

# THE REPRESENTATIONS OF TAME HEREDITARY ALGEBRAS

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## §1. INTRODUCTION

An (associative, not necessarily commutative) ring  $R$  (with 1) is called (right) *hereditary* if each right ideal of  $R$  is projective or, equivalently, if the functor  $\text{Ext}^1(X, -)$  is right exact for any (right)  $R$ -module  $X$ . Important examples of hereditary rings are the *tensor rings* defined in the following way: Let  $S$  be an (artinian) semisimple ring, and  ${}_S M_S$  an  $S$ - $S$ -bimodule; we denote inductively  $M^{i+1} = M^i \otimes_S M$ , with  $M^0 = S$ ; then, inducing the multiplication by the tensor product  $\otimes$  the direct sum  $T(M) = \bigoplus_{i \geq 0} M^i$  becomes a ring. In this paper, we shall consider only *semiprimary* rings; for these rings, the properties of being right hereditary and left hereditary coincide. It is easy to see that the tensor ring  $T(M)$  is semiprimary if and only if  $M^i = 0$  for some  $i$ . Note that there exist examples of hereditary semiprimary rings which are not tensor rings [10]; later in this paper, we shall consider one particular class of such rings in more detail. Also, we should mention the following class of tensor rings: Let  $F, G$  be (not necessarily commutative) fields, and  ${}_F M_G$  an  $F$ - $G$ -bimodule. Writing  $S = F \times G$ ,  $M$  can be considered as an  $S$ - $S$ -bimodule, and we denote by  $R({}_F M_G)$  the tensor algebra  $T({}_S M_S)$  which, of course, is just the matrix ring  $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ . For such a

bimodule  ${}_F M_G$ , the invariant  $d({}_F M_G) = (\dim_F M) \cdot (\dim M_G)$  is known to be of importance (here,  $\dim$  denotes the ordinary vector space dimension).

An indecomposable  $R$ -module  $X$  is a (right)  $R$ -module having no non-zero submodules  $Y_1$  and  $Y_2$  such that  $X = Y_1 \oplus Y_2$ . The semiprimary ring  $R$  is said to be of *finite representation type* if there is only a finite number of (isomorphism classes of) indecomposable  $R$ -modules. In this case, the indecomposable modules are of finite length, and every (arbitrarily large) module is the direct sum of indecomposable ones [9]. We say that a hereditary semiprimary ring  $R$  is of *wild representation type* provided that there are fields  $F$  and  $G$  and a bimodule  $M = {}_F M_G$  with  $d(M) > 4$  such that the category of all  $R(M)$ -modules of finite length can be embedded as a full and exact subcategory into the category  $M_R$  of  $R$ -modules of finite length. (For arbitrary, not necessarily hereditary semiprimary rings, this definition would be too special; in addition to the full exact subcategories one has to consider also those which are representation equivalent to them). This rather technical condition has the following interpretation: If  $R$  is a finite dimensional algebra over a commutative field, then the condition is equivalent to the fact that there exists a commutative field  $k$  such that for any finite dimensional  $k$ -algebra  $A$ , the category of all  $A$ -modules of finite length can be embedded as a full and exact subcategory into  $M_R$ . Thus, it is unreasonable to expect a complete classification of all  $R$ -modules of finite length in this case. Also in the general case, the category  $M_R$  with  $R$  of wild representation type seems to contain a rather large amount of indecomposable modules of finite length. For example, for such a ring  $R$  and any natural number  $n$ , there exists  ${}_F M_G$  with  $d(M) > n$  such that  $M_{R(M)}$  can be embedded as a full and exact subcategory into  $M_R$ . If  $R$  is neither of finite nor of wild representation type, then  $R$  is called *tame*.

The hereditary semiprimary rings of finite representation type, together with all their modules, have been completely described in [4]; a more conceptual account has appeared in [5]. In particular, it has been shown that they are always tensor rings. The tame tensor rings which satisfy some duality condition (which is, for example, satisfied in the case of finite dimensional algebras) have been described in [5]; also all but a certain class of their modules (the "homogeneous" ones) were exhibited in detail there. However, the following construction will show that there are tame hereditary semiprimary rings which are not tensor rings.

Let  $F$  be a field,  $\varepsilon$  an automorphism of  $F$ , and  $\delta$  an  $(\varepsilon, 1)$ -derivation of  $F$  (that is,  $\delta : F \rightarrow F$  is additive and satisfies  $\delta(f_1 f_2) = \varepsilon(f_1) \cdot \delta(f_2) + \delta(f_1) \cdot f_2$ ; see [2]). This information leads to an  $F$ - $F$ -bimodule  $M = M(\varepsilon, \delta)$  which, as a left  $F$ -space is just  ${}_F F \oplus {}_F F$ , whereas the right  $F$ -action is given by

$$(a, b) \cdot f = (af + b \cdot \delta(f), b \cdot \varepsilon(f)) \quad \text{for } a, b, f \in F.$$

Note that the  $F$ - $F$ -submodule  $F \oplus 0$  of  $M$  and  ${}_F F_F$  are canonically isomorphic, and we shall identify them. Now we can define the ring  $\tilde{A}_n(\varepsilon, \delta)$  as the  $(n+1) \times (n+1)$ -matrix ring

$$\begin{pmatrix} F & & & & & \\ & F & & & & \\ & & F & & & \\ & & & \ddots & & \\ & & & & F & \\ & & & & & F \\ M & F & \cdots & F & F & \end{pmatrix}$$

(it contains, as a subring, the ring of all lower triangular  $(n+1) \times (n+1)$  matrices over  $F$ ). It is obvious that  $\tilde{A}_n(\varepsilon, \delta)$  is hereditary and semiprimary. For  $n = 1$ ,  $\tilde{A}_1(\varepsilon, \delta) = R({}_F M_F)$ ; thus we get a tensor ring. However, for  $n > 1$ ,  $\tilde{A}_n(\varepsilon, \delta)$  is a tensor ring only in the case that the  $(\varepsilon, 1)$ -derivation  $\delta$  is inner [2]. Indeed,  $M(\varepsilon, \delta)$

decomposes as a bimodule if and only if  $\delta(f) = \varepsilon(f)c - cf$  for a suitable  $c \in F$ .

Theorem 1. *A tame hereditary semiprimary ring is Morita equivalent to the product of a tensor ring and a finite number of rings of the form  $\tilde{A}_n(\varepsilon, \delta)$  for a suitable choice of  $n$ 's,  $\varepsilon$ 's and  $\delta$ 's.*

Next we are going to consider the question whether it is possible to describe the category of all  $\tilde{A}_n(\varepsilon, \delta)$ -modules of finite length. For  $\delta = 0$ , this has been done in [5] (where the  $\tilde{A}_n(\varepsilon, 0)$ -modules were considered as representations of the extended Dynkin diagram  $\tilde{A}_n$  with respect to a modulation using  $\varepsilon$ , and a suitable orientation; note that this explains our notation  $\tilde{A}_n(\varepsilon, \delta)$ ). On the other hand, for  $n = 1$ , the problem was solved in [7] (where the  $\tilde{A}_1(\varepsilon, \delta)$ -modules were called representations of a non-simple affine bimodule). The general case consists in a combination of these results, as the next theorem reveals. We denote by  $F[z; \varepsilon, \delta]$  the twisted polynomial ring consisting of all finite formal sums  $\sum_{t=0}^{\infty} f_t z^t$  with  $f_t \in F$ , and multiplication defined by  $zf = \varepsilon(f)z + \delta(f)$ . Evidently,  $\tilde{A}_n(\varepsilon, \delta)$  has precisely  $n+1$  simple modules  $S_0, \dots, S_n$ , which are ordered in such a way that  $\text{Ext}^1(S_i, S_{i-1}) \neq 0$  for  $1 \leq i \leq n$ ; then we have also  $\text{Ext}^1(S_n, S_0) \neq 0$ .

In what follows, always  $n > 1$ . A sequence  $(r_1, \dots, r_d)$  of integers satisfying  $0 \leq r_t \leq n$  is called *regular* if  $r_1 \neq n$ ,  $r_d \neq 0$  and  $r_{t+1} = \pi(r_t)$  for all  $t$ , where  $\pi$  is the cyclic permutation  $(1 \ 2 \ \dots \ n-1 \ 0 \ n)$ . Furthermore,  $(r_1, r_2, \dots, r_d)$  is called *preprojective* if  $r_1 = 0$ ,  $r_2 = 1$  and  $(r_2, \dots, r_d)$  is regular. Finally,  $(r_1, \dots, r_{d-1}, r_d)$  is called *preinjective* if  $r_{d-1} = n-1$ ,  $r_d = n$  and  $(r_1, \dots, r_{d-1})$  is regular.

Theorem 2. Let  $n > 1$ . The indecomposable  $\tilde{A}_n(\epsilon, \delta)$ -modules are of the following types: For every regular, preprojective, or preinjective sequence  $(r_1, \dots, r_d)$ , there is a unique indecomposable module  $X$  with a composition series

$$0 = X_0 \subset X_1 \subset \dots \subset X_d = X,$$

such that  $X_t/X_{t-1} \cong S_{r_t}$  for all  $t$ . The direct sums of the remaining indecomposable modules of finite length form an abelian exact subcategory which is equivalent to the category of  $F[z; \epsilon, \delta]$ -modules of finite length.

The equivalence of the categories is given in the following way:

The  $F[z; \epsilon, \delta]$ -module  $W$  is associated with the  $\tilde{A}_n(\epsilon, \delta)$ -module  $\underbrace{W \oplus \dots \oplus W}_{n+1}$  on which  $\tilde{A}_n(\epsilon, \delta)$  operates by the ordinary matrix operation from the right, with the additional condition that, for  $(a, b) \in M$  and  $w \in W$ , we define  $w(a, b) = wa + wbz$ . The above theorem has several consequences:

Corollary 1. The representation type of a hereditary semiprimary ring  $A$  is not determined by the quotient  $A/N(A)^2$ , where  $N(A)$  denotes the radical of  $A$ .

In fact, taking for  $F$  the field  $\mathbb{C}(z)$  of rational functions in one variable over the complex numbers, for  $\epsilon$  the identity automorphism and for  $\delta$  the ordinary differentiation of functions, one can use, as in [7], the results of [6] to show that the ring  $A = \tilde{A}_n(1, \delta)$  is wild, while the ring  $B = \tilde{A}_n(1, 0)$  is tame. But for  $n > 1$ ,  $A/N(A)^2 = B/N(B)^2$ . Furthermore, using the results of [3], the ring  $\tilde{A}_n(1, \delta)$ , where  $F$  is a differentially closed field with a differential  $\delta$ , will have only finitely many indecomposable modules of any given finite length. In contrast,  $\tilde{A}_n(1, 0)$  with any infinite field  $F$  has infinitely many non-isomorphic

indecomposable modules of length  $d$  for an infinite number of integers  $d$ .

Now, if we assume that  $F$  contains a central subfield  $k$  of finite index such that the restriction of  $\epsilon$  to  $k$  is the identity and  $\delta$  is trivial on  $k$ , then (and only then)  $\tilde{A}_n(\epsilon, \delta)$  becomes a finite dimensional  $k$ -algebra. Moreover, the  $F[z, \epsilon, \delta]$ -modules of finite length are uniserial, so that  $\tilde{A}_n(\epsilon, \delta)$  cannot be wild. Thus, in this case, the two theorems yield the following result.

Corollary 2. *Let  $k$  be a commutative field and  $A$  a hereditary finite dimensional  $k$ -algebra. Then  $A$  is tame if and only if it is Morita equivalent to the product of a tame tensor algebra and a finite number of algebras of the form  $\tilde{A}_n(\epsilon, \delta)$ .*

Let us remark that an indecomposable tame tensor algebra is determined by a modulation of an extended Dynkin diagram.

The paper is divided into 3 sections followed by a remark. In §2, two endofunctors are defined in the category  $M_A$  of all  $\tilde{A}_n(\epsilon, \delta)$ -modules of finite length, and are used then in §3 to prove Theorem 2. The proof of Theorem 1 is completed in §4. The final remark refers to a recent paper [1].

## §2. CONSTRUCTION OF FUNCTORS

In the representation theory of tensor rings, certain functors play a decisive role; see [5]. One constructs first elementary functors whose behaviour imitates that of the basic reflections in the Weyl group, and then composes them to the Coxeter functors, which correspond to the Coxeter transformations in the Weyl group. The aim of this section is to construct similar functors for the rings of the form  $A = \tilde{A}_n(\epsilon, \delta)$ . It turns out that the situation is much easier in this case: The elementary functors

which will be denoted by  $\Gamma^+$  and  $\Gamma^-$  are endofunctors, and one may work with them in the same way as one usually does with the Coxeter functors (which are here the  $(n+1)$ -powers  $\Gamma^{+(n+1)}$  and  $\Gamma^{-(n+1)}$  of the elementary functors).

We denote by  $M_A$  the category of all  $A$ -modules of finite length. If  $S_0, \dots, S_n$  is the canonical ordering of the simple  $A$ -modules introduced in §1, then for every  $A$ -module  $X$  of finite length, we denote by  $\underline{\dim} X = (x_0, \dots, x_n)$  the  $(n+1)$ -tuple in which  $x_i$  is the number of the composition factors of  $X$  isomorphic to  $S_i$ ; we shall consider  $\underline{\dim} X$  to be an element of the rational vector space  $\mathbb{Q}^{n+1}$ . Observe that  $A$  has precisely one projective simple module, namely  $S_0$ , and precisely one injective simple module  $S_n$ . Let us reiterate that, since the case  $n=1$  is known by [5] and [7], we shall always assume  $n > 1$ .

Proposition. *Let  $n > 1$ . For the ring  $A = \tilde{A}_n(\epsilon, \delta)$ , there exist functors  $\Gamma^+ : M_A \rightarrow M_A$  and  $\Gamma^- : M_A \rightarrow M_A$  with the following properties:*

(i)  $\Gamma^+ S_0 = 0$ , while for any other indecomposable  $A$ -module  $X$  (with  $\underline{\dim} X = (x_0, x_1, \dots, x_n)$ ), there is a canonical isomorphism  $\Gamma^- \Gamma^+ X \simeq X$  and

$$\underline{\dim} \Gamma^+ X = (x_1, \dots, x_n, -x_0 + x_1 + x_n).$$

(ii)  $\Gamma^- S_n = 0$ , while for any other indecomposable  $A$ -module  $Y$  (with  $\underline{\dim} Y = (y_0, \dots, y_{n-1}, y_n)$ ), there is a canonical isomorphism  $\Gamma^+ \Gamma^- Y \simeq Y$  and

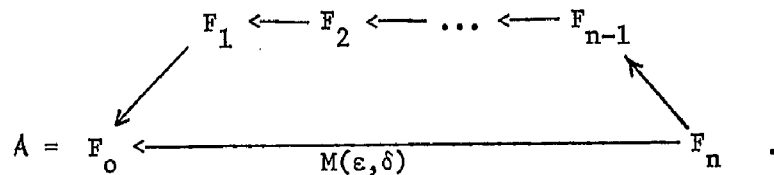
$$\underline{\dim} \Gamma^- Y = (-y_n + y_0 + y_{n-1}, y_0, \dots, y_{n-1}).$$

(iii)  $\Gamma^+$  is a right adjoint for  $\Gamma^-$ .

As a consequence, one gets again the usual properties (compare [5]:  $\Gamma^+$  is left exact;  $\Gamma^-$  is right exact; if  $X \neq S_0$  is

indecomposable, then  $\text{End}(X) \approx \text{End}(\Gamma^+ X)$  and  $\text{Ext}^1(Y, X) \approx \text{Ext}^1(\Gamma^+ Y, \Gamma^+ X)$  for all  $Y$  etc. [On the other hand, note that the dim-formulas are different from the usual ones. The dim-change is the composition of the expected reflection  $(x_0, \dots, x_n) \mapsto (-x_0 + x_1 + x_n, x_1, \dots, x_n)$  and a cyclic permutation. The reason for the appearance of the cyclic permutation lies in the fact that we use a fixed (internally defined) ordering of the simple modules  $S_i$ 's.]

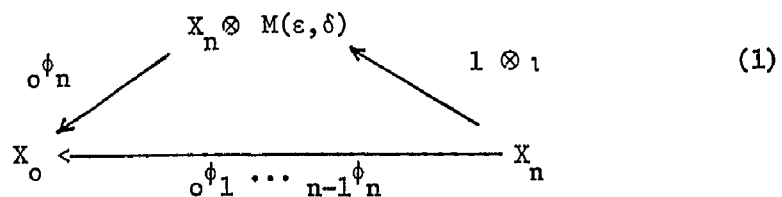
In order to facilitate the work with the  $A$ -modules we shall interpret them as the representations of the species where  $F_i = F$  for all  $i$ .



This seems to be the easiest way to get a better understanding of the internal structure of the modules and to have some graphical methods for illustration available. Recall that there is a canonical copy of  ${}_{F^F}F$  embedded into  $M(\epsilon, \delta)$ ; we denote the embedding by  $\iota$ . The representation  $\underline{X} = (X_i, j\phi_i)$  of the species  $A$  consist of  $n+1$   $F$ -vector spaces  $X_i$ ,  $n$  linear maps  ${}_{i-1}\phi_i : X_i \rightarrow X_{i-1}$  ( $1 \leq i \leq n$ ) and a linear map  ${}_0\phi_n : X_n \xrightarrow{M(\epsilon, \delta)} X_0$ . A map  $\underline{\alpha} = (\alpha_i) : (X_i, j\phi_i) \rightarrow (X'_i, j\phi'_i)$  is given by  $n+1$  linear maps  $\alpha_i : X_i \rightarrow X'_i$  such that

$$\alpha_{i-1} {}_{i-1}\phi_i = {}_{i-1}\phi'_i \alpha_i \quad (1 \leq i \leq n) \quad \text{and} \quad \alpha_0 {}_0\phi_n = {}_0\phi'_n (\alpha_n \otimes 1).$$

The full subcategory of all representations  $(X_i, j\phi_i)$  of  $A$  such that





commutes will be denoted by  $Le(A)$ . This category is equivalent to the category  $M_A$  of all  $\tilde{A}_n(\epsilon, \delta)$ -modules of finite length.

It seems to be convenient to have another description of the category  $M_A$  available; in the proof of the Proposition we will use both descriptions simultaneously.

To begin with, we need some additional notation: If  $X$  is an  $F$ -vector space, and  $\epsilon$  an automorphism of the field  $F$ , we denote by  $X\epsilon$  the twisted vector space with the scalar multiplication  $\cdot$  defined by  $x \cdot f = x \epsilon(f)$  for  $x \in X$ ,  $f \in F$ . If  $\alpha : X_F \rightarrow Y_F$  is  $F$ -linear, then the same  $\alpha$ , considered as a map  $X\epsilon \rightarrow Y\epsilon$ , is again  $F$ -linear. In particular, considering the  $F$ - $F$ -bimodule  $F\epsilon$  (with  $f_1 \cdot x \cdot f_2 = f_1 x \epsilon(f_2)$  for  $x, f_1, f_2 \in F$ ), we have  $X\epsilon = X \otimes_F F\epsilon$ . Note that there is an obvious exact sequence of  $F$ - $F$ -bimodules

$$0 \longrightarrow F \longrightarrow M(\epsilon, \delta) \xrightarrow{\pi} F\epsilon \longrightarrow 0.$$

Also, recall that for a bimodule  ${}_F N_G$  over the fields  $F$  and  $G$  with finite  $\dim_F N$ , we may define an  $G$ - $F$ -bimodule  ${}^*N = \text{Hom}_F({}_F N_G, {}_F F_F)$  (left dual!) such that there is a canonical one-to-one correspondence between the  $G$ -linear maps  $X_G \rightarrow Y_F \otimes_F N_G$  and the  $F$ -linear maps  $X_G \otimes_G ({}^*N)_F \rightarrow Y_F$  for any vector spaces  $X_G$  and  $Y_F$ . It is obvious that for an automorphism  $\epsilon$  of  $F$ , one has  ${}^*(F\epsilon) = F\epsilon^{-1}$ . Also, we will need that  ${}^*(M(\epsilon, \delta)) = F\epsilon^{-1} \otimes M(\epsilon, \delta)$ . [Proof [7]: Consider the basis of the left  $F$ -space  $N = M(\epsilon, \delta) \otimes_F F\epsilon^{-1} \otimes M(\epsilon, \delta)$  given by the elements  $u_{ij} = u_i \otimes 1 \otimes u_j + (1 \leq i, j \leq 2)$ , where  $u_1 = (1, 0)$  and  $u_2 = (0, 1) \in M(\epsilon, \delta)$ . Then,

$$\begin{aligned} u_{11} \cdot f &= \epsilon^{-1}(f) \cdot u_{11}, \\ u_{12} \cdot f &= \epsilon^{-1}\delta(f) \cdot u_{11} + f \cdot u_{12}, \\ u_{21} \cdot f &= \delta\epsilon^{-1}(f) \cdot u_{11} + f \cdot u_{21}, \\ u_{22} \cdot f &= \delta\epsilon^{-1}\delta(f) \cdot u_{11} + \delta(f) \cdot u_{12} + \delta(f) \cdot u_{21} + \epsilon(f) \cdot u_{22}. \end{aligned}$$

This shows, that  $u_{11}, u_{22}, u_{12} + u_{21}$  generate an  $F$ - $F$ -bisubmodule  $U$  such that  ${}_{F}N_F / {}_{F}U_F \cong {}_{F}F_F$ . The canonical map

$$M(\epsilon, \delta) \otimes (F\epsilon^{-1} \otimes M(\epsilon, \delta)) \longrightarrow N/U \cong {}_{F}F_F$$

induces the map

$$F\epsilon^{-1} \otimes M(\epsilon, \delta) \longrightarrow \text{Hom}_F(M(\epsilon, \delta), {}_{F}F_F) \cong {}^*M(\epsilon, \delta)$$

which is obviously injective, and is therefore an isomorphism of bimodules.]

We know that the  $A$ -modules correspond to those representations of the species

$$\begin{array}{ccccccc}
 & & F_1 & \longleftarrow & F_2 & \longleftarrow & \dots & \longleftarrow & F_{n-1} \\
 M(\epsilon, \delta) & \nearrow & & & & & & & \\
 F_0 & \xrightarrow{\hspace{10em}} & & & & & & & F_n
 \end{array}$$

with  $F_i = F$  for all  $i$  which satisfy a certain commutativity condition (note that we use now a different numbering of the indices!). Equally well, we may consider the species

$$\begin{array}{ccccccc}
 B = & & F_1 & \longleftarrow & F_2 & \longleftarrow & \dots & \longleftarrow & F_{n-1} \\
 F\epsilon^{-1} \otimes M(\epsilon, \delta) & \nearrow & & & & & & & \\
 F_0 & \xrightarrow{\hspace{10em}} & & & & & & & F_n \\
 & & & & & & & & F\epsilon^{-1}
 \end{array}$$

with  $F_i = F$  for all  $i$ : The categories of the representations are obviously equivalent under the functor  $(X_0, \dots, X_n, j\phi_i) \longmapsto (X_0\epsilon, X_1, \dots, X_n, j\phi_i)$ . Since  $F\epsilon^{-1} = {}^*(F\epsilon)$  and  $F\epsilon^{-1} \otimes M(\epsilon, \delta) = {}^*M(\epsilon, \delta)$ , the representations of  $B$  can be written in the form  $(Y_i, j\psi_i)$  with the  $F$ -vector spaces  $Y_i$  ( $0 \leq i \leq n$ ) and the linear maps

$${}_1\psi_0: Y_0 \longrightarrow Y_1 \otimes M(\epsilon, \delta), \quad {}_{i-1}\psi_i: Y_i \longrightarrow Y_{i-1} \quad (2 \leq i \leq n), \quad \text{and} \quad {}_n\psi_0: Y_0 \longrightarrow Y_n \epsilon.$$

In this way, the A-modules correspond just to those  $(Y_i, \psi_i)$  for which the diagram

$$\begin{array}{ccc}
 & Y_1 \otimes M(\epsilon, \delta) & \xrightarrow{1 \otimes \pi} & Y_1 \epsilon & \\
 1 \psi_0 \nearrow & & & & \nwarrow 1 \psi_2 \cdots n-1 \psi_n \\
 Y_0 & \xrightarrow{n \psi_0} & & & Y_n \epsilon
 \end{array} \quad (2)$$

commutes. The full subcategory of those representations of  $B$  which satisfy the condition (2) will be denoted by  $Lc'(B)$ . Obviously, it is equivalent to  $M_A$  and in this way we get an alternative description of the category of all A-modules of finite length.

In the remaining part of this section, we denote the bimodule  $M(\epsilon, \delta)$  simply by  $M$ . Recall that we have an exact sequence of bimodules  $0 \rightarrow F \xrightarrow{l} M \xrightarrow{\pi} F\epsilon \rightarrow 0$ .

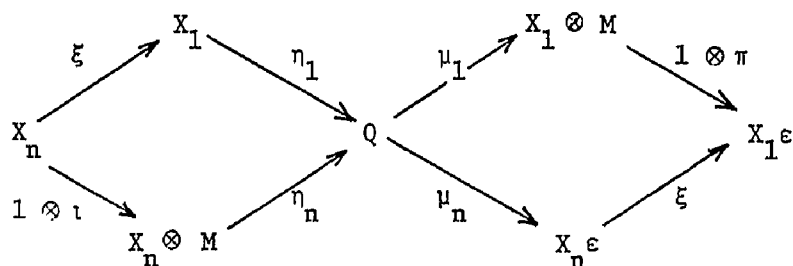
Let  $X_1, X_n$  be  $F$ -spaces, and  $\xi : X_n \rightarrow X_1$  a linear map. We claim that the following sequence is exact:

$$0 \rightarrow X_n \xrightarrow{\begin{pmatrix} \xi \\ 1 \otimes 1 \end{pmatrix}} X_1 \oplus (X_n \otimes M) \xrightarrow{\begin{pmatrix} 1 \otimes 1 & -\xi \otimes 1 \\ 0 & 1 \otimes \pi \end{pmatrix}} (X_1 \otimes M) \oplus X_n \epsilon \xrightarrow{(1 \otimes \pi, \xi)} X_1 \epsilon \rightarrow 0.$$

For, it is easy to check that the composition of any two consecutive maps is zero, and the following diagram shows that the sequence is the extension of two exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & & \xrightarrow{1 \otimes 1} & & \xrightarrow{1 \otimes \pi} & 0 \\
 & & X_1 & & X_1 \otimes M & & X_1 \epsilon \longrightarrow 0 \\
 & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \\
 0 & \longrightarrow & X_n & \longrightarrow & (X_1 \otimes M) \oplus X_n \epsilon & \longrightarrow & X_1 \epsilon \longrightarrow 0 \\
 & & \downarrow (0, 1) & & \downarrow (0, 1) & & \\
 0 & \longrightarrow & X_n & \xrightarrow{1 \otimes 1} & X_n \otimes M & \xrightarrow{1 \otimes \pi} & X_n \epsilon \longrightarrow 0
 \end{array}$$

Let  $Q$  be the image of the middle map  $\begin{pmatrix} 1 \otimes \iota & -\xi \otimes 1 \\ 0 & 1 \otimes \pi \end{pmatrix}$ , and let  $\eta_1, \eta_n, \mu_1,$  and  $\mu_n$  be the corresponding canonical maps; hence, in the diagram



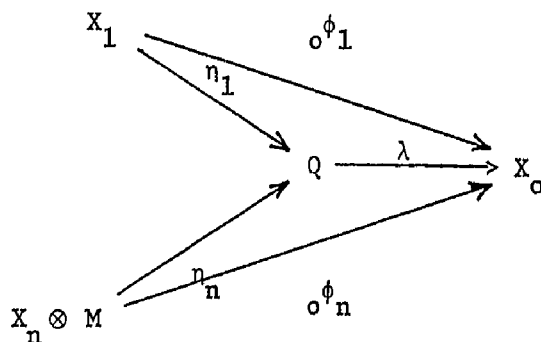
the left square is a pushout and the right square is a pullback. This pair of squares will be of importance in the sequel. The dimension of  $Q_F$  is easy to calculate:

$$\dim Q_F = \dim (X_1)_F + \dim (X_n)_F .$$

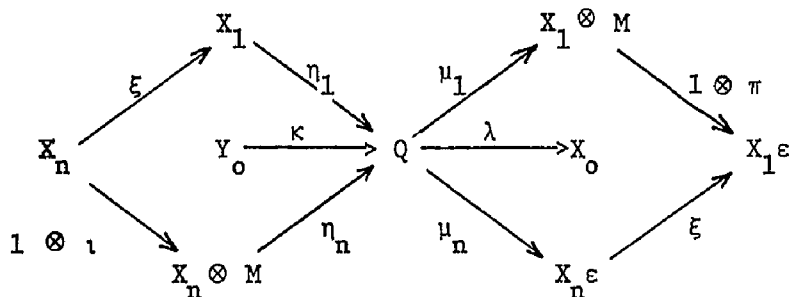
Now, we are going to define the functor

$$\Gamma^+ : M_A \approx Lc(A) \longrightarrow Lc'(B) \approx M_A$$

of the Proposition. Let  $\underline{X} = (X_i, \phi_i)$  be a representation in  $Lc(A)$ . Take the map  $\xi = \phi_1 \circ \phi_2 \circ \dots \circ \phi_n : X_n \longrightarrow X_1$  and form the above pair of squares. The commutativity condition (1) shows that we can factor the two maps  $\phi_1$  and  $\phi_n$  through  $Q$  and get



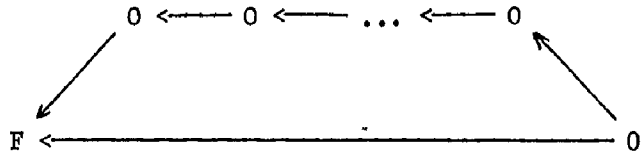
Now, define  $Y_0$  to be the kernel of the map  $\lambda$ ; thus we have



If, in addition, we define  $Y_i = X_i$  for  $i \neq 0$ ,  ${}_{i-1}\psi_1 = {}_{i-1}\phi_1$  for  $2 \leq i \leq n$ , and  ${}_{1}\psi_0 = \mu_1 k$ ,  ${}_{n}\psi_0 = \mu_n k$ , we get a representation  $\underline{Y} = (Y_i, {}_j\psi_i)$  in  $Lc^+(\mathcal{B})$ ; put  $\Gamma^+ \underline{X} = \underline{Y}$ . Note that the commutativity condition (2) follows from the commutativity of the right square above. It is obvious that this construction is functorial.

The reverse functor  $\Gamma^-$  is equally easy to construct given a representation  $\underline{Y} = (Y_i, {}_j\psi_i)$  in  $Lc^+(\mathcal{B})$ , we define  $\Gamma^- \underline{Y} = \underline{X} = (X_i, {}_j\phi_i)$  as follows. Put  $X_i = Y_i$  for  $i \neq 0$ , and  ${}_{i-1}\phi_i = {}_{i-1}\psi_i$  for  $2 \leq i \leq n$ . Then denoting the composition  ${}_{1}\phi_2 \cdots {}_{n-1}\phi_n$  of these maps by  $\xi$ , we use again for this  $\xi$  the pair of squares. This time, we factor the two maps  ${}_{1}\psi_0$  and  ${}_{n}\psi_0$  through  $Q$ , and, in this way, we get  $k : Y_0 \rightarrow Q$ . Denote the cokernel of  $k$  by  $X_0$  and the cokernel map by  $\lambda : Q \rightarrow X_0$ . In order to complete the definition of  $\Gamma^-$  we set  ${}_{0}\phi_1 = \lambda \eta_1$  and  ${}_{0}\phi_n = \lambda \eta_n$ .

Next, assume that  $\underline{X} \in Lc(A)$  is given. Then either the map  $\lambda : Q \rightarrow X_0$  (defined above) is surjective, in which case we obviously have  $\Gamma^- \Gamma^+ \underline{X} = \underline{X}$ , and also  $\dim (Y_0)_F = \dim Q_F - \dim (X_0)_F$ . Or,  $\lambda$  is not surjective, and then a copy of the representation

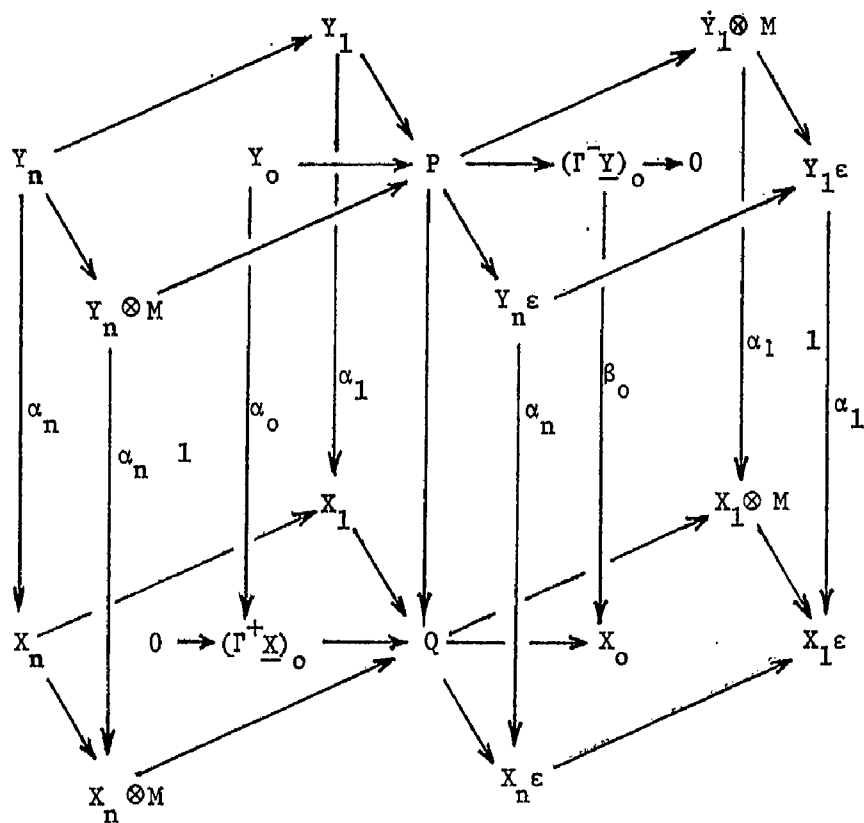


splits off  $\underline{X}$ . Thus, if  $\underline{X}$  is indecomposable, then it is simple projective (of course, conversely, this is the only simple projective representation in  $Lc(A)$ ). Similar arguments apply for  $\underline{Y} \in Lc'(B)$  and establish that either  $\Gamma^+ \Gamma^- \underline{Y} = \underline{Y}$ , or that a simple injective representation is a direct summand of  $\underline{Y}$ .

The dimension  $\underline{\dim} X$  of an  $A$ -module  $X$  was defined in terms of the canonical ordering of the simple modules  $S_i$  introduced in §1. Thus, if the  $A$ -module  $Y$  corresponds to the representation  $\underline{Y} = (Y_i, {}_j \psi_i)$  in  $Lc'(B)$  and if  $\dim (Y_i)_F = y_i$ , then  $\underline{\dim} Y = (y_1, \dots, y_n, y_0)$ . This immediately yields the  $\underline{\dim}$ -formula, since for an indecomposable representation  $\underline{X} = (X_i, {}_j \phi_i)$  in  $Lc(A)$  which is not simple projective and which satisfies  $\Gamma^+ \underline{X} = \underline{Y}$ , we have

$$\dim Y_0 = \dim Q - \dim X_0 = \dim X_1 + \dim X_n - \dim X_0.$$

To complete the proof of the Proposition, it remains to show that the functors  $\Gamma^+$  and  $\Gamma^-$  are adjoint. Let  $\underline{X} \in Lc(A)$ ,  $\underline{Y} \in Lc'(B)$  and  $\alpha = (\alpha_i) : \underline{Y} \rightarrow \Gamma^+ \underline{X}$ . In order to define the corresponding map  $\Gamma^- \underline{Y} \rightarrow \underline{X}$ , we only replace  $\alpha_0 : Y_0 \rightarrow (\Gamma^+ \underline{X})_0$  by a suitable map  $\beta : (\Gamma^- \underline{Y})_0 \rightarrow X_0$  defined by the following diagram



It is clear that, in this way, we get a bijection between  $\text{Hom}(\underline{Y}, \Gamma^+ \underline{X})$  and  $\text{Hom}(\Gamma^- \underline{Y}, \underline{X})$  which is natural in both arguments.

### §3. THE INDECOMPOSABLE $\tilde{A}_n(\epsilon, \delta)$ -MODULES

Our aim is to prove Theorem 2 of §1. Hence, we deal with the modules over a ring  $A = \tilde{A}_n(\epsilon, \delta)$  for some fixed  $n > 1$ ,  $\epsilon$  and  $\delta$ . As in the case of tame tensor rings, we introduce the notion of *defect* of a module. Thus, let  $X$  be an  $A$ -module of finite length with  $\underline{\dim} X = (x_0, \dots, x_n)$ . The defect  $\partial X$  of  $X$  is defined to be the difference between the number of simple injective and the number of simple projective composition factors of  $X$ :

$$\partial X = -x_0 + x_n.$$

First, let us formulate the following (rather trivial) assertions.

Lemma. Let  $X$  be an indecomposable  $A$ -module. If  $X \neq S_0$ , then  $\partial X = \partial \Gamma^+ X$ . If  $\Gamma^{+n} X \neq 0$ , then the lengths of  $\Gamma^{+n} X$  and  $X$  satisfy the formula

$$\text{length } \Gamma^{+n} X = \text{length } X + (n+1) \partial X .$$

Proof. The first assertion follows from

$$\partial \Gamma^+ X = -x_1 - x_0 + x_1 + x_n = -x_0 + x_n = \partial X .$$

To verify the second formula, observe that

$$\underline{\dim} \Gamma^+ X = (y_0, y_1, \dots, y_n) , \text{ where } y_i = -x_0 + x_1 + x_n \quad (0 \leq i \leq n) .$$

Thus the length of  $\Gamma^{+n} X$  equals to

$$\sum_{i=0}^n y_i = (n+1)(-x_0 + x_n) + \sum_{i=0}^n x_i = (n+1) \partial X + \text{length } X ,$$

as required.

Now, the procedure to describe all indecomposable  $A$ -modules of finite length is rather clear. Since it follows precisely the arguments used in the case of a tame tensor ring, we shall outline only the main steps; for further details, one is referred to [5] and [7].

If  $X$  is an indecomposable  $A$ -module of negative defect, then, for some  $r$ ,  $\Gamma^{+r} X = 0$ . For, otherwise one could apply  $\Gamma^{+n}$  inductively and would get non-zero modules  $\Gamma^{+nt} X$  for all  $t \in \mathbb{N}$ . The length of  $\Gamma^{+nt} X$  equals  $\text{length } X + t(n+1) \partial X$ , which has to become negative for large  $t$ ; a contradiction. If  $r$  is the least number with  $\Gamma^{+(r+1)} X = 0$ , then  $\Gamma^{+r} X \approx S_0$  (the simple projective module) and therefore  $X \approx \Gamma^{-r} S_0$ .

Consequently, the dimension type of  $X$  is of the form



$(x, \dots, x, x-1, \dots, x-1)$  with  $1 \leq S \leq n$ . These modules are called  $\underline{\quad S \quad}$  *preprojective*; they are uniquely determined by their dimension types, and are all of defect  $-1$ . Similarly, the indecomposable modules of positive defect are called *preinjective*. They are of the form  $\Gamma^{+r} S_n$ , the dimension type is of the form  $(x, \dots, x, x+1, \dots, x+1)$  for some  $S$   $\underline{\quad S \quad}$  with  $1 \leq S \leq n$ , and their defect is  $+1$ .

Now, the direct sums of indecomposable  $A$ -modules of zero defect form an abelian, exact, extension closed subcategory  $R$  of  $M_A$ ; the objects of  $R$  will be called *regular*. Some of the simple objects of  $R$  (simple in the category  $R$ , not necessarily simple  $A$ -modules) can be easily listed:  $S_1, \dots, S_{n-1}$  (these modules are actually simple  $A$ -modules) and the indecomposable  $A$ -module  $T$  of dimension type  $\underline{\dim T} = (1, 0, \dots, 0, 1)$  corresponding to the non-zero elements in  $\text{Ext}^1(S_n, S_0)$ . The  $A$ -modules from  $R$  whose composition factors (in  $R$ ) are all of the forms  $S_1, \dots, S_{n-1}$ , and  $T$ , form a serial subcategory  $U$  of global dimension 1. The indecomposable  $A$ -modules which belong to  $U$  are uniquely determined by their lowest composition factor and their length (in  $R$ ). Note that  $U$  is stable under  $\Gamma^+$  and  $\Gamma^-$ ; for example,  $\Gamma^+ T = S_{n-1}$ ,  $\Gamma^+ S_i = S_{i-1}$  (for  $2 \leq i \leq n-1$ ), and  $\Gamma^+ S_1 = T$ . Also,  $R$  is the product category of  $U$  and of the category  $H$  of all homogeneous  $A$ -modules; here, an object of  $R$  is called *homogeneous* if none of its composition factors (in  $R$ ) equals  $S_i$  ( $1 \leq i \leq n-1$ ) or  $T$ . Equivalently, an  $A$ -module  $X$  is homogeneous if the maps  $\phi_i$  ( $1 \leq i \leq n$ ) of  $\underline{X} = (X_i, \phi_i)$  are all isomorphisms. Here, of course, we consider  $X$  as a representation  $\underline{X}$  of the species  $A$ . [In order to see that  $R$  is the product category of  $U$  and  $H$ , one shows that  $\text{Ext}^1(H, S_i) = 0 = \text{Ext}^1(S_i, H)$  for all simple homogeneous objects  $H$  and  $1 \leq i \leq n-1$ . Indeed, given an exact

sequence

$$0 \longrightarrow H \longrightarrow E \longrightarrow S_i \longrightarrow 0 ,$$

one can embed  $S_i$  into  $E$  using  $\ker \phi_{i-1}$ . Similarly, given an exact sequence

$$0 \longrightarrow S_i \longrightarrow E \longrightarrow H \longrightarrow 0 ,$$

one can embed  $H$  into  $E$  using the image of  $\phi_{i+1}$ . Now applying the functor  $\Gamma^+$  to  $H$  and  $S_i$ , we get also that  $\text{Ext}^1(H, T) = 0 = \text{Ext}^1(T, H)$  for all simple homogeneous objects  $H$ .]

The category  $\mathcal{H}$  is easily seen to be equivalent to the category of all  $F[z; \epsilon, \delta]$ -modules of finite length. The equivalence functor  $M_{F[z; \epsilon, \delta]} \longrightarrow M_A$  is defined by  $W \longmapsto (X_0, \dots, X_n, \phi_i)$ , where  $X_i = W$  for all  $i$ ,  $\phi_{i-1}$  is the identity map for  $1 \leq i \leq n$ , and  $\phi_n : W \otimes M \longrightarrow W$  is given by  $\phi_n(w \otimes (a, b)) = wa + wbz$ . [Here again, we have identified  $M_A$  with  $Lc(A)$ . The description of the functor in terms of actual  $A$ -modules is given in the introduction!] For, in the case that  $\underline{X} = (X_i, \phi_i)$  is homogeneous, we may identify the different vector spaces  $X_i$  via the maps  $\phi_{i-1}$ , so that these maps become the identity maps. Thus, the only map of interest is  $\phi_n : X_n \otimes M \longrightarrow X_0$ . However, according to (1), the restriction of  $\phi_n$  to  $X_n \otimes F$  is the identity map, too. Consequently, the only invariant are the values  $\phi_n(w \otimes (0, 1))$  for  $w \in X_n = X_0$ . If we define  $w \cdot z = \phi_n(w \otimes (0, 1))$ , then we get an  $F[z; \epsilon, \delta]$ -module structure on  $X_n$ .

To complete the proof of Theorem 2, it remains to show that the indecomposable preprojective, or preinjective  $A$ -modules, as well as the indecomposable  $A$ -modules in  $U$  are uniquely characterized by the existence of a composition series

$$0 = X_0 \subset X_1 \subset \dots \subset X_d = X$$

with  $X_t/X_{t-1} \cong S_{r_t}$  for a preprojective, preinjective, or regular sequence  $(r_1, \dots, r_d)$ , respectively. Consider first the case of an indecomposable module  $X$  in  $\mathcal{U}$ . Then  $X$  has a unique composition series in  $\mathcal{U}$  the factors are  $S_i$  ( $1 \leq i \leq n-1$ ) and  $T$ . There is a unique refinement of this series to a composition series of  $A$ -modules, and the indices of the composition factors obviously form a regular sequence. Conversely, if an indecomposable  $A$ -module  $X$  has a composition series which corresponds to a regular sequence, then (calculating the defect)  $X$  has to be regular. Now, since we can embed either one of the  $S_i$ 's ( $1 \leq i \leq n-1$ ) or  $T$  into  $X$ ,  $X$  cannot belong to  $\mathcal{H}$  and therefore has to belong to  $\mathcal{U}$ . Next, consider the preprojective modules. Note that there is an inclusion

$$S_0 \hookrightarrow \Gamma^{-1}S_0 \hookrightarrow \Gamma^{-2}S_0 \hookrightarrow \dots \hookrightarrow \Gamma^{-k}S_0$$

with the factors  $\Gamma^{-(k+1)}S_0/\Gamma^{-k}S_0 \cong \Gamma^{-k}S_1$ . For, consider the sequence

$$0 \longrightarrow S_0 \longrightarrow \Gamma^{-1}S_0 \longrightarrow S_1 \longrightarrow 0,$$

and apply  $\Gamma^{-k}$ . If we consider this inclusion series for some  $X = \Gamma^{-k}S_0$ , we see that there is a unique refinement to a composition series of  $A$ -modules and that the indices of the composition factors of this series form a preprojective sequence. A similar argument works in the case of preinjective modules.

Note that the methods and results of this section are not restricted to the  $A$ -modules of finite length only, but that they can be used to deal with certain classes of  $A$ -modules of arbitrary length. For example, one can show, as in [8], that every union  $X$  of a chain

$$X^{(0)} \subset X^{(1)} \subset \dots \subset X^{(t)} \subset \dots$$

of indecomposable  $A$ -modules of finite length is again indecomposable, and that either every non-zero endomorphism of  $X$  is a monomorphism or an epimorphism, so that, in particular, the endomorphism ring  $\text{End}(X)$  of  $X$  has no zero-divisors.

#### §4. WILD RINGS

Let  $R$  be a semiprimary ring. Since we are interested only in the representation type of  $R$ , we may suppose that  $R$  is indecomposable (that is,  $R$  cannot be written as the product of two rings), and basic (every simple factor ring is a field). Let  $N$  be the radical of  $R$ .

Lemma. *Let  $f, g$  be orthogonal primitive idempotents such that  $F = fRf$  and  $G = gRg$  are fields. Then there is a full and exact embedding of the category of  $R_{(FfNg_G)}$ -modules of finite length into  $\text{mod}_R$ .*

Proof. Let  $h = 1-f-g$ ; thus  $f, g, h$  form a complete set of orthogonal idempotents. Hence  $R$  can be written in the form

$$R = \begin{pmatrix} F & fNg & fNh \\ gNf & G & gNh \\ hNf & hNg & hRh \end{pmatrix}.$$

and any module  $M_R$  can be decomposed into the direct sum of abelian group  $M = Mf \oplus Mg \oplus Mh$ , on which those matrices operate. Let  $(X_F, Y_G, \phi : X_F \otimes_{FfNg_G} \rightarrow Y_G)$  be an  $R_{(FfNg_G)}$ -module. We define an  $R$ -module  $M$  in the following way: Let  $Mf = X$ ,  $Mg = Y$ , and  $Mh = X \otimes fNh$ , and let the scalar multiplication be given by the maps

$$Mf \otimes fNg = X \otimes fNg \xrightarrow{\phi} Y = Mg,$$

$$Mf \otimes fNh = X \otimes fNh \xrightarrow{\text{id}} Mh,$$

$$Mh \otimes hNg = X \otimes fNh \otimes hNg \xrightarrow{1 \otimes \text{mult}} X \otimes fNg \xrightarrow{\phi} Y = Mg;$$

all the other maps be zero. It is easy to verify that, in this way, we get an  $R$ -module and that the respective functor is a full and exact embedding.

This shows, in particular, that in the case that  $R$  is tame we have  $d({}_F fNg_G) \leq 4$  for all such idempotents  $f$  and  $g$ .

Assume now, in addition, that  $R$  is hereditary. Then for any primitive idempotent  $e$ ,  $eRe$  is a field. Let  $e_1, \dots, e_m$  be a complete set of orthogonal primitive idempotents. The product  $\prod F_i$  of the fields  $F_i = e_i R e_i$  is a subring of  $R$  which complements the radical:  $N \oplus \prod F_i = R$ . If  ${}_R M_R$  is an  $R$ - $R$ -bimodule, we denote by  ${}_i M_j$  the submodule  ${}_i M_j = e_i M e_j$ . Obviously,  $M = \bigoplus {}_i M_j$ . Usually, we will consider  ${}_i M_j$  as an  $F_i$ - $F_j$ -bimodule. Assuming that  $R$  is tame, we know that  $d({}_i N_j) \leq 4$ .

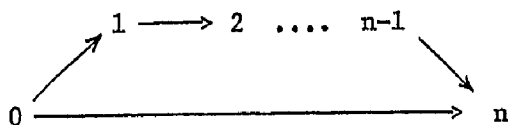
On the other hand, if  $d({}_i N_j) \leq 3$  for all  $i, j$ , then  $R$  has to be a tensor ring. Namely, the only way  $R$  can fail to be a tensor ring is that for some  $i, j$  the extension

$$0 \longrightarrow {}_i (N^2)_j \longrightarrow {}_i N_j \longrightarrow {}_i (N/N^2)_j \longrightarrow 0 \quad (3)$$

does not split as a sequence of  $F_i$ - $F_j$ -bimodules. But then necessarily  $\dim_{F_i} ({}_i N_j) \geq 2$ , and  $\dim ({}_i N_j)_{F_j} \geq 2$ ; thus  $d({}_i N_j) \geq 4$ . Therefore, we may assume that there exists a pair  $i, j$  with  $d({}_i N_j) = 4$  such that the corresponding sequence (3) does not split. Necessarily, we have

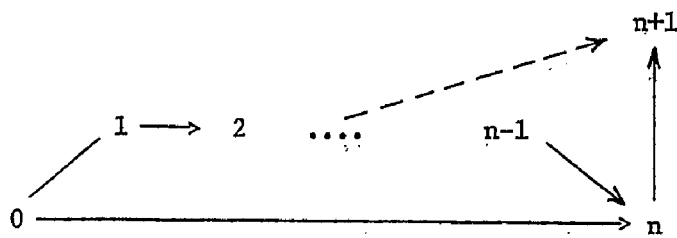
$(d({}_i (N^2)_j)) = 1$ , and  $d({}_i (N/N^2)_j) = 1$ . Let  $i = i_0, i_1, \dots, i_n = j$  be a sequence of maximal length such that  ${}_{i_{t-1}} N_{i_t} \neq 0$  for  $1 \leq t \leq n$ .

changing the indices (replacing  $i_t$  by  $t$ ), we are in the situation



such that for all bimodules  ${}_{i-1}N_i$ , we have  $d({}_{i-1}N_i) = 1$ . (For, using the multiplication map, the tensor product  ${}_{0}N_1 \otimes \dots \otimes {}_{n-1}N_n$  is canonically embedded into  ${}_0(N^2)_n$ .) In the case that the idempotents  $e_0, \dots, e_n$  form a complete set of orthogonal idempotents, we are just in the situation of a ring of the form  $A_n(\epsilon, \delta)$ . For,  $d({}_0N_n) = 4$  and the fact that  ${}_0N_n$  is not a simple bimodule, imply readily that  ${}_0N_n$  is of the form  $M(\epsilon, \delta)$  for some  $\epsilon, \delta$  [7].

Thus, we may assume that there is  $e_{n+1}$  with  ${}_{i}N_{n+1} \neq 0$  for some  $i$ . (The case  ${}_{n+1}N_i \neq 0$  can be treated similarly.) Let  $i$  be the largest possible number with  ${}_{i}N_{n+1} \neq 0$ . In case  $i = n$ , we are in the situation

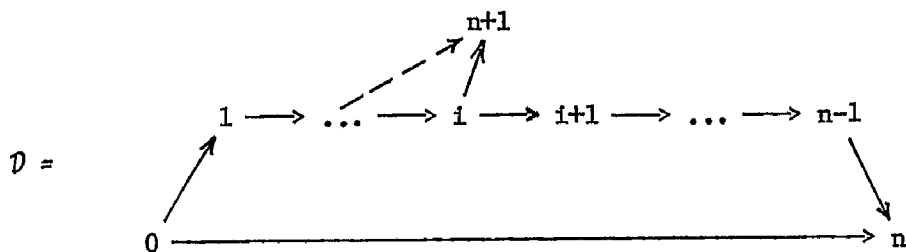


possibly with some additional arrows (indicated by  $--->$ ). It is easy to see that there is a full and exact embedding of the category of representations of the species  $C$

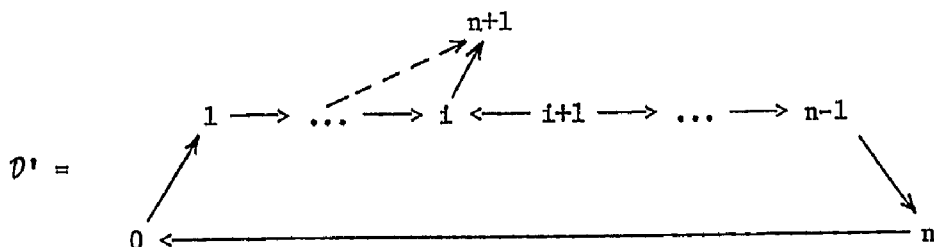
$$F_0 \xrightarrow{{}_0N_n} F_n \xrightarrow{{}_nN_{n+1}} F_{n+1},$$

(taking for the maps  ${}_{i}N_{i-1}$  ( $1 \leq i \leq n-1$ ) the identity maps, and for the additional arrows  $--->$  the zero maps). But  $C$  is wild: Consider the (unique) indecomposable representations  $\underline{X}$  with dimension type  $(t, t+1, 0)$  and  $\underline{Y}$  with dimension type  $(t+1, t, tb)$ , where  $b = \dim ({}_nN_{n+1})_{F_{n+1}}$  for some fixed  $t$ . Then it is obvious that there are no homomorphisms  $\underline{X} \rightarrow \underline{Y}$  or  $\underline{Y} \rightarrow \underline{X}$  (except zero), that the rings of

endomorphisms of both  $\underline{X}$  and  $\underline{Y}$  are fields, and that  $\text{Ext}^1(\underline{Y}, \underline{X})$  is arbitrary large (depending on  $t$ ). It is well-known (see for example [7], lemma 1.5) that there exists a full and exact embedding of  $L(\text{Ext}^1(\underline{Y}, \underline{X})^L)$  into  $L(C)$ .



can be reduced to the previous situation using the functor  $\Gamma^{-(n-1)}$ . Namely, define a functor from the category of representations of



into the category of representations of  $\mathcal{D}$ , which coincides on the circuit with  $\Gamma^{-(n-1)}$  and is the identity elsewhere. This functor kills only a certain number of injective modules, but the full and exact subcategory which is embedded into  $L(\mathcal{D}')$  is mapped bijectively into  $L(\mathcal{D})$ .

In combination with Theorem 2, this completes the proof of Theorem 1.

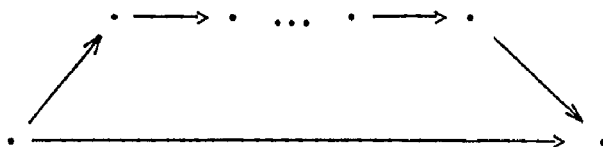
#### §5. REMARK

Since this paper deals in some detail with the relationship between hereditary semiprimary rings and tensor rings, a particular class of tensor rings should be mentioned which attracted some interest lately. In a recent paper [1], M. Auslander and M. I. Platzek develop parts of a

general representation theory of hereditary artin algebras, and they stress the fact that the techniques used in their paper are applicable for all hereditary artin algebras, not just for those associated with a  $k$ -species (those called, in this paper, tensor rings). However, at the beginning of the proof of one of the main theorems (4.1), the authors introduce the following property for a hereditary artin algebra  $R$

(P) If  $S_0, S_1, S_2$  are non-isomorphic simple modules such that  $\text{Ext}^1(S_1, S_0) \neq 0$  and  $\text{Ext}^1(S_2, S_0) \neq 0$ , then  $\text{Hom}(P(S_1), P(S_2)) = 0$ , where  $P(S_i)$  denotes the projective cover of  $S_i$ ,

and work with it throughout the proof. In fact, this property is equivalent to the property that the ring is a tensor ring whose diagram does not contain any circuit of the form



Indeed, the diagram of  $R$  is constructed in the following way: The points correspond to the simple  $R$ -modules, and the arrow  $i \rightarrow j$  means that  $\text{Ext}^1(S_i, S_j) \neq 0$ . Of course, the condition (P) just excludes circuits of the form mentioned above. However, if this type of circuits is excluded, then the ring  $R$  is in fact a tensor ring: Let  $e_i$  be pairwise orthogonal idempotents with  $P(S_i) = e_i R$ . Then  $\text{Ext}^1(S_i, S_j) \neq 0$  is equivalent to the fact that  $e_i(N/N^2)e_j \neq 0$ , and  $\text{Hom}(P(S_j), P(S_i)) \neq 0$  is equivalent to the fact that  $e_i R e_j \neq 0$  (or, to  $e_i N e_j \neq 0$  if  $S_i \neq S_j$ ). The condition (P) therefore can be rephrased as follows:

(P') For any  $i, j$ , only one of  $e_i(N^2)e_j$  and  $e_i(N/N^2)e_j$  can be non-zero.



This, of course, immediately implies that for any  $i, j$ , the exact sequence

$$0 \longrightarrow e_i(N^2)e_j \longrightarrow e_iNe_j \longrightarrow e_i(N/N^2)e_j \longrightarrow 0$$

splits, and therefore  $R$  is a tensor ring.

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