INFINITE DIMENSIONAL REPRESENTATIONS OF FINITE DIMENSIONAL HEREDITARY ALGEBRAS (*)

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This conference is concerned with abelian groups and their relations to modules, and it seems to be appropriate to present here some parts of a general representation theory for finite dimensional hereditary algebras, since it will turn out that for certain finite dimensional hereditary algebras, the socalled tame ones, the modules behave rather similar to abelian groups, or to modules over a principal ideal domain, although there occur some new complications.

The recent progress in the representation theory of finite dimensional algebras was limited mainly to the modules of finite length, and one would be interested to know in which way the structure of the modules of finite length determines the behaviour of arbitrary modules. Two results of this type are known: A finite dimensional algebra R is said to be of finite representation type provided there are only finitely many indecomposable modules of finite length. Now, if R is of finite representation type, then any module is the direct sum of modules of finite length [35, 4], and one knows since Azumaya [8] that such a decomposition is unique up to isomorphism. On the other hand, if R is not of finite representation type, then Auslander has shown that there exist indecomposable modules which are not of finite length [5]. This however is a mere existence proof, and does not reveal a concrete description of such a module. In this paper, we want to use the existing knowledge on modules of finite length in order to develop a general structure theory for modules of arbitrary length. In order to do so, we will restrict the investigation to the rather narrow class of finite dimensional algebras which are hereditary, since for them at least some classes of modules of finite length are well understood.

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Throughout the paper, we fix a commutative base field k; algebras will be defined over k, and associative with 1, modules will usually be right modules. Recall that an algebra R is said to be hereditary provided submodules of projective modules are projective again. In spite of being narrow, this class of algebras contains some finite dimensional algebras which are of great interest, also in applications. We give some examples:

First, let R_n be the ring of all $(n+1)\times(n+1)$-matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ & * & & 0 \\ & 0 & \ddots & \\ & & & * \end{pmatrix}$$

with entries in k. Then R_n has precisely n projective simple modules, and the R_n-modules without projective simple direct summand correspond to $(n+1)$-tuples $(V, V_i)_{1\leqslant i\leqslant n}$ where V_i, for $1\leqslant i\leqslant n$, is a subspace of V. Thus, classifying R_n-modules is the same as classifying the possible position of n subspaces in a vector space. This shows that the problem of classifying R_n-modules is of importance in geometry. It is known that for $n\leqslant 3$, R_n is of finite representation type. Of great importance has been the case $n = 4$, the socalled 4-subspace problem. The methods developped by Gelfand and Ponomarev [25] in order to give a complete list of the indecomposable R_4-modules of finite length, had a great influence, since it was possible to copy them for the other finite dimensional hereditary algebras of « tame » type [9, 18, 14]. In the cases $n\geqslant 5$, the classification of the indecomposable R_n-modules of finite length seems to be rather hopeless, but, at least, one knows some classes of modules.

Next, consider the problem of classifying n-tuples of linear transformations $\varphi_i\colon V\to W$, $1\leqslant i\leqslant n$, where V, W are vector spaces. Note, these are just modules over the ring

$$\begin{pmatrix} k & k^n \\ 0 & k \end{pmatrix},$$

where k^n is the n-dimensional vector space. (For any bimodule ${}_FM_G$, the matrices of the form $\begin{pmatrix} f & m \\ 0 & g \end{pmatrix}$ with $f\in F$, $g\in G$, $m\in M$, obviously form a ring, which usually will be denoted by $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$.) For $n = 2$, these objects $(V, W_1, \varphi_1, \varphi_2)$ are usually called Kronecker modules, since Kronecker [26] has given a complete classification of those of finite length, thus solving a problem raised by Weierstrass. The relation to modules over a principal ideal domain, here $k[X]$, can be seen in this case very easily: if one of the two maps φ_1, φ_2 is an identity map, say we consider the Kronecker module $(V, V, \mathrm{id}, \varphi)$, then this Kronecker module can be identified with the $k[X]$-module (V, φ), where X operates on V via the linear transformation φ. Kronecker modules may be used very effectively in solving differential equations [23]. Also, they have turned out to be of great importance in perturbation theory. There, under the name of «systems», an extended theory of infinite dimensional Kronecker modules has been developped. Our aim is to incorporate these results into a general theory of modules over a finite dimensional hereditary algebra. We will comment on these investigations at the end of the introduction further.

Also, we may consider other bimodules ${}_F M_G$. As long as F and G are semi-simple, and all three F, G, M are finite-dimensional over k, we obtain a finite dimensional hereditary algebra $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$. For example, denoting by $\mathbb{R}$ the real numbers, by $\mathbb{H}$ the quaternions, the algebra

$$\begin{pmatrix} \mathbb{R} & \mathbb{H} \\ 0 & \mathbb{H} \end{pmatrix}$$

seems to be of interest, since the modules without simple projective direct summand correspond to pairs $(U_{\mathbb{R}}, V_{\mathbb{H}})$ where $V_{\mathbb{H}}$ is an $\mathbb{H}$-vector space and $U_{\mathbb{R}} \subseteq V_{\mathbb{R}}$ a real subspace. A complete classification of the finite dimensional modules has been given in [17]; here, we have to consider modules over the principal ideal domain $\mathbb{R}[X, Y]/(X^2 + Y^2 + 1)$. Note that the corresponding algebra $\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ with $\mathbb{C}$ the complex numbers, is of finite representation type.

After having given some examples of finite dimensional hereditary algebras of interest, we note that a general representation theory for finite dimensional hereditary algebras also should give some insight into properties of modules over an arbitrary finite dimensional algebra R'. In fact, given R', there always exists some finite dimensional hereditary algebra R, and a full and exact embedding of the category of R'-modules into the category of R-modules. Thus, there seems to be good reason to propel the knowledge on representations of finite dimensional hereditary algebras.

Why is it of interest to consider infinite dimensional representations? We believe that the only reason for the usual restriction in dealing with finite dimensional algebras to consider only modules of finite dimensional algebras to consider only modules of finite length, is the fact that modules of finite length are easier to handle. Of course, as soon as one deals with general rings or algebras, a similar restriction to modules of finite length would be considered as inappropriate, since the ring itself would not fall any longer into the class of modules considered, thus one usually considers finitely generated modules. However, we note that the concept of a finitely generated module has some anomalies: if we consider a full exact embedding of some module category $\mathfrak{M}_{R'}$ into $\mathfrak{M}_R$, the image of a finitely generated R'-module may no longer be a finitely generated R-module. Thus, it seems to be reasonable to consider always also those R-modules which in some full exact subcategory which is also a module category, become finitely generated. A typical example would be the Kronecker module $(k[X], k[X], \mathrm{id}, \cdot X)$. Of course, in dealing with finitely generated modules over a general ring, the injective envelopes of the finitely generated modules are important. They are usually no longer finitely generated, but are for certain types of rings, for example noetherian rings, well behaved. In a suitable subcategory, the injective envelope of the Kronecker module $(k[X], k[X], \mathrm{id}, \cdot X)$ is just the Kronecker module $Q = (k(X), k(X), \mathrm{id}, \cdot X)$, with $k(X)$ being the field of rational functions in one variable. We will see that this Kronecker module Q plays a dominant role, it will be characterized as the unique indecomposable « torsionfree divisible » module. Thus, there always will be certain important infinite dimensional representations, and we will see that the investigation of these modules also gives some new insight into the behaviour of the modules of finite length. For example, in 5.6, we will give a new interpretation of a well-known invariant in the tame case, the socalled defect. As a final argument for the necessity to consider arbitrary, not necessarily finite dimensional modules, we should mention the fact that the applications in perturbation theory depend on the knowledge of infinite dimensional Kronecker modules.

Now, we want to give a survey of the main results of the paper. Let R be a finite dimensional hereditary algebra. We start with two classes of modules of finite length, the indecomposable preprojective and the indecomposable preinjective modules. They can be defined by applying a construction due to Auslander to the indecomposable projective or injective modules (see 1.A and B), and, as the name suggests, their behaviour is rather similar to that of projective, or injective modules, respectively. In fact, one may characterize them in terms of relative projectivity and relative injectivity. A module X will be called *preprojective* provided every non-zero submodule splits off a non-zero preprojective direct summand of finite length. The structure of these preprojective modules may be rather complicated. For a module X, let $\mathcal{T}X$ be the intersection of the kernels of all maps $X \to P$

with P indecomposable preprojective, and define by transfinite induction $\mathfrak{P}^{\lambda}X = \bigcap_{\mu<\lambda} \mathfrak{P}^{\mu}X$, for λ limit ordinal, and $\mathfrak{P}^{\lambda}X = \mathfrak{P}\,\mathfrak{P}^{\lambda-1}X$ otherwise. Then X is preprojective if and only if $\mathfrak{P}^{\lambda}X = 0$ for some ordinal λ, and theorem 2.6 asserts that for any ordinal λ, and any indecomposable preprojective module P, there exists a module X with $\mathfrak{P}^{\lambda}X = P$. Note that this behaviour is similar to that of p-groups with respect to the Ulm subgroups. If we define dually $\mathfrak{J}X$ as the sum of the images of all maps $I \to X$ with I indecomposable injective, then there is no need to iterate this process: according to 3.3, we have $\mathfrak{J}(X/\mathfrak{J}X) = 0$. If we call X *preinjective* provided $\mathfrak{J}X = X$, then the preinjective modules are just the direct sums of indecomposable preinjective modules; in particular, they are direct sums of modules of finite length. Also, according to 3.3, the maximal preinjective submodule $\mathfrak{J}X$ of X is always a pure submodule, and we may ask whether it is a direct summand. As we will see, this depends on the representation type of R (see 1.C). If R is tame, then $\mathfrak{J}X$ is always a direct summand (theorem 3.7), if R is wild, there exists X such that $\mathfrak{J}X$ is not a direct summand (theorem 3.9).

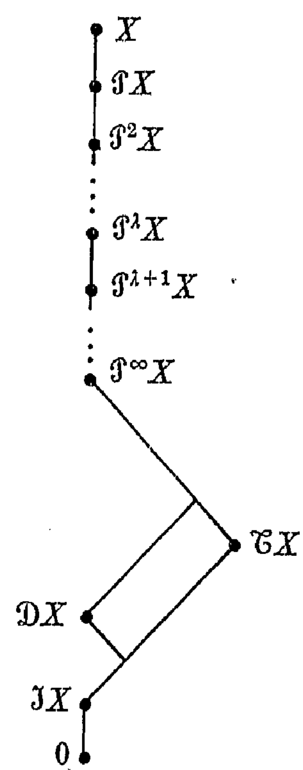
Call a module *regular* provided it has no indecomposable preprojective or preinjective direct summands. The investigation of regular modules will be done only in case R is of tame representation type. Here, however, we will see the strict analogy to the theory of abelian groups.

Given a module X, let $\mathfrak{T}X$ be the sum of all submodules U of X of finite length such that U has no indecomposable preprojective direct summand. Call X *torsion* provided $\mathfrak{T}X = X$, and *torsionfree* provided $\mathfrak{T}X = 0$. The torsion regular modules form an abelian subcategory $\mathfrak{T}$ which is the product of categories $\mathfrak{T}_t$ ($t \in T$) each of which is equivalent to the category of torsion modules over some ring $H_n(D)$ of $n \times n$-matrices of the form

$$\begin{pmatrix} D & D & \cdots & D \\ M & D & \ddots & D \\ \vdots & & \ddots & \\ M & \cdots & M & D \end{pmatrix}$$

where $D = D_t$ is a (not necessarily commutative) discrete valuation ring with maximal ideal M (theorem 4.4). For all but at most three t, we have $n = 1$; thus, for these t, the modules in $\mathfrak{T}_t$ can be considered as torsion modules over a discrete valuation ring, and therefore behave like p-groups. The rings D_t, for the various $t \in T$, are not independent. In fact we prove in (6.8) that a suitable matrix ring over D_t is the completion of a subring of some division ring E which only depends on R. The index set T is always infinite; if k is algebraically closed, then $T = \mathbb{P}_1(k)$, the projective line over k. Note that $\mathbb{P}_1(k)$ has one more element as the corresponding index set for the primary decomposition of $k[X]$-modules. Since $\mathfrak{T}$ is an abelian category, we may speak of simple objects in $\mathfrak{T}$, these are modules of finite length, and we call them *simple regular*.

A module Y will be called *divisible*, provided $\operatorname{Ext}(X, Y) = 0$ for any simple regular module X, or, equivalently, for any module X without indecomposable preinjective direct summand (4.7). Any module X contains a unique maximal divisible submodule $\mathfrak{D}X$, and $\mathfrak{D}X$ is a direct summand of X.



Given a simple regular module S, there exists a unique indecomposable module S^ω in $\mathfrak{T}$ which is divisible and contains S as a submodule. Modules of the form S^ω will be called *Prüfer modules*, since they are similar to the Prüfer groups in abelian group theory. We will show that an indecomposable module which is not of finite length, is either a Prüfer module or torsionfree regular (4.8).

Also, there exists a unique indecomposable torsionfree divisible module Q. Its endomorphism ring is a division ring, and Q is finite dimensional over $\operatorname{End}(Q)$ (theorems 5.3 and 5.7). The divisible modules are direct sums of indecomposable divisible modules, and indecomposable divisible are the indecomposable preinjective modules, the Prüfer modules, and Q (5.4).

In the last section, we will consider *torsionfree rank one* modules. By definition, these are submodules X of Q such that Q/X is torsion regular. In certain cases, for example if the base field k is algebraically closed, we will give a complete classification of the torsionfree rank one modules using equivalence classes of height functions $T \to \mathbb{N}_0 \cup \cup \{\infty\}$, similar to the classification of the torsionfree rank one abelian groups. Here however, we see the fundamental difference to the abelian group case which stems from the additional element in the index set T: the endomorphism ring of a torsionfree rank one module does not have to be an order in End (Q), but may be finite dimensional over k (6.5). This we use, for R being tame (!), in order to show that there exists a finite dimensional hereditary algebra R' of wild representation type, and a full and exact embedding of $\mathfrak{M}_{R'}$ into $\mathfrak{M}_R$ (6.9). Also note that $E = \text{End}\,(Q)$ is the division ring mentioned before.

We add a remark on the methods we use. Besides the Auslander construction « dual of transpose », and the corresponding « almost split exact » sequences, which we will call Auslander Reiten sequences, we will need partial Coxeter functors and some facts about growth numbers. In the case of tame representation type, we make extensive use of the material collected in the tables of [14]. In addition, we will need two general results for modules over finite dimensional algebras, namely a characterization of pure submodules, and a result on finite dimensional direct summands. These two results will be presented in sections 1.F and 1.G.

Let us come back to those papers inspired by problems in perturbation theory, and which deal with infinite dimensional Kronecker moduler over $\mathbb{C}$. We refer to the appendix of [2] for a well presented example which shows the use of Kronecker modules: there, the differential operator $\varphi = d^4/dt^4$ on a certain subspace V of the Hilbert space $W = L^2([-1, 1])$ is considered, with V depending on four given linearly independent boundary conditions. In this way, one obtains the Kronecker module (V, W, i, φ), with i the inclusion, and a change of the boundary conditions being interpreted as a perturbation. This development seems to have been started by Aronszajn [1], and in a joint paper with Fixman [3], some algebraic foundations were laid. Some of the results of [3] are in fact valid for arbitrary finite dimensional algebras (see sections 1.F and G of our paper), but it seems to be interesting to note that the proof of the equivalence of (i) and (ii) of 1.F in [3] and [38] uses topological considerations, whereas we invoke Auslander Reiten sequences. The main result of Aronszajn and Fixman gives the structure of the divisible Kronecker modules, and the fact that the maximal divisible submodule $\mathfrak{D}X$ always is a direct summand (see Cor. 2 of 4.7, and 5.4). These investigations were continued by Fixman and his students. In [19], Fixman considers torsionfree rank one Kronecker modules and gives the complete classification in terms of height functions on $\mathbb{P}_1(\mathbb{C})$ (see 6.5), the groups Ext (X, Y) for X, Y of rank $\leqslant 1$ have been determined in [20]. In [21], Fixman and Zorzitto prove the purity criterion 2.2, Cor. 3 for Kronecker modules (note that the result in the form we state is valid also in the wild case, in contrast to a remark at the end of the introduction of [21]). Finally, we should mention the work of Okoh [27, 28, 29, 30] treating pure simple modules, and the question of decomposing a given Kronecker module as the direct sum of modules of finite length. We do not touch the last questions in our paper. We should point out that two of our results in the tame case seem to be new also in the special situation of Kronecker modules, namely the construction of arbitrarily large preprojective modules given in 2.6, and the fact that the category of Kronecker modules id « Wild »: according to 6.9, any $\mathbb{C}$-algebra which is generated by less then $\aleph_t$ elements, $\aleph_t$ the first strongly inaccessible cardinal number, occurs as the full endomorphism ring of a suitable Kronecker module.

Section 1 contains some preliminaries which are needed in the course of the paper. *Starting with section 2, we assume that R is a (twosided indecomposable) finite dimensional hereditary algebra which is not of finite representation type. Beginning with 4.3, we assume, in addition, that R is of tame representation type.*

1. **Preliminaries.**

We want to collect some basic results which will be used throughout in the paper.

First, some words about notation and terminology. If R is a ring, we denote by rad R the Jacobson radical of R. We usually will work with a finite dimensional hereditary algebra R which is defined over some fixed commutative field k. Being interested only in the R-modules, we may assume that R is a basic algebra, that is, $R/\text{rad}\, R$ is the product of division rings. In case $R/\text{rad}\, R$ is a division ring, we call R a local algebra. Usually, we will assume that R is twosided indecomposable, that is, R does not contain any central idempotents besides 0 and 1.

Modules will usually be right R-modules, and module homomorphisms will be written on the opposite side of the scalars, thus for (right) modules X, Y, Z, and homomorphisms $f: X \to Y$, $g: Y \to Z$, the composition of f and g will be denoted by gf. An indecomposable module X is always assumed to be non-zero. The length of a module X is denoted by $|X|$. If X and Y are isomorphic, we will use the symbol $X \approx Y$. Note that sometimes the word «module» will mean «isomorphism class of module». The category of R-modules will be denoted by $\mathfrak{M}_R$.

In dealing with hereditary rings, we will denote Ext^1 just by Ext.

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ denote the natural numbers, the natural numbers with 0, the integers, the reals, the complex numbers, and the quaternions, respectively.

A. The Auslander construction and Auslander Reiten sequences. For arbitrary finite dimensional k-algebras, M. Auslander has introduced a construction which seem to be of ever increasing importance. Let M be a right R-module of finite length, and let $P_1 \xrightarrow{f} P_2 \to M \to 0$ be the first two terms of a minimal projective resolution of M (of course, in case R is in addition hereditary, as we will assume later, the map f has to be a monomorphism, and $0 \to P_1 \xrightarrow{f} P_2 \to M \to 0$ is the complete minimal resolution of M). Applying the functor $* = \mathrm{Hom}_R(\ ,R_R)$, we obtain a map $f^*: P_2^* \to P_1^*$ of left R-modules, whose cokernel will be denoted by $\mathrm{Tr}\, M$, it is a left R-module. Similarly, starting with a left R-module N of finite length, and two terms of the minimal projective resolution of N, we map apply now $* = \mathrm{Hom}_R(\ ,{}_RR)$, and obtain as cokernel a right R-module denoted by $\mathrm{Tr}\, N$. If we use now the ordinary duality $D = \mathrm{Hom}_k(\ ,k)$, we obtain from the left R-module $\mathrm{Tr}\, M$ a right R-module $AM = D\,\mathrm{Tr}\, M$. If we first apply D to the right module M, and then Tr to the left module DM, we obtain $A^{-1}M = \mathrm{Tr}\, DM$, again a right R-module. In general, the constructions A and A^{-1} are not really functorial: given a homomorphism $f: M \to N$ between R-modules of finite length, we can define Af only by using an appropriate lifting which is not necessarily uniquely given, thus Af is only defined up to maps factoring through injective modules, and similarly, A^{-1} is only defined up to maps factoring through projective modules. However, in case R is hereditary, both A and A^{-1}, are functorial.

The main property of the constructions A and A^{-1} is the following: Let M be an indecomposable module of finite length. Then AM is either zero or indecomposable again (and again of finite length), and $AM = 0$ if and only if M is projective. If $AM \neq 0$, then M can be recovered from AM by using A^{-1}, since $A^{-1}AM \approx M$. Similarly, $A^{-1}M$ is either zero, or indecomposable and of finite length, $A^{-1}M = 0$ iff M is injective, and if $A^{-1}M \neq 0$, then $AA^{-1}M \approx M$. This shows that A and A^{-1} are very useful in constructing indecomposable modules of finite legth.

Given a non split exact sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

of modules, then we call $X \to Y \to Z$ an *Auslander Reiten sequence*, provided it satisfies the following two properties:

(i) If $\gamma: X \to X'$ is a map which is not a split monomorphism, then there exists $\gamma': Y \to X'$ with $\gamma = \gamma' \alpha$.

(ii) If $\delta: Z' \to Z$ is a map which is not a split epimorphism, then there exists $\delta': Z' \to Y$ with $\delta = \beta\delta'$.

The existence of such sequences was established by M. Auslander and I. Reiten in [7]. In fact they have shown the following: If X is an indecomposable module of finite length which is not injective, then there exists an Auslander Reiten sequence $X \to Y \to Z$ with $Z \approx A^{-1}X$ (note that all the modules X, Y, Z are of finite length, however, the properties (i) and (ii) are valid for arbitrary modules X', Z' which are not necessarily of finite length). Similarly, if Z is an indecomposable module of finite length which is not projective, then there exists an Auslander Reiten sequence $X \to Y \to Z$ with $X \approx AZ$.

On the other hand, if $X \to Y \to Z$ is an Auslander Reiten sequence, then X and Z both are indecomposable, and if for two Auslander Reiten sequences $X \to Y \to Z$ and $X' \to Y' \to Z'$, we have $X \approx X'$ or $Z \approx Z'$, then the sequences themselves are isomorphic. This shows that Auslander Reiten sequences are unique.

Now assume, in addition, that R is hereditary. Then we have noted above, both A and A^{-1} are functors, and the application of A gives a bijection

$$\mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(AX, AY)\,,$$

provided X has no non-zero projective direct summand, whereas the application of A^{-1} gives a bijection

$$\mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(A^{-1}X, A^{-1}Y)\,,$$

provided Y has no non-zero injective direct summand. In this way, we may identify for modules without non-zero projective direct summand the rings End (X) and End (AX). Also, we note the following:

LEMMA: Let R be a finite dimensional hereditary k-algebra. Let X, Y be modules of finite length, then

$$\dim_k \operatorname{Hom}(Y, AX) = \dim_k \operatorname{Ext}(X, Y).$$

B. Indecomposable preprojective modules and indecomposable preinjective modules. We assume that R is a finite dimensional hereditary algebra. If s is the number of simple modules, there are also just s indecomposable projective modules, and s indecomposable injective modules. If P is indecomposable projective, then End (P) is a division ring. We write $P \to P'$ for P, P' indecomposable projective modules, provided $\operatorname{Hom}(P, P') \neq 0$, and, in this way, we obtain a partial ordering on the set of indecomposable projective modules.

If X is a module of finite length, then we call X preprojective, provided $A^m X = 0$ for some $m \in \mathbb{N}$. If X is in addition indecomposable, then this is equivalent to the fact that $X \approx A^{-i}P$ for some $i \in \mathbb{N}_0$, and some indecomposable projective module P. If we assume that X, Y are modules of finite length, with X indecomposable, and Y preprojective, then the existence of a non-zero map $X \to Y$ implies that also X is preprojective. In particular, submodules of preprojective modules of finite length are preprojective again. Also, if $0 \to X \to Y \to Z \to 0$ is an exact sequence with modules of finite length, and X and Z both are preprojective, then Y is preprojective.

If X is a module of finite length, then we call X preinjective, provided $A^{-m}X = 0$ for some $m \in \mathbb{N}$. For an indecomposable module X of finite length, this means that $X \approx A^i I$ for some $i \in \mathbb{N}_0$, and some indecomposable injective module I. It is clear that we have the dual properties to the preprojective case. In particular, the class of preinjective modules of finite length is closed under quotients and extensions.

If R is a twosided indecomposable, then either there exists a nonzero module which is both preinjective and preprojective, then all modules of finite length are both preinjective and preprojective, and therefore there is only a finite number of indecomposable modules of finite length, thus R is of finite representation type. Or else, the $2s$ countable series of indecomposable modules $A^i P$, $A^i I$ with $i \in \mathbb{N}_0$, and P indecomposable projective, I indecomposable injective, are mutually disjoint.

In the second case, it is known that there exist also indecomposable modules of finite length which are neither preprojective nor preinjective. We call a module of finite length regular, provided it does not have any non-zero preprojective or preinjective direct summand. It is clear, that the class of regular modules of finite length is closed under extensions. Also, if X, Y are regular modules of finite length, and $\varphi: X \to Y$ is a homomorphism, then the image of φ is regular again.

Next, we want to show that there is the possibility to calculate effectively the action of A on the indecomposable modules of finite length. Let $P_1, \ldots, P_s$ be the indecomposable projective modules. For any module X of finite length, we introduce its *dimension vector* $\mathit{dim}\, X$ as follows: Since $\operatorname{End}(P_i)$ is a division ring, we see that $\operatorname{Hom}(P_i, X)_{\operatorname{End}(P_i)}$ is a vector space, and let

$$(\mathit{dim}\, X)_i = \dim \operatorname{Hom}(P_i, X)_{\operatorname{End}(P_i)}.$$

Clearly, this is just the number of composition factors in a given composition series of X which are isomorphic to the simple factor module of P_i. Thus, $\mathit{dim}\, X$ is an s-tuple of elements of $\mathbb{N}_0$, and it will be convenient to consider $\mathit{dim}\, X$ as an element of the vector space $\mathbb{R}^s$. Note that the sum of the components of $\mathit{dim}\, X$ is equal to the length $|X|$ of X.

Some of the indecomposable modules are characterized uniquely by their dimension vector. In fact, if X and Y are indecomposable modules with $\mathit{dim}\, X = \mathit{dim}\, Y$, and X is either preprojective or preinjective, then it follows that $X \approx Y$.

We come to the calculation of the effect of the Auslander construction. Namely, there exists a regular transformation c on $\mathbb{R}^s$, called the *Coxeter transformation* for R such that for any module X of finite length, and without projective direct summand, one has

$$\mathit{dim}\, AX = c(\mathit{dim}\, X),$$

and similarly, for Y of finite length and without injective direct summand, one has

$$\mathit{dim}\, A^{-1}Y = c^{-1}(\mathit{dim}\, Y).$$

We will recall in the next section that c leaves invariant a quadratic form.

C. The representation type of a finite dimensional hereditary algebra. Let R be again finite dimensional and hereditary, and assume, in addition, that R is basic. Let $P_1, \ldots, P_s$ be the indecomposable projective modules. Since R is basic, $R_R \approx \bigoplus_{i=1}^{s} P_i$. As a consequence, we see that $R/\operatorname{rad} R = \prod_{i=1}^{s} F_i$, with F_i being isomorphic to the endomorphism ring $\operatorname{End}(P_i)$ which is a division ring. We may consider $\operatorname{rad} R/(\operatorname{rad} R)^2$ as $R/\operatorname{rad} R$ - $R/\operatorname{rad} R$-bimodule, and decompose it as the direct sum of submodules ${}_iM_j$, where $\prod F_i$ acts on ${}_iM_j$ on the left via F_i and on the right via F_j. The *species* of R is given by the collection $(F_i, {}_iM_j)_{1\leq i,j\leq s}$. The species determines nearly completely the representation theory of R, we derive from it certain invariants: the oriented diagram of R is given by s points, with arrow from the point i to the point j provided ${}_iM_j \neq 0$, and we add to such an arrow the pair of numbers $(\dim_{F_i}({}_iM_j), \dim ({}_iM_j)_{F_j})$. Note that the oriented diagram does not have any loops or oriented circuits. In particular, for fixed points i, j, there is only one of the two possible arrows $i \to j$, and $j \to i$. If we replace the arrows by edges, we obtain the *diagram* of R. The ring R also determines a quadratic form q on $\mathbb{R}^s$ as follows: Let $f_i = \dim_k F_i$, and $m_{ij} = \dim_k ({}_iM_j)$. Then we put

$$q(x_1, \dots, x_s) = \sum_i f_i x_i^2 - \sum_{i,j} m_{ij} x_i x_j .$$

Note that this quadratic form (up to a scalar multiple) is uniquely defined by the diagram of R, in particular, it does not depend on the orientation. It is this quadratic form, which is invariant under the Coxeter transformation: for all $\boldsymbol{x} \in \mathbb{R}^s$, we have $q(c\boldsymbol{x}) = q(\boldsymbol{x})$.

It is known that R is of finite representation type if and only if q is positive definite. In case of R being twosided indecomposable, we call R to be of *tame representation type* provided q is semidefinite but not definite, and, we call R to be of *wild representation type* provided q is indefinite.

D. The tame case. Assume that R is a twosided indecomposable, finite dimensional hereditary algebra of tame representation type, and let $P_1, \dots, P_s$ be the indecomposable projective modules. There are only 16 different cases for the oriented diagram of R and we will list these cases together with further information at the end of this section.

In this situation, the quadratic form q vanishes precisely on a one-dimensional subspace of $\mathbb{R}^s$ which is generated by a vector $\boldsymbol{h} = (h_i)$ with coefficients $h_i \in \mathbb{N}$, such that at least one of the $h_i = 1$. We have listed $\boldsymbol{h}$ in the table, it is uniquely defined by the stated properties. An indecomposable R-module X with *dim* X a multiple of $\boldsymbol{h}$ will be called *homogeneous* (1).

There exists a non-zero linear form $\delta: \mathbb{R}^s \to \mathbb{R}$ which is invariant under c, called the *defect*, and which we normalize by the conditions that $\delta(P_i) \in \mathbb{Z}$, for all $1 \leq i \leq s$, and $\delta(P_i) = -1$ for at least one i (2).

(1) Note that this differs from the use of the notion « homogeneous » in [14].
(2) In [14], the defect had not been normalized.

In this way, δ is uniquely defined by the oriented diagram, and it is easy to check that the values $\delta(P_i)$ do not even depend on the orientation, but only on the diagram. We have listed the vector with components $-\delta(P_i)$ in the table.

Let us note the main properties of the defect. If X is an indecomposable module of finite length, then $\delta(\dim X)$ which we will just denote by δX, determines whether X is preprojective, preinjective, or regular. Namely, $\delta X < 0$ if and only if X is preprojective, and $\delta X > 0$ if and only if X is preinjective. Also, again under the assumption that X is indecomposable of finite length, $|\delta X| \leqslant 6$.

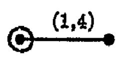
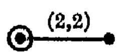
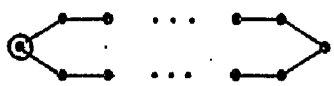
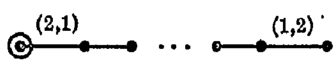
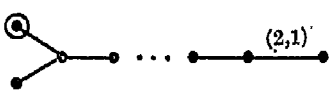
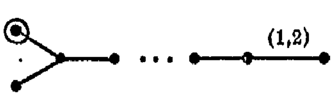
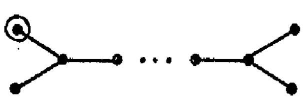
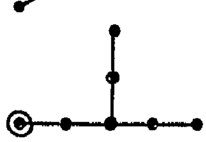
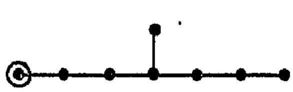
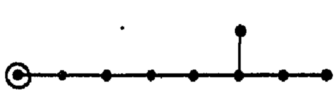
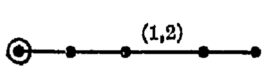
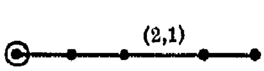
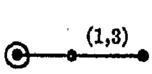
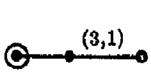
Of great importance is the fact that in the tame case the regular modules of finite length form an abelian exact subcategory $\mathfrak{t}$, and that one knows the structure of this category completely. Since $\mathfrak{t}$ is abelian, we may consider the simple objects in $\mathfrak{t}$ and call them *simple regular*. Thus a non-zero module S is simple regular if it belong to $\mathfrak{t}$, and if it does not have a proper non-zero submodule which belongs to $\mathfrak{t}$. If X is in $\mathfrak{t}$, the sum of the simple regular submodules of X is called the *regular socle* of X, the length n of a regular composition sequence, that is

$$0 = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n = X$$

with X_i/X_{i-1} simple regular for all i, is called the *regular length* of X. Given S simple regular, and $n \in \mathbb{N}$, there exists a unique indecomposable regular module S^n with regular socle S and regular length n, and every indecomposable regular module is of the form S^n, for some S and some n. In particular, every indecomposable regular module has a unique regular composition series.

Let us consider now the set of simple regular modules. On this set, A operates with finite orbits, and all but at most three orbits are one element sets. Let T be the set of orbits. If S and S' are simple regular, then $\operatorname{Ext}(S, S') \neq 0$ if and only if $S' = AS$. Thus, the category $\mathfrak{t}$ decomposes as the direct sum of categories $\mathfrak{t}_t$, where t runs through the set T, and an indecomposable regular module with regular composition series given by X_i belongs to $\mathfrak{t}_t$ if and only if one, and therefore all of the regular composition factors X_i/X_{i-1} belong to $\mathfrak{t}_t$.

For S simple regular in $\mathfrak{t}_t$, let n_t be the smallest natural number with $A^{n_t}S \approx S$. Note that S^{n_t} is always homogeneous, whereas the modules S^i, with $1 \leqslant i < n_t$ are not homogeneous. Note that for S simple regular in $\mathfrak{t}_t$, S itself is homogeneous, iff $n_t = 1$, iff all modules in $\mathfrak{t}_t$ are homogeneous. In this case, we call t homogeneous. The numbers n_t, for t non-homogeneous, are listed in the table with their precise multiplicity.

type	diagram	h	$(-\delta P_i)_i$	n_t	e_t
$\tilde{A}_{11}$		21	12	—	—
$\tilde{A}_{12}$		11	11	—	—
$\tilde{A}_{npq}$		$1\,{}^{1\cdots1}_{1\cdots1}\,1$	$1\,{}^{1\cdots1}_{1\cdots1}\,1$	$p+1, q+1$	1, 1
$\tilde{B}_n$		$11\cdots11$	$12\cdots21$	n	1
$\tilde{C}_n$		$12\cdots21$	$11\cdots11$	n	2
$\widetilde{BC}_n$		$22\cdots21$	$12\cdots22$	n	1
$\widetilde{BD}_n$		${}^1_1 2\cdots22$	${}^1_1 2\cdots21$	$2, n-1$	1, 2
$\widetilde{CD}_n$		${}^1_1 2\cdots21$	${}^1_1 2\cdots22$	$2, n-1$	1, 1
$\tilde{D}_n$		${}^1_1 2\cdots2{}^1_1$	${}^1_1 2\cdots2{}^1_1$	$2, 2, n-2$	1, 1, 1
$\tilde{E}_6$		1 2 12321	1 2 12321	2, 3, 3	1, 1, 1
$\tilde{E}_7$		2 1234321	2 1234321	2, 3, 4	1, 1, 1
$\tilde{E}_8$		3 12345642	3 12345642	2, 3, 5	1, 1, 1
$\tilde{F}_{41}$		12321	12342	2, 3	1, 1
$\tilde{F}_{42}$		12342	12321	2, 3	2, 1
$\tilde{G}_{21}$		121	123	2	1
$\tilde{G}_{22}$		123	121	2	3

In the last section, we will need, for P preprojective of defect -1, and S simple regular in some $\mathfrak{t}_t$, the dimension

$$e_t = \dim_{\mathrm{End}(S^{n_t})} \mathrm{Hom}\,(P, S^{n_t}),$$

and, for t non-homogeneous, we have listed this dimension in the table. Note that it does not depend on the particular orientation: For, a change of orientation does not change the dimension vector of S^{n_t}, and therefore not the dimension $\dim \mathrm{Hom}\,(P, S^{n_t})_{\mathrm{End}(P)}$ However, a functor infering a change of orientation, does not change neither $\mathrm{End}\,(P)$ nor $\mathrm{End}\,(S^{n_t})$, thus it keeps e_t.

A description of T, for general base field k, seems to be difficult. We note that T always is infinite. In case k is algebraically closed, then T can be identified, in a natural way, with the points of the projective line $\mathbb{P}_1(k)$ over k. For $k = \mathbb{R}$, possible parametrizations of T have been considered in the appendix of [14], one always may choose one of the sets $\mathbb{P}_1(\mathbb{C})$, $\mathbb{P}_2(\mathbb{R})$ or the hemisphere

$$\{(x, y, z) \in R \mid x^2 + y^2 + z^2 = 1,\ z \geqslant 0\}.$$

Note that T is determined by the bimodule ${}_FM_G$ listed in the tables of [14], and we note that the functor Γ from the category of $\begin{pmatrix} F & M \\ O & G \end{pmatrix}$ modules into the category of R-modules mentioned there has the following property: there exists an indecomposable projective module P with endomorphism ring F, which we will call *distinctive*, such that for any $\begin{pmatrix} F & M \\ O & G \end{pmatrix}$-module $(U, V, \varphi\colon U_n \otimes {}_FM_G \to V_G)$ with endomorphism ring E, we have ${}_EU_F \approx {}_E\mathrm{Hom}\,(P, \Gamma(U, V, \varphi))_F$. One point corresponding to a distinctive indecomposable projective module is encircled in the table on the previous page.

E. The functor Ext. We will use the standard properties of Ext, in particular the long exact sequences. Note that there are the following canonical isomorphism:

(*a*) $\mathrm{Ext}\,(X, \prod Y_i) \approx \prod \mathrm{Ext}\,(X, Y_i)$, and

(*b*) $\mathrm{Ext}\,(\bigoplus X_i, Y) \approx \prod \mathrm{Ext}\,(X_i, Y)$,

where X, Y are R-modules, and $(X_i)_i$, $(Y_i)_i$ are families of R-modules. If, in addition, X is finitely presented, then also

(*c*) $\mathrm{Ext}\,(X, \bigoplus Y_i) \approx \bigoplus \mathrm{Ext}\,(X, Y_i)$.

In particular, (*c*) is valid in case R is a finite dimensional algebra and X is of finite length.

F. Purity. We insert here a general result on pure submodules which is valid for finite dimensional algebras, and which will be used throughout in the paper.

Let R be a finite dimensional algebra. In dealing with modules of infinite length, one has to realize that there is a new type of non split extensions. Consider a module X with a submodule V, such that both V and X/V are of infinite length. If there exist submodules $0 \subseteq U \subset V \subseteq W \subset X$ with W/U of finite length such that V/U is not a direct summand of W/U, then V cannot be a direct summand of X. In this case, the fact that the submodule V of X does not split off, is explained by the fact that a certain extension of modules of finite length does not split. There are however examples of submodules $V \subset X$ which are not direct summands, which cannot be explained in this way. We will see such examples in the course of the paper.

THEOREM: *Let R be a finite dimensional algebra. Let V be a submodule of the module X. Then the following properties are equivalent:*

(i) *For any submodule W of X with $V \subseteq W$ and W/V of finite length, V is a direct summand of W.*

(ii) *For any submodule U of X with $U \subseteq V$ and V/U of finite length, V/U is a direct summand of X/U.*

(iii) *For any submodules U, W of X; with $U \subseteq V \subseteq W$ and W/U of finite length, V/U is a direct summand of W/U.*

In this case, we will call V a *pure* submodule of X.

PROOF OF THEOREM: It is clear that both (i) and (ii) imply (iii). Thus assume (iii). Let $V \subseteq W$ with W/V of finite length. We may assume that W/V is indecomposable. Let $A(W/V) \xrightarrow{\alpha} Y \xrightarrow{\beta} W/V$ be an Auslander Reiten sequence ending with W/V. If we assume that V is not a direct summand of W, then the canonical projection $\pi: W \to W/V$ is not a split epimorphism, thus there exists $\pi': W \to Y$ with $\pi = \beta\pi'$. There exists $\pi'': V \to A(W/V)$ making the left square commutative

$$\begin{array}{ccccccccc}
0 \to & A(W/V) & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & W/V & \to 0 \\
 & \uparrow{\scriptstyle \pi''} & & \uparrow{\scriptstyle \pi'} & & \uparrow{\scriptstyle \mathrm{id}} & \\
0 \to & V & \to & W & \xrightarrow{\pi} & W/V & \to 0 .
\end{array}$$

Let U be the kernel of π''. Then we can factor π'' over V/U and π'

over W/U, and obtain the following diagram

$$\begin{array}{ccccccccc}
0 & \to & A(W/V) & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & W/V & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \to & V/U & \to & W/U & \to & W/V & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \to & V & \to & W & \xrightarrow{\pi} & W/V & \to & 0 .
\end{array}$$

According to our assumption, the middle exact sequence splits. But this implies that the upper exact sequence, being induced from a split exact sequence, is also split, contrary to the assumption of being an Auslander Reiten sequence. This contradiction shows that V has to be a direct summand of W, thus we have proved (i).

In order to prove (ii), we proceed dually. Given $U \subseteq V$ with V/U of finite length, we see that we may assume V/U indecomposable. If V/U is not a direct summand of X/U, V/U cannot be injective, thus there exists an Auslander Reiten sequence $V/U \xrightarrow{\alpha} Z \xrightarrow{\beta} A^{-1}(V/U)$, and a lifting of the inclusion $\gamma\colon V/U \to X/U$ to $\gamma'\colon Z \to X/U$, with $\gamma = \gamma'\alpha$. We get γ'' and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & V/U & \xrightarrow{\alpha} & Z & \xrightarrow{\beta} & A^{-1}(V/U) & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma''} & & \\
0 & \to & V/U & \to & X/U & \to & X/V & \to & 0 .
\end{array}$$

Now, the image of γ'' is of the form W/V for some $V \subseteq W \subseteq X$, and W/V is of finite length. But then we can use our assumption (iii) which shows that V/U is a direct summand of W/U, and therefore the upper sequence splits being induced from a split exact sequence. Again, this is a contradiction, and therefore V/U has to be a direct summand of X/U.

We note some elementary properties of pure submodules. Let $U \subseteq V \subseteq X$.

(a) If U is pure in X, then U is pure in V.

(a') If V is pure in X, then V/U is pure in X/U.

(b) If U is pure in V, and V pure in X, then U is pure in X.

(b') If U is pure in X, and V/U is pure in X/U, then V is pure in X.

For the proof of (*b*), let $U' \subseteq U$, with U/U' of finite length. Since U is pure in V, there exists C' in V with $U/U' \oplus C'/U' = V/U'$. Now V/C' is of finite length, thus using that V is pure in X, there exists C in X with $V/C' \oplus C/C' = X/C'$. But then $U/U' \oplus C/U' = X/U'$.

Similarly, for the proof of (*b'*), let $V \subseteq V' \subseteq X$ with V'/V of finite length. Then there is C' with $V/U \oplus C'/U = V'/C$, and since then C'/U is of finite length, there is C with $U \oplus C = C'$. Thus $V \oplus C = V'$.

Direct summands are examples of pure submodules. However, in contrast to the set of direct summands of a fixed module, the set of pure submodules of a fixed module is closed under unions of filterted families:

(*c*) Let $(V_i)_{i\in I}$ be a filtered family of pure submodules of X. Then $\bigcup_{i\in I} V_i$ is pure in X.

Proof: Let $V = \bigcup_{i\in I} V_i \subseteq W$, with W/V of finite length. There exist submodules C of finite length with $C + V = W$. Choose such a C with minimal length. Consider the various $C \cap V_i$, there is some $i \in I$ with $C \cap V_i$ maximal, say $i = 0$. Since V_0 is pure in X, there is C' with $C + V_0 = C' \oplus V_0$. Thus

$$C + V = C + V_0 + V = C' + V_0 + V = C' + V\,,$$

and therefore, according to the minimality of C, we have $|C| \leqslant |C'|$. But then $C + V_0 = C' \oplus V_0$ implies $C \cap V_0 = 0$. Since we have choosen $C \cap V_0$ maximal under the $C \cap V_i$, we see $C \cap V_i = 0$ for all i, and therefore

$$C \cap V = C \cap \Big(\bigcup_{i\in I} V_i\Big) = \bigcup_{i\in I} (C \cap V_i) = 0\,.$$

Thus $C \oplus V = W$, which shows that V is a direct summand of W.

An immediate consequence is the following type of examples:

(*d*) Let X_i, $i \in I$, be a family of modules. Then $\bigoplus_{i\in I} X_i \subseteq \prod_{i\in I} X_i$ is a pure submodule.

Proof: For any finite subset $I' \subseteq I$, we know that $\bigoplus_{i\in I'} X_i = \prod_{i\in I'} X_i$ is a direct summand of $\prod_{i\in I} X_i$, and $\bigoplus_{i\in I} X_i$ is the sum of the filtered family $\Big\{\bigoplus_{i\in I'} X_i \,|\, I' \subseteq I \text{ finite}\Big\}$.

We end this section with two remarks which will not be used in the paper, but which are perhaps of independent interest.

REMARK 1: The theorem which has been stated for finite dimensional algebras remains true, with the same proof, for the socalled artin algebras (rings which are finitely generated over a central artinian ring), but is no longer true for artinian rings. In fact, let F be a differentially closed field with derivation δ, let $S = F[T; \delta]$ be the twisted polynomial ring in one variable T, and let M be the sub F-F-bimodule of S generated by 1 and T. Note that $\dim_F M = \dim M_F = 2$, thus

$$R = \begin{pmatrix} F & M \\ 0 & F \end{pmatrix}$$

is an artinian ring. Consider the right R-modules $X = S \oplus S$, and its submodule $V = TS \oplus TS$, where the R-action is given by ordinary matrix operation. Note that X/V is of finite length, and V is not a direct summand of X, thus condition (i) of the theorem is not satisfied. On the other hand, it is rather easy to see that $V \subseteq X$ satisfies condition (ii) of the theorem, since for $U \subseteq V$ with V/U indecomposable of finite length, either V/U is preinjective and therefore a direct summand of X/U, or regular and then injective in the category of regular modules, thus again it is a direct summand of X/U. (We have used here the fact that the representation theory of R is rather similar to the case of hereditary finite dimensional algebras of tame type, with similar notions of preinjectivity and regularity, see [31], section 7.)

REMARK 2: For a general ring R, P. M. Cohn [12] has called a submodule V_R of a module X_R pure, provided for any left module ${}_RM$, the induced map $V_R \otimes_R M \to X_R \otimes_R M$ is a monomorphism. In the case of a finite dimensional algebra R, this notion coincides with the previously considered one. In fact, for any right artinian ring R, a submodule $V_R \subseteq X_R$ is Cohn pure if and only if it satisfies the condition (i) of the theorem.

PROOF: In [12], Cohn has shown that $V \subseteq X$ is Cohn pure, if and only if given elements $x_i \in X$, $r_{ij} \in R$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, with $\sum_i x_i r_{ij} \in V$ for all j, there exist elements $v_i \in V$, $1 \leqslant i \leqslant n$, such that

$$\sum_i (x_i - q_i) r_{ij} = 0 \qquad \text{for all } j\,.$$

Now assume, $V \subseteq X$ satisfies the condition (i), and let $x_i \in X$, $r_{ij} \in R$ with $\sum_i x_i r_{ij} \in V$. Let $W = V + \sum_i x_i R$, then W/V is of finite length, thus there exists a direct decomposition $W = V \oplus C$. Write $x_i =$

$= v_i + c_i$, with $v_i \in V$, $c_i \in C$. Then $\sum_i (x_i - v_i) r_{ij}$ belongs to $C \cap V = 0$.

Conversely, assume $V \subseteq X$ is Cohn pure, and let $V \subseteq W \subseteq X$ with W/V of finite length. Let $\pi\colon W \to W/V$ be the canonical projection, and let

$$\bigoplus_{j=1}^{m} R_R \xrightarrow{\alpha} \bigoplus_{i=1}^{n} R_R \xrightarrow{\beta} W/V \to 0$$

be a finite presentation of W/V. Denote the canonical base vectors $(0 \dots 0\,1\,0 \dots 0)$ of $\bigoplus R_R$ by e_i, thus α is given by a matrix (r_{ij}) with $r_{ij} \in R$, such that $\alpha(e_j) = \sum_i e_i r_{ij}$. Let $x_i = \beta'(e_i)$, where $\beta'\colon \bigoplus_{i=1}^{n} R_R \to W$ is a fixed lifting of β. Clearly, $\sum_i x_i r_{ij} = \beta' \alpha(e_j)$ belongs to V, since $\pi\beta'\alpha = 0$. Thus, there are elements $v_i \in V$ with $\sum (x_i - v_i) r_{ij} = 0$ for all j. Define $\gamma\colon \bigoplus_{i=1}^{n} R_R \to W$ by $\gamma(e_i) = x_i - v_i$. Then

$$\gamma\alpha(e_j) = \sum_i (x_i - v_i) r_{ij} = 0$$

shows that γ factors over β, thus there is $\gamma'\colon W/V \to W$ with $\gamma = \gamma'\beta$, and consequently, $\pi\gamma'$ is the identity of W/V, thus π splits.

G. Direct summands of finite length. In this paper, we will consider mainly modules of infinite length, presupposing a good knowledge about the modules of finite length. It is natural to ask whether a given module will have a direct summand of finite length, and therefore even an indecomposable direct summand of finite length. Also, one may ask the stronger question whether the module is even a direct sum of indecomposable modules of finite length. If R is an artinian ring of finite representation type, then any module can be decomposed as the direct sum of indecomposable modules of finite length ([35], see also [4]). Thus, in dealing with modules of infinite length, there is no need to consider any longer rings of finite representation type.

Given a decomposition $X = \bigoplus_{i \in I} X_i$ where every module X_i has a local endomorphism ring, Azumaya has shown that the number of direct summands isomorphic to a given module V is an invariant of the module X [8]. In the case the X_i are of finite length, this is rather easy to see, since we will give an internal description of this invariant.

If V is a module with local endomorphism ring $\operatorname{End}(V)$, let

$$\overline{\operatorname{End}}\,(V) = \operatorname{End}(V)/\operatorname{rad}\operatorname{End}(V)$$

be the residue division ring of $\operatorname{End}(V)$. Given an arbitrary module X, let

$$\operatorname{rad}(V, X) = \{\varphi \in \operatorname{Hom}(V, X) \mid \psi\varphi \in \operatorname{rad}\operatorname{End}(V) \text{ for all } \psi \in \operatorname{Hom}(X, V)\}.$$

Note that this is an $\operatorname{End}(X)$ - $\operatorname{End}(V)$-submodule of $\operatorname{Hom}(V, X)$, thus we may form $\mathcal{C}(V, X) = \operatorname{Hom}(V, X)/\operatorname{rad}(V, X)$. Since obviously $\mathcal{C}(V, X)$ is annihilated on the right by $\operatorname{rad}\operatorname{End}(V)$, we may consider $\mathcal{C}(V, X)$ as a right $\overline{\operatorname{End}}(V)$-vector space.

We will use these vector spaces $\mathcal{C}(V, X)$, where V runs through the set $\mathfrak{V}$ of indecomposable modules of finite length, in order to prove the following theorem:

THEOREM: *Let R be a finite dimensional algebra. Given a module X, there exists a pure submodule Y which is the direct sum of indecomposable modules of finite length, such that Y/X has no indecomposable direct summand of finite length. Any two such submodules Y, Y' are isomorphic.*

In fact, we prove the following more precise result, where we denote for $\varphi \in \operatorname{Hom}(V, X)$ its residue class in $\mathcal{C}(V, X)$ by $\bar{\varphi}$.

PROPOSITION: Let R be a finite dimensional algebra. Given a module X, choose for any $V \in \mathfrak{V}$ a family $(\varphi_{iV})_{i\in I_V}$ of elements $\varphi_{iV} \in \operatorname{Hom}(V, X)$ such that $(\bar{\varphi}_{iV})_i$ is a basis of $\mathcal{C}(V, X)_{\overline{\operatorname{End}}(V)}$. Then the map

$$(\varphi_{iV})_{iV} \colon \bigoplus_{V\in\mathfrak{V}} \bigoplus_{i\in I_V} V \to X$$

is a monomorphism, the image is a pure submodule, and the cokernel has no indecomposable direct summand of finite length.

Conversely, given a pure submodule Y of X with $Y = \bigoplus_{j\in J} Y_j$, with Y_j indecomposable of finite length, and such that X/Y has no indecomposable direct summand of finite length, let

$$J_V = \{j \in J \mid Y_j \approx V\},$$

and for $j \in J_V$, choose an isomorphism $\alpha_j \colon V \to Y_j$, and denote the inclusion $Y_j \to X$ by γ_j. Then $(\overline{\gamma_j \alpha_j})_{j\in J_V}$ is a basis of $\mathcal{C}(V, X)_{\overline{\operatorname{End}}(V)}$.

PROOF: The proof will be done in several steps.

(*a*) Let $Y = \bigoplus_{j\in J} Y_j$ be a pure submodule of X, with Y_j indecomposable of finite length. Let $J_V = \{j \in J \mid Y_j \approx V\}$, and for $j \in J_V$, choose an isomorphism $\alpha_j \colon V \to Y_j$, and denote the inclusion by γ_j. Then $(\overline{\gamma_j \alpha_j})_j$ is linearly independent in $\mathcal{C}(V, X)_{\overline{\operatorname{End}}(V)}$.

Assume $\beta = \sum_j \gamma_j \alpha_j \beta_j$ belongs to rad (V, X), where j runs through a finite subset J'_V of J_V, and $\beta_j \in \operatorname{End}(V)$. Now $\bigoplus_{j \in J'_V} Y_j$ is a pure submodule of X of finite length, thus a direct summand of X. Let $\pi_j: X \to Y_j$ be the canonical projections, with $\pi_i \gamma_j = \delta_{ij}$. Since β belongs to rad (V, X), it follows that $\beta_i = \alpha_i^{-1} \pi_i \sum_j \gamma_j \alpha_j \beta_j = \alpha_i^{-1} \pi_i \beta$ belongs to rad End (V).

(*b*) Let $W = \bigoplus_{i=1}^{n} V$, with inclusions $\gamma_i: V \to W$. Then $(\bar{\gamma}_i)_i$ is a basis of $\mathcal{C}(V, W)_{\overline{\operatorname{End}(V)}}$.

By (*a*), these elements are linearly independent. Let $\varphi: V \to W$ be given. Denote by $\pi_i: W \to V$ the canonical projections. Thus $\pi_i \gamma_j = \delta_{ij}$, and $\sum_i \gamma_i \pi_i = id_W$. Therefore $\varphi = \sum_i \gamma_i \pi_i \varphi$ shows that $\bar{\varphi}$ is a linear combination of the elements $\bar{\gamma}_i$ with coefficients in $\overline{\operatorname{End}}(V)$.

(*c*) Let X be an arbitrary module, let $\varphi_i: V \to X$, $1 \leqslant i \leqslant n$ be a finite set of maps such that $(\bar{\varphi}_i)_i$ is a linearly independent set in $\mathcal{C}(V, X)_{\overline{\operatorname{End}(V)}}$. Then $(\varphi_i)_i: \bigoplus_{i=1}^{n} V \to X$ is a split monomorphism.

Proof by induction on n. For $n = 1$, the fact that φ_1 does not belong to rad (V, X) shows that there is π_1 with $\pi_1 \varphi_1 \notin$ rad End (V). Thus $\pi_1 \varphi_1$ is an automorphism and φ_1 is split mono. Now assume there is given $\varphi_1, \ldots, \varphi_n$. By induction, we may assume that

$$(\varphi_i)_{i=1}^{n-1}: \bigoplus_{i=1}^{n-1} V \to X$$

is a split mono, with image X'. Let X'' be a direct complement of X' in X, thus $X = X' \oplus X''$. We can write $\varphi_n: V \to X' \oplus X''$ as $(\varphi'_n, \varphi''_n)$ with $\varphi'_n: V \to X'$ and $\varphi''_n: V \to X''$. Note that $\bar{\varphi}'_n$ belongs to the subspace $\mathcal{C}(V, X')$ of $\mathcal{C}(V, X)_{\overline{\operatorname{End}(V)}}$, and therefore $\bar{\varphi}'_n$ is a linear combination of the basis elements $\bar{\varphi}_1, \ldots, \bar{\varphi}_{n-1}$ of $\mathcal{C}(V, X')$. This shows that φ''_n cannot belong to rad (V, X''), since otherwise $\bar{\varphi}_1, \ldots, \bar{\varphi}_n$ would not be linearly independent. Thus, the case $n = 1$ shows that $\varphi''_n: V \to X''$ is a split mono, say let $\varphi''_n(V) \oplus C = X''$. Then $X = X' \oplus \varphi''_n(V) \oplus C$ shows that the image of $(\varphi_i)_{i=1}^{n}$ which is just $X' \oplus \varphi''_n(V)$ is a direct summand of X. Also, $(\varphi_i)_{i=1}^{n}$ has to be a monomorphism.

(*d*) Let V_j, $1 \leqslant j \leqslant m$, be pairwise non-isomorphic indecomposable module of finite length. Let $\varphi_{ij}: V_j \to X$, $1 \leqslant i \leqslant n_j$, be maps with $(\bar{\varphi}_{ij})_j$ linearly independent in $\mathcal{C}(V_j, X)_{\overline{\mathrm{End}(V_j)}}$. Then

$$(\varphi_{ij})_{ij}\colon \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n_j} V_j \to X$$

is a split monomorphism.

By the previous case,

$$\varphi_j = (\varphi_{ij})_i\colon \bigoplus_{i=1}^{n_j} V_j = W_j \to X$$

is a split mono, say with retraction $\pi_j\colon X \to W_j$. We claim that the map $(\pi_j p_{j'}) \in \mathrm{End}\left(\bigoplus_j W_j\right)$ is an automorphism. However, this is clear, since $\pi_j \varphi_j = id_{W_j}$, and, for $j \neq j'$, W_j and $W_{j'}$ have no common indecomposable direct summand. But if $(\pi_j \varphi_{jk})$ is an automorphism, then $(\varphi_j)_j\colon \bigoplus W_j \to X$ is a split monomorphism.

(*e*) Now assume for $V \in \mathfrak{B}$, there is given a family $(\varphi_{iV})_{i \in I_V}$ in $\mathrm{Hom}(V, X)$ such that $(\bar{\varphi}_{iV})_i$ is a basis of $\mathcal{C}(V, X)_{\overline{\mathrm{End}(V)}}$. Now $\bigoplus_V \bigoplus_{I_V} V$ is the filtered union of the finite direct sums $\bigoplus_{V \in \mathfrak{B}'} \bigoplus_{i \in I'_V} V$ where $\mathfrak{B}'$ is a finite subset of $\mathfrak{B}$, and I'_V is a finite subset of I_V. It follows immediately that $(\varphi_{iV})_{iV}\colon \bigoplus_V \bigoplus_{I_V} V \to X$ is a monomorphism onto a pure submodule Y of X, since the restrictions of $(\varphi_{iV})_{iV}$ to $\bigoplus_{V \in \mathfrak{B}'} \bigoplus_{i \in I'_V} V$ are monomorphisms onto direct summands of X. It remains to show that X/Y has no indecomposable direct summand of finite length. Assume X/Y has a direct summand W/Y with $W/Y \approx V \in \mathfrak{B}$. Since Y is pure in X, there is an embedding $\varphi\colon V \to W$ with $W = Y \oplus \varphi(V)$. Also, since Y is pure in X, and W/Y is pure in X/Y, we know that W is pure in X. But then, according to (*a*), the set $(\bar{\varphi}_{iV})_{i \in I_V} \cup \bar{\varphi}$ is linearly independent in $\mathcal{C}(V, X)_{\overline{\mathrm{End}(V)}}$, contrary to the assumption that $(\bar{\varphi}_{iV})_{i \in I_V}$ is a basis. This proves the first part of the proposition.

(*f*) For the converse, let Y be a pure submodule of X, $Y = \bigoplus_{j \in J} Y_j$ with Y_j indecomposable of finite length, and X/Y without indecomposable direct summand of finite length. Let $J_V = \{j \in J \mid Y_j \approx V\}$, choose isomorphism $\alpha_j\colon V \to Y_j$, and let $\gamma_j\colon Y_j \to X$ be the inclusion. Let $\varphi_{jV} = \gamma_j \alpha_j$. We know from (*a*) that $(\bar{\varphi}_{jV})_{j \in J_V}$ is a linearly independent subset of $\mathcal{C}(V, X)_{\overline{\mathrm{End}(V)}}$. Extend this to a basis $(\bar{\varphi}_{iV})_{i \in I_V}$, of $\mathcal{C}(V, X)_{\mathrm{End}(V)}$, with $J_V \subseteq I_V$. By the first part of the theorem, the image of $(\varphi_{iV})_{iV}$ is a pure submodule of X, which contains Y as a direct summand.

But then $\bigoplus_V \bigoplus_{I_V \setminus J_V} V \approx \left(\bigoplus_V \bigoplus_{I_V} V\right)/Y \subseteq X/Y$ is a pure submodule, and every indecomposable direct summand of $\left(\bigoplus_V \bigoplus_{I_V} V\right)/Y$, being of finite length, is a direct summand of X/Y. Thus, since X/Y has no indecomposable direct summand of finite length, we conclude that $\left(\bigoplus_V \bigoplus_{I_V} V\right)/Y = 0$, and therefore $I_V = J_V$. This proves the last part of the proposition.

Remark: Assume X is a direct sum of indecomposable modules of finite length, say $X = \bigoplus_V \bigoplus_{I_V} V$. Then we know that the inclusions $\gamma_{iV}\colon V \to X$ give rise to a basis $(\bar{\gamma}_{iV})_i$ of $\mathcal{C}(V, X)_{\overline{\mathrm{End}}(V)}$. However, if we choose an arbitrary basis $(\varphi_{iV})_{i \in I_V}$ of $\mathcal{C}(V, X)_{\overline{\mathrm{End}}(V)}$, for every V, then the image Y of $(\varphi_{iV})_{iV}$ may be a proper submodule of X.

Example: Let R be a finite dimensional hereditary algebra of tame representation type, and let S be a simple homogeneous module. Denote by $u_n\colon S^n \to S^{n+1}$ the canonical inclusion. Now let $X = \bigoplus_{n \in N} X_n$ with $X_n = S^n$, and let

$$\varphi_n = (\mathrm{id}, u_n)\colon S^n \to S^n \oplus S^{n+1} = X_n \oplus X_{n+1} \hookrightarrow X\,.$$

From the decomposition $X = \bigoplus X_n$, we see that $\mathcal{C}(S^n, X)_{\overline{\mathrm{End}}(S^n)}$ is one-dimensional, and it is clear that $\bar{\varphi}_n \neq 0$, thus it gives a basis of $\mathcal{C}(S^n, X)_{\overline{\mathrm{End}}(S^n)}$. On the other hand, the image Y of $(\varphi_n)_n$ is a proper submodule of X. In fact $X/Y \approx S^\omega$ (for the definition of S^ω, see 4.5).

2. Preprojective modules.

2.1. Let $\mathfrak{P}$ be the set of (isomorphism classes of) indecomposable preprojective modules of finite length. Note that $\mathfrak{P}$ is partially ordered by: $P' \geqslant P$ iff there exists a chain

$$P' = P_1 \to P_2 \to \ldots \to P_{n-1} \to P_n = P$$

of non-zero maps, where all $P_i \in \mathfrak{P}$. Let $\mathfrak{A}$ be a subset of $\mathfrak{P}$ closed under predecessors (thus, if $P \in \mathfrak{A}$ and $P' \geqslant P$, then $P' \in \mathfrak{A}$). It is clear that $\mathfrak{A}$ is either finite, or $\mathfrak{A} = \mathfrak{P}$. For any module X, let $\mathcal{T}_{\mathfrak{A}} X$ be the intersection of the kernels of all maps $X \to P$ with $P \in \mathfrak{A}$.

Proposition: Let $\mathfrak{A}$ be a finite set of indecomposable preprojective modules closed under predecessors. Let X be a module. Then, for any P in $\mathfrak{A}$, there exists a (not necessarily uniquely defined) submodule U_P of X which is isomorphic to a direct sum of copies of P such that

$$X = \mathfrak{T}_{\mathfrak{A}} X \oplus \bigoplus_{P \in \mathfrak{A}} U_P .$$

In particular, $\mathfrak{T}_{\mathfrak{A}} X$ is a direct summand of X, and $\mathfrak{T}_{\mathfrak{A}} \mathfrak{T}_{\mathfrak{A}} X = \mathfrak{T}_{\mathfrak{A}} X$.

For the proof we will use a factorisation of the functor $\mathfrak{T}_{\mathfrak{A}}$, and induction on the number of elements of $\mathfrak{A}$. Before we give the proof, we want to derive two corollaries.

2.2. The functor $\mathfrak{T}_{\mathfrak{P}}$ will be denoted just by $\mathfrak{T}$, thus $\mathfrak{T}X$ is the intersection of the kernels of all maps $X \to P$ with P indecomposable preprojective.

Corollary: Let X be a module. Then $\mathfrak{T}X = X$ if and only if X has no indecomposable preprojective direct summand.

Proof: If $X = X' \oplus P$ with P indecomposable preprojective, then obviously $\mathfrak{T}X \subseteq X' \subset X$. Conversely, assume there is a non-trivial homomorphism $X \to P$ with P indecomposable preprojective. Let $\mathfrak{A}$ be the set of predecessors of P in $\mathfrak{P}$. Then $\mathfrak{A}$ is a finite set, thus we can apply the proposition: the submodule $\mathfrak{T}_{\mathfrak{A}} X$ of X has a direct complement which is the direct sum of indecomposable preprojective modules.

Corollary 2: If P is indecomposable preprojective, and X is a module with $X = \mathfrak{T}X$, then $\operatorname{Ext}(P, X) = 0$.

Proof: Let $0 \to X \to Y \to P \to 0$ be an exact sequence. Let $\mathfrak{A}$ be a finite subset of $\mathfrak{P}$ closed under predecessors and containing P. Then $\mathfrak{T}_{\mathfrak{A}} Y \subseteq X$, since $Y/X \approx P$, and, on the other hand, $X = \mathfrak{T}_{\mathfrak{A}} X \subseteq \mathfrak{T}_{\mathfrak{A}} Y$, thus $X = \mathfrak{T}_{\mathfrak{A}} Y$, and therefore X is a direct summand in Y.

Corollary 3: Let X be a submodule of the module Y. Assume that X has no indecomposable preprojective direct summand, and that every indecomposable submodule of Y/X of finite length is preprojective. Then X is pure in Y.

Proof: Let U be a submodule of Y containing X such that U/X is of finite length. By assumption, the indecomposable direct summands of U/X have to be preprojective, since they are submodules of Y/X. Let $\mathfrak{A}$ be the set of predecessors of the indecomposable direct summands of U/X in $\mathfrak{P}$. By construction of $\mathfrak{A}$, $\mathfrak{T}_{\mathfrak{A}}U \subseteq X$, and, since $\mathfrak{T}_{\mathfrak{A}}X = X$, we see that even $\mathfrak{T}_{\mathfrak{A}}U = X$. By proposition 2.1, $X = \mathfrak{T}_{\mathfrak{A}}U$ is a direct summand of U.

2.3 Lemma: Let $\mathfrak{A}$ be a finite set of indecomposable preprojective R-modules, closed under predecessors. Then there exists a finite dimensional hereditary algebra B and adjoint functors $S^-: {}_B\mathfrak{M} \to {}_R\mathfrak{M}$ and $S^+: {}_R\mathfrak{M} \to {}_B\mathfrak{M}$ with natural transformation $\gamma_X: S^-S^+X \to X$ such that

(1) γ_X is a monomorphism, for all R-modules X.

(2) γ_X is an isomorphism, whenever X has no direct summand isomorphic to any P in $\mathfrak{A}$.

(3) $S^+P = 0$ for $P \in \mathfrak{A}$.

(4) If $P \in \mathfrak{P} \setminus \mathfrak{A}$, and all predecessors of P belong to $\mathfrak{A}$, then S^+P is simple projective.

For a proof, we refer to [6] and [34] (for tensor algebras, this has been established in [14]).

Proof of Proposition 2.1: By induction on the number of elements in $\mathfrak{A}$. If $\mathfrak{A}$ is empty, nothing has to be shown. If $\mathfrak{A}$ is non-empty, let Q be a module in $\mathfrak{A}$ which is not the predecessor of any other module in $\mathfrak{A}$. Let $\mathfrak{B} = \mathfrak{A} \setminus \{Q\}$. By induction, we know that X is of the form $X = \mathfrak{T}_{\mathfrak{B}}X \oplus \bigoplus_{P \in \mathfrak{B}} U_P$, for some submodules U_P being direct sums of copies of P. Now $\mathfrak{T}_{\mathfrak{A}}X$ is contained in $\mathfrak{T}_{\mathfrak{B}}X$, and there is an exact sequence of the form

$$0 \to \mathfrak{T}_{\mathfrak{A}}X \xrightarrow{m} \mathfrak{T}_{\mathfrak{B}}X \xrightarrow{f} \prod_{i \in I} Q$$

where m in the inclusion map, and I is some index set. Now we use the lemma above for $\mathfrak{B}$. The functor S^+ which we obtain is a right adjoint functor, thus it is left exact and commutes with products. As a consequence, we get the exact sequence

$$0 \to S^+\mathfrak{T}_{\mathfrak{A}}X \xrightarrow{S^+m} S^+\mathfrak{T}_{\mathfrak{B}}X \xrightarrow{S^+f} \prod_{i \in I} S^+Q$$

of B-modules. Since S^+Q is a simple B-module, the ring $B/\mathrm{Ann}\, S^+Q$ is a simple artinian ring (where $\mathrm{Ann}\, M$ denotes the annihilator of the module M), and therefore $\prod_{i \in I} S^+Q$ and also the image of S^+f are direct sums of copies of S^+Q. Thus, we obtain an exact sequence

$$0 \to S^+\mathfrak{T}_{\mathfrak{A}}X \xrightarrow{S^+m} S^+\mathfrak{T}_{\mathfrak{B}}X \xrightarrow{g} \bigoplus_{j \in J} S^+Q \to 0$$

for some index set J. Note that this sequence splits since S^+Q and therefore also $\bigoplus_{j\in J} S^+Q$ are projective modules. As a split exact sequence, it stays exact when we apply the functor S^-. This gives us the following commutative diagram with exact first row

$$\begin{array}{ccccccc} 0 \to S^-S^+\mathfrak{T}_{\mathfrak{A}}X & \xrightarrow{S^-S^+m} & S^-S^+\mathfrak{T}_{\mathfrak{B}}X & \xrightarrow{S^-g} & \bigoplus_{j\in J} S^-S^+Q \to 0 \\ \downarrow \gamma_{\mathfrak{T}_{\mathfrak{A}}X} & & \downarrow \gamma_{\mathfrak{T}_{\mathfrak{B}}X} & & \\ \mathfrak{T}_{\mathfrak{A}}X & \xrightarrow{m} & \mathfrak{T}_{\mathfrak{B}}X\,. & & \end{array}$$

Since $\mathfrak{T}_{\mathfrak{B}}X$ does not have any direct summand isomorphic to some $P\in\mathfrak{B}$, we know that $\gamma_{\mathfrak{T}_{\mathfrak{B}}X}$ is an isomorphism. Now the map $(S^-g)\gamma^{-1}_{\mathfrak{T}_{\mathfrak{B}}X}$ can be extended to X, since $\mathfrak{T}_{\mathfrak{B}}X$ is a direct summand of X, and therefore it has to vanish on $\mathfrak{T}_{\mathfrak{A}}X$ by the definition of $\mathfrak{T}_{\mathfrak{A}}$. This implies that the monomorphism $\gamma_{\mathfrak{T}_{\mathfrak{A}}X}$ in fact has to be an isomorphism. Since both $\gamma_{\mathfrak{T}_{\mathfrak{A}}X}$ and $\gamma_{\mathfrak{T}_{\mathfrak{B}}X}$ are isomorphisms, we see that m is a split monomorphism with cokernel isomorphic to $\bigoplus_{j\in J} S^-S^+Q$. Note that $S^-S^+Q \approx Q$. Let U_Q be a complement of $P_{\mathfrak{A}}X$ in $P_{\mathfrak{B}}X$. Then $U_Q \approx \bigoplus_{j\in J} Q$, and

$$X = \mathfrak{T}_{\mathfrak{B}}X \oplus \bigoplus_{P\in\mathfrak{B}} U_P = \mathfrak{T}_{\mathfrak{A}}X \oplus U_Q \oplus \bigoplus_{P\in\mathfrak{B}} U_P = \mathfrak{T}_{\mathfrak{A}}X \oplus \bigoplus_{P\in\mathfrak{B}} U_P\,.$$

This finishes the proof.

Corollary: The image of the natural map $\gamma_X\colon S^-S^+X \to X$ is just $\mathfrak{T}_{\mathfrak{A}}X$.

Proof: Consider the commutative diagram

$$\begin{array}{ccc} S^-S^+\mathfrak{T}_{\mathfrak{A}}X & \xrightarrow{S^-S^+u} & S^-S^+X \\ \downarrow \gamma_{\mathfrak{T}_{\mathfrak{A}}X} & & \downarrow \gamma_X \\ \mathfrak{T}_{\mathfrak{A}}X & \xrightarrow{u} & X\,. \end{array}$$

Since $\mathfrak{T}_{\mathfrak{A}}X$ has no direct summand isomorphic to $P\in\mathfrak{A}$, it follows from the lemma that $\gamma_{\mathfrak{T}_{\mathfrak{A}}X}$ is an isomorphism. On the other hand, $X = \mathfrak{T}_{\mathfrak{A}}X \oplus \bigoplus_{P\in\mathfrak{A}} U_P$ for submodules U_P which have the property that $S^-S^+U_P = 0$. As a consequence, also S^-S^+u is an isomorphism, and therefore u and γ_X have the same image.

2.4. We have seen that for any finite subset $\mathfrak{A}$ of $\mathfrak{P}$ closed under predecessors, the submodule $\mathfrak{T}_{\mathfrak{A}}X$ is a direct summand of X, and that $X/\mathfrak{T}_{\mathfrak{A}}X$ is a direct sum of indecomposable preprojective modules. For $\mathfrak{T} = \mathfrak{T}_{\mathfrak{P}}$, the situation is rather different. We will see that $\mathfrak{T}X$ does not have to be a direct summand of X, nor that $X/\mathfrak{T}X$ has to be a direct sum of indecomposable modules. An example of the latter is given by the module $\prod_{P\in\mathfrak{P}} P$ which clearly satisfies $\mathfrak{T}\left(\prod_{P\in\mathfrak{P}} P\right) = 0$ without however being a direct sum of indecomposable modules.

On the other hand, assume for some module X, the submodule $\mathfrak{T}X$ is a direct summand, say $X = \mathfrak{T}X \oplus C$ for some submodule C. Since $\mathfrak{T}C = 0$, we see that $\mathfrak{T}X = \mathfrak{T}\mathfrak{T}X \oplus \mathfrak{T}C = \mathfrak{T}\mathfrak{T}X$. Thus, a necessary condition for $\mathfrak{T}X$ to be a direct summand is that $\mathfrak{T}\mathfrak{T}X = \mathfrak{T}X$. The main result below will be that not only $\mathfrak{T}^2 \neq \mathfrak{T}$, but that all functors $\mathfrak{T}^\lambda$ with λ an arbitrary ordinal, are different. Here, $\mathfrak{T}^\lambda$ is defined inductively as follows:

Let X be a module. Define $\mathfrak{T}^0X = X$. If λ is an ordinal, and $\mathfrak{T}^\mu X$ is defined for all $\mu < \lambda$, then let $\mathfrak{T}^\lambda X = \bigcap_{\mu<\lambda} \mathfrak{T}^\mu X$ in case λ is a limit ordinal, and $\mathfrak{T}^\lambda X = \mathfrak{T}\mathfrak{T}^{\lambda-1}X$ in case λ is not a limit ordinal. Finally, let $\mathfrak{T}^\infty X = \bigcap \mathfrak{T}^\lambda X$ where the intersection is taken over all ordinals what so ever.

It is clear that $\mathfrak{T}^\infty\mathfrak{T}^\infty = \mathfrak{T}^\infty$. For, given any module X, there is some ordinal λ such that $\mathfrak{T}^\infty X = \mathfrak{T}^\lambda X$, since any chain of submodules has to stop eventually. But then $\mathfrak{T}\mathfrak{T}^\lambda X = \mathfrak{T}^\lambda X$, and therefore

$$\mathfrak{T}^\infty\mathfrak{T}^\infty X = \mathfrak{T}^\infty\mathfrak{T}^\lambda X = \mathfrak{T}^\lambda X = \mathfrak{T}^\infty X .$$

We list some elementary properties of the functors $\mathfrak{T}^\lambda$, with λ an ordinal, or $\lambda = \infty$.

(a) If $f: X \to Y$ is a homomorphism, then $f(\mathfrak{T}^\lambda X) \subseteq \mathfrak{T}^\lambda Y$.

(b) For any module X, we have $\mathfrak{T}^\lambda(X/\mathfrak{T}^\lambda X) = 0$.

The proof of both (a) and (b) is by transfinite induction and rather straight forward. An immediate consequence of these two properties is the following useful fact:

(c) Assume X is a submodule of Y, and $\mathfrak{T}^\lambda(Y/X) = 0$. Then $\mathfrak{T}^\lambda Y \subseteq X$.

Finally, we mention that $\mathfrak{T}^\lambda$ always commutes with direct sums:

(d) If X_i is a family of modules, then $\mathfrak{T}^\lambda(\bigoplus X_i) = \bigoplus \mathfrak{T}^\lambda X_i$. Again, the proof is by transfinite induction.

2.5. In the proof of the next theorem we will need the following lemma:

Lemma: Let P, P' be indecomposable preprojective modules. Then there exists an exact sequence

$$0 \to P \to Q \to Q' \to 0$$

where Q and Q' are both preprojective of finite length, such that $\mathrm{Hom}(Q, P') = 0$.

Proof: We will index the set $\mathfrak{P}$ of indecomposable preprojective modules by $\mathbf{N}$ in the following way: let $P_1, \ldots, P_s$ be the indecomposable projective modules numbered in such a way that $\mathrm{Hom}(P_i, P_j) \neq 0$ implies $i \leqslant j$. For $n \in \mathbf{N}$, and $1 \leqslant i \leqslant s$, let $P_{ns+i} = A^{-n} P_i$, where A refers to the Auslander construction. It is clear that in this way $\mathrm{Hom}(P_i, P_j) \neq 0$ implies $i < j$ for all i, j. If

$$0 \to P_i \xrightarrow{\alpha_i} X_i \to P_{i+s} \to 0$$

is an Auslander Reiten sequence, then the indecomposable direct summands of X_i are of the form P_j with $i < j < i + s$.

In order to prove the lemma, fix $P = P_i$. We show that for every $n \in \mathbf{N}$, there exists an exact sequence

$$0 \to P \xrightarrow{\alpha} Q \to Q' \to 0$$

such that the indecomposable direct summands of Q are of the form P_j with $j > n$, and with Q' preprojective of finite length. For $n \leqslant i$, we can take just the Auslander Reiten sequence starting with P. Now assume we have such a sequence for some n. Let $Q = \bigoplus_j P_j^{m_j}$, thus $m_j \neq 0$ only for some $j > n$. Consider the monomorphism

$$P \xrightarrow{\alpha} Q = \bigoplus_j P_j^{m_j} \xrightarrow{(\alpha_j)} \bigoplus_j X_j^{m_j},$$

its cokernel is an extension of Q' by $\bigoplus P_{j+s}^{m_j}$, and therefore preprojective of finite length. The indecomposable direct summands of $\bigoplus_j X_j^{m_j}$ are of the form P_k with $k > n + 1$.

2.6. Theorem: *Let λ be an ordinal, and P an indecomposable preprojective module. Then there exists a module $X = X(\lambda, P)$ such that $\mathfrak{F}^{\lambda} X \approx P$.*

Proof: By induction on λ. Thus fix λ and P, and assume we have constructed $X(\mu, Q)$ for all ordinals $\mu < \lambda$, and any indecomposable preprojective module Q.

First, assume λ is a limit ordinal. Fix modules $X_\mu = X(\mu, P)$, and isomorphisms $\varphi_\mu \colon \mathfrak{F}^\mu X_\mu \to P$, for all $\mu < \lambda$. Define $X = X(\lambda, P)$ and α by the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & \bigoplus\limits_{\mu<\lambda} \mathfrak{F}^\mu X_\mu & \to & \bigoplus\limits_{\mu<\lambda} X_\mu & \to & \bigoplus\limits_{\mu<\lambda} X_\mu/\mathfrak{F}^\mu X_\mu \\
& \Big\downarrow {\scriptstyle (\varphi_\mu)_\mu} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}} \\
0 \to & P & \to & X & \to & \bigoplus\limits_{\mu<\lambda} X_\mu/\mathfrak{F}^\mu X_\mu
\end{array}$$

where the maps in the first row are the canonical ones. We have $\mathfrak{F}^\lambda\Big(\bigoplus\limits_{\mu<\lambda} X_\mu/\mathfrak{F}^\mu X_\mu\Big) = \bigoplus\limits_{\mu<\lambda} \mathfrak{F}^\lambda(X_\mu/\mathfrak{F}^\mu X_\mu) = 0$, thus $\mathfrak{F}^\lambda X \subseteq \alpha(P)$. Conversely, consider for $\nu < \lambda$ the commutative diagram

$$\begin{array}{ccc}
\mathfrak{F}^\nu X_\nu & \to & X_\nu \\
\Big\downarrow {\scriptstyle \iota_\nu} & & \Big\downarrow {\scriptstyle \iota_\nu} \\
\bigoplus\limits_{\mu<\lambda} \mathfrak{F}^\mu X_\mu & \to & \bigoplus\limits_{\mu<\lambda} X_\mu \\
\Big\downarrow {\scriptstyle (\varphi_\mu)_\mu} & & \Big\downarrow \\
P & \xrightarrow{\alpha} & X
\end{array}$$

where ι_ν denotes the inclusion of the ν-th summand. We may regard the right vertical map $X_\nu \to X$ as an inclusion, thus X_ν may be considered as a submodule of X in such a way that $\mathfrak{F}^\nu X_\nu = \alpha(P)$. Consequently $\mathfrak{F}^\nu X \supseteq \mathfrak{F}^\nu X_\nu = \alpha(P)$, and therefore also $\mathfrak{F}^\lambda X = \bigcap\limits_{\nu<\lambda} \mathfrak{F}^\nu X \supseteq \alpha(P)$. Together with the previously derived inclusion, we see that $\mathfrak{F}^\lambda X = \alpha(P)$.

Next, assume $\lambda = \mu + 1$, where μ is an ordinal or $\mu = 0$. For any indecomposable preprojective module Q, we fix some $X(\mu, Q)$. If $Q' = \bigoplus\limits_{i=1}^{n} Q_i$ with Q_i indecomposable preprojective, we introduce

$$X(\mu, Q') = \bigoplus_{i=1}^{n} X(\mu, Q_i)\,,$$

and we see that also in this case, $\mathfrak{F}^\mu X(\mu, Q') \approx Q'$. For conveniency, we use $\mathbf{N}$ as index set for $\mathfrak{P}$, say $\mathfrak{P} = \{P_n \mid n \in \mathbf{N}\}$. According to 2.5, we may choose exact sequences

$$0 \to P \xrightarrow{\beta_n} Q_n \to Q_n' \to 0$$

with Q_n, Q'_n preprojective of finite length and $\operatorname{Hom}(Q_n, P_n) = 0$. Denote by $\alpha_n: Q_n \to X(\mu, Q_n)$ a monomorphism mapping onto $\mathfrak{T}^\mu X(\mu, Q_n)$, and by Z_n its cokernel. Thus we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & P & \xrightarrow{\beta_n} & Q_n & \longrightarrow & Q'_n & \to & 0 \\
 & & & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \delta_n} & & \\
0 & \to & P & \xrightarrow{\gamma_n} & X(\mu, Q_n) & \to & Y_n & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z_n & \xrightarrow{\mathrm{id}} & Z_n & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We define $X = X(\lambda, P)$ and α by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \bigoplus_{n\in\mathbb{N}} P & \xrightarrow{\oplus \gamma_n} & \bigoplus_{n\in\mathbb{N}} X(\mu, Q_n) & \to & \bigoplus_{n\in\mathbb{N}} Y_n & \to & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \to & P & \xrightarrow{\alpha} & X & \xrightarrow{\varepsilon} & \bigoplus_{n\in\mathbb{N}} Y_n & \to & 0
\end{array}$$

where σ is the summation map. Applying $\mathfrak{T}^\mu$, we get $\mathfrak{T}^\mu Z_n = 0$, since $Z_n = X(\mu, Q_n)/\mathfrak{T}^\mu X(\mu, Q_n)$. Thus $\mathfrak{T}^\mu Y_n \subseteq \delta_n(Q'_n) \approx Q'_n$. Therefore

$$\mathfrak{T}^\mu\Big(\bigoplus_{n\in\mathbb{N}} Y_n\Big) = \bigoplus_{n\in\mathbb{N}} \mathfrak{T}^\mu Y_n \subseteq \bigoplus_{n\in\mathbb{N}} \delta_n(Q'_n) \approx \bigoplus_{n\in\mathbb{N}} Q'_n .$$

Since $\mathfrak{T}\Big(\bigoplus_{n\in\mathbb{N}} Q'_n\Big) = 0$, we also have

$$\mathfrak{T}^\lambda\Big(\bigoplus_{n\in\mathbb{N}} Y_n\Big) = \mathfrak{T}^{\mu+1}\Big(\bigoplus_{n\in\mathbb{N}} Y_n\Big) = \mathfrak{T}\mathfrak{T}^\mu\Big(\bigoplus_{n\in\mathbb{N}} Y_n\Big) = 0 .$$

As a consequence, $\mathfrak{T}^\lambda X \subseteq \alpha(P)$.

Consider for some $m \in \mathbb{N}$ the commutative diagram

$$\begin{array}{ccc}
P & \xrightarrow{\gamma_m} & X(\mu, Q_m) \\
\downarrow{\scriptstyle \iota_m} & & \downarrow{\scriptstyle \iota_m} \\
\bigoplus_{n\in\mathbb{N}} P & \xrightarrow{\oplus \gamma_n} & \bigoplus X(\mu, Q_n) \\
\downarrow{\scriptstyle \sigma} & & \downarrow \\
P & \xrightarrow{\alpha} & X
\end{array}$$

As in the previous case, we may suppose that the right vertical map is an inclusion map $X(\mu, Q_m) \subseteq X$, and, in this way, $\alpha(P) = \gamma_m(P) \subseteq \subseteq \mathfrak{T}^\mu X(\mu, Q_m) \subseteq \mathfrak{T}^\mu X$. Now suppose there is given some homomorphism $\varphi\colon \mathfrak{T}^\mu X \to P_m$, for some $m \in \mathbb{N}$. Then its restriction to $\mathfrak{T}^\mu X(\mu, Q_m) \approx Q_m$ has to be zero, since $\operatorname{Hom}(Q_m, P_m) = 0$. Thus its restriction to $\alpha(P)$ is zero, since $\alpha(P) \subseteq \mathfrak{T}^\mu X(\mu, Q_m)$. This shows that $\alpha(P)$ is contained in the kernel of any homomorphism φ from $\mathfrak{T}^\mu X$ into an indecomposable preprojective module, thus $\alpha(P)$ is contained in $\mathfrak{T}\mathfrak{T}^\mu X = \mathfrak{T}^{\mu+1} X = = \mathfrak{T}^\lambda X$. Together with the previously established reverse inclusion, we conclude that $\alpha(P) = \mathfrak{T}^\lambda X$. This finishes the proof.

2.7. A module X will be called *preprojective* in case it satisfies the following equivalent conditions:

(i) $\mathfrak{T}^\infty X = 0$,

(ii) There exists an ordinal λ with $\mathfrak{T}^\lambda X = 0$,

(iii) Every non-zero submodule U of X has a direct summand which is indecomposable preprojective.

Note that for modules of finite length, and also for indecomposable modules, this notion reduces to the previous use of the notion « preprojective ».

Proof of the equivalence: Clearly, (i) and (ii) are equivalent. Also, if $\mathfrak{T}^\infty X = 0$, then $\mathfrak{T}^\infty U = 0$ for any submodule U of X. If U has no indecomposable preprojective direct summand, then $\mathfrak{T} U = U$ by 2.2, and therefore also $\mathfrak{T}^\infty U = U$, thus $U = 0$. This proves the implication (i) $\Rightarrow$ (iii). On the other hand, assume (iii) for X. Since any transfinite chain of submodules of X has to stop eventually, there is an ordinal λ with $\mathfrak{T}^\lambda X = \mathfrak{T}^\infty X$. If this submodule is non-zero, then by assumption it will split off an indecomposable preprojective direct summand, and therefore $\mathfrak{T}^{\lambda+1} X \subset \mathfrak{T}^\lambda X$, impossible. Thus $\mathfrak{T}^\lambda X = 0$, which shows (ii).

Proposition: The class of preprojective modules is closed under submodules, products, and extensions.

Proof: If $U \subseteq X$, then also $\mathfrak{T}^\infty U \subseteq \mathfrak{T}^\infty X$, thus if X is preprojective, also U is preprojective. If $0 \to X \to Y \to Z \to 0$ is an exact sequence, and X and Z are preprojective, then $\mathfrak{T}^\infty Z = 0$ implies $\mathfrak{T}^\infty Y \subseteq X$, and therefore $\mathfrak{T}^\infty Y = \mathfrak{T}^\infty \mathfrak{T}^\infty Y \subseteq \mathfrak{T}^\infty X = 0$. Finally, let X_i be a family of modules. We claim that $\mathfrak{T}^\lambda(\prod X_i) \subseteq \prod \mathfrak{T}^\lambda X_i$ for all ordinals λ. Again, we have to check it only for $\lambda = 1$, using transfinite induction, since intersections commute with products. Now $\prod \mathfrak{T} X_i$ is the intersection of the maps of the form

$$\prod X_i \xrightarrow{\pi_j} X_j \xrightarrow{\varphi} P$$

with P indecomposable preprojective, where π_j denotes a canonical projection. As a consequence, $\mathfrak{I}(\prod X_i)$ being the intersection of the kernels of all maps $\prod X_i \to P$ with P indecomposable preprojective, is contained in $\prod \mathfrak{I} X_i$. Thus, if all the X_i are preprojective, then

$$\mathfrak{I}^\infty(\prod X_i) \subseteq \prod \mathfrak{I}^\infty X_i = 0\,,$$

thus $\prod X_i$ is preprojective.

3. Preinjective modules.

3.1. Let $\mathfrak{J}$ be the set of (isomorphism classes of) indecomposable preinjective modules. Note that $\mathfrak{J}$ is partially ordered by: $I \geqslant I'$ iff there exists a chain

$$I = I_1 \to I_2 \to \ldots \to I_{n-1} \to I_n = I'$$

of non-zero maps, where all $I_i \in \mathfrak{J}$. Let $\mathfrak{B}$ be a subset of $\mathfrak{J}$ closed under successors (thus, if $I \in \mathfrak{B}$ and $I \geqslant I'$, then $I' \in \mathfrak{B}$). Again, it is clear that $\mathfrak{B}$ is either finite or $\mathfrak{B} = \mathfrak{J}$. For any module X, let $\mathfrak{I}_{\mathfrak{B}} X$ be the sum of the images of all maps $I \to X$, with $I \in \mathfrak{B}$. In the case of $\mathfrak{B} = \mathfrak{J}$, we denote the functor $\mathfrak{I}_{\mathfrak{J}}$ just by $\mathfrak{I}$, thus $\mathfrak{I}X$ is the sum of the images of all maps $I \to X$ with I indecomposable preinjective.

Proposition: Let $\mathfrak{B}$ be a finite set of indecomposable preinjective modules closed under successors. Let X be a module. Then $\mathfrak{I}_{\mathfrak{B}} X$ is a direct summand of X. For any $I \in \mathfrak{B}$, there exists a (not necessarily uniquely defined) submodule V_I of X which is a direct sum of copies of I, such that $\mathfrak{I}_{\mathfrak{B}} X = \bigoplus_{I \in \mathfrak{B}} V_I$. If $\mathfrak{B}'$ is a subset of $\mathfrak{B}$ closed under successors, then $\mathfrak{I}_{\mathfrak{B}'} X = \bigoplus_{I \in \mathfrak{B}'} V_I$.

The proof will be rather similar to the corresponding result for the functors $\mathfrak{I}_{\mathfrak{A}}$. Again, we will use a factorisation of the functor $\mathfrak{I}_{\mathfrak{B}}$.

3.2. Lemma: Let $\mathfrak{A}$ be a finite set of indecomposable preinjective R-modules, closed under successors. Then, there exists a finite dimensional hereditary algebra B and adjoint functors $S^-: {}_R\mathfrak{M} \to {}_B\mathfrak{M}$ and

$S^+: {}_B\mathfrak{M} \to {}_R\mathfrak{M}$ with natural transformation $\varphi_X: X \to S^+S^-X$ such that

(1) φ_X is an epimorphism, for all R-modules X.

(2) φ_X is an isomorphism, whenever X has no direct summand isomorphic to any $I \in \mathfrak{A}$.

(3) $S^-I = 0$ for all $I \in \mathfrak{A}$.

(4) If $I \in \mathfrak{F} \setminus \mathfrak{A}$, and all successors of I belong to $\mathfrak{A}$, then S^-I is simple injective.

Again, for a proof, we refer to [6] and [34].

Proof of Proposition 3.1: By induction on the number of elements in $\mathfrak{B}$ If $\mathfrak{B}$ is empty, nothing has to be shown. Otherwise, choose some Y in $\mathfrak{B}$ which is not the successor of any other element in $\mathfrak{B}$, and let $\mathfrak{A} = \mathfrak{B} \setminus \{Y\}$. By induction, we know that $\mathfrak{J}_{\mathfrak{A}}X$ is a direct summand of X, and of the form $\mathfrak{J}_{\mathfrak{A}}X = \bigoplus_{I \in \mathfrak{A}} V_I$, where each V_I is a direct sum of copies of I. We apply the lemma to $\mathfrak{A}$, and obtain an algebra B, functors S^- and S^+, and a natural transformation φ.

Let $u: \mathfrak{J}_{\mathfrak{A}}X \to X$ be the inclusion map. There is a direct sum $\bigoplus_J Y$ of copies of Y, and a homomorphism $\alpha: \bigoplus_J Y \to X$ such that the homomorphism $(\alpha, u): \bigoplus_J Y \oplus \mathfrak{J}_{\mathfrak{A}}X \to X$ maps onto $\mathfrak{J}_{\mathfrak{B}}X$, thus we obtain an exact sequence

$$\bigoplus_J Y \oplus \mathfrak{J}_{\mathfrak{A}}X \xrightarrow{(\alpha, u)} X \to X/\mathfrak{J}_{\mathfrak{B}}X \to 0\,.$$

Apply the right exact functor S^-, we get

$$\bigoplus_J S^-Y \xrightarrow{S^-\alpha} S^-X \to S^-(X/\mathfrak{J}_{\mathfrak{B}}X) \to 0\,,$$

where we have used that direct sums of copies of I, with I in $\mathfrak{A}$, vanish under S^-. However, S^-Y is simple, thus there is a subset J' of J such that the restriction of $S^-\alpha$ to $\bigoplus_{J'} S^-Y$ is a monomorphism. Also, S^-Y is injective, and since B is a finite dimensional algebra, any direct sum of injective modules is injective, thus we obtain a split exact sequence

$$0 \to \bigoplus_{J'} S^-Y \to S^-X \to S^-(X/\mathfrak{J}_{\mathfrak{B}}X) \to 0\,,$$

which gives under S^+ the split exact sequence

$$0 \to \bigoplus_{J'} S^+S^-Y \to S^+S^-X \to S^+S^-(X/\mathfrak{I}_{\mathfrak{B}}X) \to 0\,.$$

However, by induction, the natural epimorphism $\varphi_X\colon X \to S^+S^-X$ splits and has as kernel just $\mathfrak{I}_{\mathfrak{A}}X$ and thus induces an isomorphism $X/\mathfrak{I}_{\mathfrak{A}}X \to S^+S^-X$. Since we will show that $X/\mathfrak{I}_{\mathfrak{B}}X$ has no direct summand isomorphic to any $I \in \mathfrak{B}$, we see that we have the following commutative square with vertical maps isomorphisms

$$\begin{array}{ccc} S^+S^-X & \to & S^+S^-(X/\mathfrak{I}_{\mathfrak{B}}X) \\ \uparrow & & \uparrow \varphi_{X/\mathfrak{I}_{\mathfrak{B}}X} \\ X/\mathfrak{I}_{\mathfrak{A}}X & \to & X/\mathfrak{I}_{\mathfrak{B}}X \end{array} .$$

Thus, the lower map also is a split epimorphism with kernel isomorphic to $\bigoplus_{J'} S^+S^-Y \approx \bigoplus_{J'} Y$. This shows that $\mathfrak{I}_{\mathfrak{B}}X$ is a direct summand of X, and that the inclusion $\mathfrak{I}_{\mathfrak{A}}X \subseteq \mathfrak{I}_{\mathfrak{B}}X$ splits with cokernel $\bigoplus_{J'} Y$. Thus, there is a direct complement V_Y of $\mathfrak{I}_{\mathfrak{A}}X$ in $\mathfrak{I}_{\mathfrak{B}}X$, which is a direct sum of copies of Y.

It remains to be seen that $X/\mathfrak{I}_{\mathfrak{B}}X$ has no direct summand isomorphic to any I in $\mathfrak{B}$. In fact, we show that $\mathfrak{I}_{\mathfrak{B}}(X/\mathfrak{I}_{\mathfrak{B}}X) = 0$. Thus, let $\alpha\colon I \to X/\mathfrak{I}_{\mathfrak{B}}X$ be a homomorphism. We get an induced exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{I}_{\mathfrak{B}}X & \to & X & \to & X/\mathfrak{I}_{\mathfrak{B}}X & \to & 0 \\ & & \uparrow \mathrm{id} & & \uparrow \alpha' & & \uparrow \alpha & & \\ 0 & \to & \mathfrak{I}_{\mathfrak{B}}X & \to & X' & \to & I & \to & 0\,. \end{array}$$

Now $\mathfrak{I}_{\mathfrak{B}}X$ is an epimorphic image of a direct sum $\bigoplus_J I_j$ of indecomposable preinjective modules I_j belonging to $\mathfrak{B}$, and there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{I}_{\mathfrak{B}}X & \to & X' & \to & I & \to & 0 \\ & & \uparrow \beta & & \uparrow \beta' & & \uparrow \mathrm{id} & & \\ 0 & \to & \bigoplus_J I_j & \to & X'' & \to & I & \to & 0\,. \end{array}$$

Since I is of finite length, X'' decomposes into the direct sum of some of the I_j and a module X''' which is an extension of the form

$$0 \to \bigoplus_{j \in J'} I_j \to X''' \to I \to 0 ,$$

with J' finite. But then X''' itself is a direct sum of indecomposable preinjective modules belonging to $\mathfrak{B}$, since all I_j and I belong to $\mathfrak{B}$, and $\mathfrak{B}$ is closed under successors. Thus, X'' is a direct sum of elements of $\mathfrak{B}$, and therefore $\alpha'\beta'(X'') \subseteq \mathfrak{I}_{\mathfrak{B}}X$. But then $\alpha = 0$.

This finishes the proof.

Corollary: *The kernel of the natural map* $\varphi_X: X \to S^+ S^- X$ *is just* $\mathfrak{I}_{\mathfrak{A}}X$.

3.3. Theorem: *Let X be an arbitrary module. Then $\mathfrak{I}X$ is a pure submodule of X, with $\mathfrak{I}(X/\mathfrak{I}X) = 0$, and $\mathfrak{I}X$ is a direct sum of indecomposable preinjective modules.*

Proof: We use $\mathbb{N}$ as index set for $\mathfrak{S}$, say $\mathfrak{I} = \{\mathfrak{S}_n | n \in \mathbb{N}\}$, in such a way that $I_i \leqslant I_j$ implies $i \leqslant j$. Let $\mathfrak{I}_n = \mathfrak{I}_{\{I_1, \dots, I_n\}}$. Then

$$\mathfrak{I}_n X = \mathfrak{I}_{n-1} X \oplus V_n$$

with V_n a direct sum of copies of I_n Thus

$$\mathfrak{I}X = \bigcup_{n \in \mathbb{N}} \mathfrak{I}_n X = \bigoplus_{n \in \mathbb{N}} V_n ,$$

that is, $\mathfrak{I}X$ is a direct sum of copies of the various I_n. Since $\mathfrak{I}X = \bigcup_{n \in \mathbb{N}} \mathfrak{I}_n X$ is the union of a chain of direct summands of X, we see that $\mathfrak{I}X$ has to be pure in X.

Finally, assume $\mathfrak{I}(X/\mathfrak{I}X) \neq 0$. Then $X/\mathfrak{I}X$ splits off an indecomposable preinjective summand, thus there is a submodule Y of X containing $\mathfrak{I}X$ such that $Y/\mathfrak{I}X$ is indecomposable preinjective. Since $\mathfrak{I}X$ is pure in X, and $Y/\mathfrak{I}X$ is of finite length, $\mathfrak{I}X$ is a direct summand of Y, say $Y = \mathfrak{I}X \oplus Y'$ for some submodule Y' of Y. Now Y' is an indecomposable preinjective submodule of X, and therefore $Y' \subseteq \mathfrak{I}X$, a contradiction.

3.4. Modules X with $\mathfrak{I}X = X$ will be called *preinjective*, thus these are just the direct sums of indecomposable preinjective modules.

Proposition: The class of preinjective modules is closed under quotients, direct sums and extensions.

PROOF: If $X \to Y$ is surjective, and $\mathfrak{J}X = X$, then also Y is generated by the images of maps $I \to Y$ with I indecomposable preinjective, thus $\mathfrak{J}Y = Y$. Let $0 \to X \to Y \to Z \to 0$ be exact, and X and Z preinjective. Since X is preinjective, $X = \mathfrak{J}X \subseteq \mathfrak{J}Y$. Thus there is a surjective map $Z = Y/X \to Y/\mathfrak{J}Y$. Since Z is preinjective, also $Y/\mathfrak{J}Y$ is preinjective, by the first part of the proof. Thus $Y/\mathfrak{J}Y = \mathfrak{J}(Y/\mathfrak{J}Y) = 0$, by 3.3. This shows that $Y = \mathfrak{J}Y$. It is obvious that direct sums of preinjective modules are preinjective, again.

REMARK: The direct product of preinjective modules does not have to be preinjective.

Consider for example $\prod_{I \in \mathfrak{J}} I$. We claim that $\mathfrak{J}\left(\prod_{I \in \mathfrak{J}} I\right) = \bigoplus_{I \in \mathfrak{J}} I$. For, if I' is an indecomposable preinjective module, and $\mathfrak{B}$ is the set of successors of I', then any homomorphism $\varphi\colon I' \to \prod_{I \in \mathfrak{J}} I$ maps into $\prod_{I \in \mathfrak{B}} I = \bigoplus_{I \in \mathfrak{B}} I \subseteq \bigoplus_{I \in \mathfrak{J}} I$, thus $\mathfrak{J}\left(\prod_{I \in \mathfrak{J}} I\right) \subseteq \bigoplus_{I \in \mathfrak{J}} I$. The reverse inclusion is trivial.

3.5. We want to derive an easy corollary from the fact that $\mathfrak{J}X$ is always pure in X.

COROLLARY: *Let X be of finite length without non-zero preinjective direct summands. Let Y be preinjective. Then* $\operatorname{Ext}(X, Y) = 0$.

PROOF: Let $0 \to Y \to Z \to X \to 0$ be an exact sequence. Then $Y = \mathfrak{J}Y \subseteq \mathfrak{J}Z$, and conversely also $\mathfrak{J}Z \subseteq Y$, since $\mathfrak{J}(Z/Y) = \mathfrak{J}X = 0$. Thus, $\mathfrak{J}Z = Y$. However, since $\mathfrak{J}Z$ is pure in Z, and $Z/\mathfrak{J}Z = X$ is of finite length, $\mathfrak{J}Z$ is a direct summand of Z. Thus the given sequence splits.

3.6. We are interested to know under what conditions $\mathfrak{J}X$ is not only pure in X, but even a direct summand of X. The following theorem gives a necessary and sufficient condition on the algebra A in order to have the property that for any A-module X, the submodule $\mathfrak{J}X$ is a direct summand. This condition will be used to single out all those algebras A which have this splitting property.

THEOREM: *The following conditions are equivalent for a finite dimensional hereditary algebra A:*

(i) *For any A-module X, the submodule $\mathfrak{J}X$ is a direct summand of X.*

(ii) *If X is an A-module with a submodule U of finite length, and if $\mathfrak{J}(X) = 0$, $\mathfrak{J}(X/U) = X/U$, then X is of finite length.*

Proof: (i) $\Rightarrow$ (ii). Assume there exists a module X not of finite length, and a submodule U of finite length, such that $\mathfrak{I}X = 0$, $\mathfrak{I}(X/U) = X/U$. We want to construct a module Y such that $\mathfrak{I}Y$ is not a direct summand of Y. Let $X/U = \bigoplus_{j \in J} I_j$ with I_j indecomposable preinjective. For $j \in J$, let

$$0 \to V_j \to W_j \to I_j \to 0$$

be a non-split exact sequence with V_j preinjective of finite length. Such sequences do exist, for example let $V_j = AI_j$ and take the Auslander Reiten sequence. Let

$$V = \bigoplus_{j \in J} V_j, \qquad W = \bigoplus_{j \in J} W_j, \qquad \alpha = \bigoplus \alpha_j, \qquad \beta = \bigoplus \beta_j,$$

and consider the induced exact sequence

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & \xrightarrow{\text{id}} & U & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \iota} & & \\
0 & \to & V & \xrightarrow{\alpha'} & Y & \xrightarrow{\beta'} & X & \to & 0 \\
 & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\
0 & \to & V & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & X/U & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where ι is the inclusion, and π the projection.

Since V is a direct sum of indecomposable preinjective module, and $\alpha'(V) \subseteq V$, we have $\alpha'(V) \subseteq \mathfrak{I}Y$. On the other hand, $\mathfrak{I}(Y/\alpha'(V)) = \mathfrak{I}(X) = 0$, thus also $\mathfrak{I}Y \subseteq \alpha'(V)$. This shows that $\mathfrak{I}Y = \alpha'(V)$. Now assume $\mathfrak{I}Y$ is a direct summand of Y, then there is a splitting map $\sigma: X \to Y$ with $\beta'\sigma$ the identity of X. Consider the diagram

$$(*) \qquad \begin{array}{ccccccccc}
0 & \to & U & \to & X & \xrightarrow{\pi} & X/U & \to & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \pi'\sigma} & & \downarrow{\scriptstyle \text{id}} & & \\
0 & \to & V & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & X/U & \to & 0 .
\end{array}$$

The right square $\beta\pi'\sigma = \pi\beta'\sigma = \pi$ commutes, thus there exists $\varphi: U \to V$

making the left square commutative. Note that this diagram shows that the exact sequence

$$0 \to U \to X \xrightarrow{\pi} X/U \to 0$$

is induced from the lower exact sequence in $(*)$ by φ. However, U is a module of finite length, thus $\varphi\colon U \to V = \bigoplus_{j\in J} V_j$ maps into a finite direct sum, say into $\bigoplus_{j\in J'} V_j$ with J' a finite subset of J. Let $J'' = J \setminus J'$. The lower exact sequence of $(*)$ splits accordingly into the two exact sequences

$$0 \to \bigoplus_{j\in J'} V_j \to \bigoplus_{j\in J'} W_j \to \bigoplus_{j\in J'} Q_j \to 0$$

and

$$0 \to \bigoplus_{j\in J''} V_j \to \bigoplus_{j\in J''} W_j \to \bigoplus_{j\in J''} Q_j \to 0\,.$$

We conclude that therefore X has a direct summand isomorphic to $\bigoplus_{j\in J''} Q_j$. In particular, $\mathfrak{J}X \neq 0$, contrary to our assumption.

(ii) $\Rightarrow$ (i). Let X be a module. Assuming (ii), we want to construct a complement of $\mathfrak{J}X$ in X. Consider the set $\mathfrak{U}$ of submodules U of X with the following two properties

(*a*) $\mathfrak{J}X \cap U = 0$, and

(*b*) $\mathfrak{J}(X/(\mathfrak{J}X + U)) = 0$.

The set is non-empty, since $0 \in \mathfrak{U}$. The set $\mathfrak{U}$ also is inductive: let $(U_j)_j$ be a chain of submodules belonging to $\mathfrak{U}$. The union $\bigcup U_j$ clearly satisfies (*a*). In order to prove (*b*), let I be indecomposable preinjective. Then

$$\begin{aligned}\operatorname{Hom}(I, X/(\mathfrak{J}X + \textstyle\bigcup U_j)) &= \operatorname{Hom}(I, \varinjlim X/(\mathfrak{J}X + U_j)) \\ &= \varinjlim \operatorname{Hom}(I, X/(\mathfrak{J}X + U_j)) = 0\,.\end{aligned}$$

Consequently, we may choose U maximal in $\mathfrak{U}$. We claim that U is a complement of $\mathfrak{J}X$ in X. Assume $\mathfrak{J}X + U \neq X$. Let $\mathfrak{J}X + U \subset Y' \subseteq X$ such that $Y'/(\mathfrak{J}X + U)$ is of finite length. Let $\mathfrak{J}(X/Y') = Y/Y'$ with $Y' \subseteq Y \subseteq X$. We can apply (ii) to the module $Y/(\mathfrak{J}X + U)$ with submodule $Y'/(\mathfrak{J}X + U)$, since

$$\mathfrak{J}(Y/(\mathfrak{J}X + U)) \subseteq \mathfrak{J}(X/(\mathfrak{J}X + U)) = 0 \quad \text{and} \quad \mathfrak{J}(Y/Y') = Y/Y'\,.$$

Thus, we see that $Y/(\mathfrak{J}X + U)$ is of finite length. If we apply $\operatorname{Hom}(-, \mathfrak{J}X)$ to the exact sequence

$$0 \to \mathfrak{J}X + U \xrightarrow{\iota} Y \to Y/(\mathfrak{J}X + U) \to 0 ,$$

where ι denotes the inclusion map, we get an exact sequence

$$\operatorname{Hom}(Y, \mathfrak{J}X) \xrightarrow{\operatorname{Hom}(\iota, \mathfrak{J}X)} \operatorname{Hom}(\mathfrak{J}X + U, \mathfrak{J}X) \to \operatorname{Ext}\big(Y/(\mathfrak{J}X + U), \mathfrak{J}X\big) .$$

By 3.4, the last term is zero, thus $\operatorname{Hom}(\ ,\mathfrak{J}X)$ is surjective. In particular, if we take the projection map $\pi\colon \mathfrak{J}X \oplus U \to \mathfrak{J}X$ with kernel U, we see that ι extends to a map $\pi'\colon Y \to \mathfrak{J}X$. Let U' be its kernel. Then $\pi = \pi'|\mathfrak{J}X \oplus U$ shows that the restriction of π to U is zero, thus $U \subseteq U'$, and that the restriction of π' to $\mathfrak{J}X$ is the identity, thus $Y = \mathfrak{J}X \oplus U'$. Moreover, since $X/(\mathfrak{J}X \oplus U') = X/Y$ satisfies

$$\mathfrak{J}(X/(\mathfrak{J}X \oplus U')) = 0 ,$$

we see that U' belongs to $\mathfrak{U}$. This contradicts the maximality of U, and therefore concludes the proof.

3.7. Theorem: *Let A be a finite dimensional hereditary algebra of tame representation type, and X an A-module. Then $\mathfrak{J}X$ is a direct summand of X.*

This is an immediate consequence of the previous criterion and the following lemma.

Lemma: Let A be a finite dimensional hereditary algebra of tame type. Let X be an A-module, and U a submodule of X of finite length which is the direct sum of t indecomposable modules. Assume $\mathfrak{J}X = 0$, and $\mathfrak{J}(X/U) = X/U$. Then X/U is the direct sum of at most $6t$ indecomposable preinjective modules. In particular, X/U, and therefore also X, are of finite length.

Remark: It will be clear from the proof that the factor 6 in the formulation of the lemma can be improved in most cases. Actually, the case that X/U is the direct sum of $6t$ indecomposable modules can only occur in case of type $\tilde{\mathbb{E}}_8$. On the other hand, if A is of type $\tilde{\mathbb{A}}_{12}$, $\tilde{\mathbb{A}}_n$ or $\tilde{\mathbb{C}}_n$, then the assumptions of the lemma imply that X/U is the direct sum of at most t indecomposable modules.

Proof of the lemma: Decompose $X/U = \bigoplus_{j \in J} Q_j$ with Q_j indecomposable preinjective. Let J' be a subset of J with precisely $6t+1$, elements. Consider the exact sequence induced by the inclusion $\bigoplus_{j\in J'} Q_j \subseteq \subseteq \bigoplus_{j\in J} Q_j$,

$$\begin{array}{ccccccccc} 0 & \to & U & \to & X' & \to & \bigoplus_{j\in J'} Q_j & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ 0 & \to & U & \to & X & \to & \bigoplus_{j\in J} Q_j & \to & 0\,. \end{array}$$

Then X' is of finite length, thus we can calculate its defect $\delta X'$. Since U is the direct sum of t indecomposable modules, $\delta U \geqslant -6t$. Since Q_j is indecomposable preinjective, $\delta Q_j \geqslant 1$ for all $j \in J'$, thus

$$\delta X' = \delta U + \sum_{j\in J'} \delta Q_j \geqslant -6t + 6t + 1 = 1\,.$$

As a consequence, X' splits off an indecomposable preinjective direct summand. Since X' can be considered as a submodule of X, we see that $0 \neq \mathfrak{J}X' \subseteq \mathfrak{J}X$, contrary to our assumption. This shows that J can have at most $6t$ elements.

3.8. Now assume that the algebra R is of wild representation type. In this case, we have to work with growth numbers. Recall that we denote by A the Auslander construction. Also recall that for any module X of finite length, there is defined a vector $\mathit{dim}\, X$ in $\mathbb{R}^s$, where s is the number of simple R-modules, as follows: we choose a fixed ordering $P_1, \ldots, P_s$ of the indecomposable projective modules, and the i-th component of $\mathit{dim}\, X$ is given by the dimension

$$\dim \operatorname{Hom}(P_i, X)_{\operatorname{End}(P_i)}\,.$$

Clearly, the length $|X|$ of X is just the sum of the components of $\mathit{dim}\, X$.

Proposition: Let R be a twosided indecomposable finite dimensional hereditary algebra of wild representation type. Then there exists a real number $g > 1$ such that for any indecomposable injective module I, the limit

$$\lim_{n\to\infty} \frac{1}{g^n} \cdot \mathit{dim}\, A^n I$$

in $\mathbb{R}^s$ exists, and has no zero component.

Note that this implies that also the limit $\lim_{n\to\infty} 1/g^n |A^n I|$ exists and is non-zero. Thus, both $\mathit{dim}\, A^n I$ and $|A^n I|$ grow exponentially with n.

The number g measures the growth rate, and will be called the *growth number* of R.

For the proof, we refer to a forthcoming paper [33]. The case $s = 2$ has been considered in [31]; as an illustration of the general result, it seems to be valuable to consider this case $s = 2$ in more detail: in fact, we can give an explicit formula both for g and the limit $\lim_{n\to\infty} 1/g^n \, \dim A^n I$.

Thus, let R be of the form

$$R = \begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$$

where F, G are division rings, and ${}_F M_G$ is a bimodule, everything finite dimensional over the commutative field k which operates centrally. Let $a = \dim M_G$, $b = \dim {}_F M$. The algebra R is of wild representation type if and only if $ab > 4$. An R-module is given by two vector spaces U_F, V_G, and a linear map $\varphi\colon U_F \otimes {}_F M_G \to V_G$, its dimension type is $\dim(U, V, \varphi) = (\dim U_F, \dim V_G) \in \mathbb{R}^2$. If X is an R-module of finite length and without non-zero projective direct summands, then $\dim AX = c \dim X$, where c is the linear transformation of $\mathbb{R}^2$ given by the matrix $\begin{pmatrix} ab-1 & -b \\ a & -1 \end{pmatrix}$. Thus, in order to study the behaviour of A, we have to look at c. In fact, assume the limit $x' = \lim_{n\to\infty} 1/g^n \dim A^n X$ exists, and is non-zero, for some indecomposable module X of finite length. Then x' is an eigenvector of c with eigenvalue g. For, X cannot be preprojective, since $A^n X \neq 0$ for all n, thus $\dim A^n X = c^n \dim X$, and therefore for $x = \dim X$,

$$cx' = \lim_{n\to\infty} \frac{1}{g^n} c^{n+1} x = g \cdot \lim_{n\to\infty} \frac{1}{g^{n+1}} c^{n+1} x = gx'.$$

Thus, we have to determine the eigenvalues of c, and to decide whether one of those is the growth number of R.

Now, the eigenvalues of c are

$$g = \frac{1}{2}\left(ab - 2 + \sqrt{(ab)^2 - 4ab}\right) \quad \text{and} \quad \frac{1}{g} = \frac{1}{2}\left(ab - 2 - \sqrt{(ab)^2 - 4ab}\right),$$

they are real if $ab \geqslant 4$, and different, if $ab \neq 4$. This shows that we can have a growth number > 1 only in case $ab > 4$. An eigenvector for g is given by

$$x_1 = \begin{pmatrix} ab + \sqrt{(ab)^2 - 4ab} \\ 2a \end{pmatrix}, \quad \text{for } \frac{1}{g} \quad \text{by} \quad x_2 = \begin{pmatrix} ab - \sqrt{(ba)^2 - 4ab} \\ 2a \end{pmatrix}.$$

Now assume that $ab > 4$. In this case it is clear that for a vector $x \in \mathbb{R}^2$, $\lim_{n\to\infty} (1/g^n) c^n x$ always exists, and that it has positive components if and only if x belongs to $\mathbb{R}_+ x_1 + \mathbb{R} x_2$, where $\mathbb{R}_+ = \{r \in \mathbb{R} | r > 0\}$. Now the dimension types of the indecomposable injective modules are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}b\\1\end{pmatrix}$, and it is easy to check that both belong to $\mathbb{R}_+ x_1 + \mathbb{R} x_2$. This shows that g is the growth number of R, and that $\lim_{n\to\infty} 1/g^n \dim A^n I$, for I indecomposable injective, is a multiple of x_1.

3.9. Theorem: *Let R be a finite dimensional hereditary algebra of wild representation type. Then there exists an R-module X such that $\mathfrak{J}X$ is not a direct summand of X.*

Proof: We may assume that R is twosided indecomposable. We want to construct a module X as the union of a proper chain of submodules X_m of finite length,

$$X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_m \subset \ldots \bigcup X_m = X\,,$$

such that $\mathfrak{J}X_m = 0$ and $\mathfrak{J}(X_m/X_0) = X_m/X_0$ for all m, thus also $\mathfrak{J}X = 0$, $\mathfrak{J}(X/X_0) = X/X_0$. As a consequence, the assertion will follow from 3.6. In the proof, we will need two results on exponentially growing sequences which will be derived in 3.10.

We use $\mathbb{N}$ as index set for $\mathfrak{J}$ as follows: let $I_1, \ldots, I_s$ be the indecomposable injective modules such that $\operatorname{Hom}(I_i, I_j) \neq 0$ implies $i \geq j$. For $n \in \mathbb{N}$, $1 \leq i \leq s$, let $I_{ns+i} = A^n I_i$. Then, for all i, j, we have that $\operatorname{Hom}(I_i, I_j) \neq 0$ implies $i \geq j$. Consider the sequence $(a_n)_n$ with $a_n = |I_n|$. If g is the growth number of R, then

$$\lim_{n\to\infty} \frac{1}{g^n} a_{ns+i} > 0\,,$$

for all $1 \leq i \leq s$, thus, by 3.10.(2), there is i with $1 \leq i \leq s$, such that for all c, $N \in \mathbb{N}$, there exists $n \geq N$ with $a_j > a_{ns+i} + c$ for all $j > ns + i$. Let $I = I_i$. Thus we have:

(*) For any c, $N \in \mathbb{N}$, there exists $n \geq N$ such that for any indecomposable preinjective module I', with $I' \not\approx A^n I$, and $\operatorname{Hom}(I', A^n I) \neq 0$, we have $|I'| > |A^n I| + c$.

For, let $I' = I_j$ for some $j \in \mathbb{N}$. Since $\operatorname{Hom}(I_j, A^n I) \neq 0$, we have $j \geq ns + i$, and since $I_j \not\approx A^n I$, we have even $j > ns + i$. Consequently, $|I'| = a_j > a_{ns+i} + b = |A^n I| + b$.

Let P be the indecomposable projective module with $P/\operatorname{rad} P \approx$ $\approx \operatorname{soc} I$. Let g be the growth number of R. Since $g > 1$, there is $u \in \mathrm{N}$ with $g^u \geq 2$.

Let $X_0 \approx P$. We are going to construct X_m by induction on m, with the following properties:

(1) $X_{m-1}/X_0 \subset X_m/X_0$ splits,

(2) $X_m/X_{m-1} \approx A^{n_m} I$, with $n_m \geq um$,

(3) $\Im X_m = 0$.

Assume, we have constructed X_m. Denote by $\alpha: X_0 \to X_m$ the inclusion map, by $\beta: X_m \to Q$ the cokernel of α. By induction, we have

$$Q = \bigoplus_{j=1}^{m} A^{n_j} I, \qquad \text{with } n_j \geq uj.$$

Consider the sequence $(b_n)_n$ with $b_n = \dim \operatorname{Hom}(P, A^n I)$. By 3.8, we know that $\lim_{n\to\infty} b_n/g^n > 0$. Thus, we may apply 3.10.(3), to the sequence $(b_n)_n$ and the natural numbers $n_1, \ldots, n_m$ in order to obtain a natural number $N > u(m+1)$ such that

$$b_{n+1} > \sum_{j=1}^{m} b_{n-n_j} \qquad \text{for all } n \geq N.$$

Ler $c = |X_m|$. Applying $(*)$ to c, N, we find $n = n_{m+1} \geq N$ with the following property:

(a) If I' is indecomposable preinjective, $I' \not\approx A^{n_{m+1}} I$, and

$$\operatorname{Hom}(I', A^{n_{m+1}} I) \neq 0, \qquad \text{then} \qquad |I'| > |A^{n_{m+1}} I| + |X_m|.$$

Note that

$$b_{n_{m+j}+1} = \dim_k \operatorname{Hom}(P, A^{n_{m+1}+1} I) = \dim_k \operatorname{Ext}(A^{n_{m+1}} I, P),$$

using 1.A. And,

$$b_{n_{m+1}-n_j} = \dim_k \operatorname{Hom}(P, A^{n_{m+1}-n_j} I) = \dim_k \operatorname{Hom}(A^{n_{m+1}-n_j} I, I) =$$
$$= \dim_k \operatorname{Hom}(A^{n_{m+1}} I, A^{n_j} I),$$

where we use the fact that $\dim_k \operatorname{Hom}(P, Y) = \dim_k \operatorname{Hom}(Y, I)$ for any module Y, and a property of the Auslander construction, see 1.A. Consequently, we conclude that

$$(b)\quad \dim_k \operatorname{Ext}(A^{n_{m+1}}I, P) > \dim_k \operatorname{Hom}(A^{n_{m+1}}I, Q).$$

We apply the functor $\operatorname{Hom}(A^{n_{m+1}}I, -)$ to the given exact sequence

$$0 \to X_0 \xrightarrow{\alpha} X_m \xrightarrow{\beta} Q \to 0,$$

and we get the connecting homomorphism

$$\delta\colon \operatorname{Hom}(A^{n_{m+1}}I, Q) \to \operatorname{Ext}(A^{n_{m+1}}I, X_0).$$

According to (b), the map δ is not surjective, thus there exists an exact sequence

$$E\colon\ 0 \to X_0 \xrightarrow{\gamma} Y \xrightarrow{\pi} A^{n_{m+1}}I \to 0$$

which is not in the image of δ. We form the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & X_0 & \xrightarrow{\alpha} & X_m & \xrightarrow{\beta} & Q & \to & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \to & Y & \xrightarrow{\alpha'} & X_{m+1} & \to & Q & \to & 0 \\
 & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} & & & & \\
 & & A^{n_{m+1}}I & \xrightarrow{\mathrm{id}} & A^{n_{m+1}}I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and we may suppose that $\gamma'\colon X_m \to X_{m+1}$ is an inclusion. In this way, we have constructed X_{m+1}, and the properties (1) and (2) for X_{m+1} are clear from the construction.

It remains to be seen that $\mathfrak{J}X_{m+1} = 0$. By 3.1, we have to exclude the case that there is an embedding $\varphi\colon I' \to X_{m+1}$ for some indecomposable preinjective module I'. Since φ does not map into X_m according to $\mathfrak{J}X_m = 0$, we see that $\pi'\varphi \neq 0$. Thus $\operatorname{Hom}(I', A^{n_{m+1}}I) \neq 0$. On the other hand, since I' embeds into X_{m+1}, we have

$$|I'| \leqslant |X_{m+1}| = |A^{n_{m+1}}I| + |X_m|.$$

It follows from property (a) that $I' \approx A^{n_{m+1}}$. Since any non-zero endomorphism of an indecomposable preinjective module is an automorphism, $\pi'\varphi$ has to be an isomorphism, and therefore π' splits. Consequently, we get a map $\varrho: X_{m+1} \to X_m$ with $\varrho\gamma'$ being the identity map of X_m. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \to & X_0 & \xrightarrow{\gamma} & Y & \xrightarrow{\pi} & A^{n_{m+1}}I & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varrho\alpha'} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \to & X_0 & \xrightarrow{\alpha} & X_m & \xrightarrow{\beta} & Q & \to & 0\,. \end{array}$$

the left square being commutative according to $\varrho\alpha'\gamma = \varrho\gamma'\alpha = \alpha$. Thus there exists $\psi: A^{n_{m+1}}I \to Q$ making the right square commutative, and therefore the lower sequence E is induced from the upper sequence by ψ, that is, $E = \delta(\psi)$, contrary to our assumption. This shows that there can be no embedding $\varphi: I' \to X_{m+1}$ with I' indecomposable preinjective, and therefore $\mathfrak{J}X_{m+1} = 0$.

3.10. It remains to derive those results on exponentially growing sequences which were used in the proof of Theorem 3.9. Let $(a_n)_n$ be a sequence of real numbers. Note that $\lim\limits_{n\to\infty} a_n/g^n = a$ if and only if for any $\varepsilon > 0$ there exists N with $g^n(a-\varepsilon) < a_n < g^n(a+\varepsilon)$ for all $n \geqslant N$.

(1) Let $\lim\limits_{n\to\infty} a_n/g^n = a > 0$, with $g > 1$. Let $c \in \mathbb{N}$. Then there exists N with $a_{n+1} > a_n + c$ for all $n \geqslant N$.

Proof: First, choose some natural m with $(a/2)(g-1)g^m \geqslant c$. Let

$$0 < \varepsilon < \frac{1}{2}\cdot\frac{g-1}{g+1}\,.$$

There exists $N \geqslant m$ such that $g^n(a-\varepsilon) < a_n < g^n(a+\varepsilon)$ for all $n \geqslant N$. Thus, for $n \geqslant N$, we have

$$\begin{aligned} a_{n-1} - a_n &> g^{n+1}(a-\varepsilon) - g^n(a+\varepsilon) = g^n[a(g-1) - \varepsilon(g+1)] \\ &> g^n\cdot\frac{a}{2}(g-1) \geqslant g^m\cdot\frac{a}{2}(g-1) \geqslant c\,. \end{aligned}$$

(2) Let $(a_n)_n$ be a sequence, let $s \in \mathbb{N}$. Assume

$$\lim_{n\to\infty} \frac{a_{ns+i}}{g^n} > 0$$

for some $g > 1$, and all $1 \leqslant i \leqslant s$. Then there exists i, with $1 \leqslant i \leqslant s$, such that for $c, N \in \mathbb{N}$, there exists $n \geqslant N$ with $a_j > a_{ns+i} + c$ for all $j > ns + i$.

PROOF: According to (1), choose for any $c \in \mathbb{N}$, some N_c such that $a_{u+s} > a_u + sc$ for all $u \geqslant N_c$. If we fix such an u, with $u \geqslant N_c$, it is easy to see that one of the s elements u' between u and $u + s - 1$ satisfies $a_{u'+t} > a_{u'} + c$ for all $1 \leqslant t \leqslant s-1$. Call this element u_c, thus we have $a_{u_c+t} > a_{u_c} + c$, for all $1 \leqslant t \leqslant s-1$. In the infinite set $U_c = \{u_c | u \geqslant N_c\}$ there have to be infinitely many elements which are congruent modulo s, thus there is some $i(c)$, with $1 \leqslant i(c) \leqslant s$ such that the set $U_{c,i(c)} = \{u_c | c \geqslant N_c,\ u_c \equiv i(c) \pmod s\}$ is infinite. Again, for infinitely many c, the elements $i(c)$ have to be equal, say equal to i. In this way, we have found i. Now, let c and $N \in \mathbb{N}$ be given. We may replace c by a larger number c' satisfying $i(c') = i$, thus we may assume $i(c) = i$. Now choose an element $ns + i$ in $U_{c,i}$ such that $n \geqslant N$. We claim that this satisfies the property

$$a_j > a_{ns+i} + c \qquad \text{for all } j > ns + i\,.$$

Let $j > ns + i$ be given, let $t = j - (ns + i)$. If $1 \leqslant t \leqslant s-1$, then

$$a_j = a_{ns+i+t} > a_{ns+i} + c\,, \qquad \text{since } ns + i \in U_c\,.$$

If $t \geqslant s$, say $t = ms + t'$ with $1 \leqslant m$, $0 \leqslant t' \leqslant s-1$

$$a_j = a_{ns+i+ms+t'} > a_{ns+i+t} + msc \geqslant a_{ns+i} + c\,,$$

since $ns + i + t \geqslant N_c$, and using the previous case. This finishes the proof.

(3) Assume $\lim\limits_{n\to\infty} a_n/g^n > 0$ for some $g > 1$. Let $g^u \geqslant 2$ for some natural number u. Assume, we have selected natural numbers $n_j \geqslant uj$ for $1 \leqslant j \leqslant m$. Then there exists $N \geqslant u(m+1)$ such that for all $n \geqslant N$, we have $a_{n+1} > \sum\limits_{j=1}^{m} a_{n-n_j}$.

PROOF: Let $a = \lim\limits_{n\to\infty} a_n/g^n$. For $\varepsilon = a\cdot(g-1)/(g+1)$, there exists N' with $g^n(a-\varepsilon) < a_n < g^n(a+\varepsilon)$ for all $n \geqslant N'$. Let N be the maximum of $u(m+1)$ and the various $N' + n_j$ with $1 \leqslant j \leqslant m$. We claim that this N works. Thus, let $n \geqslant N$ be given, let $n = ut + r$ with

$u \in \mathbb{N}$, and $r < u$. Then we have

$$\sum_{j=1}^{m} a_{n-n_j} < (a+\varepsilon)\sum_{j=1}^{m} g^{n-n_j} \leqslant (a+\varepsilon)\sum_{j=1}^{m} g^{n-uj} = (a+\varepsilon)\sum_{j=1}^{m} g^{u(t-j)+r} \leqslant$$

$$\leqslant (a+\varepsilon)\, g^r \cdot \sum_{j=0}^{t-1} g^{uj} = (a+\varepsilon)\, g^r \frac{g^{ut}-1}{g^u-1} \leqslant (a+\varepsilon)\, g^r g^{ut} = (a+\varepsilon)\, g^n =$$

$$= (a-\varepsilon)\, g \cdot g^n = (a-\varepsilon)\, g^{n+1} < a_{n+1}\,.$$

4. **Torsion regular modules.**

4.1. A module of finite length will be called *torsion* if it has no non-zero preprojective direct summand. For an arbitrary module X, let $\mathfrak{T}X$ be the submodule of X generated by the submodules of finite length which are torsion. This submodule $\mathfrak{T}X$ will be called the torsion submodule of X, and X will be said to be *torsion* if $\mathfrak{T}X = X$, and *torsionfree* if $\mathfrak{T}X = 0$.

Note that if X_1, X_2 are torsion submodules of X of finite length, then no non-zero direct summand of $X_1 + X_2$ can be preprojective, since otherwise there would be a non-zero homomorphism from one of the modules X_1, X_2 into an indecomposable preprojective module. Thus, the set of torsion submodules of finite length is filtered. As a consequence, if a submodule of finite length is contained in $\mathfrak{T}X$, then there is a torsion submodule of X of finite length containing Y. Also note that a torsion module cannot have a non-zero preprojective direct summand. For, assume X is torsion, $X = P \oplus X'$ with P indecomposable preprojective. Then P is a direct summand in every submodule which contains P. Thus there does not exist a torsion submodule of finite length containing P, a contradiction. As a consequence, we see that we have the following inclusions, for any module X

$$\mathfrak{J}X \subseteq \mathfrak{T}X \subseteq \mathfrak{F}X\,.$$

Theorem: *Let X be a module. Then $\mathfrak{T}X$ is a pure submodule, and* $\mathfrak{T}(X/\mathfrak{T}X) = 0$.

Proof: We have seen that $\mathfrak{T}X$ has no indecomposable preprojective direct summand. If we show that every indecomposable submodule of $X/\mathfrak{T}X$ of finite length is preprojective, then we can conclude from Cor. 3 in 2.2 that $\mathfrak{T}X$ is pure in X, and we also know that $\mathfrak{T}(X/\mathfrak{T}X) = 0$. Thus, let $\mathfrak{T}X \subset U \subseteq X$, with $U/\mathfrak{T}X$ of finite length. Let U' be a submodule of X of finite length with $U = U' +$

$+\mathcal{T}X$. Now $U \cap \mathcal{T}X$ is a submodule of finite length of $\mathcal{T}X$, and, therefore contained in a torsion submodule U'' of finite length. Since $U' + U''$ is of finite length and not contained in $\mathcal{T}X$, it cannot be torsion, thus $U' + U''$ has an indecomposable direct summand P which is preprojective, say $U' + U'' = P \oplus V$ for some submodule V. Since U'' is torsion, $\operatorname{Hom}(U'', P) = 0$, thus U'' has to be included in V. Consequently

$$U/\mathcal{T}X \approx U'/(U' \cap U'') \approx (U' + U'')/U'' \approx P \oplus (V/U'')$$

shows that $U/\mathcal{T}X$ has an indecomposable preprojective direct summand.

We note the following consequence:

COROLLARY: *If $\mathcal{T}X$ is of finite length, then $\mathcal{T}X$ is a direct summand of X.*

If $\mathcal{T}X$ is not of finite length, then $\mathcal{T}X$ does not have to be a direct summand of X, even in the tame case.

COROLLARY 2: *The class of torsion modules is closed under quotients, extensions, and direct sums. The class of torsionfree modules is closed under submodules, extension and products.*

As we have seen, the extension of a torsion module by a torsionfree module is always pure; however, it does not have to split. Let us consider also the other way: it is rather trivial to construct nonsplit extensions of a torsion module by a torsionfree module. However, if such an extension is pure, then it splits:

LEMMA: Let Y be a pure submodule of X. Assume Y is torsionfree, and X is torsion. Then Y is a direct summand of X.

PROOF: We claim that the torsion submodule $\mathcal{T}X$ of X is a direct complement. First, we show that $\mathcal{T}X \cap Y = 0$. For, let U be a submodule of $\mathcal{T}X \cap Y$ of finite length. There exists a torsion submodule U' of finite length containing U. Now since Y is pure in X, we see that Y is a direct summand of $Y + U'$, say $Y + U' = Y \oplus U''$. Since U' is torsion, the projection map $U' \subseteq Y + U' = Y \oplus U'' \to Y$ into the torsionfree module Y is zero, thus $U' \subseteq U''$. Thus $U = U \cap Y \subseteq U' \cap Y \subseteq U'' \cap Y = 0$. This shows $\mathcal{T}X \cap Y = 0$. On the other hand, let V/Y be a torsion submodule of finite length of X/Y, with $Y \subseteq V \subseteq X$. Again, using that Y is pure, we see that there is a submodule V' in X with $Y \oplus V' = V$. Since $V' \approx V/Y$ is torsion, $V' \subseteq \mathcal{T}X$. Thus also $\mathcal{T}X + Y = X$.

4.2. A module X will be called *regular* if X has no non-zero preprojective or preinjective direct summand. This is equivalent to the fact that X has no indecomposable preprojective or preinjective direct summand, and also to the fact that $\mathfrak{J}X = 0$ and $\mathfrak{T}X = X$. Now, if a module X satisfies $\mathfrak{J}X = 0$, then for any submodule U of X, we have $\mathfrak{J}U = 0$, and if X satisfies $\mathfrak{T}X = X$, then for any quotient Q of X, we have $\mathfrak{T}Q = Q$. Thus, if X is regular, a submodule U of X is regular if and only if $\mathfrak{T}U = U$, and a quotient module Q of X is regular if and only if $\mathfrak{J}Q = 0$. And similarly, if X and Y are regular, and $\varphi: X \to Y$ is a homomorphism, then the image of φ is regular. This proves the first part of the following assertion, the remaining ones being equally clear.

PROPOSITION: The class of regular modules is closed under images of homomorphisms, extensions, and direct sums.

Also, we want to recall from 1.G a construction, which in our case associates to every module a regular module. Note that for indecomposable preprojective modules P, any maximal pure submodule U_P of a module X which is a direct sum of copies of P is in fact a direct summand of X, according to 2.1. Thus:

LEMMA: Let X be a module. For any indecomposable projective module P, let U_P be a maximal direct summand of X which is a direct sum of copies of P. Then $X/\left(\mathfrak{J}X \bigoplus_{P\in\mathfrak{P}} U_P\right)$ is regular.

4.3. In contrast to the investigation of preprojective and preinjective modules, the study of regular modules will be limited to the tame case, the general case being hard to attack. From now on, we will assume that R is a finite dimensional hereditary algebra of tame representation type.

PROPOSITION: Let R be a twosided indecomposable finite dimensional hereditary algebra of tame type. Let P be an indecomposable projective module with defect -1. Then, for any regular module X, there exists a submodule U of X which is isomorphic to a direct sum of copies of P, such that X/U is torsion regular.

PROOF: By transfinite induction, we will construct submodules U_λ with $U_{\lambda+1}/U_\lambda \approx P$ for any ordinal λ, and $U_\lambda = \bigcup_{\mu<\lambda} U_\mu$ for any limit ordinal λ, such that X/U_λ is regular, for any λ. The construction will stop as soon as X/U_λ is torsion regular.

First, we show: if Y is torsionfree regular, and $\mathrm{Hom}(P, Y) = 0$, then $Y = 0$. For, $\mathrm{Hom}(P, Y) = 0$ implies that one particular simple module does not appear as composition factor of Y, thus Y can be considered as module over a finite dimensional hereditary algebra R' which is a proper factor algebra of R. Now R' will be of finite representation type, and therefore Y is a direct sum of modules of finite length. However, a torsionfree regular module has no non-zero direct summand of finite length whatsoever. Thus we see that $Y = 0$.

Now assume Y is regular, but not torsion. We claim that there is a submodule V of Y, with $V \approx P$, such that Y/V is regular, again.

Proof: By assumption, $Y' = Y/\mathcal{T}Y \neq 0$ thus there exists a nonzero homomorphism $\alpha' \colon P \to Y'$. Let V' be the image of α'. Since Y' is torsionfree, $\delta V' < 0$. Thus $\delta(\operatorname{Ker} \alpha') = \delta P - \delta V' \geqslant 0$, and therefore $\operatorname{Ker} \alpha' = 0$, that is, α' is a monomorphism, and $V' \approx P$. Assume Y'/V' has an indecomposable preinjective direct summand, say $V' \subset W' \subset Y'$ with W'/V' indecomposable preinjective. Then $\delta W' = \delta V' + \delta(W'/V') = -1 + \delta(W'/V') \geqslant 0$, contrary to the fact that Y' is torsionfree. Since Y'/V' as a quotient of Y' also has no indecomposable preprojective direct summand, we conclude that Y'/V' is regular. The homomorphism $\alpha' \colon P \to Y' = Y/\mathcal{T}Y$ can be lifted to a homomorphism $\alpha \colon P \to Y$. Since α' is a monomorphism, also α is a monomorphism, and, in addition, the image V of α satisfies $V \cap \mathcal{T}Y = 0$. Thus, Y/V is an extension of the regular module $(V + \mathcal{T}Y)/V \approx \mathcal{T}Y$ by the regular module $Y/(V + \mathcal{T}Y) \approx Y'/V'$, and therefore regular.
Let $U_0 = 0$. If U_λ is defined, with X/U_λ regular, but not torsion, then we can use the previous consideration for $Y = X/U_\lambda$, and we get $U_\lambda \subseteq U_{\lambda+1} \subset X$ such that $U_{\lambda+1}/U_\lambda \approx P$, and $X/U_{\lambda+1}$ regular. If U_μ is defined for all $\mu < \lambda$, let $U_\lambda = \bigcup_{\mu<\lambda} U_\mu$. Then X/U_λ has no preprojective direct summands. Assume there is $U_\lambda \subset W \subseteq X$ with W/U_λ indecomposable preinjective. Let W' be of finite length with $U_\lambda + W' = W$. Since W' is of finite length, there is $\mu < \lambda$ with $U_\lambda \cap W' = U_\mu \cap W'$. Thus

$$(U_\mu + W')/U_\mu \approx W'/(W' \cap U_\mu) = \\ = W'/(W' \cap U_\lambda) \approx (U_\lambda + W')/U_\lambda = W/U_\lambda$$

shows that also X/U_μ would have an indecomposable preinjective submodule, impossible. Thus, also in this case, X/U_λ is regular.

Since P is projective, it follows from $U_{\lambda+1}/U_\lambda \approx P$ that $U_{\lambda+1}$ is the direct sum of U_λ and a copy of P, thus any U_λ is the direct sum of copies of P. This finishes the proof.

4.4. We are going to give a complete description of the full subcategory of torsion regular modules. In order to do so, we need some concepts.

A not necessarily commutative ring D will be called a *discrete valuation ring* if it is a local ring without zero divisors such that the powers M^i ($i \in \mathbb{N}$) of the maximal ideal M are the only left ideals and the only right ideals. A discrete valuation ring D is said to be *complete* if the canonical map $D \to \varprojlim D/M^i$ is an isomorphism.

Now assume D is a discrete valuation ring with maximal ideal M. We denote by $H_n(D)$ the ring

$$H_n(D) = \begin{pmatrix} D & D & \cdots & D \\ M & D & & D \\ \vdots & & \ddots & \vdots \\ M & \cdots & M & D \end{pmatrix}$$

consisting of all $n \times n$-matrices with entries in D such that the entries below the main diagonal belong to M (note that this is a semiperfect hereditary noetherian prime ring). An $H_n(D)$-module is called *torsion* if any element of it is annihilated by some non-zero twosided ideal of $H_n(D)$, and therefore by an ideal of the form

$$\begin{pmatrix} M^i & \cdots & M^i \\ \vdots & & \\ M^i & \cdots & M^i \end{pmatrix}.$$

Clearly, when considering torsion $H_n(D)$-modules, we may replace D by its completion $\hat{D} = \varprojlim D/M^i$ and consider torsion $H_n(\hat{D})$-modules.

THEOREM: *Let R be a finite dimensional hereditary algebra of tame representation type. Then the full subcategory of torsion regular modules of the category of all R-modules is an exact abelian subcategory, and it is the product of categories $\mathfrak{T}_t$ ($t \in T$) each of which is equivalent to the category of torsion modules over a ring of the form $H_{n_t}(D_t)$, where $n_t \in \mathbb{N}$, and D_t is a complete discrete valuation ring.*

REMARK 1: The index set T is equal to the corresponding index set in the case of regular modules of finite length, see 1.*D*. In particular, T is always infinite, and, in case the base field is algebraically closed, $T = \mathbb{P}_1(k)$, the projective line over k.

REMARK 2: The number n_t is equal to the number of simple $H_{n_t}(D_t)$-modules, and these correspond to those modules of finite length in $\mathfrak{T}_t$ which are simple regular. Thus n_t is equal to the corresponding number in the case of regular modules of finite length, see 1.*D*. In particular, for all but at most three t, we have $n_t = 1$.

Remark 3: The ring D_t is constructed as follows: Let S be a simple regular module in $\mathfrak{T}_t$, and S_n the indecomposable regular module of finite length with regular length n and $\operatorname{Hom}(S_n, S) \neq 0$. Then there is a chain of epimorphisms

$$\ldots \to S_n \to S_{n-1} \to \ldots \to S_1$$

which gives rise to a chain of ring surjections

$$\ldots \to \operatorname{End}(S_n) \to \operatorname{End}(S_{n-1}) \to \ldots \to \operatorname{End}(S_1)\,.$$

Let $D_t = \varprojlim \operatorname{End}(S_n)$. It is clear that D_t is a complete discrete valuation ring.

The rings D_t, for the various $t \in T$, are not independent. We will see later that a suitable matrix ring over D_t is the completion of a subring of some fixed division ring.

Proof of the theorem: We know that the category of regular modules of finite length is an exact abelian subcategory. Let X, Y be arbitrary torsion regular modules and $\varphi\colon X \to Y$ a homomorphism. First, we want to show that the kernel U of φ is torsion regular again. Since U is a submodule of X, it has no non-zero preinjective submodules, thus we have to show that U is generated by regular submodules of finite length. Let U'' be a submodule of U of finite length, let X' be a regular submodule of X of finite length containing U'', and let Y' be a regular submodule of Y of finite length containing $\varphi(U)$. Let φ' be the restriction of φ to X', then $\varphi'\colon X' \to Y'$ is a homomorphism between regular modules of finite length, thus the kernel U' of φ' is regular of finite length, and by construction, $U'' \subseteq U'$. Thus, the kernel of φ is torsion regular. Next, let $\pi\colon Y \to V$ be the cokernel of φ. Clearly, $\mathcal{T}V = V$, since $\mathcal{T}Y = Y$. Thus, we have to exclude the possibility that V contains a non-zero preinjective submodule V'' of finite length. If it does, let Y'' be a submodule of Y of finite length with $\pi(Y'') = V''$, and let X'' be a submodule of X of finite length with $\varphi(X'') = \varphi(X) \cap Y''$. Note that this is just the kernel of the restriction π'' of π to Y''. Thus, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\pi} & V & \to & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ X'' & \xrightarrow{\varphi''} & Y'' & \xrightarrow{\pi''} & V'' & \to & 0\,. \end{array}$$

Let X' be regular of finite length with $X'' \subseteq X' \subseteq X$, let Y' be regular of finite length with $\varphi(X') + Y'' \subseteq Y' \subseteq Y$, and let V' be the cokernel, of the restriction $\varphi' : X' \to Y'$ of φ. Since X' and Y' both are regular of finite length, also V' is regular of finite length. On the other hand, the inclusion map $V'' \to V$ factors through V'. Since V'' is preinjective, and V' is regular, it follows that $\mathrm{Hom}(V'', V') = 0$, thus the inclusion $V'' \to V$ is the zero-map, and therefore $V'' = 0$. Thus, we have shown that V has no non-zero preinjective submodule, and therefore V is torsion regular. Since the class of torsion regular modules is closed also under direct sums, it follows that the full subcategory of all torsion regular modules is an exact abelian subcategory of the category of R-modules.

Next, we note that for any torsion regular module X, there is an exact sequence

$$\bigoplus_{V \in \mathfrak{V}} V \to \bigoplus_{U \in \mathfrak{U}} U \to X \to 0$$

where all $U \in \mathfrak{U}$, $V \in \mathfrak{V}$ are indecomposable regular modules of finite length. For, let $\mathfrak{U}$ be the set of indecomposable regular submodules of X of finite length. Then the submodules U in $\mathfrak{U}$ generate X, thus the canonical map $\bigoplus_{U \in \mathfrak{U}} U \to X$ is surjective. Now the kernel of this map is again torsion regular, thus we can do the same construction with X replaced by the kernel and, in this way, we obtain the asserted exact sequence.

Now fix some $t \in T$. Let $S(i)$, $1 \leqslant i \leqslant n = n_t$ be the simple regular modules in $\mathfrak{T}_t$, such that $AS(i) \approx S(i+1)$, for all i, with $S(n+1) = S(1)$. Denote by $S(i)_m$ the indecomposable regular module in $\mathfrak{T}_t$ of regular length m such that $\mathrm{Hom}\,(S(i)_m, S(i)) \neq 0$. Let $Y_m = \bigoplus_{i=1}^{n} S(i)_m$, and let $Y_{m+1} \to Y_m$ be the canonical epimorphism with kernel the regular socle of T_{m+1}. Then the sequence of epimorphisms

$$\ldots \to T_{m+1} \to T_m \to \ldots \to T_1$$

induces a chain of ring surjections

$$\ldots \to \mathrm{End}\,(T_{m+1}) \to \mathrm{End}\,(T_m) \to \ldots \to \mathrm{End}(T_1)\,,$$

the inverse limit will be denoted by

$$H = \varprojlim_m \mathrm{End}\,(T_m)\,.$$

We may lift the unique set of orthogonal primitive idempotents of

End (T_1) to a complete set $e_1, \ldots, e_n$ of orthogonal idempotents of H such that

$$e_i H e_i = \varprojlim_m \operatorname{End}\big(S(i)_m\big) .$$

Note that the radical J of H is the kernel of the canonical map $H \to \operatorname{End}(T_1)$; it is generated by elements ${}_ih_{i+1} \in e_i H e_{i+1}$ such that the image of ${}_ih_{i+1}$ in End (T_2) is a non-zero map $S(i+1)_2 \to S(i)_2$. Let $D = e_1 H e_1$. Using the elements ${}_ih_{i+1}$, we may identify all $e_i H e_i$ with D, and it is clear that in this way, we identify H with $H_n(D)$. Let $H(i)_m = e_i H / e_i J^m$, then this is an indecomposable torsion H-module of length n, and with top composition factor $H(i)_1$. Clearly, any torsion H-module is the cokernel of a map

$$\bigoplus_{i,m} \bigoplus_{\alpha_{im}} H(i)_m \to \bigoplus_{i,m} \bigoplus_{\beta_{im}} H(i)_m ,$$

with α_{im}, β_{im} being cardinal numbers. It should be clear that in this way we obtain an equivalence between the category of torsion H-modules, and $\mathfrak{T}_t$, with $H(i)_m$ corresponding to $S(i)_m$.

4.5. By the previous theorem, the investigation of torsion regular modules completely is reduced to the study of torsion $H_n(D)$-modules, with D a complete discrete valuation ring. Namely, any torsion regular module X is the direct sum of its maximal submodules belonging to $\mathfrak{T}_t$, and this decomposition is unique (this is the precise analogue of the primary decomposition in abelian group theory), and instead of considering a torsion regular module in $\mathfrak{T}_t$, we may consider its corresponding $H_{n_t}(D_t)$-module. Note that a regular torsion module X in $\mathfrak{T}_t$ is of finite length if and only if its corresponding $H_{n_t}(D_t)$-module is of finite length, and the length of the $H_{n_t}(D_t)$-module which corresponds to X has been called the *regular length* of X; clearly, it is just the length of X when considered as an object of the abelian category of all torsion regular modules. The torsion regular modules of regular length 1 are just the *simple regular* modules. Now, for any regular torsion module X, the submodule of X generated by the simple regular submodules of X may be called the *regular socle* of X, it is a direct sum of simple regular modules. In this way, we get an ascending chain of inclusion

$$0 = U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq U_{n+1} \subseteq \ldots \subseteq X$$

such that U_{n+1}/U_n is the regular socle of X/U_n, the regular socle sequence; note that $X = \bigcup_{n \in \mathbb{N}} U_n$.

We want to single out a special type of torsion regular modules which are of grant importance. If S is simple regular, let S^n be the indecomposable torsion regular module of regular length n such that $\mathrm{Hom}(S, S^n) \neq 0$, or, equivalently, S^n is of regular socle length n, and its regular socle is isomorphic to S.

There is a sequence of embeddings

$$S = S^1 \subset S^2 \subset \ldots \subset S^n \subset S^{n+1} \subset \ldots$$

and we denote by $S^\omega = \bigcup_{n \in \mathbb{N}} S^n$ the direct limit. Since any automorphism of S^n extends to an automorphism of S^{n+1}, we see that S^ω is (up to isomorphism) uniquely defined. The modules of the form S^ω will be called *Prüfer modules*, since they are the direct analogue of the Prüfer groups. They have been introduced in [32]. Note that the only regular submodule of S^ω of regular length n is S^n with its canonical inclusion in S^ω. Thus S^ω has a unique chain of regular submodules, and this chain is just the regular socle sequence. In particular, S^ω is indecomposable.

If S belongs to $\mathfrak{T}_t$, then it is clear that the $H_{n_t}(D_t)$-module corresponding to S^ω is just the injective envelop of the $H_{n_t}(D_t)$-module corresponding to S. As a consequence, we see that $\mathrm{Ext}(X, S^\omega) = 0$ for any module in $\mathfrak{T}_t$. But since for torsion regular modules X, Y belonging to different subcategories $\mathfrak{T}_t$ and $\mathfrak{T}_{t'}$ $(t \neq t')$, we have always $\mathrm{Ext}(X, Y) = 0$, we conclude:

LEMMA: $\mathrm{Ext}(X, S^\omega) = 0$ for any torsion regular module X.

Also, we note that any non-zero torsion $H_n(D)$-module, D a discrete valuation ring, has a direct summand which is either of finite length, or the injective envelop of a simple $H_n(D)$-module. As a consequence, we get

LEMMA 2: A non-zero torsion regular module has an indecomposable direct summand which is either of finite length, or a Prüfer module.

The ring $H_n(D)$, D a discrete valuation ring, is noetherian, thus a direct sum of injective modules is injective again. As a consequence, it follows that $\mathrm{Ext}(X, Y) = 0$ for X torsion regular, and Y an arbitrary direct sum of Prüfer modules. However, also the converse is true:

LEMMA 3: Let X be torsion regular, and assume $\mathrm{Ext}(S, X) = 0$ for all simple regular modules. Then X is a direct sum of Prüfer modules.

PROOF: Assume there is given an $H_n(D)$-module X' with $\operatorname{Ext}(S', X') = 0$ for all simple $H_n(D)$-modules S'. Then it is clear that X' is injective. Thus, if X' is a torsion $H_n(D)$-module, then it is a direct sum of $H_n(D)$-modules which are injective envelops of simple $H_n(D)$-modules. The correspondence between torsion $H_n(D)$-modules and torsion regular R-modules gives the result.

4.6. A module Y will be called *divisible* iff $\operatorname{Ext}(S, Y) = 0$ for all simple regular modules S. The previous section shows that a torsion regular module is divisible if and only if it is the direct sum of Prüfer modules. Also, according to 3.5, any preinjective module is divisible. However, our interest will lie more in the divisible regular modules.

PROPOSITION: The class of divisible modules is closed under quotients, extensions and direct sums. The torsion part of a divisible module is divisible.

PROOF: The first assertion follows from the fact that for an epimorphism $Y \to Y'$, and any module X, the induced map $\operatorname{Ext}(X, Y) \to \to \operatorname{Ext}(X, Y')$ is surjective. Also, if $0 \to Y' \to Y \to Y'' \to 0$ is an exact sequence, and X is a module, then there is an exact sequence

$$\operatorname{Ext}(X, Y') \to \operatorname{Ext}(X, Y) \to \operatorname{Ext}(X, Y'') .$$

Thus, if Y' and Y'' both are divisible, also Y is divisible. Next, any simple regular module S is of finite length, thus for any family $(Y_i)_i$ of modules we have $\operatorname{Ext}(S, \oplus Y_i) = \oplus \operatorname{Ext}(S, Y_i)$. Thus, if all Y_i are divisible, also $\oplus Y_i$ is divisible. Finally, assume Y is divisible, and apply the long exact sequence for $\operatorname{Hom}(S, H)$ to the exact sequence $0 \to \mathfrak{T}Y \to Y \to Y/\mathfrak{T}Y \to 0$. We get an exact sequence

$$\operatorname{Hom}(S, Y/\mathfrak{T}Y) \to \operatorname{Ext}(S, \mathfrak{T}Y) \to \operatorname{Ext}(S, Y) .$$

The last term is zero by assumption, the first is zero since $Y/\mathfrak{T}Y$ is torsionfree. Thus, $\mathfrak{T}Y$ is divisible.

As a consequence, for any module X, there exists the maximal divisible submodule, which we denote by $\mathfrak{D}X$, and which contains any divisible submosule of X. A module X will be called *reduced* if $\mathfrak{D}X = 0$.

LEMMA: For any module X, we have

$$\mathfrak{J}X \subseteq \mathfrak{D}X \subseteq \mathfrak{I}^{\infty}X , \qquad \text{and} \qquad \mathfrak{D}(X/\mathfrak{D}X) = 0 .$$

Proof: Since preinjective modules are divisible, $\mathfrak{I}X \subseteq \mathfrak{D}X$. Next, we want to see that a divisible module cannot have an indecomposable preprojective direct summand P. For given P, there exists a simple regular module S with $\mathrm{Hom}(P, S) \neq 0$, but then also $\mathrm{Ext}(A^{-1}S, P) \neq 0$ by 1.A. Since with S also $A^{-1}S$ is simple regular, we see that P itself is not divisible, thus P cannot be a direct summand of a divisible module. The last assertion follows from the fact that an extension of divisible modules is divisible again.

4.7. Proposition: Let Y be divisible. If X is any module with $\mathfrak{I}X = 0$, then $\mathrm{Ext}(X, Y) = 0$.

Proof: First, consider the case that Y is preinjective. According to 3.7, $\mathrm{Ext}(X, Y) = 0$ in this case, since for an exact sequence $0 \to Y \to Z \to X \to 0$, it follows from $\mathfrak{I}Y = Y$, $\mathfrak{I}X = 0$ that $\mathfrak{I}Z = Y$, thus the sequence splits.

Since for general Y, $\mathfrak{I}Y$ is a direct summand of Y, we have

$$\mathrm{Ext}(X, Y) = \mathrm{Ext}(X, \mathfrak{I}Y) \oplus \mathrm{Ext}(X, Y/\mathfrak{I}Y) = \mathrm{Ext}(X, Y/\mathfrak{I}Y),$$

and $Y/\mathfrak{I}Y$ is divisible again, we may assume that Y is regular.

With Y also $\mathfrak{T}Y$ and $Y/\mathfrak{T}Y$ are divisible regular. Thus, we may assume that Y is either torsion regular or torsionfree regular, since there is an exact sequence

$$\mathrm{Ext}(X, \mathfrak{T}Y) \to \mathrm{Ext}(X, Y) \to \mathrm{Ext}(X, Y/\mathfrak{T}X).$$

We consider first the case that X and Y both are torsion. By the previous theorem, we may consider the corresponding question for torsion $H_n(D)$-modules, where D is a discrete valuation ring. But if Y' is an $H_n(D)$-module with $\mathrm{Ext}(S', Y') = 0$ for any simple $H_n(D)$-module, then clearly Y' is injective, and therefore $\mathrm{Ext}(X', Y') = 0$ for any $H_n(D)$-module, in particular, any torsion $H_n(D)$-module.

Next, let X be torsion, and Y torsionfree. Since we want to show that $\mathrm{Ext}(X, Y) = 0$, let

$$0 \to Y \xrightarrow{\alpha} Z \xrightarrow{\beta} X \to 0$$

be an exact sequence, where we may assume that α is an inclusion. Consider the restriction $\beta'\colon \mathfrak{T}Z \to X$ of β. Since $\mathfrak{T}Z$ and X both are torsion regular, also kernel and cokernel of β' are torsion regular. The kernel of β' is $\mathfrak{T}Z \cap Y$. Since this is a submodule of the torsionfree module Y, we conslude that $\mathfrak{T}Z \cap Y = 0$. Thus, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{T}Z & \xrightarrow{\text{id}} & \mathcal{T}Z & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \beta'} & & \\
0 & \to & Y & \xrightarrow{\alpha} & Z & \xrightarrow{\beta} & X & \to & 0 \\
 & & \downarrow{\scriptstyle \text{id}} & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & Z/\mathcal{T}Z & \xrightarrow{\beta'} & X/\beta(\mathcal{T}Z) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Assume the cokernel $X/\beta(\mathcal{T}Z)$ of β' is non-zero. Since this is a torsion regular module, there is a simple regular module S and an embedding $\gamma\colon S \to X/\beta(\mathcal{T}Z)$. Thus, we get an induced exact sequence

$$\begin{array}{ccccccccc}
0 & \to & Y & \to & Z/\mathcal{T}Z & \xrightarrow{\beta'} & X/\beta(\mathcal{T}Z) & \to & 0 \\
 & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \to & Y & \to & Z' & \xrightarrow{\beta''} & S & \to & 0
\end{array}$$

However, according to $\operatorname{Ext}(S, Y) = 0$, we know that β'' splits, thus there is an embedding $S \to Z'$, which together with γ' gives an embedding of the torsion module S into the torsionfree module $Z/\mathcal{T}Z$, impossible. Thus $X/\beta(\mathcal{T}Z) = 0$, and therefore β' is an isomorphism. Thus β splits.

Alltogether, we know that $\operatorname{Ext}(X, Y) = 0$ for Y divisible regular, and X torsion regular. Also, if X is the direct sum of indecomposable preprojective modules, say $X = \bigoplus P_i$, and Y is regular, then

$$\operatorname{Ext}(\bigoplus P_i, Y) = \prod \operatorname{Ext}(P_i, Y) = 0\,,$$

according to 1.E, and 2.2. But if X is an arbitrary module with $\mathfrak{I}X = 0$, then there is a submodule U which is the direct sum of indecomposable preprojective modules, such that X/U is regular, by Lemma 4.2, and according to 4.3, there is a submodule $U \subseteq V \subseteq X$ such that V/U is a direct sum of copies of a fixed indecomposable projective module, and such that X/V is torsion regular. By the previous considerations, we know that for Y divisible regular,

$$\operatorname{Ext}(U, Y) = 0\,,\ \operatorname{Ext}(V/U, Y) = 0 \quad \text{and} \quad \operatorname{Ext}(X/V, Y) = 0\,,$$

thus also $\operatorname{Ext}(X, Y) = 0$.

Finally, we may apply this result to $\operatorname{Ext}(Y/\mathfrak{T}Y, \mathfrak{T}Y)$. Since $\mathfrak{T}Y$ is divisible regular, and $Y/\mathfrak{T}Y$ is torsionfree, $\operatorname{Ext}(Y/\mathfrak{T}Y, \mathfrak{T}Y) = 0$, thus $\mathfrak{T}Y$ is a direct summand of Y. This finishes the proof.

Remark: Note that the result cannot be improved. The only modules Y with $\operatorname{Ext}(X, Y) = 0$ for *all* modules X are the injective modules, thus we need a restriction on the modules X. Now any module is the direct sum of indecomposable preinjective modules and a module X satisfying $\mathfrak{I}X = 0$, and given any indecomposable preinjective module I, there are only few modules Y with $\operatorname{Ext}(I, Y) = 0$.

Corollary: *Let Y be divisible. Let X be a module with a submodule X' such that $\mathfrak{I}(X/X') = 0$. Then, any homomorphism $\varphi'\colon X' \to Y$ has an extension $\varphi\colon X \to Y$.*

Proof: Consider the induced exact sequence

$$\begin{array}{ccccccccc}
0 & \to & X' & \xrightarrow{\alpha} & X & \to & X/X' & \to & 0\\
 & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi''} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \to & Y & \xrightarrow{\alpha'} & Z & \to & X/X' & \to & 0\,,
\end{array}$$

where α denotes the canonical inclusion map. Since $\operatorname{Ext}(X/X', Y) = 0$, the bottom sequence splits, thus there is $\beta\colon Z \to Y$ with $\beta\alpha' = 1_Y$, and therefore $\varphi = \beta\varphi''$ satisfies $\varphi\alpha = \beta\varphi''\alpha = \beta\alpha' p' = p'$.

Corollary 2: *Any module is the direct sum of a divisible module and a reduced module.* Any divisible module is the direct sum of indecomposable preinjective modules, Prüfer modules and a divisible torsionfree module.

Proof: Let X be a module. Then $X \approx \mathfrak{I}X \oplus X/\mathfrak{I}X$ by 3.7. Now $\operatorname{Ext}(X/\mathfrak{D}X, \mathfrak{D}X/\mathfrak{I}X) = 0$, since $\mathfrak{I}(X/\mathfrak{D}X) \approx \mathfrak{D}(X/\mathfrak{D}X) = 0$, and $\mathfrak{D}X/\mathfrak{I}X$ is divisible regular. Thus $X/\mathfrak{I}X \approx \mathfrak{D}X/\mathfrak{I}X \oplus X/\mathfrak{D}X$, and therefore $X \approx \mathfrak{I}X \oplus \mathfrak{D}X/\mathfrak{I}X \oplus X/\mathfrak{D}X$, the first two summands being divisible, the last one reduced.

If X is divisible, then $\operatorname{Ext}(X/\mathfrak{T}X, \mathfrak{T}X/\mathfrak{I}X) = 0$, since $X/\mathfrak{T}X$ is regular, and $\mathfrak{T}X/\mathfrak{I}X$ is divisible regular. Thus $X \approx \mathfrak{I}X \oplus \mathfrak{T}X/\mathfrak{I}X \oplus$

$\oplus X/\mathfrak{T}X$. Note that $\mathfrak{I}X$ is the direct sum of indecomposable preinjective modules, and that the divisible torsion regular module $\mathfrak{T}X/\mathfrak{I}X$ is a direct sum of Prüfer modules.

4.8. Proposition: Let X be a module without non-zero direct summands of finite length. Then X is the direct sum of Prüfer modules and a torsionfree regular module.

Proof: Since X has no indecomposable preprojective or preinjective direct summand, X is regular. Let $\mathfrak{T}X$ be its torsion part. We can write $\mathfrak{T}X$ as the direct sum of Prüfer modules and a submodule U which has no submodule isomorphic to a Prüfer module. Assume $U \neq 0$. By Lemma 2 of 4.5, there is a direct summand V of U which is indecomposable and of finite length. Now V is a direct summand of $\mathfrak{T}X$, and $\mathfrak{T}X$ is pure in X, thus V is pure in X. However, since V is of finite length, this implies that V is a direct summand of X, impossible. Thus $U = 0$, and $\mathfrak{T}X$ is a direct sum of Prüfer modules. However, a direct sum of Prüfer modules is divisible, and therefore $\operatorname{Ext}(X/\mathfrak{T}X, \mathfrak{T}X) = 0$, by 4.6. This shows that X is the direct sum of $\mathrm{T}X$ and the torsionfree regular module $X/\mathfrak{T}X$.

Corollary: *Let X be indecomposable. Then either X is of finite length, or X is a Prüfer module, or X is torsionfree regular.*

We may use the proposition above in order to give some hints on the structure of an arbitrary module X. If we choose for any indecomposable module Y of finite length a maximal pure submodule U_Y of X which is a direct sum of copied of Y, then we know that $X/\oplus U_Y$ has no non-zero direct summands of finite length, thus we may apply the proposition. This shows that X as the extension of a module which is the direct sum of indecomposable modules of finite length by a module which is a direct sum of Prüfer modules and a torsionfree regular module.

5. Torsionfree divisible modules.

Again, we assume that the ring R is of tame representation type. In this section, we want to study torsionfree divisible modules, and possible embeddings of torsionfree modules into torsionfree divisible modules.

5.1. Lemma: Let X, Y be torsionfree divisible, and $X' \subseteq X$, $Y' \subseteq Y$ submodule with X/X' and Y/Y' torsion regular. Then any homomorphism $\varphi' : X' \to Y'$ has a unique extension $\varphi : X \to Y$. And, if φ' is an isomorphism, then any extension φ is an isomorphism.

Proof: Denote the inclusion by $\alpha\colon X' \to X$, $\beta\colon Y' \to Y$. Given $\varphi'\colon X' \to Y'$, we know from 4.7 that the map $\beta\varphi'\colon X' \to Y$ has an extension $\varphi\colon X \to Y$.

The extension is unique, since the zero homomorphism $X' \to Y'$ has as extension only the zero homomorphism $X \to Y$. For, assume we have $\varphi\colon X \to Y$, with $\varphi\alpha = 0$, then φ factors over the cokernel X/X' of α. However, X/X' is torsion, and Y is torsionfree, thus $\mathrm{Hom}\,(X/X', Y) = 0$.

Now assume, φ' is an isomorphism and take $\varphi\colon X \to Y$ with $\varphi\alpha = \varphi'\beta$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \xrightarrow{\alpha} & X & \to & X/X' & \to & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \to & Y' & \xrightarrow{\beta} & Y & \to & Y/Y' & \to & 0 \end{array}$$

Since φ' is an isomorphism, we know that $\ker\varphi \approx \ker\varphi''$, and $\mathrm{cok}\,\varphi \approx \approx \mathrm{cok}\,\varphi''$. However, since both X/X' and Y/Y' are torsion regular, also the kernel and the cokernel of φ'' are torsion regular. Thus $\ker\varphi$, being a torsion regular submodule of the torsion-free module X, has to be zero. Thus, φ is a monomorphism. However, the exact sequence

$$0 \to X \xrightarrow{\varphi} Y \to \mathrm{cok}\,\varphi \to 0$$

splits, since X is torsionfree divisible, and $\mathrm{cok}\,\varphi$ is torsion regular, thus Y has a submodule isomorphic to $\mathrm{cok}\,\varphi$. Since Y is torsion-free, this implies $\mathrm{cok}\,\varphi = 0$, thus φ is also surjective. This shows that φ is an isomorphism.

5.2. If S is simple regular, then $\mathrm{End}\,(S)$ is a division ring, thus, for any module X, $\mathrm{Ext}\,(S, X)$ may be considered as a vectorspace over $\mathrm{End}\,(S)$, and we denote its dimension by

$$e_{SX} = \dim \mathrm{Ext}\,(S, X)_{\mathrm{End}\,(S)}\,.$$

Proposition: Let P be indecomposable preprojective. Then there exists a torsionfree divisible module X with an exact sequence

$$0 \to P \to X \to \bigoplus_{S} \bigoplus_{e_{SP}} S^{\omega} \to 0\,,$$

where S runs through all simple regular modules.

For the proof, we need the following lemma:

LEMMA: Let P be indecomposable preprojective, S simple regular. Assume there exists an exact sequence

$$0 \to P \to X \to \bigoplus_m S \to 0 ,$$

where X has no submodule isomorphic to S. Then $m \leqslant e_{SP}$. Conversely, for $m \leqslant e_{SP}$ there exists such an exact sequence with X preprojective.

PROOF OF THE LEMMA (Compare [36]). First, assume such a sequence

$$E\colon 0 \to P \to X \to \bigoplus_m S \to 0$$

is given. The inclusions $u_i\colon S \to \bigoplus_m S$ $(1 \leqslant i \leqslant m)$ give elements Eu_i^* in $\operatorname{Ext}(S, P)$. If $m > e_{SP}$, then there are endomorphisms φ_i of S, not all zero, with

$$0 = \sum_{i=1}^{m} Eu_i^* \varphi_i^* = E\Big(\sum_{i=1}^{m} u_i \varphi_i\Big)^* = E(\varphi_i)_i^* .$$

Thus, the induced sequence

$$\begin{array}{ccccccccc}
0 & \to & P & \to & X & \to & \bigoplus_m S & \to & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle (\varphi_i)_i} & & \\
0 & \to & P & \to & X' & \to & S & \to & 0
\end{array}$$

splits, and we get an embedding of S into X'. Since $(\varphi_i)_i$ is a monomorphism, X' is a submodule of X, thus we have an embedding of S into X.

Conversely, take an exact sequence

$$E\colon 0 \to P \to X \xrightarrow{\pi} \bigoplus_m S \to 0$$

such that the induced elements Eu_i^* $(1 \leqslant i \leqslant m)$ are linearly independent in $\operatorname{Ext}(S, P)_{\operatorname{End}(S)}$. This is possible provided $m \leqslant e_{SP}$. Let X' be an indecomposable direct summand of X. If X' would be preinjective, then $\pi(X') = 0$ since it is a preinjective submodule of the regular module $\bigoplus_m S$. Thus X' is contained in the kernel P of X, impossible, since P is preprojective. Similarly, if X' is regular, then the restriction π' of π to X' is a map between torsion regular modules, thus the kernel of π' is a torsion regular submodule of P, and therefore zero. This shows that π' is a monomorphism, and consequently, X' being an indecomposable regular submodule of $\bigoplus_m S$, has to be isomorphic to S. In this way, we get homomorphisms $\varphi_i\colon S \to S$, $1 \leqslant i \leqslant m$, not all zero, such that the induced exact sequence $E(\varphi_i)_i^*$ splits. Thus $0 = E(\varphi_i)_i^* = \sum_{i=1}^m Eu_i^* \varphi_i^*$ gives a contradiction to the choice of E.

Proof of the proposition: Let P be a fixed indecomposable preprojective module. For any simple regular module S, let

$$E'_S\colon 0 \to P \to X'_S \to \bigoplus_{e_{SP}} S \to 0$$

be an exact sequence with X_S preprojective. Fix an embedding of S into the corresponding Prüfer module S^ω, and choose an exact sequence E_S which induces E'_S, say

$$\begin{array}{ccccccccc} E'_S\colon & 0 \to & P & \to & X'_S & \to & \bigoplus_{e_{SP}} S & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ E_S\colon & 0 \to & P & \to & X_S & \to & \bigoplus_{e_{SP}} S^\omega & \to & 0\,. \end{array}$$

Note that here we use that $\operatorname{Ext}(S^\omega, P)$ maps surjectively onto $\operatorname{Ext}(S, P)$. Finally, let E be the exact sequence

$$E\colon 0 \to P \hookrightarrow X \xrightarrow{\pi} \bigoplus_S \bigoplus_{e_{SP}} S^\omega \to 0$$

given by

$$E = (E_S)_S \in \prod_S \operatorname{Ext}\Big(\bigoplus_{e_{SP}} S^\omega, P\Big) = \operatorname{Ext}\Big(\bigoplus_S \bigoplus_{e_{SP}} S^\omega, P\Big)\,.$$

First, we show that X is torsionfree. Since P and X/P are without non-zero preinjective direct summands, the same is true for X. Assume, X has a submodule U of finite length which is simple regular, say isomorphic to T. Thus $U \subset X \xrightarrow{\pi} X/P$ maps into the T-component of the regular socle of the torsion regular module X/P. But the inverse image of the T-component of the regular socle of X/P under π is just X'_T.

Thus X'_T has a submodule isomorphic to T which is impossible, since X'_T is preprojective.

Next, we claim that X is divisible. Let T be simple regular. We want to show that $\operatorname{Ext}(T, X) = 0$, thus assume there is given an exact sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} T \to 0 .$$

We form the induced exact sequence with respect to π,

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & P & \xrightarrow{\text{id}} & P & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & T \to 0 \\ & & \downarrow \pi & & \downarrow \pi' & & \downarrow \text{id} \\ 0 \to & \bigoplus_S & \bigoplus_{e_{SP}} S^{\omega} & \to & Y' & \xrightarrow{\beta'} & T \to 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Now β' splits, thus

$$Y' = \bigoplus_S \bigoplus_{e_{SP}} S^{\omega} \oplus T .$$

Consequently, Y' has $\left(\bigoplus_{e_{TP}} T\right) \oplus T$ as submodule, and we consider the corresponding induced exact sequence

$$\begin{array}{ccccccc} 0 \to & P & \to & Y & \xrightarrow{\pi'} & Y' & \to 0 \\ & \uparrow \text{id} & & \uparrow & & \uparrow & \\ 0 \to & P & \to & Y'' & \to & \left(\bigoplus_{e_{TP}} T\right) \oplus T & \to 0 . \end{array}$$

By the previous lemma, Y'' contains a submodule isomorphic to T, thus also Y has a submodule, say U, with $U \approx T$. Now U cannot be contained in $\alpha(X)$, since we have seen that $X \approx \alpha(X)$ is torsion-free. Thus, the restriction β' of β to U is non-zero. Since $\beta' \colon U \to T$ is a map between simple regular modules, it has to be an isomorphism, thus β splits.

5.3. Theorem: *There exists a unique indecomposable torsionfree divisible module Q.*

Its endomorphism ring is a division ring.

PROOF: Let P be an indecomposable projective module of defect -1. By the last proposition, there exists a torsionfree divisible module Q with P as submodule, such that Q/P is torsion regular.

Let U be a submodule of Q of finite length and defect -1. Since Q/P is torsion regular, there exists a submodule V with $U + P \subseteq V \subseteq Q$ such that V/P is of finite length, and both V/P and Q/V are torsion regular. Now note that

$$\delta(V/U) = \delta V - \delta U = \delta P + \delta(V/P) - \delta U = -1 + 0 + 1 = 0 .$$

Assume that V/U has an indecomposable preinjective direct summand, say V'/U. Then $\delta V' = \delta(V'/U) + \delta(U) \geqslant 0$ shows that V' is a submodule of Q of finite length which is not preprojective, a contradiction to the fact that Q is torsionfree. Thus, no indecomposable direct summand of V/U is preinjective, and since $\delta(V/U) = 0$, we conclude that V/U is regular. Since both V/U and Q/V are torsion regular, also Q/U is torsion regular.

Next, we show that the endomorphism ring of Q is a division ring. $0 \neq \varphi\colon Q \to Q$ be a homomorphism. Note that $\varphi(P) \neq 0$, since otherwise the torsion module Q/P would map into the torsionfree module Q non-trivially. Since $\varphi(P)$ is a non-zero submodule of Q, its defect is < 0, thus the kernel of the restriction map $\varphi'\colon P \to \varphi(P)$ of φ has defect $\geqslant 0$, and therefore is zero. Thus, $\varphi'\colon P \to \varphi(P)$ is an isomorphism. Since $\varphi\colon Q \to Q$ is an extension of this φ', and since both Q/P and $Q/\varphi(P)$ are torsion regular, we conclude from 5.1, that φ is an isomorphism. Thus End (Q) is a division ring, and, in particular, Q is indecomposable.

Finally, we have to show that Q is unique. Assume there is given some other indecomposable torsionfree divisible module Q'. According to 4.3, we may choose a submodule U' of Q' which is isomorphic to the direct sum of copies of P such that Q'/U' is torsion regular. Let $\alpha\colon P \to Q$, $\beta\colon U' \to Q'$ be the inclusions. Let $\varphi'\colon P \to U'$, $\psi'\colon U' \to P$ be maps with $\psi'\varphi' = 1_P$: According to 5.1, we can extend φ' and ψ', and get maps $\varphi\colon Q \to Q'$, $\psi\colon Q' \to Q$ such that $\varphi\alpha = \beta\varphi'$, $\psi\beta = \alpha\psi'$. Then $\psi\varphi\colon Q \to Q$ is an extension of $\psi'\varphi' = 1_P$, and therefore an isomorphism. Thus φ maps Q isomorphically onto a direct summand of Q'. Since Q' is indecomposable, we conclude that $\varphi\colon Q \to Q'$ is an isomorphism. This finishes the proof of the theorem.

From now on, we will denote by Q a fixed indecomposable torsionfree divisible module.

5.4. LEMMA: A torsionfree divisible module is a direct sum of copies of Q.

PROOF: Fix an indecomposable projective module of defect -1. Let X be torsionfree divisible. According to 4.3, there exists a submodule U of X with X/U torsion regular, such that U is isomorphic to a direct sum of copies of P. We can embed P into Q such that Q/P is torsion regular, thus we can embed U into a direct sum V of copies of P such that V/U is torsion regular. By 5.1, we can extend the identity of U to an isomorphism $X \to V$. This proves the lemma.

Combining this with 4.7, we conclude:

THEOREM: *A divisible module is the direct sum of indecomposable preinjective modules, Prüfer modules, and copies of Q.*

5.5. THEOREM: *Any torsionfree module X can be embedded into a direct sum Y of copies of Q such that the quotient Y/X is torsion regular.*

REMARK: Given X, the number of copies of Q in a direct decomposition of such a Y is an invariant of X, and will be called the (torsionfree) *rank* of X.

PROOF: Let X be torsionfree. According to 4.2 and 4.3, there exist submodules $U' \subseteq U \subseteq X$ such that U' is a direct sum of indecomposable preprojective modules, U/U' is a direct sum of copies of a fixed projective module, and X/U is torsion regular. Since U/U' is projective, U' is a direct summand of U, thus U is a direct sum of indecomposable preprojective modules, say $U = \bigoplus_{i \in I} P_i$, with P_i indecomposable preprojective. According to 5.2, we find an embedding $P_i \hookrightarrow Y_i$ with Y_i torsionfree divisible, and torsion regular quotient Y_i/P_i. Let $\alpha\colon U = \bigoplus_{i\in I} P_i \to \bigoplus_{i \in I} Y_i = Y$ be the canonical inclusion map. According to 4.7, there exists an extension $\beta\colon X \to Y$ of α to X, since Y is divisible, and X/U is regular. Thus, in the following diagram the left square commutes

$$\begin{array}{ccccccccc}
0 & \to & U & \hookrightarrow & X & \to & X/U & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \varphi} & & \\
0 & \to & U & \xrightarrow{\alpha} & Y & \to & Y/\alpha(U) & \to & 0\,,
\end{array}$$

and there exists φ making the right square commutative. Now the kernel of β is equal to the kernel of φ. But this kernel is torsion regular, since X/U and $Y/\alpha(U)$ both are torsion regular. Thus $\ker(\beta) = 0$, since it is a torsion submodule of the torsionfree module X. Also, the cokernel of β equals the cokernel of φ, and therefore is torsion regular. Thus, we have found a monomorphism $\beta: X \to Y$ with torsion regular cokernel, such that Y is torsionfree divisible. This finishes the proof of the theorem, since 5.4 shows that $Y = \bigoplus_I Q$ for some set I.

In order to verify the remark, note that if X is embedded into Y and Y' with Y/X and Y'/X both torsion regular, and Y and Y' direct sums of copies of Q, then $Y \approx Y'$ according to 5.1. The general Krull-Remak-Schmidt-Azumaya theorem can be applied, since the endomorphism ring of Q is local, thus the number of indecomposable direct summands in any direct decomposition of $Y = \bigoplus_I Q$ is fixed.

5.6. PROPOSITION: *For any preprojective module P of finite length, the rank of P is equal to the negative of the defect $\delta(P)$ of P.*

PROOF: Take an embedding $P \subseteq Y$ with Y/P torsion regular, and $Y = \bigoplus_I Q$, for some index set I. Since P is of finite length, P is contained in some $\bigoplus_{I'} Q$ with $I' \subseteq I$ a finite subset. However, Y/P has $\bigoplus_{I \setminus I'} Q$ as direct summand, and since Q is torsionfree, and Y/P torsion, we conclude $I' = I$, thus I is finite. Let P' be a fixed indecomposable projective submodule of Q of defect -1, thus Q/P' is torsion regular. Let $U = \bigoplus_I P' \subseteq \bigoplus_I Q = Y$. Since $(P + U)/U$ is a submodule of Y/U of finite length, and Y/U is torsion regular, there exists $P + U \subseteq V \subseteq Y$ such that V/U is regular of finite length. Note that V/P is the kernel of the canonical map $Y/P \to Y/V$, thus, since both Y/P, Y/V are torsion regular, also the kernel V/P is torsion regular. We calculate defects: V/U and V/P are regular, thus $\delta(V/U) = 0 = \delta(V/P)$, thus

$$\delta(P) = \delta(P) + \delta(V/P) = \delta(V) = \delta(V/U) + \delta(U) = \delta(U),$$

and $\delta(U)$ is the product of the cardinality of I with $\delta(P') = -1$. The assertion follows, since the cardinality of I is just the rank of P, by definition of the rank.

5.7. We have seen in 5.3, that the endomorphism ring E of Q is a division ring, thus we may consider Q as E-vector space ${}_EQ$.

THEOREM: *The vector space ${}_{\mathrm{End}(Q)}Q$ is finite dimensional.*

REMARK: We will give an explicite formula for the dimension of ${}_EQ$. Namely, let $R = \bigoplus_{i=1}^{n} P_i$, with P_i indecomposable projective. Then

$$\dim {}_EQ = -\sum_{i=1}^{n} \delta(P_i)\,.$$

Note that this dimension is not invariant under Morita equivalence. This is clear, since under a categorical equivalence, the absolute dimension of an endomorphism ring over the base field is invariant, whereas the absolute dimension of the module may vary.

PROOF OF THE THEOREM: Let $R = \bigoplus_{i=1}^{n} P_i$, with P_i indecomposable projective. The elements of Q can be identified with homomorphism $R_R \to Q_R$, thus we have

$$Q \approx \operatorname{Hom}(R, Q) = \operatorname{Hom}\Big(\bigoplus_{i=1}^{n} P_i, Q\Big) \approx \bigoplus_{i=1}^{n} \operatorname{Hom}(P_i, Q)\,,$$

and these isomorphisms, and the last decomposition are those of left E-vector spaces, where E operates on Q from the left. We will prove:

If P is indecomposable projective, of defect $-d$, and if $(\varphi_i)_{1\leqslant i\leqslant d}$: $P \to \bigoplus_{i=1}^{d} Q$ is an embedding with torsion regular cokernel, then $\varphi_1, \dots, \varphi_d$ is a basis of ${}_E\operatorname{Hom}(P, Q)$.

If we denote by $\pi\colon \bigoplus_{i=1}^{d} Q \to Z$ the cokernel of $(\varphi_i)_i$, we have the following exact sequence

$$0 \to P \xrightarrow{(\varphi_i)_i} \bigoplus_{i=1}^{d} Q \xrightarrow{\ \pi\ } Z \to 0$$

First, we show that the elements φ_i are linearly independent. Thus, assume there are elements α_i in E with $\sum_{i=1}^{n} \alpha_i \varphi_i = 0$. Consider the map $(\alpha_i)_i\colon \bigoplus_{i=1}^{n} Q \to Q$, the assumption implies that it factors over π, thus there is $\alpha\colon Z \to Q$ with $(\alpha_i)_i = \alpha\pi$. However, α belongs to $\operatorname{Hom}(Z, Q)$, and $\operatorname{Hom}(Z, Q) = 0$ since Z is torsion and Q torsionfree. Thus, $\alpha_i = 0$ for all i.

Next, we have to show that the elements φ_i generate $\operatorname{Hom}(P, Q)$. Thus, let $\varphi\colon P \to Q$ be given. Since Q is torsionfree divisible, and the cokernel Z of $(\varphi_i)_i$ is torsion regular, we can extend to $\bigoplus_{i=1}^{d} Q$ along $(\varphi_i)_i$, thus there is a map $(\alpha_i)_i\colon \bigoplus_{i=1}^{d} Q \to Q$ with $(\alpha_1, \dots, \alpha_n)\cdot\begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_d\end{pmatrix} = \varphi$, thus φ is of the form $\varphi = \sum_{i=1}^{d} \alpha_i\varphi_i$ with α_i in E.

Examples: It seems to be of interest to study the division ring $E = \operatorname{End}(Q)$ further. Here are some examples.

In case the base field k is algebraically closed, we obtain $E = k(t)$, the function field in one variable. More general, in case of type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, we always obtain $E = F(t)$ with $F = \operatorname{End}(P)$ for some indecomposable projective module P, whereas in type $\tilde{A}_n$, we will get a twisted ring of the form $F(t; \varepsilon)$, with ε an automorphism of F. In base of the ring $\begin{pmatrix}\mathbb{R} & \mathbb{H}\\ 0 & \mathbb{H}\end{pmatrix}$, we obtain the function field $\mathbb{R}(t_1, t_2)$ with $t_1^2 + t_2^2 = -1$ [17].

6. Torsionfree rank 1 modules.

Again, we assume that R is a twosided indecomposable finite dimensional hereditary algebra of tame representation type. As we know, there exists a unique indecomposable torsionfree divisible module Q, and its endomorphism ring will be denoted by E. Recall that a module X is said to be torsionfree of rank 1, if X can be embedded into Q with torsion regular quotient. Examples are the preprojective modules of finite length and defect -1. The remaining ones are regular, since we will see that a torsionfree rank 1 module has to be indecomposable. The regular torsionfree rank 1 modules are the ones we are mainly interested in.

6.1. Proposition: A torsionfree rank 1 module is indecomposable, its endomorphism ring is a subring of E.

Proof: It suffices to prove the last assertion, since a subring of E cannot contain non-trivial idempotents. Now assume X is torsionfree rank 1, say with embedding $\alpha\colon X \to Q$ with torsion regular quotient. Then, any endomorphism of X can be extended uniquely to an endomorphism of Q, according to 5.1. In this way, we obtain a ring embedding $\operatorname{End}(X) \to \operatorname{End}(Q) = E$.

Proposition 2: Let X, Y be torsionfree rank 1 modules, and $\varphi\colon X \to Y$ a non-zero homomorphism. Then φ is a monomorphism, and $Y/\varphi(X)$ is torsion regular.

Proof: We may assume that X and Y are submodules of Q such that Q/X and Q/Y both are torsion regular. Given a homomorphism there is an extension $\varphi'\colon Q \to Q$, according to 4.8, and therefore also an induced homomorphism $\varphi''\colon Q/X \to Q/Y$

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Q & \to & Q/X & \to & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \to & Y & \to & Q & \to & Q/Y & \to & 0\,. \end{array}$$

If $\varphi \neq 0$, then φ' is an automorphism. Thus, φ is a monomorphism, and the cokernel of φ is isomorphic to the kernel of φ''. However, the kernel of φ'' is torsion regular, since both Q/X and Q/Y are torsion regular.

Corollary: *A torsionfree rank* 1 *module has no proper torsionfree quotient.*

Proof: Assume X is torsionfree rank 1, and U is a submodule with X/U torsionfree. According to 5.5, we can embed X/U into a direct sum of copies of Q. If $X/U \neq 0$, we therefore obtain a non-zero homomorphism $X \to Q$ with kernel containing U. By the previous proposition, this is impossible in case $U \neq 0$. Thus either $U = 0$ or $U = X$.

6.2. Given a torsionfree rank 1 module X, we will consider the different embeddings of X into Q.

Lemma: Let X_1, X_2 be submodules of Q which are torsionfree rank 1 modules. If $X_1 \cap X_2 \neq 0$, then $X_1 \cap X_2$ and $X_1 + X_2$ are torsionfree rank 1 modules.

Proof: Let U be a non-zero submodule of $X_1 \cap X_2$ of finite length. Let $U \subseteq V \subseteq Q$ with $V/U = \mathfrak{J}(Q/U)$. Then, according to 3.6 and 3.7, the submodule V is of finite length. Now $\mathfrak{J}(Q/X_i) = 0$, thus applying $\mathfrak{J}$ to the canonical epimorphism $Q/U \xrightarrow{\pi} Q/X_i$ shows that $\pi(V/U) = \pi(\mathfrak{J}(Q/U)) \subseteq \mathfrak{J}(Q/X_i) = 0$, thus $V \subseteq X_i$, for $i = 1, 2$, and therefore $V \subseteq X_1 \cap X_2$. On the other hand, Q has no proper torsionfree quotient, thus $Q/V = \mathfrak{T}(Q/V)$, and therefore Q/V is torsion regular. With Q/V and Q/X_i also X_i/V is torsion regular, and therefore also the intersection $(X_1/V) \cap (X_2/V) = (X_1 \cap X_2)/V$ and the union $(X_1/V) + (X_2/V) = (X_1 + X_2)/V$ are torsion regular.

REMARK: The assumption $X_1 \cap X_2 \neq 0$ is also necessary in order to have $X_1 + X_2$ torsionfree rank 1, since we have seen that torsionfree rank 1 modules are indecomposable. Note that it is easy to construct examples of torsionfree rank 1 modules X_1, X_2 in Q with $X_1 \cap X_2 = 0$.

6.3. A preprojective submodule of X of finite length and defect -1 will be called a *peg* in X.

LEMMA: Let X be a regular torsionfree rank 1 module, and P a preprojective module of finite length and defect -1. Then X has a peg isomorphic to P.

PROOF: The easiest way to obtain such a peg is by using 2.4. Let $\mathfrak{A}$ be the set of predecessors of P, and S^+, S^- the corresponding functors. Since X is regular, $S^-S^+X \approx X$. Also, S^+X has to have composition factors of any type. For, otherwise S^+X would decompose as a direct sum of modules of finite length, and then the same would be true for $X \approx S^-S^+X$, but X is torsionfree regular. Now S^+P is simple projective, thus there is a non-zero homomorphism $\varphi: S^+P \to S^+X$. Applying S^-, we obtain a non-zero homomorphism $S^-\varphi: P \approx S^-S^+P \to S^-S^+X \approx X$. Let U be the image of $S^-\varphi$ in X. Then $\delta(U) \leqslant -1$, since X is regular, thus the kernel of $S^-\varphi$ has defect $\geqslant 0$ and therefore has to be 0. This shows that $S^-\varphi$ is a monomorphism and therefore U is a peg isomorphic to P.

If we fix a peg P in Q, and consider only the torsionfree rank 1 modules X in Q which contain P, then we obtain a complete lattice with respect to intersection and summation, and this lattice is isomorphic to the lattice of torsion regular modules of Q/P. In this way, we obtain representatives of all isomorphism types of regular torsionfree rank 1 modules. For, given a regular torsionfree rank 1 module, we can choose a peg U in X which is isomorphic to P, and we can extend the monomorphism $U \approx P \subseteq Q$ to an embedding of X into Q, and then the image will contain P. Of course, different pegs in X give rise to different submodules of Q.

We also note that given two pegs U_1, U_2 in a torsionfree rank 1 module X, there exists a peg U in X with $U_1 + U_2 \subseteq U$. For, let U be the submodule of X with $U_1 + U_2 \subseteq U$ such that $U/(U_1 + U_2) = \mathfrak{J}(X/(U_1 + U_2))$. Then, according to 3.6 and 3.7, we know that U is of finite length. But X/U and X/U_1 both are torsion regular, thus U/U_1 is regular, and therefore $\delta(U) = \delta(U_1) = -1$.

6.4. In order to be able to investigate torsionfree rank one modules further, we have to consider Q in more detail.

Given a torsion regular module Y, let $Y = \bigoplus_{t \in T} Y_t$ with Y_t in $\mathfrak{T}_t$, and denote by $s_t(Y)$ the regular length of the regular socle of Y_t. In case Y_t is in addition divisible, $s_t(Y)$ is just the number of Prüfer modules in a direct sum decomposition of Y_t. As a consequence, given Y torsion regular and divisible, and U a regular submodule of finite length, then $s_t(Y) = s_t(Y/U)$ for all t. It follows that for P a peg of Q, the numbers

$$e_t = s_t(Q/P)\,, \qquad t \in T\,,$$

are actually independent of P. For, given two pegs P_1, P_2 of Q, there exists a peg P with $P_1 + P_2 \subseteq P$, and then

$$s_t(Q/P_1) = s_t(Q/P) = s_t(Q/P_2)\,.$$

Recall that for S simple regular in $\mathfrak{T}_t$, we denote by n_t the smallest natural number with $A^{n_t}S \approx S$. Let $\mathfrak{S}_t$ be the set of simple regular modules in $\mathfrak{T}_t$.

Lemma: Let P be preprojective of finite length, and $t \in T$. Then

$$e_t = \frac{-1}{\delta(P)} \sum_{S \in \mathfrak{S}_t} \dim \operatorname{Ext}(S, P)_{\operatorname{End}(S)}\,.$$

If S is a fixed simple regular module in $\mathfrak{T}_t$, then

$$e_t = \frac{-1}{\delta(P)} \dim_{\operatorname{End}(S^{n_t})} \operatorname{Hom}(P, S^{n_t})\,.$$

Proof: In order to prove the first equality, we note that both $\delta(P)$ and $e_{SP} = \dim \operatorname{Ext}(S, P)_{\operatorname{End}(S)}$ are additive on direct sums in P, thus we may suppose that P is indecomposable. However, then we get from 5.2 that there exists an embedding $P \subseteq X$ with X torsion-free divisible such that $s_t(X/P) = \sum_{S \in \mathfrak{S}_t} e_{SP}$. According to 5.6, we know that X is the direct sum of $-\delta(P)$ copies of Q, thus $s_t(X/P) = -\delta(P)\, e_t$. This shows the first equality.

We can rewrite this formula as

$$e_t = \sum_{i=0}^{t^-} \dim_{\operatorname{End}(A^iS)} \operatorname{Hom}(P, A^iS)\,,$$

using the fact that $e_{SP} = \dim_{\operatorname{End}(AS)} \operatorname{Hom}(P, AS)$, according to 1.A.

However, the $A^i S$, $0 \leqslant i < n_t$, are just the regular composition factors of S^{n_t}. Since P is preprojective, the functor Hom $(P, -)$ is exact on exact sequences of regular modules, thus

$$\dim_k \operatorname{Hom}(P, S^{n_t}) = \sum_{i=0}^{n_t - 1} \dim_k \operatorname{Hom}(P, A^i S) .$$

The second assertion now follows from the fact that End $(S^{n_t}) \approx$ $\approx$ End $(A^i S)$ for all i.

Note that we have listed e_t, for t non-homogeneous, in the table in section 1.D; however, the remaining e_t may be non-trivial, also.

EXAMPLE: Let us consider an example in more detail. Consider the bimodule ${}_{\mathbb{C}}M_{\mathbb{C}} = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}_{\mathbb{C}} = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\overline{\mathbb{C}}}$, where the right action of $\mathbb{C}$ on the second summand is given by complex conjugation: if we denote $x = (1, 0)$, $y = (0, 1)$, then for $c \in \mathbb{C}$, $xc = cx$, $yc = \bar{c}y$. Representations of the ring $R = \begin{pmatrix} \mathbb{C} & M \\ 0 & \mathbb{C} \end{pmatrix}$ can be given in the form (U, V, α, β), where U, V are two $\mathbb{C}$-vector spaces, $\alpha\colon U \to V$ is $\mathbb{C}$-linear, and $\beta\colon U \to V$ is $\mathbb{C}$-anti-linear. In [16], the indecomposable modules of finite length have been determined, and we recall the description of the simple regular ones. Let $\mathbb{R}_+ = \{r \in \mathbb{R}, r > 0\}$, $\mathbb{R}_- = \{r \in \mathbb{R} | r < 0\}$, and $\mathbb{C}_+ = \mathbb{R} \times \mathbb{R}_+$. For $a \in \mathbb{R}_+ \cup \{0\}$, let $S_a = (\mathbb{C}, \mathbb{C}, 1, \cdot a)$, and let $S_\infty = (\mathbb{C}, \mathbb{C}, 0, 1)$. Also, for $c \in \mathbb{R}_- \cup \mathbb{C}_+$, let

$$S_c = \left(\mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, 1, \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}\right).$$

Then these modules S_c with $c \in \mathbb{R} \cup \mathbb{C}_+ \cup \{\infty\}$, form a complete set of pairwise non-isomorphic simple regular modules, and their endomorphism rings are as follows: End $(S_0) =$ End $(S_\infty) = \mathbb{C}$, End $(S_a) = \mathbb{R}$ for $a \in \mathbb{R}_+$, End $(S_a) = \mathbb{H}$ for $a \in \mathbb{R}_-$, and End $(S_c) = \mathbb{C}$ for $c \in \mathbb{C}_+$.

It is now easy to calculate the possible numbers e_t. Note that the indecomposable projective module $P = (0, \mathbb{C}, 0, 0)$ has defect -1, thus $e_t = \dim_{\mathrm{End}(S_t)} \operatorname{Hom}(P, S_t)$, and therefore $e_t = 1$ for $t \in \mathbb{R}_- \cup \{0\} \cup \{\infty\}$, and $e_t = 2$ for $t \in \mathbb{R}_+ \cup \mathbb{C}_+$.

In addition, we will consider the module $Q = (E, E, 1, \cdot x)$, where $E = \mathbb{C}(x, {}^-)$ is the twisted function field in one variable: let $\mathbb{C}[x, {}^-]$ be the twisted polynomial ring with respect to complex conjugation, its elements are the polynomials $\sum x^i c_i$ with coefficients $c_i \in \mathbb{C}$, with ordinary addition, and with multiplication induced by $xc = \bar{c}x$. The quotient field of $\mathbb{C}[x, {}^-]$ is $E = \mathbb{C}(x, {}^-)$, its elements are of the form $f^{-1}g$, where $f, g \in \mathbb{C}[x, {}^-]$ and $f \neq 0$. If $U_{\mathbb{C}}$ is a subspace of $E_{\mathbb{C}}$, let

$X(U)$ be the submodule $X(U) = (U, U + Ux, 1, \cdot x)$ of Q. We embed $\mathbb{C}$ into E as the set of constant polynomials, and therefore $P = (0, \mathbb{C}, 0, 0)$ is a submodule of Q. It is clear that Q is torsionfree and that Q/P is torsion, thus P is a peg in Q, and Q is torsionfree rank one. We claim that Q is the unique indecomposable torsionfree divisible module. This follows from the fact that Q/P is divisible and that the regular socle of $(Q/P)_t$, for any $t \in \mathbb{R} \cup \mathbb{C}_+ \cup \{\infty\}$ is of regular length $\geqslant e_t$.

We want to describe submodules $P \subseteq X_t \subseteq Q$ with X_t/P being the direct sum of e_t copies of S_t, thus X_t/P has to be the regular socle of $(Q/P)_t$. The modules X_t will be of the form $X_t = X(U_t)$ with U_t a $\mathbb{C}$-subspace of $E_{\mathbb{C}}$, and we will use the symbol $U_t = \langle ... \rangle$ in order to describe a suitable basis of U_t.

For $a \in \mathbb{R}_+$, let $U_a = \langle (x-a)^{-1}, (x+a)^{-1} i \rangle$. Note that we have the following two equalities

$$(x-a)^{-1}x = (x-a)^{-1}a + 1\,, \qquad (x+a)^{-1}ix = (x+a)^{-1}ia - i\,,$$

thus $X(U_t)/P \approx S_a \oplus S_a$.

For $t = 0$, let $U_0 = \langle x^{-1} \rangle$, for $t = \infty$, let $U_\infty = \langle 1 \rangle$. Then, clearly $X(U_0)/P \approx S_0$, and $X(U_\infty)/P \approx S_\infty$.

For $c \in \mathbb{R}_-$, let $U_c = \langle (x^2 - c)^{-1}, (x^2 - c)^{-1}x \rangle$. Then the equality

$$(x^2 - c)^{-1}x^2 = (x^2 - c)^{-1}c + 1$$

show that $X(U_c)/P \approx S_c$.

Finally, for $c \in \mathbb{C}_+$, let

$$U_c = \langle (x^2 - c)^{-1}, (x^2 - c)^{-1}x, (x^2 - \bar{c})^{-1}xi, -(x^2 - \bar{c})^{-1}i\bar{c} \rangle\,.$$

We have to use besides the previous equality, the following two equalities

$$(x^2 - \bar{c})^{-1}xix \quad = -(x^2 - \bar{c})^{-1}i\bar{c} - i\bar{c}\,,$$

$$-(x^2 - \bar{c})^{-1}i\bar{c}x = (x^2 - \bar{c})^{-1}xic\,,$$

and therefore, $X(U_c)/P \approx S_c \oplus S_c$.

We call Q *multiplicity-free*, in case $e_t = 1$ for all $t \in T$. According to the calculation of e_t for t non-homogeneous, we see that in case of type $\tilde{C}_n$, $\widetilde{BD}_n$, $\tilde{F}_{42}$ and $\tilde{G}_{22}$, the module Q can never be multiplicity-free. In the other cases, it will depend on the bimodule which determines the homogeneous component of $\mathfrak{T}$.

Proposition: If R is of type $\widetilde{CD}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_{41}$ or $\tilde{G}_{21}$, and if $R/\text{rad}\, R$ is commutative, then Q is multiplicity-free. If R is a k-algebra of type $\tilde{A}_{12}$ or $\tilde{A}_n$, and $R/\mathrm{rad}\, R$ is a product of copies of k, then Q is multiplicity-free.

Proof: For t non-homogeneous, in all these cases we know that $e_t = 1$. Thus, consider t homogeneous. There exists a dimodule ${}_FM_G$ and a full and exact embedding Γ from the category of representations of ${}_FM_G$ into $\mathfrak{M}_R$ such that any simple regular R-module in $\mathfrak{T}_t$ with t homogeneous, is of the form (U, V, φ) with $(U, V, \varphi\colon U \otimes \otimes M \to G)$ a simple regular representation of ${}_FM_G$. According to 1.D, the bimodule ${}_FM_G$ listed in the tables of [14] satisfies

$$ {}_E\mathrm{Hom}\,(P, (U, V, \varphi))_F \approx {}_EU_F\,, $$

where P is a fixed distinctive projective R-module with endomorphism ring F, and $E = \mathrm{End}\,(U, V, \varphi) = \mathrm{End}\,((U, V, \varphi))$. Thus, it suffices to check that $\dim {}_EU = 1$ for any simple regular representation (U, V, φ) of ${}_FM_G$ with endomorphism ring E.

We claim that in all cases mentioned in the proposition, the bimodule ${}_FM_G$ is actually of the form $({}_FF_F)^2$. This is clear for $\tilde{D}_n$, $\tilde{E}_n$, and, under the additional assumption $k = F$, also for $\tilde{A}_{12}$ and $\tilde{A}_n$. In the case $\widetilde{CD}_n$, $\tilde{F}_{41}$, and $\tilde{G}_2$, there exists a division ring F' containing F, such that ${}_FM_G$ is of the form ${}_FF'_F$ or ${}_F(F'/F)_F$. Thus, if we assume, as we do, that F' is commutative, then again the bimodule is of the form $({}_FF_F)^2$.

It remains to determine the simple regular representations of $({}_FF_F)^2$, thus we consider just simple regular Kronecker modules over the commutative field F. Besides the representation

$$ F_F \overset{0}{\underset{\mathrm{id}}{\rightrightarrows}} F_F\,, $$

with endomorphism ring F, the simple regular Kronecker modules are of the form

$$ F(x)_F \overset{\mathrm{id}}{\underset{x}{\rightrightarrows}} F(x)_F\,, $$

where $F(x)$ is a finite extension field of F with primitive element x. The endomorphism ring of this Kronecker module is just $F(x)$. Thus, we see thar for any simple regular representation (U, V, φ) of $({}_FF_F)^2$ with endomorphism ring E, we have $\dim {}_EU = 1$. This finishes the proof.

Corollary: *If the base field is algebraically closed, then Q is multiplicity-free.*

Proof: Under the assumption that the case field is algebraically closed, the only possible types are $\tilde{A}_{12}$, $\tilde{A}_n$, $\tilde{D}_n$ and $\tilde{E}_n$, and $R/\text{rad}\, R$ is a product of copies of the base field.

6.5. We will need conditions in order to decide when two torsionfree rank one modules X_1 and X_2 are isomorphic. If X_1, X_2 are contained in Q and contain the peg P, we will compare the quotient modules X_1/P and X_2/P. These are torsion regular modules, and we will use the following definition: the torsion regular modules Y_1, Y_2 will be called *equivalent*, provided there exists submodules $V_1 \subseteq Y_1$, $V_2 \subseteq Y_2$ of finite length such that $\dim V_1 = \dim V_2$, and $Y_1/V_1 \approx Y_2/V_2$.

Lemma: Let X be a torsionfree rank one module with two pegs P_1, P_2, with $P_1 \approx P_2$, then X/P_1 and X/P_2 are equivalent torsion regular modules.

Proof: Choose a peg U of X with $P_1 + P_2 \subseteq U$. Then

$$\dim U/P_1 = \dim U - \dim P_1 = \dim U - \dim P_2 = \dim U/P_2 .$$

Thus, the submodules U/P_1 of X/P_1 and U/P_2 of X/P_2 show that X/P_1 and X/P_2 are equivalent.

The converse is true in case Q is multiplicityfree. More general, we will introduce the notion of a clean torsionfree rank one module. Let $Z = \bigoplus_{t \in T} Z_t$ be a torsion regular and divisible module, with Z_t being the direct sum of e_t Prüfer modules in $\mathfrak{T}_t$ (where $e_t \in \mathbb{N}$). Then a regular submodule Y of Z is called *clean* provided $Y = \bigoplus_{t \in T} Y_t$, with Y_t being the direct sum of e_t indecomposable modules of equal regular length h_t, for some $h_t \in \mathbb{N}_0 \cup \{\omega\}$. It is clear that there is a bijection between clean submodules of Z, and functions $h\colon T \to \mathbb{N}_0 \cup \{\omega\}$. Such functions will be called *height functions*.

A peg P of a torsionfree rank one module X will be said to be a *clean peg* provided X/P is a clean torsion regular submodule of Q/P. Note that in case Q is multiplicity-free, then any peg of a torsionfree rank one module is clean. In particular, this is true over an algebraically closed base field.

Theorem: *Let X_1, X_2 be torsionfree rank one modules, with pegs $P_1 \subseteq X_1$, $P_2 \subseteq X_2$ such that $P_1 \approx P_2$. Assume one of the modules X_1, X_2 has a clean peg.*

Then $X_1 \approx X_2$ if and only if X_1/P_1 and X_2/P_2 are equivalent torsion regular modules.

Proof: It remains to be seen that for X_1/P_1 equivalent to X_2/P_2, we have $X_1 \approx X_2$. Choose modules $P_1 \subseteq U_1 \subseteq X_1$, $P_2 \subseteq U_2 \subseteq X_2$ of finite length such that $\dim U_1/P_1 = \dim U_2/P_2$ and $X_1/U_1 \approx X_2/U_2$. Let, without loss of generality, P be a peg of X_1 such that X_1/P is a clean torsion regular submodule of Q/P. The submodule $(U_1+P)/P$ of finite length of the clean submodule X_1/P of Q/P can be embedded into a clean submodule U_1'/P of finite length of Q/P such that $U_1+P \subseteq U_1' \subseteq X_1$, and then it is clear that X_1/U_1' is a clean submodule of Q/U_1'. Under a fixed isomorphism $\alpha: X_1/U_1 \to X_2/U_2$, the submodule U_1'/U_1 will be mapped onto some submodule $\alpha(U_1'/U_1) = U_2'/U_2$ with $U_2 \subseteq U_2' \subseteq X_2$. Then

$$\begin{aligned} \dim U_1' &= \dim P_1 + \dim U_1/P_1 + \dim U_1'/U_1 \\ &= \dim P_2 + \dim U_2/P_2 + \dim U_2'/U_2 = \dim U_2', \end{aligned}$$

thus $U_1' \approx U_2'$. Now X_1 is the unique extension inside Q of U_1' by the clean torsion regular module X_1/U_1'. Since $U_1' \approx U_2'$ and $X_1/U_1' \approx X_2/U_2'$, it follows that also $X_1 \approx X_2$. For, let α be an automorphism with $\alpha(U_1') = U_2'$, and consider the t-component $(X_1/U_1')_t$. If $(X_1/U_1')_t$ is the direct sum of e_t indecomposable modules of regular length h_t, then α has to map $(X_1/U_1')_t$ onto the unique submodule of Q/U_2' which is the direct sum of e_t indecomposable modules of regular length h_t, and this is just the t-component $(X_2/U_2')_t$ of X_2/U_2'. Thus, $\alpha(X_1) = X_2$.

Remark: Note that this theorem gives a complete classification of torsionfree rank one modules having a fixed clean peg P. The isomorphism classes of such modules correspond bijectively to equivalence classes of height functions, which can be described completely combinatorially. Call two height functions $h, h': T \to \mathbb{N}_0$ equivalent, provided the following three conditions are satisfied:

(i) $h(t) = \infty$ iff $h'(t) = \infty$,

(ii) the set $A = \{t \mid h(t) \neq h'(t)\}$ is finite, and

(iii) $\sum\limits_{\substack{S \in \mathfrak{S}_t \\ t \in A}} e_{SP} \dim S^{h(t)} = \sum\limits_{\substack{S \in \mathfrak{S}_t \\ t \in A}} e_{SP} \dim S^{h'(t)}$.

It is also clear that one can similarly decide when two torsionfree one rank modules with different clean pegs are isomorphic.

6.6. Proposition: Let P be a clean peg of a torsionfree rank one module X. Let $P \subseteq X' \subseteq X$ be the submodule with X'/P being the divisible part of X/P. Let $\varphi: P \to X$ be a homomorphism. Then φ extends to an endomorphism of X if and only if $\varphi(P) \subseteq X'$.

For the first part of the proof, we need the following auxiliary result:

LEMMA: Let $t \in T$. Let $U \subseteq V$ be two preprojective modules of finite length with defect -1, such that $V/U \approx \bigoplus_{S \in \mathfrak{S}_t} \bigoplus^{e_{SU}} S^n$, for some n. Then, for any homomorphism $\varphi\colon V \to S^n$, with $S \in \mathfrak{S}_t$, we have $\varphi(U) = 0$.

PROOF: By induction on n. Let $n = 1$. By assumption, there are given e_{SU} linearly independent elements $\varphi_1^S, \ldots, \varphi_{e_{SU}}^S$ in ${}_{\mathrm{End}(S)}\mathrm{Hom}(V, S)$, for any $S \in \mathfrak{S}_t$, thus $\dim_{\mathrm{End}(S)}\mathrm{Hom}(V, S) \leqslant e_{SU}$. However, according to Lemma 6.4, we have both

$$\sum_{S \in \mathfrak{S}_t} e_{SU} = e_t\,,$$

and

$$\sum_{S \in \mathfrak{S}_t} \dim_{\mathrm{End}(S)}\mathrm{Hom}(V, S) = \sum_{S \in \mathfrak{S}_t} \dim_{\mathrm{End}(AS)}\mathrm{Hom}(V, AS) = \sum_{S \in \mathfrak{S}_t} e_{SV} = e_t\,,$$

and therefore, for all $S \in \mathfrak{S}_t$, $\dim_{\mathrm{End}(S)}\mathrm{Hom}(V, S) = e_{SU}$. Thus, the elements $\varphi_1^S, \ldots, \varphi_{e_{SU}}^S$ form a basis of ${}_{\mathrm{End}(S)}\mathrm{Hom}(V, S)$. Thus, given $\varphi\colon V \to S$, there are endomorphisms α_i of S with $\varphi = \sum \alpha_i \varphi_i^S$, and thus the kernel of φ contains the intersection of the kernels of the φ_i^S, thus it contains U.

Now assume we know the assertion for $n-1$. Let U' be the inverse image of $\bigoplus_{S \in \mathfrak{S}_t} \bigoplus^{e_{SU}} S$ under the given epimorphism $V \to \bigoplus_{S \in \mathfrak{S}_t} \bigoplus^{e_{SU}} S^n$, thus $V/U' \approx \bigoplus_{S \in \mathfrak{S}_t} \bigoplus^{e_{SU}} (A^{-1}S)^{n-1}$. There are given e_{SU} linearly independent elements in ${}_{\mathrm{End}(S)}\mathrm{Hom}(U', S)$, thus $e_{SU} \leqslant \dim_{\mathrm{End}(S)}\mathrm{Hom}(U', S) = e_{A^{-1}S, U'}$. In order to see that we have equality, we argue as in the previous case: According to 6.4, we have both

$$\sum_{S \in \mathfrak{S}_t} e_{SU} = e_t \qquad \text{and} \qquad \sum_{S \in \mathfrak{S}_t} e_{A^{-1}S, U'} = e_t\,.$$

Now assume there is given $\varphi\colon V \to S^n$. Let $\pi\colon S^n \to S^n/S$ be the canonical projection, thus we can apply induction to the map

$$\pi\varphi\colon V \to S^n/S = (A^{-1})\,S)^{n-1}\,,$$

and conclude that $\pi\varphi(U') = 0$. If we denote by φ' the restriction of φ to U', then we see that φ' maps nto the kernel S of π. However, applying the case $n = 1$ to the inclusion $U \subseteq U'$ with $U'/U \approx \bigoplus_{S \in \mathfrak{S}_t} \bigoplus^{e_{SU}} S$, and the map $\varphi'\colon U' \to S$, we see that $\varphi(U) = \varphi'(U) = 0$.

PROOF OF THE PROPOSITION: First, let α be an endomorphism of X. We want to show that $\alpha(P) \subseteq X'$.

Thus, let X^t be the submodule with $P \subseteq X^t \subseteq X$ such that $X^t/P = (X/P)_t$, for any $t \in T$. Given α, consider the map α_t given by

$$X^t \subseteq X \xrightarrow{\alpha} X/\sum_{t' \neq t} X^{t'} \approx (X/P)_t .$$

If $h_t(X/P)$ is finite, we can apply the lemma to α_t, and obtain $\alpha_t(P) = 0$, and thus $\alpha(P) \subseteq \sum_{t' \neq t} X^{t'}$. Denote by I the set $\{t \in T | h_t \text{ finite}\}$. Then we get that

$$\alpha(P) \subseteq \bigcap_{t \in I} \sum_{t' \neq t} X^{t'} = X' .$$

This proves the first part of the proposition.

Conversely, assume $\varphi \colon P \to X$ maps into X', and we may assume $\varphi \neq 0$. If we embed X into Q, there exists an automorphism α of Q with $\alpha|P = \varphi$. We want to show that $\alpha(X) \subseteq X$. Since P and $\alpha(P)$ are contained in X', we can choose a peg U of X' with $P + \alpha(P) \subseteq U$. Now consider $(\alpha(X) + U)/P$. This is an extension of U/P by $(\alpha(X) + U)/U$. For the first module we have $(U/P)_t = 0$ for $t \in I = \{t \in T | h_t \text{ finite}\}$, since U/P is a submodule of X'/P. The second module

$$(\alpha(X) + U)/U \approx \alpha(X)/(\alpha(X) \cap U)$$

is an epimorphic image of $\alpha(X)/\alpha(P) \approx X/P$, thus we see that for $t \in I$, the indecomposable direct summands of $((\alpha(X) + U)/U)_t$ have a length $\leqslant h_t$, and then the same is also true for the indecomposable direct summands of $((\alpha(X) + U)/P)_t$. This shows that for $t \in I$,

$$((\alpha(X) + U)/P)_t \subseteq (X/P)_t .$$

But for $t \notin I$, we have the similar inclusion, since $(X/P)_t = (Q/P)_t$ in this case. Thus, we conclude that $\alpha(X) + U \subseteq X$, and, in particular, $\alpha(X) \subseteq X$.

COROLLARY 1: *Let P be a clean peg of a torsionfree rank one module. Let $P \subseteq X' \subseteq X$ be the submodule with X'/P being the divisible part of X/P. Then, for any endomorphism α of X, we have $\alpha(X') \subseteq X'$, and the restriction map* End $(X) \to$ End (X') *is an isomorphism.*

PROOF: We may consider both End (X) and End (X') as subrings of End (Q) by choosing a fixed embedding of X into Q and extending the endomorphism of X and X' to endomorphisms of Q. Then, we have to show that $\mathrm{End}(X) = \mathrm{End}(X')$.

Let α be a non-zero element of $\mathrm{End}(X)$. By the previous proposition, $\alpha(P) \subseteq X'$. Applying now the proposition to the clean peg P of X', we see that $\alpha|P$ is the restriction of an element $\beta \in \mathrm{End}(X')$, but then $\alpha = \beta$ since they coincide on P. This shows $\mathrm{End}(X) \subseteq \subseteq \mathrm{End}(X')$, and similarly, one gets also the other inclusion.

COROLLARY 2: *Let X be torsionfree rank one, with endomorphism ring D. Let P be a clean peg in X and assume, X/P is divisible. Then the D-module ${}_D\mathrm{Hom}(P, X)$ is free of rank one, generated by the inclusion map $P \hookrightarrow X$.*

PROOF: Let $u: P \to X$ be the inclusion map. Define

$$ {}_D D \to {}_D\mathrm{Hom}(P, X) \qquad \text{by } \alpha \mapsto \alpha u\,. $$

Then this is a monomorphism, since with α also αu is a monomorphism, and therefore non-zero. By the proposition, this map is also surjective.

6.7. THEOREM: *Let X be a torsionfree rank one module with endomorphism ring D. Assume, P is a clean peg in X. If X/P is reduced, the D is a division ring which is a finite dimensional k-algebra. If X/P is not reduced, then* $\mathrm{End}(X)$ *is a left order in E.*

PROOF: First, assume that X/P is reduced. Let α be a non-zero endomorphism of X. Then Proposition 6.6 shows that $\alpha(P) \subseteq P$. Thus, the restriction map $\mathrm{End}(X) \to \mathrm{End}(P)$ exists, and it is clear that this is a monomorphism. Thus $\mathrm{End}(X)$ is a subring of the finite dimensional k-algebra $\mathrm{End}(P)$ which is a division ring, and therefore itself a finite dimensional k-algebra and a division ring.

For the converse, we fix an embedding of X into Q, and identify $\mathrm{End}(X)$ with the subring $\{\varphi \in E \mid \varphi(X) \subseteq X\}$ of the endomorphism ring $E = \mathrm{End}(Q)$. By assumption, X/P is not reduced, thus there is $t_0 \in T$ with $X^{t_0}/P = (X/P)_{t_0} = (Q/P)_{t_0}$. Let S be simple regular in $\mathfrak{T}_{t_0}$, and assume the dimension type of $S^{n_{t_0}}$ is $\dim S^{n_{t_0}} = s_0 \cdot \boldsymbol{h}$. Also, we assume that for $t \in T$, the indecomposable direct summands of $(X/P)_t$ are of regular length h_t.

Now, let α be a non-zero element of E. Choose a peg U of Q containing $P + \alpha(P)$. Note that we may assume that $\dim U/P$ is a multiple of $s_0 \cdot \boldsymbol{h}$. For, any indecomposable regular submodule of Q/P is of the form S'^m for some simple regular module S' in some $\mathfrak{T}_{t'}$, and if we choose some natural number m' with $m \leqslant m' s_0 \cdot n_{t'}$, then $S'^m \subseteq \subseteq S'^{m' s_0 \cdot n_{t'}}$, and $\dim S'^{m' s_0 \cdot n_{t'}}$ is obviously a multiple of $s_0 \cdot \boldsymbol{h}$. Thus, we can replace the indecomposable direct summands of U/P by some larger regular submodules of Q/P, and, in this way, we obtain a larger peg $U' \supseteq U$ such that $\dim U'/P$ is a multiple of $s_0 \cdot \boldsymbol{h}$.

Since $\dim U/P$ is a multiple of $s_0 \cdot \boldsymbol{h}$, there is some peg V of X with $P \subseteq V \subseteq X^{t_0}$ such that $\dim V/P = \dim U/P$. Then also $\dim V = \dim U$, and therefore $V \approx U$. Thus, there exists an automorphism $\beta: Q \to Q$ with $\beta(U) = V$. We claim that $\beta(X) \subseteq X$, and $\beta\alpha(X) \subseteq X$.

In order to show that $\beta(X) \subseteq X$, consider $\beta(X + U)/P$. This is an extension of V/P by $\beta(X + U)/V$. Now β induces an isomorphism from $(X + U)/U$ onto $\beta(X + U)/V$, and $(X + U)/U \approx X/(X \cap U)$ is an epimorphic image of X/P. Thus, for $t \in T$, with $t \neq t_0$, it follows from $(V/P)_t = 0$ that

$$\big(\beta(X + U)/P\big)_t \approx \big(\beta(X + U)/V\big)_t \approx \big(X/(X \cap U)\big)_t$$

is an epimorphic image of $(X/P)_t$, and therefore the indecomposable direct summands of $\big(\beta(X + U)/P\big)_t$ have regular length $\leqslant h_t$, consequently, $\big(\beta(X + U)/P\big)_t \subseteq (X/P)_t$. But the last inclusion is trivially valid also for $t = t_0$, since in this case $(X/P)_{t_0} = (Q/P)_{t_0}$. Thus we conclude that $\beta(X + U) \subseteq X$, in particular, $\beta(X) \subseteq X$.

In order to show that $\beta\alpha(X) \subseteq X$, consider $\big(\beta\alpha(X) + V\big)/P$. This is an extension of V/P by $\big(\beta\alpha(X) + V\big)/V \approx \beta\alpha(X)/\big(\beta\alpha(X) \cap V\big)$. The last module is an epimorphic image of $\beta\alpha(X)/\beta\alpha(P) \approx X/P$, since

$$\beta\alpha(P) \subseteq \beta\alpha(X) \cap \beta(U) = \beta\alpha(X) \cap V\,.$$

Then, again, for $t \in T$ with $t \neq t_0$, it follows from $(V/P)_t = 0$ that $\big((\beta\alpha(X) + V)/P\big)_t$ is an epimorphic image of $(X/P)_t$, and therefore the indecomposable direct summands of $\big((\beta\alpha(X) + V)/P\big)_t$ are of regular length $\leqslant h_t$, thus $\big((\beta\alpha(X) + V)/P\big)_t \subseteq (X/P)_t$. Again, we use that the last inclusion is also valid for $t = t_0$, since $(X/P)_{t_0} = (Q/P)_{t_0}$, thus $\beta\alpha(X) + V \subseteq X$, and therefore $\beta\alpha(X) \subseteq X$.

Thus, we have shown that $\alpha = \beta^{-1}(\beta\alpha)$ with β and $\beta\alpha$ two elements of E satisfying $\beta(X) \subseteq X$, and $\beta\alpha(X) \subseteq X$, and therefore belonging to $\operatorname{End}(X) = \{\varphi \in E \mid \varphi(X) \subseteq X\}$. This shows that $\operatorname{End}(X)$ is a left order in E.

6.8. THEOREM: *Let $t \in T$ be a fixed element. Let P be a peg of Q, and let X be torsionfree rank one module with $P \subseteq X \subseteq Q$ such that $(X/P)_t = 0$, and $(X/P)_{t'} = (Q/P)_{t'}$ for all $t' \neq t$. We may consider* $\operatorname{End}(X)$ *as a subring of* $\operatorname{End}(Q)$. *Then, the canonical map* $\operatorname{End}(X) \to \operatorname{End}(Q/X)$ *is an embedding and is the completion with respect to the powers of the radical of* $\operatorname{End}(X)$.

Proof: Let $E = \operatorname{End}(X) \subseteq \operatorname{End}(Q)$. First, we note that the canonical map $E \to \operatorname{End}(Q/X)$ is a monomorphism. For, if $\alpha \in E$, then the induced map in $\operatorname{End}(Q/X)$ is zero if and only if $\alpha(Q) \subseteq X$, but this is possible only for $\alpha = 0$.

Let U_i be the submodule with $P \subseteq U_i \subseteq Q$ such that U_i/P is the i-the regular socle of $(Q/P)_t$, and let $X_i = X + U_i$, for $i \in \mathrm{N}_0$. Note that $X = X_0$. According to the Corollary 1 of 6.6, the endomorphism rings $\operatorname{End}(X_i)$, considered as subrings of $\operatorname{End}(Q)$, all coincide with E. In particular, for any $\alpha \in E$, we have $\alpha(X_i) \subseteq X_i$. Let $\pi_i: X_i \to X_i/X$ be the projection.

We claim that for all i, the map $D \to \operatorname{End}(X_i/X)$ with $\alpha \mapsto \bar{\alpha}$ such that $\bar{\alpha}\pi_i = \pi_i\alpha$, is surjective. Now U_i is a clean peg of X_i with $X_i/U_i \approx X/P$ divisible, thus according to Corollary 2 of 6.6, the left D-module ${}_D\operatorname{Hom}(U_i, X_i)$ is free with generator the inclusion $u_i: U_i \to \to X_i$. The exact sequence

$$0 \to X \to X_i \xrightarrow{\pi} X_i/X \to 0$$

gives rise to an epimorphism $\operatorname{Hom}(U_i, X_i) \to \operatorname{Hom}(U_i, X_i/X)$, since $\operatorname{Ext}(U_i, X) = 0$. Note that this is a left D-module homomorphism, thus $\operatorname{Hom}(U_i, X_i/X)$ as a left D-module is generated by the image $\pi_i u_i$ of u_i in $\operatorname{Hom}(U_i, X_i/X)$. Thus, any $\varphi: U_i \to X_i/X$ is of the form $\varphi = \bar{\alpha}\pi_i u_i = \pi_i \alpha u_i$ with $\alpha \in D$. Now, let β be an element of $\operatorname{End}(X_i/X)$. Then $\beta\pi_i u_i$ belongs to $\operatorname{Hom}(U_i, X_i/X)$, thus there exists $\alpha \in D$ with $\beta\pi_i u_i = \pi_i\alpha u_i$. The map $\beta\pi_i - \pi_i\alpha: X_i \to X_i/X$ vanishes on U_i, thus it induces a map $X_i/U_i \to X_i/X$. However, X_i/U_i and X_i/X are torsion regular modules, X_i/X belongs to $\mathfrak{T}_t$, and $(X_i/U_i)_t = 0$, thus the only homomorphism $X_i/U_i \to X_i/X$ is the zero map. This shows that $\beta\pi_i - \pi_i\alpha = 0$, thus $\beta = \bar{\alpha}$.

Next, let I_i be the kernel of the map $D \to \operatorname{End}(X_i/X)$, let $I = I_1$. We claim that I is the Jacobson radical of D. On the one hand, $D/I \approx \operatorname{End}(X_1/X)$ is a finite dimensional semi-simple algebra, thus $\operatorname{rad} D \subseteq I$. In order to see the converse, let $\alpha \in I$, thus $\alpha(X_1) \subseteq X$. Since $\bar{\alpha}$ is an endomorphism of Q/X which vanishes on the regular socle X_1/X of Q/X, it maps the i-th regular socle into the $(i-1)$-th regular socle, thus $\alpha(X_i) \subseteq X_{i-1}$, and therefore $\alpha + 1$ induces the identity endomorphism of X_i/X_{i-1}. Let $\beta = (\alpha + 1)^{-1}$ in $\operatorname{End}(Q)$. Assume $\beta(P) \nsubseteq X$. Then we can choose $i \in \mathrm{N}$ with $\beta(P) \subseteq X_i$, $\beta(P) \nsubseteq X_{i-1}$. However, since $\alpha + 1$ induces the identity on X_i/X_{i-1}, we see that $(\alpha + 1)\beta(P) \nsubseteq X_{i-1}$, contrary to the assumptions $(\alpha + 1)\beta = 1$ and $P \subseteq \subseteq X_0 \subseteq X_{i-1}$. This shows that $\beta(P) \subseteq X$, and therefore β belongs to D. Thus, any element of I is quasi-regular, and therefore the ideal I is contained in the Jacobson radical of D.

Finally, note that the subset $\{\varphi \in \mathrm{End}\,(Q/X) | \varphi(X_i/X) = 0\}$ is some power of the radical of End (Q/X), say $\mathrm{rad}^{i'}$ End (Q/X). Then $\mathrm{rad}^{i'} D$, maps into $\mathrm{rad}^{i'}$ End (Q/X), and therefore $\mathrm{rad}^{i'} D \subseteq I_i$. But conversely, one also sees easily that $I_i \subseteq \mathrm{rad}^{i'} D$, thus the ideals I_i are powers of rad D.

It follows that

$$\mathrm{End}\,(Q/X) = \varprojlim \mathrm{End}\,(X_i/X) \approx \varprojlim D/I_i = \varprojlim D/\mathrm{rad}^{i'} D$$

is the completion of D with respect to the powers of the radical of D.

REMARK: Let $t \in T$, and choose some simple regular module S in $\mathfrak{T}_t$, and let S^∞ be the Prüfer module with regular socle S. We always can choose a peg P in Q such that for the torsionfree rank one module X with $P \subseteq X \subseteq Q$ and $(X/P)_t = 0$, $(X/P)_{t'} = (Q/P)_{t'}$, for $t' \neq t$, the factor module Q/X is the direct sum of e_t copies of S^∞, and therefore

$$\mathrm{End}\,(Q/X) = M_{e_t}(\hat{D}_t)\,, \qquad \text{where } \hat{D}_t = \mathrm{End}\,(S^\infty)\,.$$

This shows, that the various complete discrete valuation rings $\hat{D}_t$ which determine the category $\mathfrak{T}$ of torsion regular modules, are related to each other in the following way: for any t, there exists a subring of $E = \mathrm{End}\,(Q)$ such that its radical completion is just $M_{e_t}(\hat{D}_t)$.

6.9. THEOREM: *There exists an infinite set $\{X_i | i \in I\}$ of torsionfree rank one modules with the following properties:*

(*a*) End (X_i) *is a division ring, finite dimensional over* k.

(*b*) Hom $(X_i, X_j) = 0$ *for* $i \neq j$.

(*c*) Ext $(X_i, X_j) \neq 0$ *for all* i, j.

PROOF: As index set I, we choose any infinite partition of the set of homogeneous types, such that all $i \in I$ are infinite subsets of T.

Pet P be a fixed peg in Q, and let $Y = \bigoplus_{t \in T} Y_t$ be the regular socle of Q/P. If i is a subset of T, let X_i be the submodule of Q with $P \subseteq X_i$, such that $X_i/P = \bigoplus_{t \in i} Y_t$. By construction, any such module X_i is torsionfree of rank 1, and satisfies the condition of Proposition 6.6, thus (*a*) is satisfied.

Let i, j be disjoint infinite subsets of T, and assume there is a homomorphism $\varphi\colon X_i \to X_j$. Then $(\varphi(P) + P)/P$ is contained in some $\bigoplus_{t \in j'} Y_t$ with j' a finite subset of j, thus $\varphi(P)$ is contained in $X_{j'} \subseteq X_j$.

The induced homomorphism $X_i/P \to X_j/X_{j'}$ has to be zero, since $X_i/P = \bigoplus_{t\in i} Y_t$, $X_j/X_{j'} \approx \bigoplus_{t\in j\setminus j'} Y_t$ and $i \cap j = \emptyset$. Thus $\varphi(X_i) \subseteq X_{j'}$. However, this implies that $\varphi = 0$, since X_i is torsionfree regular, and $X_{j'}$ is preprojective. This proves condition (b).

Next, consider the exact sequence

$$0 \to P \to X_j \to Y_j \to 0\,,$$

for j a subset of T. It induces an exact sequence

$$\operatorname{Hom}(X_i, Y_j) \to \operatorname{Ext}(X_i, P) \to \operatorname{Ext}(X_i, X_j)\,.$$

Let us assume that i, j are disjoint infinite subsets of T. Then, we want to construct an exact sequence as follows: Let $i = i_1 \dot\cup i_2$, with i_1 and i_2 both being infinite. Then, denoting by u the various inclusion maps, the following sequence

$$0 \to P \xrightarrow{(u,-u)} X_{i_1} \oplus X_{i_2} \xrightarrow{\binom{u}{u}} X_i \to 0$$

is exact, and therefore belongs to $\operatorname{Ext}(X_i, P)$. Assume it lies in the image of $\operatorname{Hom}(X_i, Y_j)$, then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & P & \to & X_{i_1} \oplus X_{i_2} & \to & X_i & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \to & P & \to & X_j & \to & X_i & \to & 0\,. \end{array}$$

However, by the previous consideration, any homomorphism $X_{i_1} \to X_j$ and $X_{i_2} \to X_j$ is zero, thus $\mu' = 0$, a contradiction. Thus $\operatorname{Ext}(X_i, X_j) \neq 0$ for disjoint infinite subsets of T.

Finally, we want to show that $\operatorname{Ext}(X_i, X_i) \neq 0$, where i is infinite and contains only homogeneous elements $t \in T$. Construct X_i' with $P \subseteq X_i' \subseteq Q$ such that X_i'/P is the second regular socle of $\bigoplus_{t\in i} (Q/P)_t$. Then $X_i \subseteq X_i'$, and, since any $y \in i$ is homogeneous, we have $X_i'/X_i \approx \approx Y_i$. The exact sequence $0 \to P \to X_i \to Y_i \to 0$ gives rise to the exact sequence

$$\operatorname{Hom}(P, X_i) \to \operatorname{Ext}(Y_i, X_i) \to \operatorname{Ext}(X_i, X_i)\,.$$

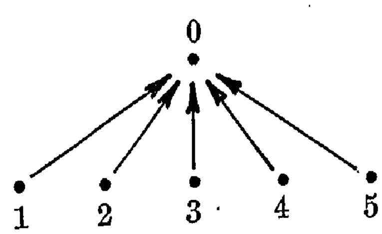
Now, we have an element in $\operatorname{Ext}(Y_i, X_i)$, given by the module X_i'. Assume it lies in the image of $\operatorname{Hom}(P, X_i)$. Thus, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & P & \to & X_i & \to & Y_i & \to & 0\,. \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \to & X_i & \to & X_i' & \to & Y_i & \to & 0\,.
\end{array}$$

Now there is a finite subset i' of i with $\varphi(P) \subseteq X_{i'}$. Let $\pi\colon X_i \to X_i/X_{i'}$ be the projection. Then $\pi\varphi = 0$, however, the sequence induced from $0 \to X_i \to X_i' \to Y_i \to 0$ by π does not split. This shows that there is an element in $\operatorname{Ext}(Y_i, X_i)$ which is not in the image of $\operatorname{Hom}(P, X_i)$, and therefore $\operatorname{Ext}(X_i, X_i) \neq 0$.

COROLLARY: *There exists a finite dimensional hereditary k-algebra R' of wild representation type, and a full and exact embedding of $\mathfrak{M}_{R'}$ into $\mathfrak{M}_R$.*

PROOF: We work with the set $\{X_i \mid i \in I\}$ of torsionfree rank 1 modules with the properties stated in the theorem. In fact, we need only a subset of I with six elements, say $\{0, 1, \ldots, 5\} \subseteq I$. For $i \in I$. let $F_i = \operatorname{End}(X_i)$. Now $\operatorname{Ext}(X_i, X_j)$ is an F_j-F_i-bimodule, thus a left $F_j \otimes_k F_i$-module. Since $F_j \otimes_k F_i$ is a finite dimensional k-algebra, we see that $\operatorname{Ext}(X_i, X_j)$ has a simple F_j-F_i-submodule E_{ij}, and E_{ij} is finite dimensional over k. Let $E^*_{ij} = \operatorname{Hom}_k(E_{ij}, k)$, this is an F_i-F_j-bimodule. Now consider the species with underlying graph



division rings F_i $(0 \leqslant i \leqslant 5)$, and bimodules E^*_{i0} $(1 \leqslant i \leqslant 5)$. According to [31], 1.5, there is a full and exact embedding of the category of representations of this species into $\mathfrak{M}_R$. Or, if we denote by R' the tensor algebra over this species, then R' is a finite dimensional hereditary algebra, and there is a full and exact embedding of $\mathfrak{M}_{R'}$ into $\mathfrak{M}_R$. However, this species, and therefore the ring R' is of wild representation type.

REMARK: As a consequence, we see that there is a finite extension field K of k such that the category of modules over the free associative algebra $K\langle x, y\rangle$ in two variables over K can be embedded as a full and exact subcategory of $\mathfrak{M}_R$, and thus the category of modules over any K-algebra which is generated over K by less then $\aleph_\iota$ elements ($\aleph_\iota$ the first strongly inaccessible cardinal number), see [31].

In fact, as the proof reveals, one may choose for K an extension field of k which is contained in one of the division rings End (P_i), where P_i is an indecomposable projective R-module.

This behaviour may be brought in the following suggestive form « tame finite dimensional hereditary algebras are Wild », (the small « *t* » in tame refers to tameness with respect to modules of finite length, the capital « *W* » in Wild refers to wildness with respect to arbitrary modules which are not necessarily of finite length).

Note that this is in sharp contrast to the situation in abelian groups or, more general, in principal ideal domains, as considered in [11]. There it was shown, on the one hand, that all kinds of pathological behaviour can occur. However, in dealing with modules over an integral domain E with quotient field F, a ring as simple as $F\times F$ cannot occur as endomorphism ring of an E-module, as was pointed out by Corner [13] in the case of $E = \mathbb{Z}$.

Added in proof.

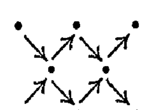
In this paper, we have shown that for a twosided-indecomposable finite dimensional hereditary algebra R of tame representation type, one particular infinite dimensional R-module is of great importance and seems to dominate the whole structure theory of R-modules: namely the unique indecomposable torsionfree divisible R-module Q. This module can be characterized in a different way: it is the only infinite dimensional R-module with endomorphism ring a division ring which is finite dimensional as a vector space over its endomorphism ring. (C. M. Ringel: *The spectrum of a finite dimensional algebra*, to appear in: Proc. Antwerp. Conf., Marcel Dekker). Note that this characterization does not refer to any technical notion such as « torsionfree » or « divisible ». For any ring R, the set of isomorphism classes of R-modules with End (M_R) a division ring and ${}_{\mathrm{End}(M_R)}M$ finite dimensional should be considered as the « spectrum » of R (this seems to be an appropriate generalisation of the notion of the prime spectrum of a commutative ring to not necessarily commutative rings). The result mentioned above shows that for a finite dimensional hereditary algebra of tame representation type, one has a complete knowledge of its spectrum.

Also, the endomorphism ring E of the unique indecomposable torsionfree divisible module Q incorporates a great deal of information about the category of R-modules. At the end of section 5, we have given some examples concerning the structure of E. Note that it follows from section 5 of [14] that one only has to consider the bimodule cases $\tilde{A}_{11}$ and $\tilde{A}_{12}$. The algebras of type $\tilde{A}_{12}$ are of the form $\begin{pmatrix} F & M \\ O & G \end{pmatrix}$, with ${}_FM_G$ a bimodule with $\dim {}_FM = \dim M_G = 2$. If ${}_FM_G$ is not simple, then $F = G$ and $M = M(\varepsilon, \delta)$ for some automorphism ε of F and some ε-1-derivation (see [31]), and then $E = F(t; \varepsilon, \delta)$, the quotient field of the twisted polynomial ring $F[t; \varepsilon, \delta]$. In particular, for $M = F \oplus F$, with canonical bimodule action, $E = F(t)$. If F, G are commutative, $F \supseteq H$, $G \supseteq H$, with $[F{:}H] = [G{:}H] = 2$ and ${}_FM_G = F \otimes_H G$, then E is the quotient field of the free product $F *_H G$ (note that E is uniquely determined since $F *_H G$ satisfies a polynomial identity). Finally, let us consider the case $\tilde{A}_{11}$. Then we have division rings $G \subset F$ with $\dim_G F = 4$, and the algebra is given by $\begin{pmatrix} G & F \\ O & F \end{pmatrix}$. For example, if $G = \mathbb{R}$, $F = \mathbb{H}$, then we have noted that E is the quotient field of $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$, and therefore commutative, whereas for $G = \mathbb{Q}$, $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, we obtain the (non-commutative!) quotient ring of $\mathbb{Q}\langle x, y\rangle/(xy + yx, x^2 + 2y^2 - 3)$ (see a forthcoming joint paper with V. Dlab: *Homogeneous representations of tame species*).

It has been asked frequently whether the structure theory developed in this paper is restricted to finite dimensional hereditary algebras or whether it may carry over to other finite dimensional algebras. Let us single out two other classes of finite dimensional algebras of tame representation type which have a similar structure theory: the tame algebras with radical square zero and the algebras corresponding to the « crucial » quivers with commutativity relations as studied by M. Loupias (*Indecomposable representations of finite ordered sets*, in: *Representations of Algebras*, Springer Lecture Notes, 488 (1975), 201-209) and others. For brevity, let us call the algebras of the second type Loupias algebras. It is easy to see that for any Loupias algebra S, there exists a tame finite dimensional hereditary algebra R, a finite number of finite dimensional S-modules $X_1, \dots, X_s$, and a finite number of finite dimensional R-modules $Y_1, \dots, Y_r$ such that the category of S-modules without direct summand of the form X_i is equivalent to the category of R-modules without direct summand of the form Y_i. This of course shows immediately that the infinite dimensional S-modules behave in the same way as the infinite dimensional R-modules; in particular, we have again a unique module Q_S with End (Q_S) a division ring and ${}_{\mathrm{End}(Q_S)}Q$ a finite dimensional vector space. Of course, in all cases it is easy to construct this module directly.

Similarly, in the case of a tame finite dimensional algebra S with radical square zero, there exists a tame finite dimensional hereditary algebra R such that the category of S-module and the category of R-modules are stably equivalent (that is, equivalent modulo maps which factor through projective modules). Here again, the structure theory developed for infinite-dimensional R-modules carries over to the infinite dimensional S-modules. Note, however, one important change: the S-module corresponding to our favorite R-module Q may no longer belong to the spectrum of S, since it may have non-zero nilpotent endomorphisms. For example, consider the 3-dimensional local commutative algebra $S = k[x, y]/(x^2, y^2, xy)$. The category of S-modules is stably equivalent to the category of modules over $R = \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$. The indecomposable torsionfree divisible R-module corresponds under this stable equivalence to the S-module $Q = k(t) \oplus k(t)$ with $(a, b)x = (0, a)$, $(a, b)y = (0, at)$. Note that End (Q_S) is not even artinian (but a local semiprimary ring with radical square zero), however, ${}_{\mathrm{End}(Q_S)}Q$ is a module of finite length.

It is easy to construct other finite dimensional algebras of tame type for which the representation theory essentially reduces to the study of modules over a tame finite dimensional hereditary algebra. However, there do exist tame finite dimensional algebras which behave rather different, even ones which correspond to a quiver with commutativity condition. For example, the quiver



with commutative square is tame, but has (at least) a countable family of infinite dimensional modules M with End (M) the field of rational functions in one variable, and ${}_{\mathrm{End}(M)}M$ finite dimensional. The representation theory of this commutative quiver does not seem to be dominated by any one of these modules.

Finally, let us come back to the question why we are interested in *infinite* dimensional representations when dealing with *finite* dimensional algebras. We gave several reasons in the introduction of this paper, but we should mention one additional situation where these modules occur rather naturally. If we assume that the given finite dimensional algebra R is not of finite representation type, then the study of finite dimensional R-modules will mainly be the study of continuous series of R-modules, according to the positive answer to the second Brauer-Thrall conjecture. Of course, one would like to reduce the problem of studying series of R-modules to problems concerning individual modules. The usual device is to replace R by an extension ring, say $R \otimes k[t]$, which however no longer is artinian; (see for example DROZD: *Tame and wild matrix problem* (Russian), in: Matrix Problem, Kiev (1977), 104-114). However, there is another possibility, namely to consider suitable infinite dimensional R-modules which, in some sense, parametrize the given series of finite dimensional R-modules. An account of this approach is outlined in the Antwerp paper mentioned above.

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