4 matrices D and D'. Also, by Lemma B, we can assume that the third diagonal element of D' is zero. Let

$$\mathbf{x} = (x_1, x_2, x_3, x_4, 1,0,0,0)^T = \begin{pmatrix} I & D' \\ 0 & I \end{pmatrix} (0,0,0,0,1,0,0,0)^T$$

and

$$\mathbf{y} = (y_1, y_2, 0, y_4, 0,0,1,0)^T = \begin{pmatrix} I & D' \\ 0 & I \end{pmatrix} (0,0,0,0,0,0,1,0)^T.$$

Then $C\mathbf{x}$, $C(\mathbf{x}\cdot i)$, $C\mathbf{y}$ and $C(\mathbf{y}\cdot i)$ belong to a two-dimensional **C**-subspace $W'_{\mathbf{C}}$ of $W_{\mathbf{C}}$. However, the following calculation shows that this is impossible. Indeed,

$$C\mathbf{x} = (cx_3 + dx_4, x_2 + dx_3 - cx_4, x_1 + x_3, -x_2, 0,0,1,0)^T,$$

$$C(\mathbf{x}\cdot i) = (-cx_4 + dx_3, x_1 - dx_4 - cx_3, -x_2 - x_4, -x_1, 0,1,0,-1)^T,$$

$$C\mathbf{y} = (dy_4, y_2 - cy_4, y_1, -y_2, c, d, 1, 0)^T$$

and

$$C(\mathbf{y}\cdot i) = (-cy_4, y_1 - dy_4 + q, -y_2 - y_4, -y_1, d, -c, 0, 0).$$

Now, $W'_{\mathbf{C}}$ is generated by $C\mathbf{x}$ and $C(\mathbf{x}\cdot i)$. Hence, comparing the last four coordinates, we get

$$C\mathbf{y} = [C\mathbf{x}](c + di + 1) + [C(\mathbf{x}\cdot i)](d - ci)$$

and

$$C(\mathbf{y}\cdot i) = [C\mathbf{x}](d - ci) + [C(\mathbf{x}\cdot i)](-c - di).$$

Thus, the first, third and fourth coordinates of Cy yield

$$dy_4 = cx_1 - dx_2 + cx_3 + dx_4, \tag{1}$$

$$y_1 = x_1 + (c+1)x_3 - dx_4, \tag{2}$$

$$-y_2 = -x_2 + dx_3 + cx_4, \tag{3}$$

and the second, third and fourth coordinates of $C(\mathrm{y}\cdot i)$ yield

$$y_1 - dy_4 + q = -cx_1 + dx_2, \tag{4}$$

$$-y_2 - y_4 = dx_3 + cx_4, \tag{5}$$

$$-y_1 = -cx_3 + dx_4. \tag{6}$$

Taking the linear combination

$$(1) - 2c(2) - 2d(3) - (4) + 2d(5) - (2c+1)(6),$$

we get $-q=0$, a contradiction.

The proof is completed. ■

As a consequence of the results in this section we have established the Theorem. Indeed, given a real $2m\times 2n$ matrix, it is obviously $\mathbf{C}$-similar to a zero-augmented product of $\mathbf{C}$-indecomposable matrices. By Lemma A in combination with Proposition 1, every $\mathbf{C}$-indecomposable matrix is of type (i) or (ii), or is a square matrix. The $\mathbf{C}$-indecomposable square matrices are, by Lemma B, extensions of $\mathbf{C}$-irreducible ones. The latter are described in Proposition 2. Thus, taking into account Proposition 3, the types (iii) and (iv) exhaust all possible $\mathbf{C}$-indecomposable square matrices. At the same time, the argument shows that all these types are $\mathbf{C}$-indecomposable and pairwise non-$\mathbf{C}$-similar. Also, it follows that the decomposition is unique.

3. TRANSLATION OF THE PROBLEM

Our problem asks for a classification of all $\mathbf{R}$-linear transformations ψ between two $\mathbf{C}$-vector spaces $V_{\mathbf{C}}$ and $W_{\mathbf{C}}$ of dimensions n and m, respectively, i.e., for a classification of all $\mathbf{R}$-linear transformations φ between the $\mathbf{R}$-vector spaces $V_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\approx V_{\mathbf{R}}$ and $\mathrm{Hom}_{\mathbf{C}}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}, W_{\mathbf{C}})\approx W_{\mathbf{R}}$ of dimensions $2n$

and $2m$, respectively, subject to the following definition of equivalence:

Two $\mathbf{R}$-linear transformations ψ: $V_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\to\mathrm{Hom}_{\mathbf{C}}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}, W_{\mathbf{C}})$ and $\psi': V'_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\to\mathrm{Hom}_{\mathbf{C}}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}, W'_{\mathbf{C}})$ are similar if there exist regular $\mathbf{C}$-linear transformations

$$q: V'_{\mathbf{C}}\to V_{\mathbf{C}} \quad \text{and} \quad p: W_{\mathbf{C}}\to W'_{\mathbf{C}}$$

such that $\mathrm{Hom}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}},p)\psi(q\otimes 1)=\psi'$.

Using the natural isomorphism

$$\mathrm{Hom}_{\mathbf{R}}(V_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}},\ \mathrm{Hom}_{\mathbf{C}}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}, W_{\mathbf{C}}))\approx\mathrm{Hom}_{\mathbf{C}}(V_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}, W_{\mathbf{C}}),$$

we see that our problem asks for a classification of all $\mathbf{C}$-linear transformations φ between the $\mathbf{C}$-vector spaces $V_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}$ and $W_{\mathbf{C}}$ subject to the following condition: φ and φ' are equivalent if there exist isomorphism $\eta: V_{\mathbf{C}}\to V'_{\mathbf{C}}$ and $\xi: W_{\mathbf{C}}\to W'_{\mathbf{C}}$ such that

$$\begin{array}{ccc} V_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{C}_{\mathbf{C}} & \xrightarrow{\varphi} & W_{\mathbf{C}} \\ \downarrow{\scriptstyle \eta\otimes 1\otimes 1} & & \downarrow{\scriptstyle \xi} \\ V'_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{C}_{\mathbf{C}} & \xrightarrow{\varphi'} & W'_{\mathbf{C}} \end{array}$$

commutes (in comparison with the earlier notation, $\xi=p,\eta=q^{-1}$). For later use, we remark that it is easy to verify that the elements

$$s=1\otimes 1-i\otimes i \quad \text{and} \quad t=1\otimes 1+i\otimes i$$

form a basis of the left $\mathbf{C}$-vector space ${}_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}$ and that

$$si=1\otimes i+i\otimes 1=is \quad \text{and} \quad ti=1\otimes i-i\otimes 1=-it.$$

Thus

$${}_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}\approx{}_{\mathbf{C}}\mathbf{C}_{\mathbf{C}}\oplus{}_{\mathbf{C}}\mathbf{C}_{\bar{\mathbf{C}}},$$

where the first summand is generated by s and the second one by t, and the right action of $\mathbf{C}$ on the second summand is given by the conjugation $^{-}$; thus $(as+bt)c=acs+b\bar{c}t$ for all a, b, $c\in\mathbf{C}$.

In this way, we have translated our problem to the classification of all indecomposable representations of the extended Dynkin diagram

$$\tilde{\mathbf{A}}_{12} = \bullet \overset{(2,2)}{\text{———}} \bullet$$

with the bimodule ${}_{\mathbf{C}}\mathbf{C}_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}$, i.e., of the oriented species

$$\mathbf{C} \xrightarrow{{}_{\mathbf{C}}\mathbf{C}_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}} \mathbf{C}.$$

The general theory of representations of extended Dynkin diagrams can be found in [4]. Let us summarize the results of [4] required in the proof of Lemmas A and B. Denote by $\mathfrak{L}$ the (Abelian) category of all its representations, i.e. of all the triples (V, W, φ) consisting of $\mathbf{C}$-vector spaces $V_{\mathbf{C}}$ and $W_{\mathbf{C}}$ and a $\mathbf{C}$-linear transformation

$$\varphi : V_{\mathbf{C}} \otimes ({}_{\mathbf{C}}\mathbf{C}_{\mathbf{R}} \otimes {}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}) \to W_{\mathbf{C}}$$

together with the maps

$$(\eta, \xi) : (V_{\mathbf{C}}, W_{\mathbf{C}}, \varphi) \to (V'_{\mathbf{C}}, W'_{\mathbf{C}}, \varphi'),$$

where η: $V_{\mathbf{C}} \to V'_{\mathbf{C}}$ and ξ: $W_{\mathbf{C}} \to W'_{\mathbf{C}}$ are $\mathbf{C}$-linear transformations such that $\xi\varphi = \varphi'(\eta \otimes 1)$. For every $\mathbf{X} = (V, W, \varphi) \in \mathfrak{L}$ define the dimension type $\mathbf{dim}\, X$ by $\mathbf{dim}\, X = (\dim V_{\mathbf{C}}, \dim W_{\mathbf{C}})$.

Now, denoting by $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{I}_1$ and $\mathbf{I}_2$ the indecomposable projective and indecomposable injective representations in $\mathfrak{L}$, there exist endofunctors C^+ and C^- of $\mathfrak{L}$ (the so-called *Coxeter functors*) such that

$$C^{-r}\mathbf{P}_j \quad \text{and} \quad C^{+r}\mathbf{I}_j \qquad \text{for} \quad r = 0, 1, 2, \ldots \text{ and } j = 1, 2$$

are indecomposable representations in $\mathfrak{L}$ (the "discrete" ones). Moreover, for an indecomposable $\mathbf{X} \in \mathfrak{L}$ such that $C^-\mathbf{X} \neq \mathbf{0}$, or $C^+\mathbf{X} \neq \mathbf{0}$, we have

$$\mathbf{dim}(C^-X) = c^{-1}(\mathbf{dim}\, X), \quad \text{or} \quad \mathbf{dim}(C^+X) = c(\mathbf{dim}\, X),$$

respectively, where c is the Coxeter transformation on $\mathbf{R}^2$ defined by

$$c(v, w) = (3v - 2w, 2v - w)$$

for all $(v,w)\in\mathbf{R}^2$. The dimension types of $\mathbf{P}_1,\mathbf{P}_2,\mathbf{I}_1,\mathbf{I}_2$ are respectively $(1,2)$, $(0,1)$, $(1,0)$, $(2,1)$, and thus there is a bijection between *discrete* indecomposable representation in $\mathcal{L}$ and the "positive roots" of $\tilde{\mathbf{A}}_{12}$, i.e., the elements

$$c^{-q}(1,2)=(2q+1,2q+2),$$

$$c^{-q}(0,1)=(2q,2q+1)$$

$$c^{q}(1,0)=(2q+1,2q),$$

$$c^{q}(2,1)=(2q+2,2q+1),$$

with $q=0,1,2,\ldots,$ in $\mathbf{R}^2$. These elements are just all the pairs $(p+1,p)$ and $(p,p+1)$ with $p=0,1,2,\ldots$. Of course, the roots $(1,0)$ and $(0,1)$ correspond to the indecomposable representations for which the map φ is zero (and which, in the matrix form, yield the zero-augmentation). Furthermore, all other ("homogeneous") indecomposable representations $\mathbf{X}$ satisfy $\mathbf{dim}\,X=(p,p)$ for some $p=1,2,\ldots$.

This proves the following lemma which, in turn, implies Lemma A.

LEMMA A*. *For each $p=1,2,\ldots,$ there is just one indecomposable representation in $\mathcal{L}$ whose dimension type is $(p+1,p)$ and just one whose dimension type is $(p,p+1)$. The dimension type of any other indecomposable representation in $\mathcal{L}$ is (p,p), $p=1,2,\ldots$*

Denote by $\mathcal{H}$ the subcategory of $\mathcal{L}$ of all homogeneous representations. This subcategory is Abelian [4], and Theorem 1 of [7] implies

LEMMA B*. *$\mathcal{H}$ is a product of uniserial subcategories of global dimension one, each of them containing only one simple representation.*

Note that Lemma B is precisely a reformulation of Lemma B* avoiding any homological notion. The rather technical statements there just describe some basic facts about extensions of simple representations in $\mathcal{H}$. For example, if we have an exact sequence

$$\mathbf{0}\to\mathbf{X}\xrightarrow{\eta}\mathbf{Y}\xrightarrow{\epsilon}\mathbf{X}\to\mathbf{0},$$

where $\mathbf{X}$ and $\mathbf{Z}$ are simple objects of $\mathcal{H}$, then either $\mathbf{Y}$ is indecomposable (and then $\mathbf{X}$ and $\mathbf{Z}$ are isomorphic, and $\mathbf{Y}$ is uniquely determined), or there exists an isomorphism $\varphi:\mathbf{Y}\to\mathbf{X}\oplus\mathbf{Z}$ such that the diagram

$$\begin{array}{ccccc} 0\to\mathbf{X} & \overset{\eta}{\to} & \mathbf{Y} & \overset{\epsilon}{\to} & \mathbf{Z}\to 0 \\ \| & & \downarrow\varphi & & \| \\ 0\to\mathbf{X} & \overset{\nu}{\to} & \mathbf{X}\oplus\mathbf{Z} & \overset{\pi}{\to} & \mathbf{Z}\to 0. \end{array},$$

where ν is the canonical inclusion and π is the canonic projection, commutes. This proves the first assertion of Lemma B(b) with

$$\mathbf{X}=\mathbf{Z}=S,\quad \mathbf{Y}=\begin{pmatrix} S & T \\ 0 & S\end{pmatrix}\quad\text{and}\quad \varphi=\left(\begin{pmatrix} I & -B \\ 0 & I\end{pmatrix},\begin{pmatrix} I & A \\ 0 & I\end{pmatrix}\right).$$

Also, in this case, given an isomorphism $\iota:\mathbf{Z}\to\mathbf{X}$, we may replace φ by the map $\varphi+\nu\iota\epsilon$, which yields the second assertion of Lemma B(b).The assertion of Lemma B* follows immediately from the fact that, for $\mathbf{X}$, $\mathbf{Y}\in\mathcal{H}$,

$$\dim_{\mathbf{R}}\operatorname{Hom}(\mathbf{X},\mathbf{Y})=\dim_{\mathbf{R}}\operatorname{Ext}^1(\mathbf{X},\mathbf{Y}).$$

The description of simple representations in $\mathcal{H}$ is given in [7] in terms of the category $\mathfrak{M}_R$ of all R-modules of finite dimension over $\mathbf{C}$ (recall that $R=\mathbf{C}[z,\bar{\ }]$):

Lemma C*. *The functor* $T:\mathfrak{M}_R\to\mathcal{H}$ *given by*

$$T(M_R)=(M_{\mathbf{C}},M_{\mathbf{C}},\varphi),\qquad \textit{where}\quad \varphi(m\otimes s)=m\quad\textit{and}\quad \varphi(m\otimes t)=mz,$$

defines a full exact embedding. The category $\mathcal{H}$ *is the direct product of* $T(\mathfrak{M}_R)$ *and a uniserial category of global dimension one with one simple object* $\mathbf{S}$:

$$\mathbf{S}=(\mathbf{C},\mathbf{C},\varphi),\qquad \textit{where}\quad \varphi(c\otimes s)=0\quad\textit{and}\quad \varphi(c\otimes t)=\bar{c}.$$

Now, Lemma C follows from Lemma C*, because an easy calculation shows that the matrix representation given there is just the evaluation of the mapping

$$\psi: M_{\mathbf{C}}\otimes_{\mathbf{C}}\mathbf{C}_{\mathbf{R}}\to\operatorname{Hom}_{\mathbf{C}}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}},M_{\mathbf{C}})$$

which corresponds to $\varphi: M_{\mathbf{C}} \otimes_{\mathbf{C}} \mathbf{C}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}_{\mathbf{C}} \to M_{\mathbf{C}}$ of $T(\mathfrak{M}_R)$ with respect to the basis $\{\mathbf{u}_1 \otimes 1, \mathbf{u}_1 \otimes i, \mathbf{u}_2 \otimes 1, \mathbf{u}_2 \otimes i, \ldots, \mathbf{u}_d \otimes 1, \mathbf{u}_d \otimes i\}$, where $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_d\}$ is a $\mathbf{C}$-basis of M. For, if $\mathbf{u}_l z = \sum_{k=1}^{d} \mathbf{u}_k(a_{kl} + ib_{kl})$, then

$$\varphi(\mathbf{u}_l \otimes 1 \otimes 1) = \tfrac{1}{2}\left[\varphi(\mathbf{u}_l \otimes s) + \varphi(\mathbf{u}_l \otimes t)\right] = \frac{1}{2}\left[\mathbf{u}_l + \sum_{k=1}^{d} \mathbf{u}_k(a_{kl} + ib_{kl})\right]$$

and

$$\varphi(\mathbf{u}_l \otimes i \otimes 1) = -\varphi(\mathbf{u}_l \otimes i \otimes i)i \quad = \tfrac{1}{2}\left[\varphi(\mathbf{u}_l \otimes s) - \varphi(\mathbf{u}_l \otimes t)\right]i$$

$$= \frac{1}{2}\left[\mathbf{u}_l - \sum_{k=1}^{d} \mathbf{u}_k(a_{kl} + ib_{kl})\right]i = \frac{1}{2}\left[\mathbf{u}_l i + \sum_{k=1}^{d} \mathbf{u}_k(b_{kl} - ia_{kl})\right].$$

Thus, denoting by $\{\mathbf{u}_1', \mathbf{u}_1'', \mathbf{u}_2', \mathbf{u}_2'', \ldots, \mathbf{u}_d', \mathbf{u}_d''\}$ the basis of $\mathrm{Hom}_{\mathbf{C}}({}_{\mathbf{R}}\mathbf{C}_{\mathbf{C}}, M_{\mathbf{C}})$ given by $\mathbf{u}_k'(1) = \mathbf{u}_k 1/2$ and $\mathbf{u}_k''(1) = \mathbf{u}_k i/2$, we have

$$\psi(\mathbf{u}_l \otimes 1) = \mathbf{u}_l' + \sum_{k=1}^{d} (\mathbf{u}_k' a_{kl} + \mathbf{u}_k'' b_{kl})$$

and

$$\psi(\mathbf{u}_l \otimes i) = \mathbf{u}_l'' + \sum_{k=1}^{d} (\mathbf{u}_k' b_{kl} - \mathbf{u}_k'' a_{kl}).$$

Also, the matrix representation corresponding to S is given, in a similar way, by

$$\psi(\mathbf{u} \otimes 1) = \mathbf{u}' \quad \text{and} \quad \psi(\mathbf{u} \otimes i) = -\mathbf{u}''.$$

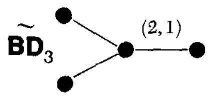
Finally, it remains to verify Lemma D. It is a routine matter to show that $x - r$, with an arbitrary non-negative real r, and $z^2 - (a + ib)$, with either $b > 0$ or $b = 0$ and $a < 0$, are irreducible elements in R, and that there is a bijection between these polynomials and the isomorphism classes of simple R-modules M with $\dim M_{\mathbf{C}} \leqslant 2$.

But, every simple R-module M satisfies $\dim M_{\mathbf{C}} \leqslant 2$. For, R is a principal ideal ring containing $\mathbf{R}$ as a central subfield, and thus $M = R/fR$, $f \in R$, is a finite-dimensional $\mathbf{R}$-module. Consequently, $E = \mathrm{End}\, M_R$ and therefore also

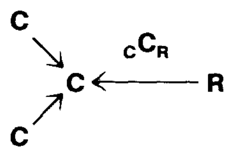
$D = \mathrm{End}_E M$ are finite-dimensional $\mathbf{R}$-algebras. Since R/I, where I is the annihilator ideal of M, can be embedded in D (in fact, $R/I \approx D$), we have $I \neq 0$. Now $I = gR$ with $g \in R$ is a maximal two sided ideal, and thus an easy calculation shows that $g = z$ or $g \in \mathbf{R}[z^2]$. From this, it follows immediately that $\dim M_{\mathbf{C}} \leqslant 2$.

The statement of Lemma D is proved in [1]. We are indebted to P. M. Cohn for pointing out that a general argument is given in his monograph on *Free Rings and Their Relations* (p. 234).

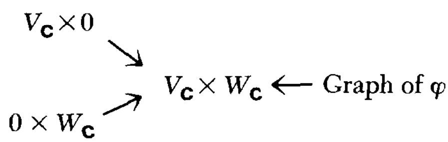
In conclusion, let us make two brief remarks. First, our problem can be also interpreted as the classification of all indecomposable representations of the Dynkin diagram



with the species



subject to certain dimension conditions:



Second, considering the Dynkin diagram $\tilde{\mathbf{A}}_{12}$ with the species

$$F \xrightarrow{{}_F F_F \oplus {}_F F_F} F,$$

our method yields immediately Kronecker's normal form of pairs of matrices over a division ring F.

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