THE SPECTRUM OF A FINITE DIMENSIONAL ALGEBRA

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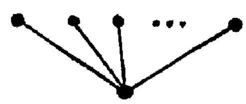
P.M. Cohn [16] once has proposed to consider the set of epimorphisms from a ring A into simple artinian rings as the "spectrum" of A, in this way generalizing the very useful notion of the prime spectrum of a commutative ring to arbitrary rings. Here, "epimorphisms" means the categorical notion, which includes besides the onto homomorphisms also, for example, localisations with respect to Øre sets. As in the commutative case, this spectrum can be considered as a topological space by means of a suitable partially ordering (given by the notion of specialisation), however, we note that it no longer has to be a compact space. The interest of this spectrum for the representation theory of A lies in the fact that we can identify the elements of the spectrum with the isomorphism classes of those A-modules ${}_AX$ for which the endomorphism ring $\mathrm{End}({}_AX)$ is a division ring such that X, considered as an $\mathrm{End}({}_AX)$-module, is a finite dimensional vector space.

We therefore will call these modules "points" (of the spectrum).

In this survey, we will be mainly interested in the case of a finite dimensional k-algebra A over some (commutative) field k. Our aim is two-fold: On the one hand, we would like to point out the importance of this spectrum in studying finite dimensional representations of A, since it turns out that certain large points of the spectrum give rise to infinite families of finite dimensional points: one should consider them as parametrizing these families. On the other hand, in due course of our report, there will turn up a strong interrelation between various parts of ring theory: we will see that the question of constructing finite dimensional modules over finite dimensional algebras leads to problems concerning finitely generated (but not finite dimensional) PI rings as, for example, rings of generic matrices, and even to problems concerning free associative algebras. The free associative algebras $k\langle x_1,\dots,x_q\rangle$ made their first appearance in the theory of finite dimensional algebras in the development of the notion of wild representation type, when one considered full exact embeddings $k\langle x_1,\dots,x_q\rangle^{\mathcal{M}} \hookrightarrow {}_A\mathcal{M}$ of the module categories, and such embeddings seem to be a very typical situation which one has to consider.

The main result of this paper will be the determination of the spectrum of a tame finite dimensional hereditary k-algebra A. We will show that in case A is twosided indecomposable, there exists a unique epimorphism from A into a simple artinian ring which is infinite dimensional over k. As a consequence, the spectrum of A is the disjoint union of countably many one-point sets and a connected partially ordered set



with one generic point and $\max(\aleph_0, |k|)$ remaining points.

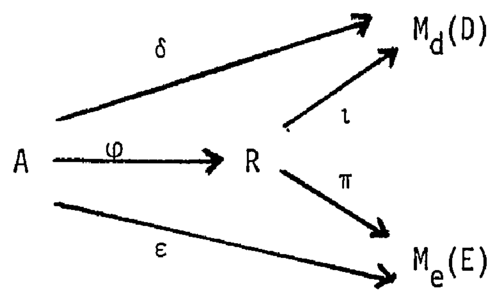
The proof of this result is given in section 6 and presupposes the structure theorems for infinite dimensional modules over a tame finite dimensional hereditary k-algebra derived in [36].

Sections 1 to 4 of this paper develop a general theory of the spectrum of a finite dimensional algebra. In section 5, the typical situation in case of a wild finite dimensional hereditary k-algebra over an algebraically closed field is exhibited.

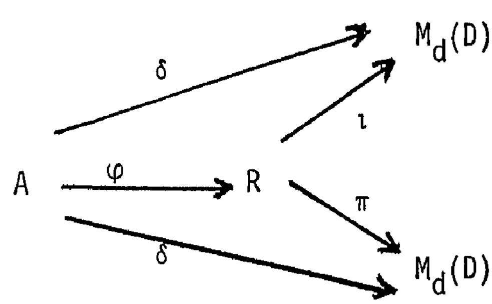
1. The Spectrum

We consider only rings with 1, and ring homomorphisms are supposed to preserve 1. We denote by $M_n(R)$ the $n \times n$ matrix ring over R. The Jacobson radical of R will be denoted by rad R. If k is a (commutative) field, a k-algebra R is, by definition, a ring R with a fixed embedding of k into the center of R. We denote the free associative k-algebra generated by $x_1,\dots,x_m$ by $k\langle x_1,\dots,x_m\rangle$.

1.1. Recall that a ring homomorphism $\varepsilon : A \to B$ is called an epimorphism provided for any ring homomorphisms $\beta,\beta' : B \to C$, satisfying $\varepsilon\beta = \varepsilon\beta'$, we may conclude $\beta = \beta'$. Examples of epimorphisms are onto ring homomorphisms, but also certain inclusions as $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Two epimorphisms $\varepsilon : A \to B$ and $\varepsilon' : A \to B'$ will be called equivalent provided there exists an isomorphism $\beta : B \to B'$ with $\varepsilon\beta = \varepsilon'$. Note that for given A, the cardinality of B with epimorphism $A \to B$ is bounded [28], thus the equivalence classes of epimorphisms $A \to B$, with A fixed, form a set. The set of all equivalence classes of epimorphisms $A \to M_d(D)$, with D a division ring, will be called the spectrum of A. In fact, we will consider the spectrum of A as a partially ordered set using the notion of specialisation: The epimorphisms $\varepsilon : A \to M_e(E)$ is called a specialisation of the epimorphism $\delta : A \to M_d(D)$ provided there exists an epimorphism $\varphi : A \to R$, where R is a subring of $M_d(D)$, with inclusion ι, and where $R/\text{rad } R$ is isomorphic to $M_e(E)$, with projection $\pi : R \to M_e(E)$ such that the following diagram commutes



<u>Proof</u>: Since reflexivity and transitivity are obvious, we only have to check that there is no non-trivial specialisation of $\delta : A \to M_d(D)$ into itself. But given the commutative diagram



with φ an epimorphism, we conclude $\iota = \pi$.

1.2. <u>Examples</u>: a) <u>Commutative rings</u>: Let A be commutative. In this case, we obtain precisely the prime spectrum (the set of prime ideals of A with partially ordering given by inclusion). For, given an epimorphism $A \to R$ with A commutative, one knows that R is commutative [39]. Therefore, for any epimorphism $\delta : A \to M_d(D)$ with D a division ring, we have $d = 1$ and D is a field. The kernel of δ is a prime ideal and depends only on the equivalence class of δ. Conversely, given a prime ideal p, of A the canonical map $A \to \mathrm{Quot}(A/p)$ is an epimorphism. It remains to note that a specialisation from $A \to \mathrm{Quot}(A/p)$ to $A \to \mathrm{Quot}(A/p')$, with p, p' prime ideals of A, exists if and only if $p' \subseteq p$.

b) Direct products. Let A be the product of two rings, $A = A_1 \times A_2$. Then the spectrum of A is the disjoint union of the spectrum of A_1 and the spectrum of A_2. For, given an epimorphism $A \to R$, the orthogonal idempotents $(1,0)$ and $(0,1)$ are mapped to orthogonal idempotents with sum 1. However, these idempotents are central and the image of the center of A lies in the center of R, thus if R is twosided indecomposable, one of the elements $(1,0)$ and $(0,1)$ goes to zero. Thus, any epimorphism $A \to M_d(D)$ with D a division ring factors over one of A_1, A_2, and also there can be no specialisation between epimorphisms which factor over different A_i.

c) Semi-simple rings: In order to determine the spectrum of a semi-simple (artinian) ring A, we only have to consider the case of a full matrix ring $M_d(D)$ with D a division ring. We claim that in this case, any epimorphisms $\delta : M_d(D) \to R$ with R not the zero ring, is an isomorphism. For, using the images of the matrix units of $M_d(D)$, we see that R is of the form $M_d(R')$ for some ring R', and $\delta = M_d(\delta')$, for some ring homomorphism $\delta' : D \to R'$. Also it is easy to see that δ' is an epimorphism.

However, this implies that δ' (and therefore δ) is an isomorphism. This shows that for a semi-simple ring A the points of the spectrum correspond bijectively to the maximal ideals of A.

d) Artinian rings: Next, let A be an artinian ring. There are some obvious epimorphisms $A \to M_d(D)$, with D a division ring, namely the projections $A \to A/m$, with m a maximal ideal. Clearly, these epimorphisms only depend on the semi-simple ring $A/\mathrm{rad}\,A$, and there is only a finite number of equivalence classes of such epimorphisms. Also, conversely, every epimorphisms $A \to M_d(D)$ with $d = 1$ is of this form (for, its kernel is a prime ideal and therefore maximal). On the other hand, for certain A there do exist non-trivial epimorphisms $A \to M_d(D)$, D a division ring, with $d \geq 2$. For example, let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, the ring of upper triangular 2×2 matrices over the field k. Then the inclusion map $A \to M_2(k)$ is an epimorphism. (This is well-known [28], but it will follow also easily from the considerations in Section 2).

e) Given a bimodule ${}_FM_G$, one may consider the ring $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ of all matrices $\begin{pmatrix} f & m \\ o & g \end{pmatrix}$ with $f \in F$, $g \in G$, $m \in M$. Let F, G be division rings, and assume $A = \begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ is a finite dimensional k-algebra for some field k. It has been shown in [35] that for any $d \in \mathbb{N}$ there exists an epimorphism $A \to M_d(D)$ with D a division ring, provided $\dim {}_FM + \dim M_G + (\dim {}_FM)(\dim M_G) \geq 12$.

f) Consider now the bimodule k^3 where k is some field, and let $A = \begin{pmatrix} k & k^3 \\ o & k \end{pmatrix}$. Let R be any k-algebra which is generated over k by x,y. Then the canonical map

$$A = \begin{pmatrix} k & k+kx+ky \\ o & k \end{pmatrix} \to M_2(R)$$

is an epimorphism. Also, if R is a k-algebra generated by $x_1,\dots,x_n$, then there is an epimorphism $A \to M_{2n+4}(R)$ given as follows:

Consider in $M_{n+2}(R)$ the unit matrix $E = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$, and the matrices

$$I = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ x_1 \cdots x_m & & & 1 & 0 \end{pmatrix}$$, and define an embedding

$A \to M_{2n+4}(R)$ by the rule that the elements $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & (100) \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & (010) \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & (001) \\ 0 & 0 \end{pmatrix}$ are mapped onto

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix},$$

respectively ([13]).

This shows that any finitely generated k-division algebra D can occur in an epimorphism $A \to M_d(D)$, with A a suitable finite dimensional k-algebra. For, let D be generated (as a division algebra) by $x_1,\dots,x_n$, and R the subalgebra generated by $x_1,\dots,x_n$. Then $R \hookrightarrow D$, and therefore also $M_d(R) \hookrightarrow M_d(D)$ is an epimorphism.

1.3 PROPOSITION. Let A be a finitely generated k-algebra, and $\delta : A \to M_d(D)$ an epimorphism with D a division ring. Then

D is a finitely generated k-division ring.

Proof: First we note that the center of A is mapped into the center of $M_d(D)$, a known property of an epimorphism [39]. Thus the image of k in $M_d(D)$ lies in the center of D with D embedded into $M_d(D)$ as the set of scalar matrices. This shows that D is a k-algebra. Next, let $a_1,\ldots,a_m$ be generators of A as a k-algebra, and $\delta(a_i) = (\alpha^i_{st})_{st}$ with $\alpha^i_{st} \in D$, $i \le s, t \le d$, $1 \le i \le m$. Let D' be the k-division subring generated by the elements α^i_{st}. Then $\delta(A) \subseteq M_d(D')$, and D' is a finitely generated k-division ring. We claim that $D' = D$. This follows from the fact that the embedding $M_d(D') \subseteq M_d(D)$ is an epimorphism, thus the identity (see 1.2.c).

1.4. Assume now that A is a finite-dimensional k-algebra over some field k. We fix a complete set $f_1,\ldots,f_n$ of primitive idempotents (thus, there are primitive orthogonal idempotents f_{st}, $1 \le s \le n$, $1 \le t \le m_s$ with $1 = \sum_{s,t} f_{st}$ such that the left A-modules Af_i and Af_{st} are isomorphic if and only if $i = s$; we call m_i the multiplicity of f_i). Given an epimorphism $\delta : A \to M_d(D)$, with D a division ring, we define its dimension vector $\underline{\dim}\,\delta \in \mathbb{Q}^n$ as follows: suppose the idempotent $\delta(f_i)$ of $M_d(D)$ can be written as the sum of b_i primitive orthogonal idempotents of $M_d(D)$, then

$$(\underline{\dim}\,\delta)_i = \frac{b_i}{a_i},$$

with $a_i = \dim_k[e_iAe_i/e_i(\text{rad } A)e_i]$. It is clear that $\underline{\dim}\ \delta$ only depends on the equivalence class of δ. The equality

$$d = \sum_i m_i b_i = \sum_i m_i a_i (\underline{\dim}\ \delta)_i$$

shows that the dimension vector of $\delta : A \to M_d(D)$, together with the datas m_i, a_i (which only depend on A) determines the number d. In contrast to d, the dimension vector is a Morita invariant.

1.5 PROPOSITION. Let A be a finite dimensional k-algebra, let D,E be division rings and assume there exists a specialisation from the epimorphism $\delta : A \to M_d(D)$ to the epimorphism $\varepsilon : A \to M_e(E)$. Then $\mathbb{Q}\ \underline{\dim}\ \delta = \mathbb{Q}\ \underline{\dim}\ \varepsilon$.

Proof: By assumption, there exists a subring $\iota : R \hookrightarrow M_d(D)$, an epimorphism $\varphi : A \to R$, and a ring surjection $\pi : R \twoheadrightarrow M_e(E)$ with kernel $J = \text{rad } R$ such that $\delta = \varphi\iota$, $\varepsilon = \varphi\pi$.

Given an idempotent f of R, let $p(f)$ be the number of summands if we write f^π as a sum of primitive orthogonal idempotents of $M_e(E)$. We claim that any idempotent of R can be written as a sum of primitve idempotents and that for any two primitive idempotents f,f' of R, we have $p(f) = p(f')$ and $Rf \approx Rf'$ as left R-modules. The first assertion follows from the fact that R/J is of finite length and that for any non-zero direct summand U of ${}_RR$, also U/JU is a non-zero direct summand of R/J.

Next, let f, f' be primitive idempotents of R, and assume $p(f) \leq p(f')$. Then Rf/Jf is isomorphic to a direct summand of Rf'/Jf', thus there are maps

$$\bar{u} : Rf/Jf \to Rf'/Jf', \quad \bar{v} : Rf'/Jf' \to Rf/Jf$$

with $\bar{u}\bar{v} = \mathrm{id}$. We can lift $\bar{u}$ to a right multiplication by some $u \in fRf'$, and $\bar{v}$ to a right multiplication by some $v \in f'Rf$. Then $uv \in fJf$, thus $f-uv$ is invertible in fRf. This shows that $u : Rf \to Rf'$ is a split monomorphism, and, since Rf' is indecomposable, even an isomorphism. Thus $Rf \approx Rf'$ and $p(f) = p(f')$. Denote by p the common value of $p(f)$, with f primitive idempotent of R.

Now write f_i^{φ} as the sum of, say d_i, orthogonal primitive idempotents of R. Then $f_i^{\varepsilon} = f_i^{\varphi\pi}$ is the sum of pd_i orthogonal primitive idempotents, thus $(\underline{\dim}\ \varepsilon)_i = pd_i a_i^{-1}$.

On the other hand, assume $1 \in R$ is the sum of s primitive orthogonal idempotents. Let f be a fixed primitive idempotent of R and $S = fRf$. Then ${}_RR \approx \bigoplus_s Rf$, thus R is isomorphic, as a ring, to $M_s(S)$. As a consequence, we have in R, and therefore in $M_d(D)$ elements which correspond to the matrix units of $M_s(S)$. This shows that $M_d(D)$ is of the form $M_s(S')$ for some ring S'. Clearly $d = sq$ for some $q \in \mathbb{N}$, and $S' = M_q(D)$. Thus, we see that any primitive idempotent of R can be written in $M_d(D)$ as the sum of q orthogonal primitive idempotents. Thus $(\underline{\dim}\ \delta)_i = qd_i a_i^{-1}$. This finishes the proof.

1.6 EXAMPLES. We give two examples in order to show that, in the situation of 1.5, not necessarily one of $\underline{\dim}\,\delta$ and $\underline{\dim}\,\varepsilon$ is an integral multiple of the other.

a) Let $k(t)$ be the field of rational functions over k in one variable, and consider the k-algebra R generated by the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$ in $M_2(k(t))$, thus

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k[t]) \mid \text{will } a\text{-}d, c \in tk[t] \right\} .$$

Let $\delta : R \to M_2(k(t))$ be the inclusion, $\varepsilon : R \to k$ the projection with kernel $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,c,d \in tk[t],\ b \in k[t] \right\}$. Then there is a specialisation from δ to ε given by the inclusion

$$\varphi : R \to \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k[t]_{(t)}) \mid a\text{-}c, c \in tk[t]_{(t)} \right\} .$$

b) Consider the algebra $R = k\langle x,y\rangle / (xy+yx)$. The center of R is the subalgebra generated by x^2 and y^2, thus R is a PI-algebra without zero divisors, and therefore an Ore domain, say with quotient field D. Let $\delta : R \to D$ be the embedding. On the other hand, there is an epimorphism $\varepsilon : R \to M_2(k(t))$, given by $x \mapsto \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It is easy to see that ε is a specialisation of δ, using the localisation $\varphi : R \to R_m$ with respect to the maximal ideal $m = \langle x^2, y^2-1\rangle$ of the center of R.

In both cases a), b), the ring R was a 2-generator k-algebra, but not finite dimensional. However, using the epimorphism $A \to M_2(R)$ of 1.2 f), where $A = \begin{pmatrix} k & k^3 \\ 0 & k \end{pmatrix}$, we obtain corresponding specialisations of epimorphisms $\delta : A \to M_d(D)$ and $\varepsilon : A \to M_e(E)$, with A a finite dimensional k-algebra.

1.7 Historical remark. As was mentioned in the introduction, it was P.M. Cohn in [16], who proposed to consider the set of equivalence classes of epimorphisms $A \to M_d(D)$, with D a division ring, as the spectrum of A, after having dealt with, in several papers, the "field spectrum", the set of equivalence classes of epimorphisms $A \to D$, D a division ring. Also, in this last case he introduced the notion of a specialisation [15]. In [9], G. Bergman pointed out that also a more general concept than that of a single specialisation, namely the so called support relation, deserves to be studied in dealing with the field spectrum of a ring; a similar concept can be introduced in the case of the spectrum itself. Of course, there are many other possible generalisations of the spectrum of a commutative ring to the noncommutative situation, and recently, P.M. Cohn [18] made some investigations into the union of the field spectrums of all matrix reduction rings of A, and, changing the view of [16], has called this the spectrum of A.

There do exist several papers concerning epimorphisms of rings [10,28,30,39]. Recall that G. Bergman [10] had conjectured that for a k-algebra A of dimension n, and any epimorphism $A \to B$, the k-algebra B should be of dimension $\leq (n-1)^2 + 1$; so that, for A

finite, there should exist only a finite number of equivalence classes of epimorphisms $A \to B$. Clearly, these conjectures do not hold. (See for example 1.2.f)).

2. Large Modules

There is a well-known criterion [39] which asserts that a ring homomorphism $A \to B$ is an epimorphism if and only if the corresponding forget functor ${}_B M \to {}_A M$ is a full embedding. Here, ${}_A M$ denotes the category of all (left) A-modules. This criterion can be used to give another interpretation of the spectrum of A in terms of certain indecomposable A-modules.

2.1 An A-module ${}_A X$ will be called a point provided its endomorphism ring $D = \mathrm{End}({}_A X)$ is a division ring, and X_D is finite dimensional (in [35], this had been called a "finite point"). Note that points always are indecomposable. We recall from [35]:

PROPOSITION. The spectrum of A can be identified with the set of isomorphism classes of points.

Proof: If $\delta : A \to M_d(D)$ is an epimorphism, with D a division ring, consider the canonical $M_d(D)$-module D^d as an A-module. Since the embedding ${}_{M_d(D)}M \hookrightarrow {}_A M$ is full, $\mathrm{End}({}_A(D^d)) = \mathrm{End}({}_{M_d(D)}(D^d)) = D$, and $\dim (D^d)_D = d$, thus ${}_A(D^d)$ is a point.

Conversely, let ${}_A X$ be a point, with $D = \mathrm{End}({}_A X)$, $d = \dim X_D$, and $B = \mathrm{End}(X_D) \approx M_d(D)$. There is a canonical map $\delta : A \to B$, with $\delta(a)$ being the left multiplication by a on X, for $a \in A$. In order to see that δ is an epimorphism, note that any B-module is of the form $\bigoplus_I {}_B X$, with I some index set.

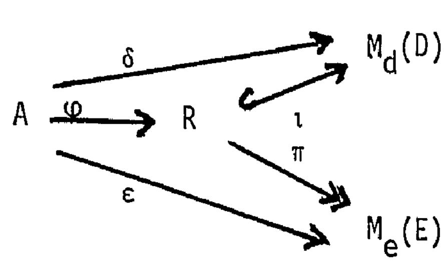
This follows from the fact that ${}_BX$ is the unique simple module of the simple artinian ring B. Now, any A-homomorphism $\bigoplus_I X \to \bigoplus_J X$ (with I,J index sets) is of the form (f_{ij}) with $f_{ij} \in ({}_AX) = D = \mathrm{End}({}_BX)$, thus (f_{ij}) is in fact a B-homomorhpism, and therefore, the embedding ${}_BM \hookrightarrow {}_AM$ is full.

It is clear that this correspondence establishes a bijection between the isomorphism classes of points and the equivalence classes of epimorphisms $A \to M_d(D)$.

<u>Corollary</u>: Let A be a local artinian ring. Then the spectrum of A consists of a single point.

<u>Proof</u>: Only the simple A-module has no non-zero nilpotent elements in its endomorphism ring, thus it is the only point.

2.2 In the sequel, it will be convenient to consider the elements of the spectrum both as being (equivalence classes of) epimorphisms and as (isomorphism classes of) points. One may reformulate the concept of a specialisation in terms of modules. We only note the following: Let ${}_AX$ be a point with corresponding epimorphism $\delta : A \to M_d(D)$, and ${}_AY$ a point with corresponding epimorphism $\varepsilon : A \to M_e(E)$, and assume there exists a specialisation from δ to ε, say given by the diagram



with epimorphism φ. Then we can consider both ${}_AX$ and ${}_AY$ as R-modules. Note that ${}_RY$ is the unique simple R-module. If we consider the R-submodules X_i of ${}_RX$, we see that for every $0 \neq x \in X$, there exist R-submodules $X_j \subsetneq X_i$ of X with $x \in X_i \setminus X_j$ and $X_i / X_j \approx {}_RY$. In particular, this means that the A-module ${}_AX$ is "covered" by factors of the form ${}_AY$. Also note that in case δ and ε are not equivalent, the R-module ${}_RX$ cannot be of finite length. For, if ${}_RX$ is of length 1, then ${}_RX \approx {}_RY$, therefore ${}_AX \approx {}_AY$, and thus δ, ε are equivalent, whereas if ${}_RX$ is of finite length > 1, then $\mathrm{End}({}_RX)$ would have non-zero nilpotent elements, which is impossible since $\mathrm{End}({}_RX) = \mathrm{End}({}_AX) = D$, a division ring.

2.3 PROPOSITION: If A is an artinian ring of finite representation type, then the spectrum of A is a finite discrete set.

Proof: Assume A is artinian and has only a finite number of indecomposable (left) A-modules $M_1,\ldots,M_m$ of finite length. One knows ([38], see also [4]) that any A-module is a direct sum of copies of these modules M_i. Thus the only possible points are those modules M_i with End (M_i) a division ring (in fact, in this case this implies that M_i is finite dimensional as an End (M_i)-module, see [4]). However, since all M_i are of finite length, there can be no proper specialisation. This proves the proposition. Since in case A is even a finite dimensional k-algebra, any $\mathrm{End}(M_i)$ is also finite dimensional over k, we have established also the following assertion.

Remark: If A is a finite dimensional k-algebra of finite representation type, and $A \to M_d(D)$ is an epimorphism, with D a division ring, then D is a finite dimensional k-algebra.

2.4. In Section 1, we have introduced the dimension vector for an arbitrary element of the spectrum of a finite dimensional algebra A, thus for any point. The extra factor a_i^{-1} may have appeared curious, however, in this way we obtain an element of $\mathbb{Q}^n$ which is (in case of a point which is of finite length) a multiple of the usual "dimension vector" defined in terms of numbers of composition factors:

LEMMA: Let A be a finite dimensional k-algebra, and let ${}_AX$ be a module of finite length with $D = \mathrm{End}({}_AX)$ a division ring. Then ${}_AX$ is a point, and if $\delta : A \to M_d(D)$ is the corresponding epimorphism, then $(\underline{\dim}\ \delta)_i \cdot \dim_k D$ is the number of composition factors isomorphic to $Af_i/(\mathrm{rad}\ A)f_i$ in ${}_AX$.

Proof: The number of composition factors of ${}_AX$ isomorphic to $Af_i/(\mathrm{rad}\ A)f_i$ is equal to the length l_i of the f_iAf_i-module f_iX and $a_il_i = \dim_k f_iX$, with $a_i = \dim_k[f_iAf_i/f_i(\mathrm{rad}\ A)f_i]$. Note that in terms of the module ${}_AX$, we have

$$(\underline{\dim}\ \delta)_i = \frac{\dim(f_iX)_D}{a_i}\ ,\ \text{thus}$$

$$\begin{aligned}(\underline{\dim}\,\delta)_i \dim_k D &= a_i^{-1} \dim(f_i X)_D \cdot \dim_k D \\ &= a_i^{-1} \dim_k f_i X = 1_i .\end{aligned}$$

If ${}_A X$ is a point with corresponding epimorphism $\delta : A \to M_d(D)$, we set $\underline{\dim}\,{}_A X = \underline{\dim}\,\delta$. (Note that this differs by a scalar from the use of the symbol "$\underline{\dim}$" in [19], whenever it was defined there; also note that $\underline{\dim}\,{}_A X$ depends on the k-structure: a change of the base field leads to a change of $\underline{\dim}\,{}_A X$, again by a scalar - this together with the result 1.5 shows that one should concentrate mainly on the element $\mathbb{Q}\,\underline{\dim}\,{}_A X$ of $\mathbb{P}_{n-1}\mathbb{Q}$ instead of the point $\underline{\dim}\,{}_A X$ in $\mathbb{Q}^n$.)

2.5. Assume now that A is finite dimensional and hereditary. The main working tool in this case are the Coxeter functors and the reflection functors ("partial Coxeter functors"). For the finite dimensional modules, two rather different constructions for the Coxeter functors are known: the original kernel-cokernel construction at least in case of a tensor algebra ([11],[19]) and the dual-of-transpose-construction ([6],[14]). If we want to deal with infinite dimensional modules, we have to use the first construction, since the usual dualities only work for finite dimensional vector spaces. We indicate the proof for tensor algebras, but note that a similar construction is possible in the general case (see [20] and [37]).

Let A be the tensor algebra of the k-species $S = (F_i, {}_iM_j)_{1 \leq i,j \leq n}$, thus the A-modules correspond to the representations $({}_iV, {}_j\varphi_i)$ of S, with ${}_iV$ a left F_i-vector space and ${}_j\varphi_i : {}_iM_j \otimes {}_jV \to {}_iV$ an F_i-linear map. (This means that we assume that A is basic, that we choose primitive orthogonal idempotents f_i and set $F_i = f_iAf_i$, and that we assume that there exists a complement ${}_iM_j$ of the $f_i(\text{rad } A)^2f_j$ in the F_i-F_j-bimodule $f_i(\text{rad } A)f_j$. Given an A-module V, let ${}_iV = f_iV$, and ${}_j\varphi_i$ the corresponding multiplication map). We call t a sink, if ${}_iM_t = 0$ for all $1 \leq i \leq n$. Note that there always will exist a sink, since the tensor algebra A is assumed to be finite dimensional. Now given a representation $V = ({}_iV, {}_j\varphi_i)$ of S, and a sink t, S_t^+V is defined by ${}_i(S_t^+V) = {}_iV$ for $i \neq t$, and the exact sequence

$$(*) \qquad 0 \to {}_t(S_t^+V) \xrightarrow{({}_t\psi_j)_j} \bigoplus_j {}_tM_j \otimes {}_jV \xrightarrow{({}_j\varphi_t)_j} {}_tV$$

with the old maps ${}_j\varphi_i$ for $i \neq t$, and the maps $({}_tM_j)^* \otimes {}_t(S_t^+V) \to {}_jV$ adjoint to ${}_t\psi_j$, where $({}_tM_j)^* = \text{Hom}({}_tM_j, k)$ denotes the F_j-F_t-bimodule dual to ${}_tM_j$. In this way, we obtain a representation of the species S_t which is obtained from S by removing the bimodules ${}_tM_j$, $1 \leq j \leq n$, and inserting the F_j-F_t-bimodules $({}_tM_j)^*$. In case t is a source (that is, ${}_tM_i = 0$ for all i), there is the dual construction S_t^-. We can consider F_t as a simple representation of S. If we assume again, that t is a sink, then F_t is projective and $S_t^+F_t = 0$. On the other hand, we recall from [19] that for any representation V of S without direct summand of the form F_t, we have $S_t^- S_t^+ V \approx V$, and $\mathrm{End}(V) \approx \mathrm{End}(S_t^+ V)$. Note that this is valid even in case V is not finite dimensional!

For finite dimensional representations, an additional dimension formula was derived in [19], and we claim that a similar formula holds for arbitrary points. Recall that for given $S = (F_i, {}_iM_j)$, the reflection s_t on $\mathbb{Q}^n$ is defined by

$$(s_t x)_i = x_i \quad \text{for} \quad i \neq t, \quad (s_t x)_t = -x_t + \sum_j \dim_{F_t}({}_tM_j) x_j \ ,$$

for $x = (x_i) \in \mathbb{Q}^n$.

<u>PROPOSITION</u>: Let A be the tensor algebra of the k-species S, and t a sink. Let X, Y be A-modules which are points.

(a) Either ${}_AX$ is simple projective, isomorphis to F_t, or else S_t^+X is a point and $\underline{\dim}\, S_t^+X = s_t\, \underline{\dim}\, X$.

(b) If there exists a proper specialisation from ${}_AX$ to ${}_AY$, then S_t^+X and S_t^+Y are points and there exists a proper specialisation from S_t^+X to S_t^+Y.

<u>Proof</u>: Note that the exact sequence (*) is a sequence of F_t-D-bimodules, where $D = \mathrm{End}(V)$. For, by the definition of a representation, ${}_j\varphi_t$ is F_t-linear, and, by the definition of an endomorphism, the canonical operation of D on the different ${}_iV$ commutes with ${}_j\varphi_i$. Thus, the right map is an F_t-D-bimodule map,

and therefore the kernel ${}_t(S_t^+V)$ is again an F_t-D-bimodule. Now assume $V = X$ is a point, and not isomorphic to F_t. Then the right map of (*) is surjective, and using the equality

$$a_i \dim {}_{F_i}({}_iM_j) = \dim {}_k({}_iM_j) = a_j \dim ({}_iM_j)_{F_j}$$

with $a_i = \dim {}_k(F_i)$, we obtain

$$\begin{aligned}(\underline{\dim}\, S_t^+X)_t &= a_t^{-1} \dim {}_tX_D \\ &= a_t^{-1}(\sum_j \dim ({}_tM_j)_{F_j} \dim {}_jX_D - \dim {}_tX_D) \\ &= \sum_j a_j^{-1} \dim {}_{F_t}({}_tM_j)\cdot\dim {}_jX_D - a_t^{-1} \dim {}_tX_D \\ &= \sum_j \dim {}_{F_t}({}_tM_j)\cdot(\underline{\dim}\, X)_j - (\underline{\dim}\, X)_t = s_t\, \underline{\dim}\, X.\end{aligned}$$

This proves (a).

In order to prove (b), assume there is given an epimorphism $\varphi : A \to R$ with a proper inclusion $\iota : R \hookrightarrow M_d(D)$ and a projection $\pi : R \to M_e(E)$ with kernel rad R such that $\varphi\iota : A \to M_d(D)$ is the epimorphism corresponding to the point ${}_AX$, and $\varphi\pi : A \to M_e(E)$ is the epimorphism corresponding to the point ${}_AY$. Using the forget functor ${}_R\mathcal{M} \to {}_A\mathcal{M}$, we can identify ${}_R\mathcal{M}$ with a full subcategory $\mathcal{A}$ of ${}_A\mathcal{M}$ which contains both ${}_AX$ and ${}_AY$, with ${}_AY$ being the unique simple object in the category $\mathcal{A}$. We claim that no object of $\mathcal{A}$ has a direct summand of the form F_t. Since $\mathcal{A}$ is closed under direct summands, F_t would otherwise be an object of $\mathcal{A}$, and therefore the unique simple object ${}_AY$.

However, every object U of $\mathcal{A}$ is covered by factors ismomorphic to ${}_AY$, thus ${}_iU = 0$ for all $i \neq t$, and therefore U is the direct sum of copies of F_t. As a consequence, ${}_AX$ being an indecomposable object of $\mathcal{A}$, would be isomorphic to F_t, and thus to ${}_AY$, contradicting the fact that we have a proper specialisation. This shows that the category $\mathcal{A}$ is mapped under S_t^+ isomorphically to a corresponding subcategory $\mathcal{A}_t$ of the category of representations of $\mathcal{S}_t$. Since ${}_AR$ is a progenerator in $\mathcal{A}$, we see that $S_t^+({}_AR)$ is a progenerator in $\mathcal{A}_t$, with endomorphism ring $\mathrm{End}(S_t^+({}_AR)) \approx \mathrm{End}({}_AR) \approx R$. Thus if we denote by A_t the tensor algebra of $\mathcal{S}_t$, then the canonical map $A_t \to \mathrm{End}(S_t^+({}_AR)_R)$ is an epimorphism which defines a specialisation from $S_t^+({}_AX)$ to $S_t^+({}_AY)$.

2.6. Again, assume that A is finite dimensional and hereditary (or even a tensor algebra for a k-species). An indecomposable module P is called <u>preprojective</u> ([19],[36]) provided there exists a sequence $t_1,\dots,t_s$ such that $S_{t_s}^+ \dots S_{t_1}^+ P$ is defined and $= 0$. It is clear that in this case P has to be of finite length and that its endomorphism ring is a division ring, thus it is a point. Also, it is uniquely determined by the numbers of the various composion factors. In fact it is the only indecomposable module X of finite length with $\underline{\dim}\, X \in \mathbb{Q}\, \underline{\dim}\, P$. This shows that there is no other point with a specilisation from P to it. By the previous result 2.5 we see that also no other point can have a specialisation to P, thus the isomorphism class of P forms a one-element component of the spectrum of A.

Similarly, an indecomposable module I is called preinjective provided there exists a sequence $t_1,\ldots,t_s$ such that $S^-_{t_s} \ldots S^-_{t_1} I$ is defined and $= 0$. Again, any preinjective module is a point of finite length, and its isomorphism class is a one-element component of the spectrum of A.

Now, if A is of finite representation type, then we know already that the spectrum of A is a finite discrete set, and we remark here that in fact all indecomposable modules are preprojective, and also preinjective [19].

If A is not of finite representation type, then there are countably many different preprojective modules, and countably many different preinjective modules. This shows:

PROPOSITION: Let A be a finite dimensional hereditary k-algebra which is not of finite representation type, then the spectrum of A has countably many one-element components. In particular, it is not compact.

3. Families of Modules

Our interest in the spectrum of the finite dimensional k-algebra A stems from the fact that the points of the spectrum seem to parametrize certain families of indecomposable modules of finite length. In particular, we will be interested in epimorphisms $A \to M_d(K)$, where K is a commutative field. We know that K is finitely generated over k, thus it has a geometrical meaning. More general, we will consider epimorphisms $A \to M_d(D)$ where D is a division ring which is finite dimensional over its center. At least in the case when k is not algebraically closed, this more general situation is definitely of importance, as the case of tame finite dimensional hereditary algebras shows (see 6.4).

3.1 PROPOSITION: Let A be a finitely generated k-algebra, and $\delta : A \to M_d(D)$ an epimorphism, where D is a division ring, finite dimensional over its center. Then there exists an order R in D such that $\delta(A) \subseteq M_d(R)$ and such that the induced map $A \to M_d(R)$ is an epimorphism, with R finitely generated as k-algebra.

Proof: We can assume that A is a subring of $M_d(D)$, with δ the inclusion. Let A be generated by $a_1,\dots,a_m$, and let $a_i = (\alpha^i_{st})_{st}$ with $\alpha^i_{st} \in D$. Let R' be the k-subalgebra of D generated by the elements α^i_{st}. Then $A \subseteq M_d(R')$ and, as we have seen in the proof of 1.3, the division ring D is generated (as a division ring) by R'. Note that D being finite dimensional over its center, implies that R' is a prime PI-ring, thus every element of D is of the form $c^{-1}r$, with $r \in R'$, and $0 \neq c \in C$, the center of R' ([31], VIII, 1.4). Since $A \subseteq M_d(D)$ is an epimorphism, every element of $M_d(D)$ satisfies a zigzag over A [28, 30] in particular this is true for the matrix units e_{ij} of $M_d(D)$. Let $e = e_{ij}$ be such a matrix unit, and take a zigzag, say $e = xYz^T$, with $x = (x_1,\ldots,x_u)$,

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1v} \\ \vdots & & \vdots \\ y_{u1} & \cdots & y_{uv} \end{pmatrix}, \quad z = (z_1,\ldots,z_v) \quad \text{where} \quad x_s, z_t \in M_d(D),$$

$y_{st} \in A$, for all s,t, and $xY \in A^u$, $zY^T \in A^v$ (here, ^{T} denotes the transpose). The elements x_s, z_t are in $M_d(D)$, thus they are of the form $x_s = c^{-1}x'_s$, $z_t = c^{-1}z'_t$ for some $x'_s, y'_t \in A$, and a fixed element $0 \neq c \in C$. Thus $e = c^{-2}(x'_1,\ldots,x'_u)Y(z'_1,\ldots,z'_v)^T$. To every matrix unit e_{ij} we obtain, in this way, a non-zero element $c_{ij} \in C$, and we denote by R the k-subalgebra of D generated by R' and the elements c_{ij}^{-1}. Let us determine the "dominion" B of A in $M_d(R)$, that is the set of elements of $M_d(R)$ determined by zigzags over A. Note that B is a subring. By construction, the matrix units belong to B, and therefore also the matrix entries of the elements of A, thus $M_d(R') \subseteq B$. However, with the scalar matrix c_{ij} also its inverse c_{ij}^{-1} belongs to B, thus $B = M_d(R)$. As a consequence [28, 30], the embedding $A \hookrightarrow M_d(R)$ is an epimorphism. It is clear that R is an order in D.

3.2 Note that for an epimorphism $A \to M_d(K)$, where A is a finite dimensional k-algebra, and K a commutative field, the induced map $A \to M_d(R')$, with R' being the ring generated by the matrix entries of the elements of A, does not have to be an epimorphism.

EXAMPLE: Let $A = \begin{pmatrix} k & k^3 \\ 0 & k \end{pmatrix}$, and $K = k(x)$, the field of rational functions in one variable. Consider the embedding $\delta : A \to M_4(K)$ which maps the elements
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & (100) \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & (010) \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & (001) \\ 0 & 0 \end{pmatrix}$ onto

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix},$$

with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. Then $R' = k[x]$.
Now it is easy to see that the embedding $A \to M_4(K)$ is an epimorphism, whereas, however the canonical map

$$A \longrightarrow M_4(k[x]) \longrightarrow M_4(k)$$

with $x \mapsto o$, is not an epimorphism. Thus $A \to M_4(R')$ cannot be an epimorphism. An example of a finitely generated k-subalgebra R of K with $\delta(A) \subseteq M_4(R)$, and such that this is an epimorphism, is given by $R = k[x,x^{-1}]$.

3.3 An important consequence whould be stressed: Again, let A be a finitely generated k-algebra. We have seen that any epimorphism $\delta : A \to M_d(D)$, with D a division ring which is finite dimensional over its center, gives rise to an epimorphism

$A \to M_d(R)$, with R a finitely generated k-algebra which is an order in D. Now R is a finitely generated k-algebra which is a PI-domain, and therefore we can use the Hilbert-Procesi-Nullstellensatz ([31], V, 1.2): any simple R-module is finite dimensional over k, and the Jacobson radical is zero. This shows that in case D is infinite dimensional over k, we obtain an infinite family of simple R-modules (all of which are finite dimensional over k), and thus we obtain an infinite family of finite dimensional A-modules which are points.

Also note that we may replace R by the localisation with respect to one additional non-zero element α in the Formanek center, and thus we may assume that R, and then also $M_d(R)$ is an Azumaya algebra ([31], VIII, 2.2.(1)). This then has the following consequence: If m is any maximal ideal of $M_d(R)$, then there exists a specialisation from the epimorphism $\delta : A \to M_d(D)$ to the epimorphism $\varepsilon : A \to M_d(R) \to M_d(R)/m$. For, in an Azumaya algebra, we can localise at the maximal ideal m, and thus we obtain an epimorphism $A \to M_d(R)_m$ through which both δ and ε factor in the appropriate way.

4. The Universal Construction Of Families Of Modules

Let k be an algebraically closed field, and A a finite dimensional k-algebra. Since k is algebraically closed, the dimension vector of a point X with endomorphism ring D is given by the simpler formula

$$(\underline{\dim}\, X)_i = \dim f_i X_D \ ,$$

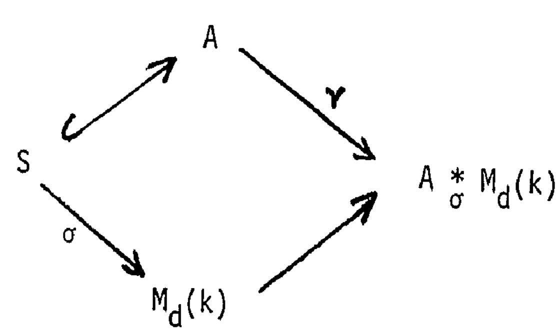
where f_i is a primitive idempotent with $Af_i/(\text{rad } A)f_i$ being the simple A-module with index i. In particular, $\underline{\dim}\, X$ belongs to $\mathbb{N}^n$. According to 1.5, we may restrict our attention to points with dimension type in some fixed $\mathbb{Q}\,\underline{d}$, with $d = (d_1,\ldots,d_n) \in \mathbb{Q}^n$, and we may assume that the entries d_i are natural numbers, not all zero. Let us sketch a well-known construction which gives a universal ring $U_{\underline{d}}$ corresponding to the dimension type $\underline{d}$.

4.1 Since k is algebraically closed, there exists a subalgebra S with $A = S \oplus \text{rad } A$. Choose a complete set $f_1,\ldots,f_n$ of primitive idempotents; in this way, we also have indexed the simple modules. Now fix an algebra homomorphism $\sigma : S \to M_d(k)$. Then k^d becomes an S-module, let d_i be the multiplicity of the i-the simple S-module. Note that the multiplicity vector $\underline{d} = (d_1,\ldots,d_n)$ determines σ up to an inner automorphism of $M_d(k)$. We call an algebra homomorphism $\varphi : A \to M_d(R)$, with R a k-algebra, to be of type σ provided $\varphi(S) \subseteq M_d(k)$ and $\varphi \mid S = \sigma$.

LEMMA: If $\varepsilon : A \to M_e(E)$ is an epimorphism with E a division ring, and $\underline{\dim}\,\varepsilon = m\underline{d}$ for some $m \in \mathbb{N}$, then there exists an isomorphism $\alpha : M_e(E) \to M_d(M_m(E))$ such that $\varepsilon\alpha$ is of type σ.

Proof: Write $1 = \sum f_{ij}$ with f_{ij} primitive orthogonal idempotents in S such that $A\, f_{ij}/(\operatorname{rad} A)f_{ij}$ is the i-th simple A-module. Let $Y = E^e$, considered as an A-E-bimodule. Then $\dim(f_{ij}\,Y)_E = (\underline{\dim}\,Y)_i = md_i$, thus we may choose E-subspaces Y_{ijs}, $1 \le s \le d_i$ such that $\dim(Y_{ijs})_E = m$ and $f_{ij}\,Y = \bigoplus_s Y_{ijs}$. We may identify $\operatorname{End}((Y_{ijs})_E)$ with $M_m(E)$ as a k-algebra, and then $\operatorname{End}((\bigoplus Y_{ijs})_E)$ with $M_d(M_m(E))$. It is clear that in this way we obtain an isomorphism $\alpha' : M_e(E) \to M_d(M_m(E))$ such that $S^{\varepsilon\alpha'}$ lies in $M_d(k) \subseteq M_d(M_m(E))$. Applying an inner automorphism α'', we can achieve that the restriction of $\varepsilon\alpha'\alpha''$ to S is equal to σ.

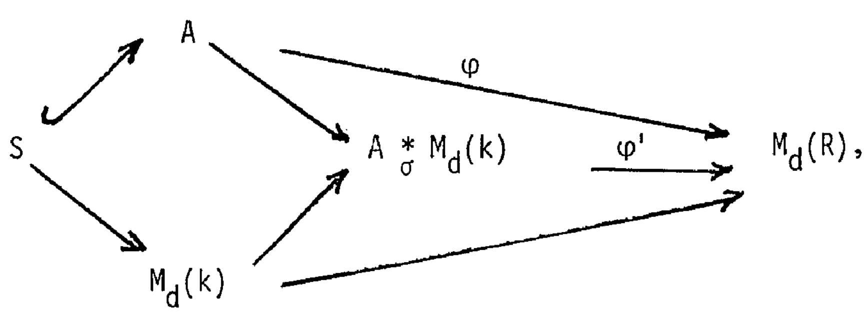
4.2. Let A be a k-algebra. S a semi-simple subalgebra and $\sigma : S \to M_d(k)$ an algebra homomorphism. Consider the free product $A \underset{\sigma}{*} M_d(k)$, that is the pushout, in the category of k-algebras (see [8,31])



Note that the images of the matrix units of $M_d(k)$ in $A \underset{\sigma}{*} M_d(k)$ make $A \underset{\sigma}{*} M_d(k)$ into a matrix ring, say

$$A \underset{\sigma}{*} M_d(k) = M_d(U_\sigma).$$

Now, given any algebra homomorphism $\varphi : A \to M_d(R)$ of type σ, there exists a unique commutative diagram of the form



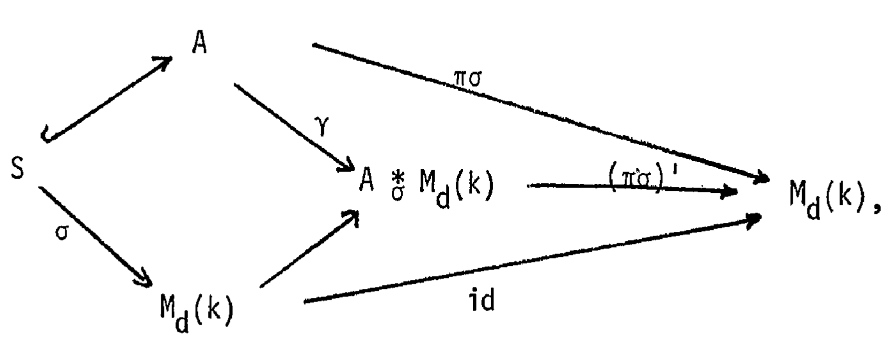
and since φ' preserves the matrix units, we see that it is of the form $M_d(\tilde{\varphi})$ with $\tilde{\varphi} : U_\sigma \to R$ a ring homomorphism. In particular, given any epimorphism $\varepsilon : A \to M_e(E)$ of dimension type a multiple of $\underline{d}$, and using an isomorphism $\alpha : M_e(E) \to M_d(M_q(E))$ such that $\varepsilon\alpha$ is of type σ, we determine an epimorphism $\widetilde{\varepsilon\alpha}$ $U_\sigma \to M_q(E)$ (with $\varepsilon\alpha$ also $(\varepsilon\alpha)'$ is an epimorphism, and then also $\widetilde{\varepsilon\alpha}$).

It is clear that U_σ only depends on the dimension vector $\underline{d} = (d_1,\dots,d_n)$, thus we will denote it also by $U_{\underline{d}}$.

4.3. The following proposition is essentially due to G. Bergman (here, we do not assume that k is algebraically closed):

<u>PROPOSITION</u>: Let A be a finite dimensional hereditary k-algebra, let S be a subalgebra of A such that $A = S \oplus \mathrm{rad}\, A$, and let $\sigma : S \to M_d(k)$ be an algebra homomorphism. Let $A \underset{\sigma}{*} M_d(k) = M_d(U_\sigma)$, then U_σ is a free ideal ring.

<u>Proof</u>: Without loss of generality, we may assume that σ is an embedding. For, let $\ker \sigma$ be generated by the central idempotend e, then $A \underset{\sigma}{*} M_d(k) = (1-e)A(1-e) \underset{\sigma'}{*} M_d(k)$, with σ' being the restriction, and with A also $(1-e)A(1-e)$ is hereditary. Now, [7] 2.5 asserts that $A \underset{\sigma}{*} M_d(k)$ is hereditary, again, and [7] 2.6 determines the structure of the projective $A \underset{\sigma}{*} M_d(k)$-modules: they are direct sums of modules obtained in the following way: let $R_0 = S$, $R_1 = A$, $R_2 = M_d(k)$, let P_i a projective R_i-module, and consider $P_i \underset{R_i}{\otimes} R$, where $R = A \underset{S}{*} M_d(k)$. Since any P_i is the direct sum of indecomposable modules, we can assume that $P = P_i$ itself is an indecomposable projective R_i-module. Thus, we can assume that there exists an idempotent $e \in M_d(k)$ such that $P = eR$, thus P is the direct sum of copies of the standard column module U_σ^d. On the other hand, R has a ringhomomorphism into a simple artinian ring, namely $(\pi\sigma)'$ with $\pi : A \to A/\mathrm{rad}\, A = S$ being the canonical projection



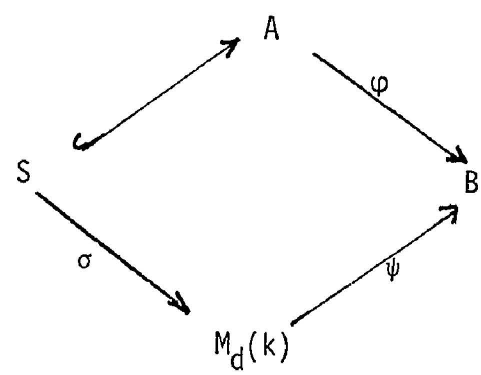
thus R is projective-trivial. The result now follows from [15], 1.4.2.

4.4 The basic finite dimensional hereditary k-algebras over an algebraically closed field are precisely the tensor algebras over quivers [25] without oriented cycles. In this case, we show that the corresponding universal rings U_σ are free algebras. Let Γ be a quiver, say with point set $\Gamma_0 = \{1,\ldots,n\}$ and arrow set Γ_1. Given $\alpha \in \Gamma_1$, denote by α' its source, by α'' its sink, thus $\alpha' \xrightarrow{\alpha} \alpha''$. Let $\underline{d} = (d_1,\ldots,d_n) \in \mathbb{N}^n$. Then we denote by $k\langle\Gamma,\underline{d}\rangle$ the free associative k-algebra generated by the variables $x_{\alpha st}$, with $\alpha \in \Gamma_1$, $1 \le s \le d_{\alpha'}$, $1 \le t \le d_{\alpha''}$, by $k[\Gamma,\underline{d}]$ the corresponding polynomial ring generated by the same set of variables, and by $k(\Gamma,\underline{d})$ the quotient field of $k[\Gamma,\underline{d}]$.

If one considers the set of representations (V_i,φ_α) with $V_i = k^{d_i}$ as the variety $\prod_\alpha \mathrm{Hom}(V_{\alpha''}V_{\alpha''})$, then $k[\Gamma,\underline{d}]$ is just its ring of regular functions. Note that in the following proposition we allow the quiver to have oriented cycles.

<u>PROPOSITION</u>: Let Γ be a quiver, and A its tensor algebra over k, with semi-simple part S. Let $\underline{d} = (d_1,\ldots,d_n) \in \mathbb{N}^n$, $d = \sum_i d_i$, and $\sigma : S \to M_d(k)$ the corresponding diagonal embedding. Then $A \underset{\sigma}{*} M_d(k) = M_d(k\langle\Gamma,\underline{d}\rangle)$.

Proof: Let us define an algebra homomorphism $\gamma : A \to M_d(k\langle\Gamma,\underline{d}\rangle)$, and verify the universal property. The diagonal embedding $\sigma : S \to M_d(k) \to M_d(k\langle\Gamma,\underline{d}\rangle)$ defines a block structure for these matrices, the blocks being $d_i \times d_j$-blocks, with $1 \le i, j \le n$. Now A is generated over S by the elements $\alpha \in \Gamma_1$. For α with $i = \alpha'$, $j = \alpha''$, let α^γ be the matrix with zeros outside the i-j-block, and the following i-j-block $(x_{\alpha st})_{st}$. This defines γ. Now assume, we have given algebra homomorphisms



with commuting diagram. Using the images of the matrix units of $M_d(k)$ under ψ, we see that B is of the form $M_d(B')$, with B' a k-algebra, and $\sigma\psi$ defines a block structure with $d_i \times d_j$-blocks for $1 \le i, j \le n$.

Also, let e_i be the idempotent of S corresponding to the vertex $i \in \Gamma_0$, thus e_i^σ is the matrix with zero outside the i-i-block, and the $d_i \times d_i$ identity matrix in the i-i-block, and similarly $e_i^{\sigma\psi}$. Then, for $\alpha \in \Gamma_1$, with $i = \alpha'$, $j = \alpha''$, we have in A the relation $\alpha = e_i \cdot \alpha \cdot e_j$, thus under φ we get

$$\alpha^{\varphi} = e_i^{\varphi}\cdot\alpha^{\varphi}\cdot e_j^{\varphi} = e_i^{\sigma\psi}\cdot\alpha^{\varphi}\cdot e_j^{\sigma\psi} ,$$

and therefore all blocks of α^{φ} but the i-j-block are zero. Let the i-j-block of α^{φ} be $(b_{\alpha st})$, with $1 \leq s \leq d_i$, $1 \leq t \leq d_j$, and define the algebra homomorphism $\tilde{\varphi} : k<\Gamma,\underline{d}> \to B$ by $x_{\alpha st}^{\tilde{\varphi}} = b_{\alpha st}$. It is clear that this gives the factorisation we are looking for, and that it is the unique solution.

4.5. Let us study in more detail points with commutative endomorphism ring. In particular, given any finite dimensional point, its endomorphism ring being a finite dimensional division ring over the algebraically closed field k, has to coincide with k. Now, given an homomorphism $\varphi : A \to M_d(K)$ say of type $\sigma : S \to M_d(k)$, with K a commutative k-field, then the factorisation through $A \underset{\sigma}{*} M_d(k) = M_d(U_\sigma)$ gives us a map $\tilde{\varphi} : U_\sigma \to K$ which vanishes on the commutator ideal I, thus it induces an algebra homomorphism $\overline{U_\sigma} = U_\sigma/I \to K$. In particular, in case we consider the tensor algebra A of the quiver Γ, and σ is of dimension type $\underline{d}$, then $\overline{U_\sigma} = k<\Gamma,\underline{d}>$/commutator ideal $= k[\Gamma,\underline{d}]$. Note that $\overline{U_\sigma}$ is a finitely generated commutative k-algebra, and its maximal ideals correspond bijectively to the algebra homomorphisms $A \to M_d(k)$ of type σ, thus the affine variety corresponding to $\overline{U_\sigma}$ can be considered as parametrizing the possible representations of A of type σ (but not their isomorphism classes!).

In a similar way, we may consider homomorphisms $\varphi : A \to M_d(D)$

of type σ , such that the k-algebra D satisfies the polynomial identidies of all $q\times q$-matrices over commutative rings, for some fixed $q \in \mathbb{N}$. This is of interest when we consider together with the dimension type $\underline{d}$ of σ also representation of multiple dimension type $m\underline{d}$, with $m \in \mathbb{N}$. The induced homomorphism $\tilde{\varphi} : U_\sigma \to D$ factors over the universal factor ring $U_{\sigma,q}$ of U_σ satisfying the polynomial identities of $q\times q$-matrices over commutative rings (see [31]); note that $\overline{U_\sigma} = U_{\sigma,1}$. In case of a quiver Γ and dimension type $\underline{d}$, we obtain $U_{\sigma,q} = k\langle\Gamma,\underline{d}\rangle_q$, the ring of generic $q\times q$-matrices in the variables $x_{\alpha st}$.

Let us come back to the finite dimensional representations $\varphi : A \to M_d(k)$ of type $\sigma : S \to M_d(k)$. Such a representation gives k^d an A-module structure. Also, to φ corresponds a unique k-homomorphism $\tilde{\varphi} : \overline{U_\sigma} \to k$, thus a maximal ideal. If we start with a maximal ideal m of $\overline{U_\sigma}$, the corresponding module structure on k^d obtained via the canonical algebra homomorphism

$$A \xrightarrow{\gamma} M_d(\overline{U_\sigma}) \longrightarrow M_d(\overline{U_\sigma})/m = M_d(k)$$

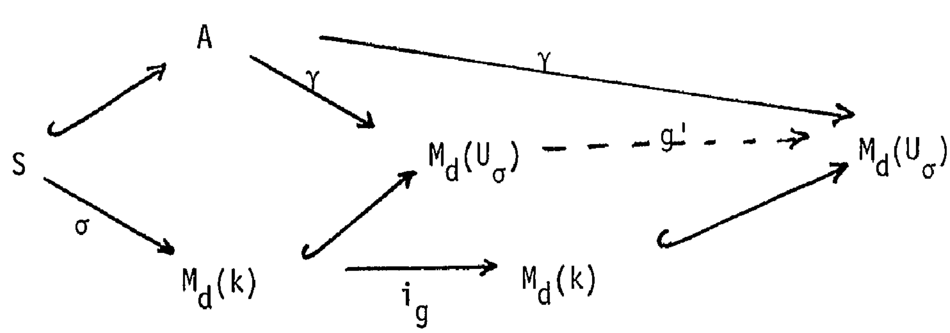
will be denoted by X_m. Note that different ideals m may give isomorphic module structures, and also note that not all modules X_m are points. Thus there are two problems: determine the set of maximal ideals m such that X_m is a point, and determine when two modules X_m and $X_{m'}$ are isomorphic. We want to reformulate these questions in terms of the operation of a reductive algebraic group operating on the affine variety spec $\overline{U_\sigma}$

4.6. Again, assume $A = S \oplus \text{rad } A$ is a finite dimensional k-algebra over an algebraically closed field k, and $\sigma : S \to M_d(k)$ an algebra homomorphism. Let

$$G_\sigma = \{g \in Gl_d(k) \mid s^\sigma g = gs^\sigma \text{ for all } s \in S\},$$

the centralizer of the image of S in $Gl_d(k)$. It is clear that $G_\sigma = \prod_i Gl_{d_i}(k)$, where $\underline{d} = (d_1,\ldots,d_n)$ is the dimension type of σ.

Note that G_σ operates on $M_d(U_\sigma)$ in the following way: any $g \in G_\sigma$ gives rise to a unique algebra endomorphism g' of $M_d(U_\sigma)$ making the following diagram commutative



where i_g denotes conjugation by g.

PROPOSITION: There exists a (unique) algebra automorphism $\hat{g}$ of U such that $g' = i_g M_d(\hat{g}) = M_d(\hat{g}) i_g$, with i_g the conjugation of $M_d(U_\sigma)$ by g.

Proof: The restriction of g' to $M_d(k)$ is the conjugation by g, thus this restriction has $i_{g^{-1}}$ as inverse. This shows that $g'i_{g^{-1}}$ and $i_{g^{-1}}g'$ both preserve the matrix units of $M_d(U_\sigma)$, and therefore $g'i_{g^{-1}} = M_d(\hat{g})$, $i_{g^{-1}}g' = M_d(\hat{\hat{g}})$ for some automorphisms $\hat{g}$, $\hat{\hat{g}}$ of U_σ. However, the restriction of $M_d(\hat{g})$ and $M_d(\hat{\hat{g}})$ to scalar matrices shows that $\hat{g} = \hat{\hat{g}}$, since $i_{g^{-1}}$ commutes with all scalar matrices.

The operations of G_σ on U_σ via $\hat{g}$, and on $M_d(U_\sigma)$ via $i_g\hat{g}$ are of great interest, in particular one should determine the rings of invariants in both cases. If we factor out the commutator ideal of U_σ, and go over to the quotient field $Q\overline{U_\sigma}$ of $\overline{U_\sigma}$, the group G_σ operates also on $Q\overline{U_\sigma}$. In general, the invariant ring $M_d(Q\overline{U_\sigma})^{G_\sigma}$ is an algebra over the field $QU_\sigma^{G_\sigma}$, and the canonical map $A \to M_d(Q\overline{U_\sigma})$ maps into $M_d(Q\overline{U_\sigma})^{G_\sigma}$. Let us show that the action of G_σ on U_σ coincides in the case of the tensor algebra of a quiver with a well-known operation [25, 26]. Let Γ be a quiver, and A its tensor algebra with semi-simple subalgebra S. Let $\underline{d}$ a dimension type, and $\sigma : S \to M_d(k)$ a corresponding embedding. We know that $\overline{U_\sigma} = k[\Gamma,\underline{d}]$, and $\gamma : A \to M_d(\overline{U_\sigma})$ is given by the rule that for an edge α with $\alpha' = i, \alpha'' = j$, the matrix $\alpha\gamma$ is zero outside the i-j-block, and with i-j-block $(x_{\alpha st})_{st}$. Now let $G_\sigma = \prod_i Gl_{d_i}(k)$ operate on $\overline{U_\sigma}$ as follows: given $g = (g_i)_i$, with $g_i \in Gl_{d_i}(k)$, and $\alpha \in \Gamma_1$, with $\alpha' = i$, $\alpha'' = j$, consider the $d_i \times d_j$ matrices $(x_{\alpha st})_{st}$ and $g_i(x_{\alpha st})_{st}g_j^{-1}$, and let $(\hat{g}x_{\alpha st})_{st} = g_i(x_{\alpha st})_{st}g_j^{-1}$. In this way we obtain an operation of G_σ on $k[\Gamma,\underline{d}]$, which satisfies

$\gamma i_g M_d(\hat{g}) = \gamma$, and therefore coincides with the action of G_σ on $\overline{U_\sigma}$ denoted in the same way.

4.7. We come back to the questions asked in 4.5. Recall that we have defined there for every maximal ideal m of $\overline{U_\sigma}$ an A-module X_m which is d-dimensional over k.

<u>PROPOSITION</u>: Let m, m' be maximal ideals of $\overline{U_\sigma}$. Then $X_m \approx X_{m'}$ if and only if m and m' belong to the same G_σ-orbit. And X_m is a point if and only if the stabilizer of m in G_σ is k^X, the diagonally embedded multiplicative group.

<u>Proof</u>: Given $g \in G_\sigma$, and a maximal ideal m of $\overline{U_\sigma}$, the commutative diagram

$$\begin{array}{ccccc} & \overset{\gamma}{\nearrow} & M_d(U_\sigma) & \xrightarrow{\pi} & M_d(\overline{U_\sigma}/m) \\ A & & \Big\downarrow i_g M_d(\hat{g}) & & \Big\downarrow i_g \\ & \underset{\gamma}{\searrow} & M_d(U_\sigma) & \xrightarrow{\pi'} & M_d(\overline{U_\sigma}/gm) \end{array}$$

with the canonical epimorphisms π, π' shows that the representations given by $\gamma\pi$ and $\gamma\pi'$ are isomorphic via the base change g. Conversely, assume there are given maximal ideals m, m' such that the corresponding representations $\gamma\pi$ and the $\gamma\pi'$ are isomorphic. Then, there exists $g \in Gl(d,k)$ such that for all $a \in A$, we have $g^{-1}a^{\gamma\pi}g = a^{\gamma\pi'}$. In particular, for $s \in S$, we have $g^{-1}s^{\sigma}g = s^{\sigma}$, since $\gamma\pi$ and $\gamma\pi'$ both are of type σ. Thus $g \in G_{\sigma}$, and it follows that $m' = gm$.

On the other hand, X_m is a point if and only if its endomorphism ring is the base field, and this is true if and only if its automorphism group is $k^{\times}$. But the automorphism group is given by the stabilizer of m in G_{σ}.

<u>COROLLARY</u>: The set of maximal ideals m with X_m a point is open in the affine variety corresponding to $\overline{U_{\sigma}}$.

<u>Proof</u>: This follows from the fact that the stabilizer dimension is semi-continous, and, as we have seen, X_m is a point if and only if the stabilizer dimension of m is 1, the smallest possible value.

Of course, the main problem now is to determine the intersection of the maximal ideals m such that X_m is not a point - that is, the ideal defining the closed subvariety of all m, with X_m not a point. In the case of a one-point quiver with some arrows (that is, A is a free associative algebra) this ideal has been determined: it is the radical of the Formanek center ([31],VIII, 2.1).

4.7. <u>REMARK</u>: The universal ring $\overline{U_d}$ for studying d-dimensional representations of a k-algebra A was considered by various authors [1,8,31,32], here $\overline{U_d} = U_d$/commutator ideal, and

$A \underset{k}{*} M_d(k) = M_d(U_d)$. Of course, in case A has a semi-simple subalgebra S, it seems reasonable to refine the construction by fixing a representation $\sigma : S \to M_d(k)$, that is a dimension type, and considering $A \underset{\sigma}{*} M_d(k) = M_d(U_\sigma)$, as we have done it here. Note that Cohn's new approach to consider a "spectrum" of a non-commutative ring, referred to in 1.8, uses these matrix reduction rings U_d.

5. Typical Situations

Let k be an algebraically closed field, Γ a quiver, and A the tensor algebra of Γ over k. As we know, the spectrum of A is the disjoint union of the subsets given by all points with dimension type in a fixed $\mathbb{Q}\,\underline{d}$, with $\underline{d} \in \mathbb{N}^n$. Thus, let us consider such a subset. Of course, we can assume that the elements $d_1,\ldots,d_n$ do not have a proper common divisor. We will denote by $\underline{\dim}\, X$ both the dimension type of a point as well as the usual dimension type of a finite dimensional representation of Γ; note that they coincide in case both are defined.

5.1 Let $k\langle x_1,\ldots,x_q\rangle$ be a free associative algebra, and $f_{\alpha st} \in k\langle x_1,\ldots,x_q\rangle$ with $\alpha \in \Gamma_1$, $1 \leq s \leq d_{\alpha'}$, $1 \leq t \leq d_{\alpha''}$. Then these elements $f_{\alpha st}$ determine a functor $T: {}_{k\langle x_1,\ldots,x_q\rangle}M \to {}_A M$ as follows: Let M be a $k\langle x_1,\ldots,x_q\rangle$-module, then $T(M) = (T_i(M), T_\alpha(M))$, with $T_i(M) = \bigoplus_{d_i} M$, and $T_\alpha(M) = (f_{\alpha st})_{st} : T_{\alpha'}(M) \to T_{\alpha''}(M)$. Also, if $\varphi : M \to M'$ is a homomorphism of $k\langle x_1,\ldots,x_q\rangle$-modules, let $T(\varphi) = (T_i(\varphi))_i$, with $T_i(\varphi) = \bigoplus_i \varphi : \bigoplus_{d_i} M \to \bigoplus_{d_i} M'$. Functors of this kind have been constructed by various authors [13, 25, 27, 35], in order to show that certain quivers are of infinite representation type, or even wild. The typical situation to be considered seems to be the following:

(i) The elements $f_{\alpha st}$ are linear polynomials, thus belong to $k \oplus \bigoplus_{i=1}^{q} k\, x_i$;

(ii) The functor T is full, and

(iii) $q = \sum_{\alpha\in\Gamma_1} d_{\alpha'}d_{\alpha''} - \sum_i d_i^2 + 1$

In this case, we will call T a typical functor. Of course, such a functor can only exist in case $q \geq o$ (note that $q = -Q(\underline{d}) + 1$, where Q is the usual quadratic form associated to the quiver Γ, see [24, 25, 26]). Note that such a functor is a full exact embedding and the $\underline{\dim}\, T(M) = m\cdot\underline{d}$, both for finite dimensional modules and for points, with $m = \dim_k M$ in case M is finite dimensional over k, and with $m = \dim M_{\mathrm{End}(M)}$ in case M is a point.

Let T be a typical functor. Consider first its restriction to ${}_{k[x_1,\ldots,x_q]}M$. The simple $k[x_1,\ldots,x_q]$-modules are 1-dimensional, thus their images under T correspond to modules with dimension vector $\underline{d}$, and all are points. This gives us a family of points of type $\underline{d}$ indexed by the q-dimensional affine space $\mathbb{A}^q$. In fact, we may consider the set of representations of Γ of dimension type $\underline{d}$ as an affine variety $\mathbb{A}^v$, with $v = \sum_{\alpha\in\Gamma_1} d_{\alpha'}d_{\alpha''}$, whose coordinate ring is $k[\Gamma,\underline{d}]$, on which the group $G_{\underline{d}} = \prod_i GL_{d_i}(k)$ operates in such a way that the orbits correspond to the isomorphism classes (see 4.6). Then, $\mathbb{A}^q$ embeds into $\mathbb{A}^v$ with respect to $(x_i)_i \mapsto (f_{\alpha st})_{\alpha st}$ as a linear subspace, and the image consists only of points with stabilizer $k^\times$, and it hits every orbit in at most one point. (The first assertion follows from the fact that the endomorphism ring $T(S)$, for S a simple $k[x_1,\dots,x_q]$-module, is k, the second assertion from the fact, that $T(S) \approx T(S')$ for S,S' simple $k[x_1,\dots,x_q]$-modules, implies $S \approx S'$.) As a consequence, the induced map $\mathbb{A}^q \times G/k^X \to \mathbb{A}^V$, given by $((x_i),\bar{g}) \to g(f_{\alpha st})_{\alpha st}$ is injective. Since both varieties have the same dimension, we see that the image is a dense subset ([12]). Also, it follows that $k(x_1,\dots,x_q)$ can be identified with the field of rational invariants $k(\Gamma,\underline{d})^{G_{\underline{d}}}$, thus the field of rational invariants $k(\Gamma,\underline{d})^{G_{\underline{d}}}$ is a rational extension of k. If we consider the partially ordered sets of all points of the spectrum of A with commutative endomorphism ring and dimension type $\underline{d}$, then we see that it has a unique maximal element, namely $T(k(x_1,\dots,x_q))$ and its endomorphism ring is precisely $k(x_1,\dots,x_q)$.

Next, let $m \in \mathbb{N}$, and consider the ring $k\langle x_1,\dots,x_q\rangle_m$ of generic $m \times m$ matrices (that is, the factor ring of $k\langle x_1,\dots,x_q\rangle$ modulo the ideal of all polynomials which vanish on $m \times m$ matrices over commutative rings). If we localise this ring with respect to non-zero element in the Formanek center, then all simple modules are m-dimensional [31], and conversely, every simple m-dimensional $k\langle x_1,\dots,x_q\rangle$-module is $k\langle x_1,\dots,x_q\rangle_m$-module. Under T, the m-dimensional $k\langle x_1,\dots,x_q\rangle$-modules give representations of Γ of dimension type $m\,\underline{d}$, and again, it is easy to see that the set of representations of Γ isomorphic to one of the form $T(S)$, with S a simple m-dimensional $k\langle x_1,\dots,x_q\rangle$-module is dense in the affine variety $\mathbb{A}^{m^2 V}$ of representations of dimension type $m\,\underline{d}$.

Again, there is a generic one, namely $T(Qk\langle x_1,\ldots,x_q\rangle_m)$, the image of the quotient division ring $Qk\langle x_1,\ldots,x_q\rangle_m$ of $k\langle x_1,\ldots,x_q\rangle_m$ (see [2, 31]). Whether the corresponding invariant ring $k[\Gamma,m\underline{d}]^{G_{m\underline{d}}}$ is rational is an open problem (see [23, 27, 31]).

Finally, note that there are additional points of dimension type in $\mathbb{Q}\,\underline{d}$. In fact, as in 1.2 , it is easy to see that for any finitely generated k-division ring D, there exists a point with dimension type in $\mathbb{Q}\,\underline{d}$ and endomorphism ring D. Of particular interest seems to be the image under T of the universal field of fractions of $k\langle x_1,\ldots,x_q\rangle$ (see [15]).

5.2 Let us consider one example in more detail. Consider the quiver $\Gamma : \bullet \xrightarrow{\alpha} \bullet \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \bullet$, and the dimension type $(1,3,2) = \underline{d}$.

Let $k[\Gamma,\underline{d}] = k[x_i,y_{ij},z_{ij} \mid 1 \leq i \leq 3,\ 1 \leq j \leq 2]$, where the x_i,y_{ij},z_{ij} are the following coordinate functions

$$\bullet \xrightarrow{X = (x_1,x_2,x_3)} \bullet \ \underset{Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}}{\overset{Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}}{\rightrightarrows}} \ \bullet$$

A maximal ideal of $k[\Gamma,\underline{d}]$ is of the form $\langle x_i-\alpha_i, y_{ij}-\beta_{ij}, z_{ij}-\gamma_{ij}\rangle = m$ with elements $\alpha_i,\beta_{ij},\gamma_{ij} \in k$, and corresponds to the representation

$$k \xrightarrow{(\alpha_1\alpha_2\alpha_3)} k^3 \underset{\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\\ \gamma_{31} & \gamma_{32}\end{pmatrix}}{\overset{\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32}\end{pmatrix}}{\rightrightarrows}} k^2$$

We want to determine an ideal I of $k[\Gamma,\underline{d}]$ whose zero set $V(I)$ in $\mathbb{A}^{15}$ is the set of all representations which are not points.

Let $I_1 = \langle\det\begin{pmatrix}YZ0\\0YZ\end{pmatrix}\rangle \cap \langle X_1,X_2,X_3\rangle$.

If m is a maximal ideal, then $I_1 \subseteq m$ if and only if $\det\begin{pmatrix}YZ0\\0YZ\end{pmatrix} \subseteq m$ or $\langle X_1,X_2,X_3\rangle \subseteq m$. The first condition $\det\begin{pmatrix}YZ0\\0YZ\end{pmatrix} \subseteq m$ is equivalent to the fact that the restriction of M_m to $\bullet \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \bullet$ decomposes, the second condition $\langle X_1,X_2,X_3\rangle \subseteq m$ is equivalent to the fact that the restriction of M_m to $\bullet \xrightarrow{\alpha} \bullet$ is the zero representation. Thus, $I_1 \nsubseteq m$ is equivalent to the fact that $M_m = (\alpha_i,\beta_{ij},\gamma_{ij})$, the map $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is a monomorphism and $\bullet \underset{(\gamma_{ij})}{\overset{(\beta_{ij})}{\rightrightarrows}} \bullet$ is indecomposable.

If $m \in \mathbb{A}^{15} \setminus V(I)$, then it is clear that M_m is a point, and the orbits of $\mathbb{A}^{15} \setminus V(I)$ under the canonical action of the group $G_{132} = GL_1(k) \times Gl_3(k) \times Gl_2(k)$ form a projective space $\mathbb{P}_2(k)$. Namely, consider the subset in $\mathbb{A}^{15}$

$$\cdot \xrightarrow{(\alpha_1,\alpha_2,\alpha_3)} \cdot \underset{\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}}{\overset{\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}}{\rightrightarrows}} \cdot$$

with $(\alpha_1,\alpha_2,\alpha_3) \neq (0,0,0)$. Then we obtain representatives of all orbits outside $V(I)$, and two such representations given by $(\alpha_1,\alpha_2,\alpha_3)$, $(\alpha_1',\alpha_2',\alpha_3')$ are isomorphic iff $k(\alpha_1,\alpha_2,\alpha_3) = k(\alpha_1',\alpha_2',\alpha_3')$. Thus, we obtain in this way also typical functors, for example $T : {}_{k<X_1,X_2>}M \to {}_AM$ given by

$$\cdot \xrightarrow{(X_1,X_2,1)} \cdot \underset{\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}}{\overset{\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}}{\rightrightarrows}} \cdot$$

Next, let $I_2 = \langle\det\begin{pmatrix}X\cdot Y\\X\cdot Z\end{pmatrix}\rangle \cap \langle 3\times 3$ minors of $(YZ)\rangle$. Note that for a maximal ideal m, the condition $\det\begin{pmatrix}X\cdot Y\\X\cdot Z\end{pmatrix} \subseteq m$ means that the images of $\alpha\beta$ and $\alpha\gamma$ are linearly dependent, whereas the fact that all 3×3 minors of (YZ) are contained in m means that the intersection of the kernels of β and γ is non-zero. Since a representation of type $(1,3,2)$ which contains an indecomposable submodule of type $(1,1,2)$ is indecomposable if and only if it does not split off a copy of $(0,1,0)$, it follows that for a maximal ideal m, we have $I_2 \not\subseteq m$ if and only if M_m is indecomposable and contains an indecomposable submodule of type $(1,1,2)$. Applying the Coxeter functor C^-, we see that the indecomposable modules of type (1,3,2) containing an indecomposable module of type (1,1,2) correspond to the indecomposable modules of type (2,3,4) containing an indecomposable module of type (0,3,4), thus they form again a projective plane $\mathbb{P}_2$. Also all these modules are points.

Let $I = I_1 + I_2$, thus $V(I) = V(I_1) \cap (V(I_2)$ is the set of maximal ideals m such that M_m the restriction to $\cdot \overset{\beta}{\underset{\gamma}{\rightrightarrows}} \cdot$ decomposes and the images of $\alpha\beta$ and $\alpha\gamma$ are linearly dependent, and this is equivalent to the fact that M_m is decomposable.

Consider finally $I_0 = I_1 \cap I_0 = \langle\det\begin{pmatrix} YZ0 \\ 0YZ \end{pmatrix}\rangle \cap \langle\det\begin{pmatrix} X\cdot Y \\ X\cdot Z \end{pmatrix}\rangle$. We have $I_0 \subseteq m$ for a maximal ideal m if and only if either $\cdot \overset{\beta}{\underset{\gamma}{\rightrightarrows}} \cdot$ decomposes, or the images of $\alpha\beta$ and $\alpha\gamma$ are linearly dependent (or both). Note that representatives of the orbits in $V(I_0) \setminus V(I_2) = V(I_1) \setminus V(I_2)$ are given by the representations

$$k \xrightarrow{(\alpha_1\alpha_2\alpha_3)} k^3 \overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}{\underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}{\rightrightarrows}} k^2$$

with $(\alpha_1,\alpha_2,\alpha_3) \neq (0,0,0)$ such that $\alpha_1\alpha_2 = \alpha_2^2$, since this is the condition for the fact that the images of $\alpha\beta$ and $\alpha\gamma$ are linearly dependent. Thus, we obtain in the orbit space $\mathbb{P}_2$ of $V(I_1)$ under G_{132} the quadric $V(x_1x_3-x_2^2)$. Similarly, we see that the representations M_m, with $m \in V(I_2) \setminus V(I_1)$, are

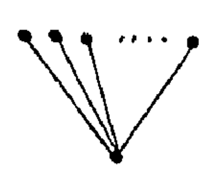
6. The Spectrum Of A Tame k-Species

6.1. In this last section, we assume that A is a finite dimensional hereditary algebra which is twosided indecomposable. Given such an algebra A with n simple modules, consider the vector space $\mathbb{Q}^n = K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $K_0(A)$ is the Grothendieck group of A (the free abelian group genereated by the simple A-modules), and given an A-module M of finite length, let $[M]$ be the corresponding element in $\mathbb{Q}^n$. Since we assume that A is hereditary, the function $b([M],[M']) = \dim_k \mathrm{Hom}_A(M,M') - \dim_k \mathrm{Ext}^1(M,M')$ is bilinear, and therefore defines a quadratic form q_A on $\mathbb{Q}^n$. It is well-known that A is of finite representation type if and only if q_A is positive definite, and A is called <u>tame</u> provided q_A is positive semi-definite.

<u>THEOREM</u>: Let A be a twosided indecomposable, finite dimensional hereditary k-algebra which is tame. Then there exists a unique point ${}_AX$ with $\dim_k X$ infinite.

Equivalently: there exists a unique equivalence class of epimorphims $\delta : A \to M_d(D)$ with $[D:k]$ infinite.

6.2. The theorem above allows us to determine completely the spectrum of a tame k-species. Denote by $P_{\aleph}$ the partially ordered set



of cardinality $\aleph$ with a unique element which specialises into all others.

<u>COROLLARY</u>: Let A be a twosided indecomposable finite dimensional hereditary k-algebra which is tame. Then the spectrum of A is the disjoint union of a countable number of one-point-sets and a component of the form $P_\aleph$ with $\aleph = \max(\aleph_0, |k|)$.

<u>Proof</u>: By 1.5, we know that the spectrum of A is the disjoint union of the sets $Sp_{\mathbb{Q}\underline{d}}$ of points with dimension type in $\mathbb{Q}\underline{d}$. If $\mathbb{Q}\underline{d}$ contains neither a Weyl root nor a null root, then $Sp_{\mathbb{Q}\underline{d}} = \emptyset$. If $\mathbb{Q}\underline{d}$ contains a Weyl root, then there exists a unique indecomposable module with dimension type in $\mathbb{Q}\underline{d}$, thus either $Sp_{\mathbb{Q}\underline{d}}$ is a one-point-set (in case this module is a point) or is empty. It is easy to determine all dimension types with $Sp_{\mathbb{Q}\underline{d}}$ a one-point-set, in particular, there are a countable number of such types (2.6). For $\underline{d}$ a null root, $Sp_{\mathbb{Q}\underline{d}}$ is of the form $P_\aleph$.

6.3. A point was defined to be a module with endomorphism ring a division ring and being finite dimensional over its endomorphism ring. If we drop the last condition, then the situation is completely different: It has been shown in [35, 36] that given any finite dimensional hereditary k-algebra A which is not of finite representation type, there exists a finite extension field k' of k such that any k'-algebra B which is generated over k' by less than $\aleph_1$ (the first strongly inaccessible cardinality) elements, can be realised as the endomorphism ring of an A-module. In particular, this applies to any division ring which is a k'-algebra and generated by less than $\aleph_1$ elements - of course, we see from theorem 6.1 that the corresponding A-module usually will be infinite dimensional over its endomorphism ring.

6.4. Assume from now on that A is a twosided indecomposable, finite dimensional, hereditary k-algebra of tame representation type. Denote the unique infinite dimensional point by ${}_AQ$, let $D = \mathrm{End}({}_AQ)$, and $\underline{d}$ the dimension type of ${}_AQ$. Note that the existence of such a module has been shown in [36], 5.3 and 5.7, the unicity will be proved below. The division ring D and the vector $\underline{d}$ are interesting invariants of the algebra A; the vector $\underline{d}$ (or, at least, the line $\mathbb{Q}\underline{d}$) depends only on the type of A and has been determined in [36] (see 5.7, and the column denoted $(-\delta P_i)_i$ in the table in 1. D).

Let us give some remarks concerning the possible structure of D. It follows from section 5 of [19] that one only has to consider the bimodule case $\tilde{A}_{11}$ and $\tilde{A}_{12}$. The algebras of type $\tilde{A}_{12}$ are of the form $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$, with ${}_FM_G$ a bimodule with $\dim {}_FM = \dim M_G = 2$. If ${}_FM_G$ is not simple, then $F = G$ and $M = M(\varepsilon,\delta)$ for some automorphism ε of F and some ε-1-derivation δ (see [35]), and then $D = F(t;\varepsilon,\delta)$, the quotient field of the twisted polynomial ring $F[t;\varepsilon,\delta]$. In particular, for $M = F \oplus F$, with canonical bimodule action, $D = F(t)$. If F,G are commutative, $F \supseteq H$, $G \supseteq H$, with $[F:H] = [G:H] = 2$ and

${}_F M_G = F \underset{H}{\otimes} G$, then D is the quotient field of the free product $F \underset{H}{*} G$ (note that D is uniquely determined since $F \underset{H}{*} G$ satisfies a polynomial identity). Finally, let us consider the case $\tilde{A}_{11}$. Then we have division rings $G \subset F$ with $\dim_G F = 4$, and the algebra is given by $\begin{pmatrix} G & F \\ 0 & F \end{pmatrix}$. For example, if $G = \mathbb{R}$, $F = \mathbb{H}$, then D is the quotient field of $\mathbb{R}[x,y] / (x^2+y^2+1)$, and therefore commutative, whereas for $G = \mathbb{Q}$, $F = \mathbb{Q}(\sqrt{2},\sqrt{3})$, we obtain the (non-commutative!) quotient ring of $\mathbb{Q}\langle x,y\rangle / (xy+yx, x^2+2y^2-3)$, see [21].

6.5. Let us recall from [36] certain notions and results concerning A-modules, with A a twosided indecomposable finite dimensional hereditary algebra of tame representation type. In 2.6, we have seen the notions of an indecomposable preprojective or preinjective module. Given any module M, the sum I(M) of all preinjective submodules is a direct sum of indecomposable preinjective submodules, and $I(M/I(M)) = 0$ ([36],3.3).
A module M is called <u>regular</u>, provided it has no indecomposable direct summand which is preprojective or preinjective; equivalently, $\mathrm{Hom}(M,P) = 0$ for P indecomposable preprojective and $\mathrm{Hom}(I,M) = 0$ for I indecomposable preinjective. The regular modules of finite length form an abelian category, the simple objects in this category are called <u>simple regular</u>. Given a module M, the sum of all submodules of finite length which are either preprojective or regular, is called its torsion submodule T(M). We have $T(M/T(M)) = 0$, [36] 4.1. If $T(M) = M$, then M

is called <u>torsion</u>; if $T(M) = 0$, then M is called <u>torsionfree</u>. Note that the torsion regular modules form an exact abelian subcategory ([36]. 4.4). Besides the indecomposable regular modules of finite length, there are additional indecomposable modules which are torsion regular, the so-called <u>Prüfer-modules</u> ([36], 4.5). Of importance is the following result: any indecomposable module which is not of finite length, is either a Prüfer module, or torsionfree regular ([36], 4.8). Finally, we mention that a module X is called <u>divisible</u> if $\mathrm{Ext}^1(S,X) = 0$ for all simple regular modules, and this is equivalent to the fact that $\mathrm{Hom}(X,S) = 0$ for all simple regular modules S. It has been shown in [36], 5.3 that there exists a unique indecomposable torsion-free divisible module Q, this is an infinite dimensional module, and it is a point [36], 5.3 and 5.7. We will show below that Q is characterised by the property of being an infinite dimensional point. For this proof, we will need two auxilliary results.

6.6 <u>LEMMA</u>: Let S be simple regular, and Y a direct sum of copies of S. Let X be a submodule of Y which has no non-zero preprojective direct summand. Then X is a direct sum of copies of S.

<u>Proof</u>: Let $Z = Y/X$, with epimorphism $\varepsilon': Y \to Z$. First, assume that Z is a direct sum of indecomposable preinjective modules. We want to show that $Z = 0$. If not, let $Z = Z' \oplus Z''$, with Z' indecomposable preinjective with projection $\pi : Z \to Z'$.

We claim that $Y = Y' \oplus Y''$, where Y' is a finite direct sum of copies of S, and Y'' is contained in the kernel of the projection $\varepsilon = \varepsilon'\pi : Y \to Z'$. For, let $Y = \bigoplus_{i \in I} Y_i$, with Y_i the image of an inclusion $\gamma_i : S \to Y$. Now ${}_{\mathrm{End}(S)}\mathrm{Hom}(S,Z')$ is of finite length thus there is a finite number of maps $\gamma_j\varepsilon$, say $1 \leq j \leq m$, such that any other $\gamma_i\varepsilon$ is a linear combination with coefficients γ_{ij} in $\mathrm{End}(S)$, say $\gamma_i\varepsilon = \sum_{j=1}^{m} \gamma_{ij}\gamma_j\varepsilon$. For $i \notin \{1,\ldots,m\}$, let Y_i'' be the image of $\gamma_i - \sum_{j=1}^{m} \gamma_{ij}\gamma_j$, and Y'' the direct sum of all Y_i'' with $i \notin \{1,\ldots,m\}$. We denote by $Y' = \bigoplus_{i=1}^{m} Y_i$, then $Y = Y' \oplus Y''$, Y' is a finite direct sum of copies of S, and $Y'' \subseteq \ker \varepsilon$.

Take now a decomposition $Y = Y' \oplus Y''$ with $Y'' \subseteq \ker \varepsilon$ and Y' of minimal length. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \xrightarrow{\varepsilon'} & Z & \to & 0 \\ & & & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\ & & & & Y/Y'' & \xrightarrow{\varepsilon''} & Z' & \to & 0 \end{array}$$

with the canonical projection π' and the induced map ε''. We denote the kernel of ε'' by W. Then, W cannot have a non-zero regular direct summand. For, we can identify Y/Y'' with Y', and an indecomposable regular submodule of Y' would be a direct summand of Y', thus if it lies in the kernel of ε, then we can use it to enlarge Y'', impossible. Thus W is a direct sum of indecomposable preprojective modules. Now $X\pi' \subseteq \ker \varepsilon'' = W$. However, since X has no indecomposable preprojective direct summand, Hom(X,W) = 0. Thus π' can be factored through ε' and gives rise to a map $\pi'' : Z \to Y/Y''$ with $\varepsilon'\pi'' = \pi'$. Since Z is a direct sum of preinjective modules, and Y/Y'' is regular, we conclude that $\pi'' = 0$, and therefore $\pi = \pi''\varepsilon'' = 0$. This contradiction shows that $Z = 0$.

Next, consider the general case, let $I(Z)$ be the submodule of Z generated by the indecomposable preinjective submodules. We obtain the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & X & \hookrightarrow & V & \longrightarrow & I(Z) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & X & \hookrightarrow & Y & \longrightarrow & Z & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z/I(Z) & = & Z/I(Z) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now, $Z/I(Z)$ is regular. For, $I(Z/I(Z)) = 0$ shows that it has no non-zero preinjective direct summand, and being a quotient of Y, it cannot have a non-zero preprojective quotient. Also $Z/I(Z)$ is generated by the images of the indecomposable summands of Y, thus it follows that $Z/I(Z)$ is torsion regular. Now, V is the kernel of a map $Y \to Z/I(Z)$, and therefore also torsion regular, and in fact then a direct sum of copies of S. This shows that we can apply the previous considerations to X considered as a submodule of V, and conclude that $I(Z) = V/X = 0$. This finishes the proof.

6.7 LEMMA: Let S be simple regular, and X a submodule of Y with Y/X a direct sum of copies of S. Then, if Y is regular, also X is regular.

Proof: If X contains a non-zero preinjective submodule, the same is true for Y. Thus, it remains to consider the case that X maps onto a non-zero preprojective module P, say $\alpha : X \twoheadrightarrow P$. Consider the induced exact sequence

$$\begin{array}{ccccccccc} 0 & \to & X & \hookrightarrow & Y & \to & Y/X & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & P & \hookrightarrow & Z & \to & Y/X & \to & 0 . \end{array}$$

Let $T(Z)$ be the torsion submodule of Z, that is the sum of all submodules of finite length which are either regular or preinjective. We claim that $Z/T(Z)$ is of finite length, and therefore a direct sum of indecomposable preprojective modules. Now the canonical map $T(Z) \hookrightarrow Z \to Y/X$ has torsion regular kernel and kokernel. The kernel is a submodule of the preprojective module P, thus zero. Denote the cokernel by W. Since it is a quotient of Y/X, it is again a direct sum of copies of S, say $W = \bigoplus_I S$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & P & \hookrightarrow & Z & \to & Y/X & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & P & \to & Z/T(Z) & \to & W & \to & 0 . \end{array}$$

If $|I|$ is infinite, or even $> \dim_{End(S)} Ext^1(S,P)$, then it is clear that we obtain in $Z/T(Z)$ a submodule isomorphic to S, but this is impossible since $T(Z/T(Z)) = 0$. Thus W, and therefore also $Z/T(Z)$, is of finite length. As a consequence, we see that Y maps onto an indecomposable preprojective module, and therefore it also has an indecomposable preprojective direct summand.

6.8 Proof of the theorem: Let ${}_AX$ be a point which is infinite dimensional over k. We want to show that X is torsionfree and divisible, it follows then from [36], 5.3 that X is uniquely determined. Now since X is indecomposable and not of finite length, it is either a Prüfer module or torsionfree regular. But the endomorphism ring of a Prüfer module is a proper discrete valuation ring, thus a Prüfer module is not a point. This shows that X is torsionfree regular. It remains to be seen that X is divisible.

Assume there is a simple regular module S with $Hom(X,S) \neq 0$. Let X_1 be the intersection of all kernels of maps $X \to S$. Note that X/X_1 is embeddable into some $\prod_I S$, with I an index set. However, since S is a point, say with corresponding epimorphism $\varepsilon : A \to M_e(E)$, we may consider S, and $\prod_I S$ as modules over $M_e(E)$, thus we can rewrite $\prod_I S = \bigoplus_J S$, for some index set J. Note that X/X_1, as a quotient of the regular module X, has no non-zero preprojective direct summand, thus according to 6.6, X/X_1 itself is a direct sum of copies of S, and therefore according to 6.7, the module X_1 is regular again. As a submodule of the torsionfree module X, it is also torsionfree. Also, the exact sequence

$$0 \to X_1 \to X \to \bigoplus S \to 0$$

shows that $\mathrm{Ext}^1(S,X_1) \neq 0$. This shows that X_1 satisfies properties similar to $X = X_0$: namely, it is torsionfree regular, and there exists a simple regular module S with $\mathrm{Ext}^1(S,X) = 0$, and therefore, there exists a simple regular module S_1 with $\mathrm{Hom}(X_1,S_1) \neq 0$. By induction, we obtain in this way a chain

$$X = X_0 \supset X_1 \supset X_2 \ \ldots\ldots$$

of proper submodules, with X_i/X_{i+1} being a direct sum of copies of some simple regular module S_i, and X_{i+1} the intersection of the kernels of all maps $X_i \to S_i$. Let $D = \mathrm{End}(X)$, a division ring. Then, it is clear that all X_i are invariant with respect to D, and consequently, X_D cannot be finite dimensional. This finishes the proof.

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THE STRUCTURE OF LOCALIZABLE ALGEBRAS HAVING FINITE GLOBAL DIMENSION

by

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Let R be a complete DVR with quotient field K, and Λ a hereditary R-order. Brumer [1] and Harada [6] independently showed that Λ is conjugate to an order of the form

$$\begin{pmatrix} d & d & \cdots & d \\ m & d & \cdots & \vdots \\ & & \ddots & \vdots \\ m & m & \cdots m & d \end{pmatrix}$$

partitioned into $n_i \times n_j$ blocks, all of whose entries belong to the indicated symbol (either d or m = rad d), where d is the unique maximal R-order in D, a finite-dimensional, central skewfield extension of K, and D_n is the quotient ring of Λ (e.g., see Reiner [13] or Roggenkamp [15]). Subsequently, Michler [9] extended the above to semiperfect HNP rings, and Jategaonkar [7] obtained a further generalization to pseudo-Dedekind rings.

A natural question to ask at this point is, can we find a "reasonable" canonical form for an arbitrary semiperfect order having finite global dimension? In a different but related direction, if Λ is a semi-perfect HNP ring, Λ is a finite intersection of a unique set of maximal orders, with each maximal order $\Gamma \supset \Lambda$ quasilocal (i.e., $\Gamma/\mathrm{rad}\,\Gamma$ simple Artin) and finitely generated projective over Λ (also see Eisenbud-Robson [5]). Moreover, for each maximal ideal M_i of Λ, there exists a unique maximal order $\Gamma_i \supset \Lambda$ such that $\mathrm{trace}\,\Gamma_{i_\Lambda} + M_i = \Lambda$. This observation was pointed out by Silver [16] for the classical case and incorporated into the following definition.

Let Λ be a semilocal ring (i.e., $\Lambda/\mathrm{rad}\,\Lambda$ semisimple Artin) with maximal two-sided ideals M_i, $i = 1, \ldots, n$. A <u>complete set of finite left localizations for</u> Λ (in the sense of Silver) is a set $\{\Lambda \to \Gamma_i,\ i = 1, \ldots, n\}$ where each map $\Lambda \to \Gamma_i$ is a finite left localization of Λ at M_i, i.e., each map $\Lambda \to \Gamma_i$ is a ring epi, $\Gamma_{i\Lambda}$ is finitely generated projective, and $\Gamma_i \otimes_\Lambda \Lambda/M_j = 0$, $\forall j \neq i$. Λ is said to be (finitely) <u>localizable</u> if each ${}_\Lambda\Gamma_i$ is also finitely generated projective. As Silver shows, each Γ_i is necessarily quasilocal and $\Lambda = \bigcap_{i=1}^{n} \Gamma_i$. Thus, semiperfect HNP rings are localizable with each localization quasilocal, and hence, a maximal order $\supset \Lambda$.

To further delineate the problems related to the earlier question, and, at the same time, to suggest a plan of attack, we ask:

1. Which semiperfect orders Λ are localizable?

2. Can we find a canonical form for the class of localizable, semiperfect orders?

The answer to 1 is unknown (to me at least!); however, there is some evidence to suggest that a reasonably large class of R-orders may indeed be localizable. Specifically, when $\mathrm{glb}\,\Lambda = n$ (expressed as Λ is n-dimensional or Λ is finite-dimensional) and Λ is semiperfect and

R-free, if each maximal order $\supset\Lambda$ is quasilocal, Λ is localizable. More generally, if $n = 2$ and $\Lambda \subset D_m$, Λ is localizable whenever $m \leq 7$.

On the other hand, a partial answer to 2 has been obtained by Keating [8], who has shown that whenever $\Lambda \subset D_n$ is a semiperfect localizable R-order, each $\Gamma_i \supset \Lambda$ is quasilocal and Λ is "nicely" tiled, i.e., $\Lambda = (\Lambda_{ij})$ where $\Lambda_{ii} = d$, $\forall i$, a fixed local R-order, and Λ_{ij} is a d-invertible, d-ideal, $\forall i, j$.

The purpose of this note is to announce several recent results of the author on the structure of finite-dimensional, localizable, semiperfect algebras which provide a solution to question 2 in the spirit of our earlier question. As succeeding sections will show, such algebras admit a rather transparent canonical form and a structure theory very reminiscent of the Brumer-Harada-Michler theory for semiperfect HNP rings.

Detailed proofs of all of these results will appear elsewhere.

Finally, I wish to express my thanks to the organizer of this conference, Professor Van Oystaeyen and his able assistants, for their generous hospitality, and to Ken Fields and Mark Ramras for many profitable conversations.

§1. THE SELF-BASIC CASE

Throughout this section and the next, all rings considered will be prime, Noetherian, semiperfect algebras. By an algebra Λ, we mean a ring Λ, finitely generated as a module over a subring R (with the same identity) contained in the center of Λ. The symbol D will always be reserved for a division ring.

Recall that when d is a maximal order over a complete DVR and $p = \operatorname{rad} d$,

$$\Lambda = \begin{pmatrix} d & \cdot & \cdot & \cdot & d \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \\ p & & & & d \end{pmatrix}$$

is the canonical form for a self-basic hereditary order. More generally, if d is n-dimensional and p is an invertible ideal of d with d/p (n-1)-dimensional, it is easy to see that Λ above is an n-dimensional localizable algebra. However, as the following simple example shows, we shall require a broader class of algebras for our purposes:

Let R be any 2-dimensional regular local ring with quotient field K, d a 2-dimensional, local R-order, and D the quotient field of d. If p and q are distinct height 1 primes of d, then

$$\Lambda = \begin{pmatrix} d & d & d & d \\ q & d & q & d \\ p & p & d & d \\ pq & p & q & d \end{pmatrix}$$

is a localizable R-order in $\Sigma = D_4$ which is 2-dimensional iff $p+q = m = \operatorname{rad} d$. Clearly, Λ is not conjugate to any triangular order in Σ since each maximal ideal of Λ is idempotent.

Motivated by the above example, we shall now describe a class of algebras which is broad enough to represent all algebras that we shall consider.

DEFINITION. For an integer $k > 1$, we shall call any sequence of integers $\delta_1, \ldots, \delta_n$ satisfying

a) $\delta_1 = 1$ and $\delta_n = k$,

b) $\delta_i < \delta_{i+1}$, $1 \leq i \leq n-1$,

c) $\delta_i \mid \delta_{i+1}$, $1 \leq i \leq n-1$,

a <u>divisor sequence</u> for k and denote it by the symbol $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_n)$.

Next, let $p_1, \ldots, p_m$ be (prime) ideals of d. For a positive integer n with divisor sequence $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_{m+1})$, we inductively define a class of subalgebras of d_n, denoted $\Delta(n, \underset{\sim}{\delta}; p_1, \ldots, p_m)$, as follows:

$$m = 1 : \Delta(n, \underset{\sim}{\delta}; p_1) = \Delta_n(d, p_1) = \begin{pmatrix} d & \cdots & d \\ & \ddots & \vdots \\ p_1 & & d \end{pmatrix}.$$

Assuming that $\Delta(n', \underset{\sim}{\delta}'; p_1, \ldots, p_\ell)$ has been defined $\forall n', \underset{\sim}{\delta}'$ and $\ell < m$, we define

$$\Delta(n, \underset{\sim}{\delta}; p_1, \ldots, p_m) = \begin{pmatrix} \Delta_m & \cdot & \cdot & \cdot & \Delta_m \\ p_m\Delta_m & \Delta_m & \cdot & & \cdot \\ \vdots & & \cdot & \cdot & \cdot \\ p_m\Delta_m & \cdot & \cdot & p_m\Delta_m & \Delta_m \end{pmatrix}$$

$$= \Delta_{n/\delta_m}(\Delta_m, p_m),$$

where $\Delta_m = \Delta(\delta_m, \underset{\sim}{\delta}'; p_1, \ldots, p_{m-1})$, $\underset{\sim}{\delta}' = (\delta_1, \ldots, \delta_m)$.

EXAMPLES. (a) $n = 6$ and $\underset{\sim}{\delta} = (1, 2, 6)$

$$\Delta(6, \underset{\sim}{\delta}; p, q) = \left(\begin{array}{cc|cc|cc} d & d & d & d & d & d \\ p & d & p & d & p & d \\ \hline q & q & d & d & d & d \\ pq & q & p & d & p & d \\ \hline q & q & q & q & d & d \\ pq & q & pq & q & p & d \end{array}\right).$$

(b) $n = 6$ and $\underset{\sim}{\delta} = (1, 3, 6)$

$$\Delta(6, \underset{\sim}{\delta}; q, p) = \left(\begin{array}{ccc|ccc} d & d & d & d & d & d \\ q & d & d & q & d & d \\ q & q & d & q & q & d \\ \hline p & p & p & d & d & d \\ pq & p & p & q & d & d \\ pq & pq & p & q & q & d \end{array}\right).$$

DEFINITION. If Λ is any finite-dimensional quasilocal algebra, Λ is called a regular algebra.

Before stating the main results of this section, we shall pause to record the following important result due to Vasconcelos [17], which underscores the role of maximal orders in the structure of localizable algebras. Specifically, the quasilocal algebras associated with these algebras are maximal orders.

THEOREM. (Vasconcelos). Any regular algebra is a maximal order in a simple algebra.

Many pertinent properties of the aforementioned class of algebras are summarized in the following:

THEOREM 1. Let d be local, $p_1, \ldots, p_m$, invertible ideals of d, n a positive integer > 1, $\underset{\sim}{\delta}$ a divisor sequence for n, and $\Delta = \Delta(n, \underset{\sim}{\delta}; p_1, \ldots, p_m)$. Then,

1. Δ is localizable iff the p_i are distinct maximal invertible ideals;

2. $\text{glb}\,\Delta = t$ iff

   a. $m \leq t$,

b. p_i is invertible mod $\sum_{j>i} p_j$, $\forall 1 \leq i \leq m-1$,

c. $\mathrm{glb}\, d/\sum_{i=1}^{m} p_i = t-m$.

In particular, whenever $\mathrm{glb}\,\Delta = t$, the p_i are distinct invertible primes of d and each d/p_i is a (t-1)-dimensional regular local algebra.

That Δ is indeed the appropriate canonical form for finite-dimensional, localizable, semiperfect algebras follows from:

THEOREM. Let $\Lambda \subset D_n$ be a self-basic, t-dimensional, localizable, semiperfect algebra. Then, there exists a t-dimensional regular local algebra $d \subset D$, distinct invertible primes $p_1, \ldots, p_m$ of d with $m \leq t$, and a divisor sequence $\underset{\sim}{\delta}$ of n, such that Λ is conjugate to $\Delta(n, \underset{\sim}{\delta}; p_1, \ldots, p_m)$.

§2. THE NON-SELF BASIC CASE

If $\Lambda \subset D_n$ is no longer self-basic, since Λ is always Morita equivalent to the basic ring of Λ, Λ is Morita equivalent to $\Delta(m, \underset{\sim}{\delta}; p_1, \ldots, p_k)$ where $m \leq n$. However, when Λ contains a self-basic, finite-dimensional, localizable, semiperfect subalgebra, we can proceed as in the hereditary case and explicitly determine a canonical form for Λ.

To this end, let Λ be an arbitrary semiperfect ring.

1. DEFINITION. Λ <u>is of type</u> k (on the left) if ${}_\Lambda\Lambda = \sum_{i=1}^{k} \oplus N_i^{n_i}$ with ${}_\Lambda N_i$ indecomposable, $\forall i$, and ${}_\Lambda N_i \not\cong {}_\Lambda N_j$, $\forall i \neq j$.

2. DEFINITION. For a localizable algebra Λ, $r_i(\Lambda)$ will denote the number of localizable algebras of type i which contain Λ, and

$n(\Lambda) = \sum_{i \geq 1} r_i(\Lambda)$, the total number of localizable algebras containing Λ.

If $\Lambda \subseteq d_n$, $r_i'(\Lambda)$ will denote the number of localizable algebras Λ' of type i which satisfy $\Lambda \subseteq \Lambda' \subseteq d_n$, and $n'(\Lambda) = \sum_{i \geq 1} r_i'(\Lambda)$.

If Λ is 1-dimensional and R a complete DVR, $r_i(\Lambda) = \binom{k}{i}$ if Λ has type k, and $n(\Lambda) = 2^k - 1$.

THEOREM. Let Λ be a self-basic, t-dimensional localizable algebra $\subset d_n$ with canonical form

$$\Delta = \Delta(n, \underset{\sim}{\delta}; p_1, \dots, p_m) = \begin{pmatrix} \Delta_m & \dots & \Delta_m \\ & \ddots & \vdots \\ p_m \Delta_m & \dots & \Delta_m \end{pmatrix}$$

where $\Delta \subset d_k$ and $n = \ell \cdot k$. Then,

1. $r_j = \sum_{d|j} \binom{\ell}{d} r_{j/d}(\Delta_m)$
2. $r_j' = \sum_{d|j} \binom{\ell-1}{d-1} r'_{j/d}(\Delta_m)$
3. $n(\Lambda) = (2^\ell - 1) n(\Delta_m)$
4. $n'(\Lambda) = 2^{\ell-1} n'(\Delta_m)$.

In particular, if $m = 2$,

1. $r_j = \sum_{d|j} \binom{\ell}{d} \binom{k}{j/d}$
2. $r_j' = \sum_{d|j} \binom{\ell-1}{d-1} \binom{k-1}{j/d - 1}$
3. $n(\Lambda) = (2^\ell - 1)(2^k - 1)$
4. $n'(\Lambda) = (2^{\ell-1})(2^{k-1})$.

If $n = \ell \cdot k$, $(\ell_1, \dots, \ell_r)$ is any partition of ℓ, d a subalgebra of D, $A = d_k$, and p a prime ideal of d, $\Lambda_k^p(\ell_1, \dots, \ell_r)$ is the subalgebra of

$\Sigma = D_n$ consisting of all matrices in $A_\ell = d_n$ of the form

$$\begin{pmatrix} A & A & . & . & . & . & A \\ pA & A & . & . & . & . & A \\ pA & pA & A & . & . & . & A \\ \vdots & & & & & & \vdots \\ pA & pA & . & . & . & . & A \end{pmatrix}$$

partitioned into $\ell_i \times \ell_j$ blocks (of pA's).

THEOREM. (Same hypotheses as the preceding). If Δ_m^i, $1 \leq i \leq k$, are the k distinct maximal orders $\supset \Delta_m$, then for each (ordered) partition of ℓ, $(\ell_1, \ldots, \ell_r)$,

$$\Lambda_k^i(\ell_1, \ldots, \ell_r) = \Lambda_k^{p_m}(\ell_1, \ldots, \ell_r) \cap (\Delta_m^i)_\ell \subseteq d_n$$

is a t-dimensional localizable algebra $\supset \Lambda$. Conversely, if Λ' is any t-dimensional localizable algebra $\supset \Lambda$ with $\Lambda' \subseteq d_n$, then there exists an integer i with $1 \leq i \leq k$ and a partition $(\ell_1, \ldots, \ell_r)$ of ℓ such that $\Lambda' = \Lambda_k^i(\ell_1, \ldots, \ell_r)$.

THEOREM. Let Λ be a t-dimensional localizable algebra contained in $\Sigma = D_n$. Then Λ contains a t-dimensional localizable algebra of type n if and only if there exists a positive integer k with $k|n$, an integer i with $1 \leq i \leq k$, and a partition of $\ell = n/k$, $(\ell_1, \ldots, \ell_r)$, such that Λ is conjugate to $\Lambda_k^i(\ell_1, \ldots, \ell_r)$.

Actually, given a fixed, self-basic, finite-dimensional localizable algebra Λ, the intermediate algebras Λ' (described above) which qualify are precisely those Λ' for which Λ'_Λ is reflexive.

PROBLEMS

1. If d is a finite-dimensional, (quasi)local, prime Noetherian ring, is d a maximal order in its quotient ring?

   Remarks: a. What Vasconcelos showed was that if d is an algebra, the answer is always yes!

   b. By Ramras [12], if d is local, d is a domain.

   c. Since reflexive ideals are projective whenever $\text{glb}\, d \leq 2$ (see Cozzens [2]), d is necessarily maximal.

   d. If the answer is yes, then all of the above results extend, mutatis mutandis, to arbitrary finite-dimensional, localizable, semiperfect, prime Noetherian rings.

2. Same assumptions as in 1. Is each reflexive (prime) ideal of d projective?

   Remarks: a. By Cozzens-Sandomierski [4], yes to 2 $\Rightarrow$ yes to 1.

   b. By Ramras [11], if R is a 3-dimensional regular local ring, Λ is an R-free, maximal R-order with $\text{glb}\, \Lambda = 3$, then reflexive ideals of Λ are projective.

3. If Λ is an arbitrary finite-dimensional, localizable algebra, is $\Lambda \cong \Lambda_1 \times \Lambda_2$, where Λ_1 is semisimple Artin and Λ_2, a (finite-dimensional, localizable) <u>semiprime</u> algebra?

   Remark: a. Whenever R is a DVR and Λ is an R-algebra, the answer is yes by Silver [16].

4. If Λ is an arbitrary 2-dimensional maximal order, is Λ p-connected, i.e., are finitely generated projective Λ-modules generators?

   Remarks: a. By Riley [14], if Λ is a quaternion order, the answer is yes.

   b. See Cozzens [3] for generalization of Riley's result.

   c. If the answer to 4 is yes, then by a trivial modification of the proof given in Ramras [12], $\Lambda \approx M_n(d)$ where d is a maximal order in a division ring.

5. If Λ is 2-dimensional and R-free with R a complete 2-dimensional regular local ring, and $\Gamma \supset \Lambda$ a maximal R-order, is Γ quasilocal?

   Remarks: a. By Ramras [10], 3.5, Γ is finitely generated projective over Λ on both sides and hence, a finite localization of Λ. In particular, $\text{glb}\,\Gamma = 2$ as well.

   b. As remarked earlier, if $\Lambda \subset D_m$ and $m \leq 7$, the answer is yes.

6. Same as 5 with 2 replaced by n.

7. If Λ is as described in 6 and Γ_1 and Γ_2 are both maximal orders $\supset \Lambda$, is Γ_1 Morita equivalent to Γ_2?

   Remarks: a. By Ramras [10], if $\text{glb}\,\Lambda = 2$, the answer is yes.

   b. For $n > 2$, many partial results have been obtained, e.g., see Ramras [11], 2.2.

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THE GENUS OF A MODULE
AND
GENERIC FAMILIES OF RINGS

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1. INTRODUCTION

A right module M over a ring R is said to have a unimodular element (UME) if there exists $u \in M$ such that uR is a direct summand of M canonically isomorphic to R. Thus, M has a UME iff there is an epimorphism $M \twoheadrightarrow R$. In general, a module M generates the category mod-R of all right R-modules iff there is an epic $M^n \twoheadrightarrow R$ for some integer $n > 0$; equivalently, M^n has a UME . In this case, we let $\gamma(M)$ denote the infimum of all such integers n, and call this the genus of M. If M does not generate mod-R, we set $\gamma(M) = \infty$. The (little) right genus of a ring R will be denoted by $g_r(R)$ and is defined to be the supremum of $\gamma(M) < \infty$ for M finitely generated in mod-R. The big right genus $G_r(R)$ is defined similarly without restriction on finite generation of M. Clearly, $g_r(R) \leq G_r(R)$, and equality holds when R is a right Noetherian ring.

A family $F = \{R_i\}_{i \in I}$ of rings is generic of (with) bound B if there exists a function $B : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all modules M if $\nu(M) < \infty$ is the minimal number of elements in any set of generators of M, then there is an epic $M^{B(\nu(M))} \to R$. The product theorem states that any product of a generic family of rings of bound B is a ring which is generic of bound B (considering a ring as a family with one member) (see Theorem 6). For example, a family of rings each of genus $\leq g$ is generic with bound $\leq g$, where g also denotes the constant function. Moreover, any family of commutative rings is generic of bound $1_{\mathbb{Z}^+}$. The 2×2 theorem (Theorem 15) states that if R is a commutative ring of genus 1, then for any faithful module M with $\nu(M) = 2$, the product M^2 has a unimodular element. Thus, by the product theorem, the 2×2 theorem holds for any product of such rings.

A ring R is right (F)PF ([4]-[7]) if every finitely generated faithful module M generates mod-R; equivalently, $\gamma(M) < \infty$. A corollary of the product theorem is that any product $R = \prod_{i \in I} R_i$ of generic family right FPF rings is right FPF. (In particular, the product any family of commutative FPF rings is FPF.) This implies that any product of self-basic right FPF rings, in particular, any product of self-basic right PF rings is right FPF.

Another corollary to the product theorem states that if $\{R_i\}_{i \in I}$ is any family of commutative rings each having the property $P(n, g)$ exist integers $n > 0$ and $g > 0$ with the property that for all $i \in I$ every finitely generated R_i-module of free rank $\geq n+1$ has genus $\leq g$, then their product R also has property $P(n, g)$. The FPF theorem is the case $P(0, 1)$.

The product theorem depends on Lemma 9: the only finitely gener- ideal of the product containing the direct sum is the unit ideal.

2. PRELIMINARIES AND EXAMPLES

If M is a right R-module, let $\nu_R(M)$, or $\nu(M)$, denote the least cardinal of any generating set. Thus, $\exists\ R^{\nu(M)} \twoheadrightarrow M$, but $\nexists\ R^{\mu} \twoheadrightarrow M$ for any cardinal $\mu < \nu(M)$. If M is a generator of mod-R, then for some integer $n > 0$, $\exists\ M^n \twoheadrightarrow R$, and we let $\gamma_R^r(M)$, or $\gamma^r(M)$, denote the least such n. When M is understood to be a right R-module, let $\gamma(M)$ denote this, and $M \in \text{Gen } R$ denotes that M is a f. g. generator.

EXAMPLE. It may happen that a ring R fails to have the invariant basis number (IBN), that is, $R^n \approx R^m$ in mod-R for integers $n \neq m$. If $R \approx R^2$ in mod-R, then $R_2 \approx \text{End}_R R^2 \approx \text{End}_R R \approx R$ as rings; also, $R \approx R^n$ for every integer $n > 0$, so every f. g. right module is cyclic. If M is any right R-module, then $M \approx M \otimes_R R \approx M \otimes_R R^n \approx M^n$, so $G^r(R) = G^r(R_2) = 1$.

Sufficient conditions for IBN are for R to have a nonzero ring map into a a (skew) field, e.g. when R is local, or commutative, or Noetherian.

Among the various equivalent conditions for a generator is that the trace ideal of the module M must be the unit ideal, where the trace ideal is defined to be the image $\text{trace}_R M$ of the canonical map

$$M^* \oplus M \to R \tag{1}$$

where $M^* = \text{Hom}_R(M, R)$ is the dual module. In the special case of a cyclic right module R/I,

$$\text{trace}_R(R/I) = {}^{\perp}IR \tag{2}$$

where

$${}^{\perp}I = \{a \in R \mid ax = 0, \forall x \in I\}. \tag{3}$$

To prove (2), use the canonical isomorphism

$$(R/I)^* \approx {}^{\perp}I \tag{4}$$

and then

$$(R/I)^* \otimes R/I \approx {}^{\perp}I \otimes R/I \twoheadrightarrow {}^{\perp}IR. \tag{5}$$

Consider any generator M of mod-R, and write $M^n = R \oplus X$. Then the dual module $(M^*)^n = R \oplus X^*$ (taking $R = R^*$ canonically), so

$$\gamma^r(M) \geq \gamma^\ell(M^*) \tag{6}$$

where $\gamma^\ell(\ )$ is the right-left symmetry of $\gamma^r(\ )$, and clearly,

$$M \text{ reflexive} \Rightarrow \gamma^r(M) = \gamma^\ell(M^*). \tag{7}$$

Thus (7) holds, e.g., for any f.g. projective module M.

I am indebted to W. Vasconcelos for the next result.

1. THEOREM. If R is a commutative ring, then $\gamma(M) \leq \nu(M)$ for any f.g. generator M.

Proof. Let $M^n \twoheadrightarrow R$. Then there exist elements $x_1, \ldots, x_n \in M$, $f_1, \ldots, f_n \in M^*$ such that $\Sigma_{i=1}^n f_i(x_i) = 1$. If $t = \nu(M)$, and if $m_1, \ldots, m_t$ generate M, then $x_i = \Sigma_{j=1}^t m_j a_{ij}$ for some $a_{ij} \in R$, $i = 1, \ldots, n$. However, $f'_j = \sum_{i=1}^n f_j a_{ij} \in M^*$, $j = 1, \ldots, t$ is such that $\Sigma_{j=1}^t f'_j(m_j) = 1$, so that $M^t \twoheadrightarrow R$ holds, that is, $\gamma(M) \le t = \nu(M)$.

2. COROLLARY. If M is a f.g. faithful projective over a commutative ring R, then (M generates mod-R and) $\gamma(M) = \gamma(M^*) \le \nu(M)$.

Proof. M generates mod-R by a theorem of Azumaya [1].

A ring A is said to be a local ring provided the equivalent conditions hold:

A has a unique maximal right ideal $J(A)$. (8)

The set $J(A)$ of nonunits is closed under subtraction. (9)

The radical $J(A)$ defines a field $A/J(A)$ (not necessarily commutative). (10)

3A. DEFINITION. A ring R is semiperfect if $R = \bigoplus_{i=1}^m e_i R$, where $e_i = e_i^2 \in R$ and $e_i R e_i$ is a local ring, $i = 1, \ldots, n$. Let $e_1 R, \ldots, e_n R$ denote a full set of representatives of isomorphy classes for $\{e_i R\}_{i=1}^m$. (Thus each $e_j R \approx$ to one and only one $e_i R$ for $i \le m$.) Then $B = e_1 R \oplus \ldots \oplus e_n R$ is the basic module of R, and $R_0 = e_0 R e_0$ is the basic ring of R, where $e_0 = e_1 + \ldots + e_n$.

3B. PROPOSITION. Every semiperfect ring R is Morita equivalent to its basic ring. Moreover, B is a direct summand of every generator of mod-R.

Proof. (See, e.g., [4], 18.26.). R is self-basic provided that $e_0 = 1$; that is, $R = R_0$, or equivalently, $R/\mathrm{rad}\,R$ is a (necessarily finite) product of fields. The basic ring R_0 of a semiperfect ring R is self-basic (loc. cit.). (The basic ring R_0 is also the left basic ring of R, since also

$R = \bigoplus_{i=1}^{m} Re_i$, etc.)

4. THEOREM. If R is a semiperfect ring, then $G(R) = \gamma(X)$ where X is the basic module. If R is self-basic, then $G(R) = 1$.

Proof. Trivial corollary of Proposition 3.

5. EXAMPLES

5.1. If $R = F_n = M_n(F)$ is the $n \times n$ matrix ring over a local ring F, then $X = e_{11}R$ is the basic module and $\gamma(X) = n$, so $G(R) = n$.

5.2. If $R = T_n(F)$ the lower triangular matrices over a local ring F, then R is self-basic, so $G(R) = 1$.

5.3. If R is a semiperfect ring, then $R/\text{rad}\, R = \prod_{i=1}^{t} M_{n_i}(D_i)$ for fields $D_1, \ldots, D_n$, and the basic module is $X = \Sigma_{i=1}^{t} e_i R$, where $e_i^2 = e_i$ maps onto the $(1,1)$ matrix unit of $M_{n_i}(D_i)$ under the canonical map $R \to R/\text{rad}\, R$, $i = 1, \ldots, n$. Clearly

$$\gamma(X) = \max\{n_i\}. \tag{13}$$

This generalizes 1 (where $t = 1$) and 2 (where each $n_i = 1$).

5.4. The product $R = \prod_{n=1}^{\infty} M_n(F)$ of the rings of 5.1, one for each n, has genus ∞, since $g^r(M_n(F)) = n$. Clearly, $G^r(R)$ or $g^r(R) = n$ for a product $R = \prod_{i \in I} R_i$ of rings implies $G^r(R_i) \leq g$ (resp. $g^r(R_i) \leq g$) for every i. (See Theorem 6, also Lemma 17, for the details.)

5.5. Moreover, any product $R = \prod_{i \in I} R_i$ of rings of genus $\leq g$ has $g(R) \leq g$. (See, e.g., Corollary 8.) Thus, any product of self-basic rings has genus 1. Moreover, $G(\mathbb{Z}^{\alpha}) = 1$ for any cardinal α.

A ring R is right pre-FPF if every f.g. faithful right ideal generates mod-R. A commutative pre-FPF ring is characterized by the requirement that finitely generated faithful ideals are projective ([7], Section 2, Corollary 1D.)

5.6. If R is a prime right pre-FPF ring, then

$$g^r(R) = \sup\{\gamma^r(I) \mid 0 \neq I \subseteq R\}.$$

For if $M \in \operatorname{Gen} R$, then there is a nonzero map $f : M \to R$, and since for a prime ring every nonzero right ideal is faithful, then $f(M)$ generates mod-R, hence, so does M. Moreover, $\gamma(M) \leq \gamma(f(M))$.

5.7. EXAMPLES OF PRIME RIGHT PRE-FPF RINGS

5.7.1. Any simple ring R. For if $I \neq 0$, the $T = \operatorname{trace}_R I$ is an ideal $\neq 0$; hence $T = R$.

5.7.2. Any right pre-Prüfer ring. This designates a ring in which any f.g. (two-sided) ideal $\neq 0$ generates mod-R. Now a f.g. right ideal $I \neq 0$ generates an f.g. ideal $RI = J$. Let $f : I^{(R)} \twoheadrightarrow J$ be the canonical epic of the direct sum of $|R|$ copies of I. Then an epic $h : J^t \twoheadrightarrow R$ implies an epic $hf : I^{(R)} \twoheadrightarrow R$, so I is a generator, and R is therefore right pre-FPF.

Refer to [5] for other results on (pre)-Prüfer rings.

5.7.3. Any Prüfer ring is FPF. This is a Goldie prime ring (GPR) in which every ideal $\neq 0$ is invertible in the quotient ring $Q = Q_{c\ell}(R)$ is the sense that

$$I^{-1}I = II^{-1} = R.$$

I^{-1} is the fractional ideal consisting of all $q \in Q$ such that $qI \subseteq R$. Clearly

$$I^{-1}I = R \iff I \text{ generates mod-}R;$$

$$II^{-1} = R \iff I \text{ is finitely generated projective in mod-}R.$$

5.7.4. Special Prüfer domains have genus ≤ 2. If R is a commutative Prüfer domain, then R is special if every f.g. ideal I can be generated by $1\frac{1}{2}$ elements in the sense that given any $a \neq 0$ in I, then there exists $b \in R$ such that $I = (a, b)$; that is, any nonzero element can be specified as one of the two generators. Not every Prüfer ring is special (as Heitman and Levy [8] showed); however, it is unknown in a Prüfer ring whether $\nu(I) \leq 2$ for all f.g. I.

A special Prüfer ring has genus ≤ 2 since given any f.g. ideal I, we have $I^2 \approx R \oplus J$ for an ideal J ([8,11]).

5.7.5. If R is a Dedekind prime ring (DPR), then $G(R) \leq 2$; moreover, $G(R) = 1$ iff R is a PIR.

This follows from the fact that if M is any generator, and $0 \neq f \in M$, then $f(M) = I$ will be a right ideal $\neq 0$ and $I^2 \approx R \oplus J$, where J is a right ideal. (See, e.g., [9-10].)

If I is any essential right ideal, then $Q = E(I) = E(R)$, where E(M) denotes hull of a module M over R. Thus, $G(R) = 1 \Rightarrow I \approx R \oplus X \Rightarrow Q = Q \oplus E(X)$. Therefore, $E(X) \neq 0$ is impossible because Q is IBN. So $X = 0$, and $I \approx R$. Thus I is principal, and hence so is every right ideal.[1]

5.7.6. <u>A semifir</u> R <u>has genus</u> 1. In R f.g. right ideals $\neq 0$ are free of unique rank. (The latter holds if R is an IBN ring.) If $M \in \mathrm{Gen}\,R$ and if $0 \neq f \in M^*$, then M f.g. $\Rightarrow f(M)$ is free, so there is an epic $M \twoheadrightarrow R$.

Semifir is right left symmetric; that is, f.g. left ideals $\neq 0$ are free of unique rank (Cohn [3]). So semifirs have right and left genus 1, and hence by the product theorem (cited in 5.5), any ring R which is a product of semifirs has $g(R) = 1$.

A <u>right Bezout domain</u> is a semifir in which every f.g. right ideal $\neq 0$ is free on one generator (= rank 1).

A <u>right fir</u> is a ring in which every right ideal is free of unique rank. A right fir R is a (left) semifir, but R need not be a left fir [3].

A principal right ideal domain is a right fir and every right ideal $\neq 0$ is free on one generator.

In the next example, the <u>torsionfree rank</u> of M is the least t such that R^t embeds in M and is denoted by tf rk M. (By definition tf rk M ≥ 0, i.e., tf rk M = 0 if tf rk M $\not\geq 1$.)

Let ℓ-k-dim R denote its left Krull dimension. Let r-K-dim R denote the right Krull dimension, and K-dim R = n if the right and left dimensions equal n.

5.7.7. A Noetherian Asano order R of K-dim n has genus $\leq n+3$, and

$$G(R) = \sup\{\gamma(M) \mid 0 \neq M \hookrightarrow R\}.$$

---

[1] If K is an essential left ideal, then one shows that $K^* \approx R$, hence $K \approx K^{**} \approx R$., that is, R is also a principal left ideal ring.

An Asano order is a Noetherian Prüfer ring, so 5.7.3 and 5.6 apply. If M is torsionfree (t.f.) and tf rk $M \geq n+3$, then $\gamma(M) = 1$ by Stafford's theorem [9, Theorem 7.2] M t.f. => $\gamma(M) \leq n+3$ since tf rk $M^{n+3} \geq n+3$. Since every right ideal M is t.f., this proves that $g(R) \leq n+3$.

5.7.8. If R is a simple Noetherian ring of ℓ-K-dimn, then $g^{\ell}(R) \leq n+2$. If $n \geq 2$, then $g^{\ell}(R) \leq \max\{g, n\}$, where $g = \sup\{\gamma(M) \mid \text{tf rk } M \leq 1\} \leq n + 2$.

Stafford's theorem asserts a t.f. finitely generated left module M of tf rk $\geq n+2$ is a generator and $\gamma(M) = 1$. Then the argument employed in 5.7.5 shows for any generator M that $\gamma(M) \leq n$ if tf rk $M \geq 2$ and $n \geq 2$, since then tf rk $M^n \geq 2n \geq n+2$.

The <u>free rank of</u> M, denoted f rk M is the smallest integer t such that M_P has a free direct summand R_P^t for every maximal ideal P.

<u>Here, and for the rest of this section</u> R <u>is a Noetherian commutative ring.</u>

Let spec(R) be the space of prime ideals of R in the Zariski topology. Thus, if $S \subseteq R$, let $V(S) = [P \in \text{spec}(R) \mid P \supseteq S]$, and decree that the closed sets in spec(R) are those of the form V(S). Then the dimension of R, dim R, is the dimension of the resulting lattice of open sets of spec(R). Clearly, the lattice of open sets is Noetherian (= satisfies the a.c.c.) iff the lattice of closed sets is Artinian, so dim R is finite iff spec(R) is both Noetherian and Artinian. Dim R is also referred to as (classical) Krull dimension of R, and,

$$\dim R[t_1, \cdots, t_n] = n + \dim R$$

where $R[t_1, \cdots, t_n]$ is the polynomial ring in n variables. This implies that any finitely generated commutative algebra A over R has finite dimension provided that R does. In particular, any finitely generated commutative ring has finite dimension. (See, for example, [2], pp.101-102.)

We let max(R) denote the subpace of spec(R) consisting of maximal ideals. Thus, max(R) consists of the closed points of spec(R). Clearly, max(R) is Noetherian if spec(R) is, and

$$\dim \max(R) \leq \dim \text{spec}(R)$$

Serre's theorem is more general than the following (see [2], pp. 172-3.)

5.8 THEOREM (Serre) Let max(R) be a disjoint union of a finite number of subspaces each of dimension $\leq n$ (e.g., $\dim R \leq n$), and let M be a direct summand of a direct sum of finitely presented modules (e.g. M projective or finitely generated). Then, if f rk M > n, then M has a unimodular element.

For the corollary, we need a lemma that R. Wiegand showed us.

5.9 LEMMA. If R is a right Noetherian ring, then G(R) = g(R).

Proof. Let M be a generator of mod-R, so there exist finitely many $f_i \in M^*$ and $m_i \in M$ such that $\Sigma_{i=1}^n f_i(m_i) = 1$. The image of $M^* \otimes_R M \to R$ is thus R, and the image of $M^{*n} \otimes_R M \to R^n$ is R^n. If $f = \mathrm{col}(f_1, \dots, f_n)$ M^{*n}, then $f(m) = (f_1(m), \dots, f_n(m))$ $\forall$ $m \in M$, and the image F of M under f generates mod-R. (If $p_i : R^n \to R$ is the ith projection, then $p_i(f(m_i) = f_i(m_i)$, so the fact that $\Sigma_{i=1}^n f_i(m_i) = 1$ shows that the trace ideal of F is R.) Since F is a submodule of a Noetherian module, F is finitely generated. Since F is an epic image of M, then $\gamma(M) \leq \gamma(F)$, and we have what we want.

5.10 COROLLARY. If R satisfies the hypothesis of Serre s theorem, then

$$G(R) \leq \max\{n, g_1\} \leq n + 1$$

where

$$g_1 = \sup\{\gamma(M) \mid f \text{ rk } M = 1\}$$

Proof. If f rk $M \geq 2$, then f rk $M^n \geq n + 1$. Since we may assume that M is finitely generated by the lemma, then $\gamma(M^n) = 1$ by the theorem, hence $\gamma(M) \leq n$. If f rk M = 1, then f rk $M^{n+1} \geq n + 1$, so $\gamma(M) \leq n + 1$, hence $g_1 \leq n+1$.

Added December 1978 Wiegand and Vasconcelos have sharpened a result of [12], namely Theorem 2.1. Assume R has dim n, and suppose for modules M and N, with N finitely generated, that for each maximal ideal P there is an epimorphism $M_P \to N_P$. Then, there is an epimorphism $M^n \to N$. Thus, when M is a generator, then $\gamma(M) \leq n + 1$. This removes the hypothesis that R be Noetherian in Corollary 5.10, that is, $G(R) \leq n+1$ for any commutative ring R of dim n. (Unpublished).

In addition, an unpublished result of D. Eisenbud states that for $R = k[x, y]$, the polynomial ring in 2 variables over a field k, $G(R) = 1$.

R. Wiegand has asked which commutative rings have the property that every generator has a faithful direct summand.

3. GENERIC RINGS

Let $\mathfrak{F} = \{R_i\}_{i \in I}$ be a family of rings, and assume there exists a function $B : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that any f.g. generator M_i of R_i satisfies the inequality

$$\gamma(M_i) \leq B(\nu(M_i)), \quad \text{for all } i. \tag{14}$$

Then $\mathfrak{F}$ is said to be <u>right generic and bounded by</u> B, or <u>right</u> B-<u>generic</u> for short. Theorem 1, any family of commutative rings is generic and bounded by $\mathrm{id}\,\mathbb{Z}^+$. If $\mathfrak{F}$ consists of a single ring R (or a class of rings all $\approx R$), hen we say that the ring R is <u>right generic and bounded by</u> B (or <u>right</u> B-<u>generic</u>) if $\mathfrak{F}$ is. (In the parenthetic statement, R is right generic and bounded by B iff $\mathfrak{F}$ is.)

6. PRODUCT THEOREM. A family $\{R_i\}_{i \in I}$ of rings is right B-generic iff the product $R = \prod_{i \in I} R_i$ is B-generic. Thus, for every $M \in \mathrm{Gen}\,R$, with $\nu(M) = n < \infty$ we have:

$$\gamma(M) = \sup\{\gamma(M_i)\}_{i \in I} \leq B(n) \tag{6.1}$$

where $M_i = Me_i$, and $e_i \in R_i$ is the identity element, $\forall i \in I$.

<u>Proof.</u> $M \in \mathrm{Gen}\,R \Rightarrow M_i \in \mathrm{Gen}\,R_i$ for each $i \in I$; hence there are epics $M_i^{\gamma} \to R_i$, in mod-R_i, where $\gamma = \sup \gamma_i \leq B(n)$; hence epics $h_i : M^{\gamma} \to R_i$ in

mod-R. The image H of the product morphism $h : M^{\gamma} \to R$ satisfies $He_i = R_i, \forall i \in I$; hence H contains their direct sum, and Lemma 9 (following) asserts that $H = R$. Thus,

$$\gamma(M) \leq \gamma \leq B(n) = B(\nu(M)).$$

However, $\gamma(M) = \gamma$ since any epic $M^t \to R$ implies an epic $M_i^t \to R_i, \forall i \in I$.

Conversely, assume $R = \prod_{i\in I} R_i$ B-generic, choose $i \in I$, and $M \in$ f.g. Gen R_i. Let $n = \nu_i(M)$, and let $M^t \twoheadrightarrow R_i$, where $t = \gamma_{R_i}(M)$. Also let $N = \prod_{j\neq i} R_j$. Then $N \oplus M^t \twoheadrightarrow R = N \oplus R_i$, and hence $(N \oplus M)^t \twoheadrightarrow R$, so $\gamma(N \oplus M) \leq t$. Note however that $(N \oplus M)^{t'} \twoheadrightarrow R$ would imply $M^{t'} \twoheadrightarrow R_i$, so actually $\gamma(N \oplus M) = t$. Moreover, $\nu_R(N \oplus M) = \nu_{R_i}(M) = n$, since: $R^n = R \oplus R^{n-1} \twoheadrightarrow (N \oplus R_i) \oplus R_i^{n-1} = N \oplus R_i^n \twoheadrightarrow N \oplus M$ (using $R \twoheadrightarrow N \oplus R_i = R$, and $R_i^n \twoheadrightarrow M$). Therefore, since R is B-generic, we have

$$t = \gamma_{R_i}(M) = \gamma_R(M) \leq B(n) = B\nu_{R_i}(M),$$

that is, $\{R_i\}$ is B-generic.

It is clear from the proof that from the statement that $R = \prod_{i\in I} R_i$ is a generic product of rings we may deduce either of the two equivalent properties:

(1) R is a generic ring (bounded, e.g., by B).

(2) $\mathfrak{F} = \{R_i\}_{i\in I}$ is a generic family (bounded, e.g., by B).

7. COROLLARY. If M is an f.g. module over a product of rings $R = \prod_{i\in I} R_i$, if $M_i = Me_i$ generates mod-R_i where $e_i : R \to R_i$ is the projection idempotent, and if $\sup\{\gamma(M_i)\}_{i\in I} = \gamma < \infty$, then M generates mod-R and $\gamma(M) = \gamma$. Thus

$$\gamma(M) = \sup\{\gamma_{R_i}(M_i)\}_{i\in I}. \tag{7.1}$$

Proof. That M generates mod-R follows from the proof of the theorem which shows that if there exists γ ∞ such that

$$\forall_{i\in I} \exists\, M_i^{\gamma} \twoheadrightarrow R_i \text{ then } \exists\, M^{\gamma} \twoheadrightarrow R. \tag{15}$$

Moreover:

$$M^{\gamma} \twoheadrightarrow R \Longrightarrow M_i^{\gamma} \twoheadrightarrow R_i; \tag{16}$$

hence (7.1) holds.

8. COROLLARY. Let $R = \prod_{i\in I} R_i$. Then

$$g^r(R) = \sup\{g^r(R_i)\}_{i\in I}. \tag{8.1}$$

Proof. Follows from the corollary and the proof of the theorem.

The next lemma completes the proof of Theorem 6.

9. LEMMA. The only f.g. right ideal H of a product $\prod_{i\in I} R_i$ of rings which contains the direct sum $\oplus_{i\in I} R_i$ is the unit ideal.

Proof. Let H be generated by elements $m^1, \ldots, m^t$, and for any $x \in R$, write $x_j = xe_j$, $\forall j \in I$. Since $e_j \in H$, $\forall j \in I$, there exist $a^{ij} \in R$, $i = 1, \ldots, t$, such that

$$e_j = \sum_{i=1}^{t} m^i a^{ij} = \sum_{i=1}^{t} m_j^i a_j^{ij}. \tag{17}$$

Let $b^i \in R$ be such that $b_j^i = a_j^{ij}$, $\forall j \in I$. Then, clearly, the element

$$m = \sum_{i=1}^{t} m^i b^i \in M \tag{18}$$

is the unit element 1 of R since by (1)

$$m_j = \sum_{i=1}^{t} m_j^i b_j^i = e_j = 1_j \tag{19}$$

for any j. Thus, M is the unit ideal.

10. EXAMPLE

10.1. If F_n be the $n \times n$ matrix ring over a local ring F, then the product $R = \prod_{n\in \mathbb{Z}^+} F_n$ is not generic, since $\gamma(M) = \infty$ for the cyclic module $M = eR$, where $e = e^2$ is the idempotent the j-th component of which is the e_{11}-matrix in F_n.

10.2. An infinite product of right PF rings is never right PF since a semiperfect ring contains no infinite sets of orthogonal idempotents. Furthermore, by Example 5.4, a product of PF rings is not necessarily FPF, e.g., $R = \prod_{n \in \mathbb{Z}^+} F_n$ is not. Nevertheless, any product of right PF rings of right genus $\leq g$ is right FPF of genus $\leq g$, according to Lemma 17 in the next section. For example:

10.3. Let $R = \prod_{i \in I} M_n(F_i)$, where F_i is a self-basic right (F)PF ring is right FPF of genus n, according to Lemma 17, since $G^r(M_n(F_i)) = n$, $\forall i \in I$, by Example 5.1.

11. COROLLARY. Let $R = \prod_{i \in I} R_i$ be a product of commutative rings such that there exists an integer $n > 0$ such that each R_i satisfies Serre's condition $P(n, g)$; that is, any finitely generated R_i-module of f rk $\geq n+1$ has a unimodular element. Then, R satisfies $P(n, g)$.

<u>Proof.</u> Let M be any finitely generated R-module of f rk $\geq n+1$. If P_i is any maximal ideal of R_i, then $P = P_i \oplus R_i'$, where $R_i' = \prod_{j \neq i} R_j$, is maximal in R, and $(M_i)_P = M_P$ has rk $\geq n+1$, so M_i has a unimodular element, that is, $\gamma(M_i) = 1$; hence $\gamma(M) = 1$ by Corollary 7.

APPLICATIONS TO FPF RINGS

A ring R is right PF provided that each faithful right R-module generates mod-R. For the background to the next result, consult [4].

12. THEOREM. (Azumaya <u>et al</u>) A ring R is right PF (pseudo-Frobenius) iff R is a semiperfect right self-injective ring with essential right socle.

These include the QF rings, the Artinian (right and left) PF rings. Any semiperfect right self-injective ring with nil radical is right PF [6].

The FPF rings include all finite products of rings each of which are Dedekind prime rings (DPR's) or QF. Also, any semiperfect ring in which every f.g. ideal is a generator (both sides). Such a ring is prime and $\approx M_n(D)$, where D is a right and left valuation ring and right duo [5]. A commutative example would be any Prüfer domain.

A ring R is CFPF if every factor ring R is FPF, e.g., any DPR. A commutative local ring R is CFPF iff R is an almost maximal valuation ring (AMVR) [7], or equivalently, every f. g. module is a direct sum of cyclic modules.

A commutative local ring R is FPF iff every faithful module M with $\nu(M) = 2$ is a direct sum of cyclics [7]. This is generalized to arbitrary products of commutative rings of genus 1 in Theorem 15 and Corollary 16. Any self-injective commutative ring is FPF [7].

13. PROPOSITION. If R is any ring and M is a generator such that $\gamma(M) = 1$ and $2 \leq \nu(M) = n < \infty$, then

$$M \approx R \oplus B/K \tag{13.1}$$

where B is an f. g. projective such that

$$R^n \approx R \oplus B. \tag{13.2}$$

Proof. $\gamma(M) = 1 \Rightarrow M \approx R \oplus X$, and $\nu(M) = n \Rightarrow M \approx R^n/K$ in mod-R; hence there exist submodules A and B of R^n such that $A \cap B = K$, $R^n = A+B$, $A/K \approx R$, and $B/K \approx X$. Since R is projective, K splits in A. Write $A = K \oplus R_1$. Then $R_1 \approx R$, and

$$R^n = A+B = K+R_1+B = R_1+B = R_1 \oplus B \tag{22}$$

since $R_1 \cap B \subseteq A \cap B \cap R_1 \subseteq K \cap R_1 = 0$. Moreover,

$$M = R^n/K = R_1 \oplus B/K \approx R \oplus B/K.$$

14. COROLLARY. If R is commutative, then in the proposition, B is a progenerator (= f. g. projective generator).

Proof. By Azumaya's theorem, all that is required is that B be faithful. But $R^n = R_1 \oplus B \Rightarrow R^n a = (Ra)^n \approx R_1 a$ for all $a \in R$ which annihilates B, and this implies $n = 1$ since $R_1 a$ is cyclic, contrary to the assumption.

15. 2×2 THEOREM. If R is FPF and commutative of genus 1, then every faithful module M with $\nu(M) = 2$ is a direct sum of two cyclics: $M \approx R \oplus R/K$.

Proof. $\gamma(M) = 1$ so the corollary applies: $M = R \oplus B/K$, where $R^2 \approx R \oplus B$, and B generates mod-R. Then $B \approx R \oplus Y$ so $R^2 \approx R^2 \oplus Y$ which means that $Y_{\mathfrak{m}} = 0$, $\forall$ maximal ideals $\mathfrak{m}$; hence $Y = 0$, and $B \approx R$, so $M = R \oplus R/K$ is a direct sum of cyclics.

We shall abbreviate the conclusion of the 2×2 Theorem by the terminology: Every faithful 2-gened module is 2-cyclic. In this case, we say the 2×2 Theorem holds.

16. COROLLARY. Any product of commutative FPF rings of genus 1 is FPF, and hence the 2×2 Theorem holds.

Proof. R is FPF and $g(R) = 1$ by the $n = 1$ case of Lemma 17 (following), so Corollary 15 applies.

17. LEMMA. Any right generic product of right FPF rings is FPF.

Proof. If M is f. g. faithful in mod-R of Corollary 7, then $M_i = Me_i$ is f. g. faithful over R_i, hence generates mod-R_i, and therefore M generates mod-R by Corollary 7.

18. COROLLARY. Any product of commutative FPF rings is FPF. Similarly for products of right FPF self-basic rings.

Proof. Both are generic families.

19. COROLLARY. Any right generic product of right PF rings is right FPF.

20. EXAMPLE

20.1. As stated in Example 10, $R = \prod_{n \in \mathbb{Z}^+} F_n$ is not generic, where F is any field, and R is not FPF even though F_n is PF, $\forall n$.

20.2. The product $R = (F_n)^a$ for any cardinal a, and fixed n, is FPF since $\{F_n\}$ is generic.

20.3. $R = \mathbb{Z}^a$ is FPF for any cardinal a.

21. THEOREM. Let $\{R_i\}_{i \in I}$ be a family of rings such that R_i is a

commutative ring of one of the following types:

(i) a Bezout domain,

(ii) a local FPF ring (e.g., any AMVR, or any self-injective local ring),

(iii) an FPF ring of genus 1,

(iv) any product of rings $\{R_i\}$ where R_i has type (i)-(iv).

Then: $R = \prod_{i \in I} R_i$, is FPF of genus 1; hence the 2×2 Theorem holds.

Proof. The rings (i)-(iii) are all FPF of genus 1; hence by Corollary 16, so are the rings in (iv); hence so is $R = \prod_{i \in I} R_i$.

A ring R (commutative) is said to be quotient-injective if its classical quotient ring $Q_{cl}(R)$ is a self-injective ring, equivalently, an injective R-module. Then R is said to be fractionally self-injective (FSI) if every factor ring of R is quotient-injective. Every FPF commutative ring R is quotient-injective, hence every CFPF commutative ring is FSI. Conversely, every FSI ring R is CFPF. (See [7,13] for these results, and the background). Now the FSI rings have been completely characterized by Vamos [14]: R is FSI iff R is a finite product of rings of the following three types: (1) AMVR; (2) Almost maximal h-local domain; (3) Almost maximal torch ring. Here, almost maximal means that every local ring of R is an AMVR; h-local means that every prime ideal is contained in only finitely many maximal ideals; and a torch ring signifies that R is directly indecomposable (= has no non-trivial idempotents), has a minimal prime ideal P such that P is a uniserial R-module $\neq 0$, with $P^2 = 0$, and R/P of type (2).

This shows that no infinite product of rings can be CFPF, that is, that product theorem for FPF rings fails for CFPF rings. (Finite products of CFPF rings are CFPF however.)

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COLOCALIZATION AT IDEMPOTENT IDEALS

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My intention here is to present a short introduction to the recent efforts to define a meaningful and usable notion of "colocalization" of associative rings with unit element and modules over such rings. These efforts were motivated, on one hand, by the fruitfulness of the notion of "localization" at a hereditary torsion theory and, on the other hand, by the hope of coming up with an additional tool which, when used together with localization, would allow us to preserve information concerning the structure of such rings and modules which is lost under localization alone. Thus arose, for example, the feeling that colocalization and localization, appropriately defined, should constitute an adjoint pair.

The various approaches to colocalization which have been considered are based on dualizations of one facit or another of the notion of a hereditary torsion theory. They can be grouped as follows:

(1) Colocalization via cotorsion radicals. This approach was initiated by Beach [71] and has been since investigated in several papers, among them Ramamurthi [73a], Katayama [74], Ramamurthi and Rutter [76], and Goel [77].

(2) Colocalization via cokernel functors. This approach was initiated by Bronn [73].

(3) Colocalization at projective modules. This approach was initiated by McMaster [75], which is based on the general approach due to Lambek. (See, for example, Lambek [73].) This approach was also used in Golan [74].

(4) Colocalization at jansian torsion theories or, equivalently, at idempotent ideals. This approach was introduced independently and more-or-less simultaneously by Kato [ta], Ohtake [77], and Sato [76] on one hand and by Golan and Miller [ta] on the other. It is based mainly on the work of Miller [74, 76] and on the attempts to generalize Morita equivalence as exemplified by Onodera [77].

Since the last-mentioned approach essentially subsumes all of the others, it is the one which I will present here.

0. Background and notation. Throughout the following R will denote an associative (but not necessarily commutative) ring with unit element 1. We will denote the category of unitary left R-modules by R-mod and the category of unitary right R-modules by mod-R. Morphisms in module categories will be written as acting on the side opposite scalar multiplication. All other maps will be written as acting on the left. If M is an R-module then the injective hull of M will be denoted by $E(M)$ and the Jacobson radical of M will be denoted by $J(M)$.

The complete brouwerian lattice of all (hereditary) torsion theories on R-mod will be denoted by R-tors. In dealing with R-tors, we will follow the notation and terminology of Golan [75]. In particular, if N is a submodule of a left R-module M and if $\tau \in$ R-tors then N will be called τ-dense [resp. τ-pure] in M if and only if M/N is τ-torsion [resp. τ-torsionfree]. With every left R-module M we can associate the largest element of R-tors relative to which M is torsionfree, denoted by $\chi(M)$, and the smallest element of R-tors relative to which M is torsion, denoted by $\xi(M)$. The unique maximal element of R-tors is $\chi = \chi(0)$ and the unique minimal element of R-tors is $\xi = \xi(0)$.

With each $\tau \in$ R-tors we have an associated localization endofunctor $Q_\tau(\_)$ of R-mod which is idempotent and left exact. Moreover, we have a natural transformation λ^τ from the identity endofunctor on R-mod to $Q_\tau(\_)$ such that for every left R-module M, $\lambda_M^\tau: M \to Q_\tau(M)$ is the localization morphism. (In Golan [75] this is denoted by $\hat{\tau}_M$.) If R_τ is the endomorphism ring of $Q_\tau(R)$ then every module of the form $Q_\tau(M)$ is canonically a left R_τ-module and R-homomorphisms between such modules are also R_τ-homomorphisms.

Among the important types of torsion theories are the stable torsion theories, namely those torsion theories for which the class of all torsion modules is closed under taking injective hulls. These torsion theories were first studied by Gabriel [62]; information about them is collected in Section 11 of Golan [75].

1. Jansian torsion theories. A torsion theory $\tau \in$ R-tors is said to be jansian if and only if the class of all τ-torsion left R-modules is closed under taking direct products. (Such theories are often called TTF-theories in the literature; they were first studied by Jans in [65].) The set of all jansian torsion theories on R-mod will be denoted by R-jans. The following results are proven, among other places, in Golan [75].

(1.1) PROPOSITION: *If* $\tau \in$ R-tors *then*

(1) τ *is jansian if and only if* R *has a unique minimal* τ-*dense left ideal* $L(\tau)$.

(2) *If* τ *is jansian then a left* R-*module* M *is* τ-*torsion if and only if* $L(\tau)M = 0$.

(3) $L(\tau)$ *is an idempotent (two-sided) ideal of* R. *Indeed, the function* $\tau \mapsto L(\tau)$ *is a bijective correspondence between* R-jans *and the set of all idempotent ideals of* R.

If $\tau \in$ R-jans then set $W(\tau) = L(\tau) \otimes_R L(\tau)$. Then $W(\tau)$ is both a left and a right R-module and we have a canonical R-homomorphism (left and right) from $W(\tau)$ to R given by $\Sigma a_i \otimes b_i \mapsto \Sigma a_i b_i$ the image of which is precisely $L(\tau)$.

(1.2) PROPOSITION: *If* $\tau \in$ R-jans *then the following conditions on a left* R-*module* M *are equivalent*:

(1) M _is_ τ-_torsion_;

(2) $\mathrm{Hom}_R(W(\tau),M) = 0$;

(3) $W(\tau) \otimes_R M = 0$.

PROOF: (1) $\Leftrightarrow$ (2): Assume that M is τ-torsion and let $\alpha: W(\tau) \to M$ be a nonzero R-homomorphism. Pick $w = \Sigma_i\, a_i \otimes b_i \in W(\tau)$. Since each $a_i \in L(\tau) = L(\tau)^2$, we can write $a_i = \Sigma_j\, c_{ij}d_{ij}$, where the c_{ij} and the d_{ij} are elements of $L(\tau)$. Therefore we have $w\alpha = (\Sigma_i\Sigma_j\, c_{ij}d_{ij} \otimes b_i)\alpha = \Sigma_i\Sigma_j\, c_{ij}(d_{ij} \otimes b_i)\alpha$. But M is τ-torsion and so by Proposition 1.1 we have $L(\tau)M = 0$. Therefore $w\alpha = 0$, proving that $\mathrm{Hom}_R(W(\tau),M) = 0$. Conversely, assume that M is a left R-module satisfying $\mathrm{Hom}_R(W(\tau),M) = 0$. If $m \in M$ then we have an R-homomorphism from $W(\tau)$ to M defined by $\Sigma a_i \otimes b_i \mapsto (\Sigma a_i b_i)m$. By assumption, this must be the 0-map and so $L(\tau)m = 0$ for every $m \in M$. Therefore $L(\tau)M = 0$ and and so M is τ-torsion.

(1) $\Leftrightarrow$ (3): Let M be a τ-torsion left R-module and assume that $\Sigma_i\, a_i \otimes b_i \otimes m_i \in W(\tau) \otimes_R M$. Then each b_i can be written as $\Sigma_j\, c_{ij}d_{ij}$, where the c_{ij} and the d_{ij} are elements of $L(\tau)$. Therefore $\Sigma_i\, a_i \otimes b_i \otimes m_i = \Sigma_i\Sigma_j\, a_i \otimes c_{ij}d_{ij} \otimes m_i = \Sigma_i\Sigma_j\, a_i \otimes c_{ij} \otimes d_{ij}m_i$. But $d_{ij}m_i = 0$ for each i and each j since $d_{ij} \in L(\tau)$ and since M is τ-torsion. Therefore $W(\tau) \otimes_R M = 0$. Conversely, assume (3). Then the R-homomorphism $\alpha: W(\tau) \otimes_R M \to L(\tau)M$ given by

$\alpha\colon \Sigma a_i \otimes b_i \otimes m_i \mapsto \Sigma a_i b_i m_i$ is an epimorphism and so, by assumption, we have $L(\tau)M = 0$. By Proposition 1.1, this implies that M is τ-torsion. □

A left R-module P is said to be *pseudoprojective* if and only if for every diagram in R-mod of the form

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle\beta} & & \\ N & \xrightarrow{\alpha} & N'' & \longrightarrow & 0 \end{array}$$

with exact row and with $\beta \neq 0$ there exists an R-endomorphism θ of P and an R-homomorphism $\psi\colon P \to N$ for which $0 \neq \theta\beta = \psi\alpha$. See Bican, Jambor, Kepka, and Nemec [75] and Bican [76]. Any idempotent ideal of R is pseudoprojective as a left R-module. Zelmanowitz [72] has defined a left R-module M to be *regular* if and only if for any $m \in M$ there exists an R-homomorphism $\alpha \in \mathrm{Hom}_R(M,R)$ satisfying $m = (m\alpha)m$. Such modules are easily seen to be pseudoprojective. Similarly, the *locally projective* modules defined by Zimmerman-Huisgen [76] are both flat and pseudoprojective.

(1.3) PROPOSITION: *The following conditions on a left R-module* P *are equivalent:*

(1) P *is pseudoprojective.*

(2) *There exists a jansian torsion theory* $\eta(P) \in$ R-jans *defined by the condition that a left R-module* M *is* $\eta(P)$-*torsion if and only if* $\mathrm{Hom}_R(P,M) = 0$ *and*

moreover having the property that $L(\eta(P)) = \Sigma\{P\alpha \mid \alpha \in Hom_R(P,R)\}$.

PROOF: (1) ⇒ (2): Let P be a pseudoprojective left R-module. Then the class of all left R-modules M satisfying the condition that $Hom_R(P,M) = 0$ is closed under taking submodules, direct products, isomorphic copies, and extensions. Thus all we are left to show is that this class is closed under taking homomorphic images. Let $\alpha:M \to M''$ be an R-epimorphism and assume that $Hom_R(P,M) = 0$. If $0 \neq \beta \in Hom_R(P,M'')$ then by the pseudoprojectivity of P there exist an endomorphism θ of P and an R-homomorphism $\psi:P \to M$ satisfying $0 \neq \theta\beta = \psi\alpha$. This implies, in particular, that $\beta \neq 0$, contradicting the choice of M. Thus we must have $Hom_R(P,M'') = 0$, proving that $\eta(P)$ exists.

Now let $H = \Sigma\{P\alpha \mid \alpha \in Hom_R(P,R)\}$. Since $L(\eta(P))$ is $\eta(P)$-dense in R, we have $Hom_R(P,R/L(\eta(P))) = 0$ and so $H \subseteq L(\eta(P))$. Assume that this inclusion is strict. Then we have an R-epimorphism $\nu:L(\eta(P)) \to L(\eta(P))/H$. We claim that $L(\eta(P))/H$ is $\eta(P)$-torsion. Indeed, assume not. If $0 \neq \beta \in Hom_R(P,L(\eta(P))/H)$ then by the pseudoprojectivity of P there exists an R-endomorphism θ of P and an R-homomorphism $\psi:P \to L(\eta(P))$ satisfying $0 \neq \psi\nu = \theta\beta$. But $P\psi \subseteq R$ implies that $P\psi \subseteq H$ and so $P\psi\nu = 0$. This yields a contradiction which establishes that indeed $L(\eta(P))/H$ is $\eta(P)$-torsion.

From the exactness of the sequence

$$0 \to L(\eta(P))/H \to R/H \to R/L(\eta(P)) \to 0$$

we then conclude that R/H is $\eta(P)$-torsion, contradicting the definition of $L(\eta(P))$. Thus we must have $H = L(\eta(P))$.

(2) $\Rightarrow$ (1): Assume (2) and let $\mu:R^{(\Omega)} \to P$ be an R-epimorphism. Set $U = L(\eta(P))^{(\Omega)}$. Then $L(\eta(P))[R^{(\Omega)}/U] = 0$ and so U is $\eta(P)$-dense in $R^{(\Omega)}$. Thus μ induces an R-epimorphism from $R^{(\Omega)}/U$ to $P/U\mu$. Since $R^{(\Omega)}/U$ is $\eta(P)$-torsion, so is $P/U\mu$, which forces $P = U\mu$. Now assume that we have a diagram of the form

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \beta} & & \\ N & \xrightarrow{\alpha} & N'' & \longrightarrow & 0 \end{array}$$

with exact row and with $\beta \neq 0$. By the projectivity of $R^{(\Omega)}$, there exists an R-homomorphism $\beta':R^{(\Omega)} \to N$ satisfying $\beta'\alpha = \mu\beta$. If μ'' is the restriction of μ to U and if β'' is the restriction of β' to U then μ'' is an epimorphism and $0 \neq \mu''\beta = \beta''\alpha$. By (2), $L(\eta(P))$ is an epimorphic image of a direct sum of copies of P and hence so is U. In particular, this implies that there exists an R-homomorphism $\zeta:P \to U$ such that $\zeta\mu''\alpha \neq 0$. Set $\theta = \zeta\mu''$ and $\psi = \zeta\beta''$. Then $0 \neq \theta\beta = \psi\alpha$, proving that P is pseudoprojective. □

In particular, we note that if $\tau \in$ R-jans then $W(\tau)$ is pseudoprojective. Moreover, a torsion theory $\tau \in$ R-tors is jansian when and precisely when there exists a pseudoprojective left R-module P for which $\tau = \eta(P)$. Indeed, Proposition 1.2 assets that if $\tau \in$ R-jans then $\tau = \eta(W(\tau))$.

Several conditions for the stability of a jansian torsion theory were given in [Golan, 75; Proposition 22.10]. We now need one more.

(1.4) PROPOSITION: A torsion theory $\tau \in$ R-jans is stable if and only if $x \in L(\tau)x$ for every $x \in W(\tau)$.

PROOF: If $\tau \in$ R-jans is stable then by [Golan, 75; Proposition 22.10] we know that $x \in L(\tau)x$ for every $x \in W(\tau)$ if and only if $W(\tau) = L(\tau)W(\tau)$, and this is an immediate consequence of the definition of $W(\tau)$. Conversely, assume that this condition holds. Let M be a τ-torsion left R-module and let $\alpha \in \mathrm{Hom}_R(W(\tau),E(M))$. If there exists an $x_0 \in W(\tau)$ for which $x_0\alpha \neq 0$ then there exists an $r \in R$ such that $0 \neq rx_0\alpha \in M$. Since $rx_0 \in W(\tau)$, we have $rx_0 \in L(\tau)rx_0$ and so there exists an $a \in L(\tau)$ satisfying $rx_0 = arx_0$. Now define an R-homomorphism $\beta: W(\tau) \to M$ by $\beta: \Sigma c_j \otimes d_j \mapsto \Sigma c_j d_j r x_0$. Then $\beta \neq 0$ since $arx_0 \in \mathrm{im}(\beta)$ and so $\mathrm{Hom}_R(W(\tau),M) \neq 0$, contradicting the assumption that M is τ-torsion. Thus we must have that $\mathrm{Hom}_R(W(\tau),E(M)) = 0$, proving that $E(M)$ is τ-torsion and hence that τ is stable. □

In particular, we note that a sufficient condition for a jansian torsion theory $\tau \in$ R-jans to be stable is that $W(\tau)$ be regular as a left R-module.

A ring R is said to be <u>left weakly regular</u> if and only if the following equivalent conditions are satisfied:

(1) For every $a \in R$ there exists an element $b \in RaR$ satisfying $a = ba$.

(2) Every left ideal of R is idempotent.

(3) R/I is flat as a right R-module for every two-sided ideal I of R.

(4) Every left ideal of R is semiprime.

Such rings have been studied by Fisher [74], Hansen [75], and Ramamurthi [73]. It is easily seen that if R is a left weakly regular ring then every jansian torsion theory on R-mod is stable.

Let us consider a more concrete example. Following Bass [60], we say that a ring R is <u>left perfect</u> if and only if every left R-module has a projective cover. Dlab [70] has shown that a ring R is left perfect if and only if it is right semiartinian and every torsion theory on the category mod-R is jansian. Moreover, he gives an example of a ring satisfying the condition that every torsion theory on mod-R is jansian but which is not right semiartinian and hence not left perfect. Another characterization of left perfect rings is given in Golan [74], where it is shown that a ring R is left perfect if and only if every torsion theory on mod-R is of the form $\eta(P)$ for some projective right R-module P.

Michler [69] has studied the idempotent ideals of left perfect rings. In particular, he has shown that a left perfect ring R has precisely 2^n idempotent ideals, where n is the number of simple components of the semisimple artinian ring R/J(R). Therefore, if R is a left perfect ring then there are only finitely-many torsion theories on mod-R and all of them are of the form $\xi(A)$, where A is a subset of a complete set of representatives of the isomorphism classes of simple right R-modules.

We now want to characterize those left perfect rings having the property that every member of R-jans is stable. To do this, we recall that a ring R is said to be <u>right local</u> if and only if all simple right R-modules are isomorphic.

(1.5) PROPOSITION: <u>The following conditions on a left perfect ring</u> R <u>are equivalent</u>:

(1) R <u>is isomorphic to a finite direct product of left perfect right local rings</u>.

(2) <u>Every member of</u> R-jans <u>is stable</u>.

PROOF: By Propositions 5.5 and 23.9 of Golan [75] we know that (1) is equivalent to the condition that for any torsion theory ρ on mod-R the class of all ρ-torsionfree right R-modules is closed under taking homomorphic images.

Since every torsion theory on mod-R is jansian, by Proposition 22.12 of Golan [75] this is equivalent to the condition that $R/L(\rho)$ is projective as a left R-module for every such ρ. But the ideals of R of the form $L(\rho)$ are just the idempotent ideals of R and these are precisely the ideals of the form $L(\tau)$ for some $\tau \in$ R-jans. Thus (1) is equivalent to the condition that $R/L(\tau)$ is projective for every $\tau \in$ R-jans. Since R is left perfect, it is in particular semiperfect and so every cyclic left R-module has a projective cover. Therefore $R/L(\tau)$ is projective as a left R-module if and only if it is flat as a left R-module. But by Proposition 22.10 of Golan [75], this is precisely equivalent to (2). □

2. Modules cotorsionfree relative to a torsion theory. If $\tau \in$ R-tors then a left R-module M will be said to be τ-cotorsionfree if and only if $\mathrm{Hom}_R(M,N) = 0$ for every τ-torsion left R-module N. The class of all τ-cotorsionfree left R-modules is clearly closed under taking homomorphic images, extensions, direct sums, and projective covers (when they exist). It is closed under taking submodules if and only if there exists a torsion theory $\tau^c \in$ R-tors satisfying the condition that a left R-module is τ^c-torsion if and only if it is τ-cotorsionfree. Since no nonzero left R-module can be both τ-torsion and τ-cotorsionfree we see that if τ^c exists then $\tau^c \wedge \tau = \xi$ in R-tors and so $\tau^c \leq \tau^{\perp}$, where $\tau^{\perp}$ is the meet pseudocomplement of τ in the brouwerian lattice R-tors. But no nonzero homomorphic image of a $\tau^{\perp}$-torsion left R-module can be τ-torsion and so every $\tau^{\perp}$-torsion left R-module is τ-cotorsionfree. Therefore we conclude that if τ^c exists then it must equal $\tau^{\perp}$.

(2.1) PROPOSITION: If $\tau \in$ R-tors then a sufficient condition for the class of τ-cotorsionfree left R-modules to be closed under taking submodules is that τ be stable.

PROOF: Let M' be a submodule of a τ-cotorsionfree left R-module M. If there exists a nonzero τ-torsion left R-module N satisfying $\mathrm{Hom}_R(M',N) \neq 0$ then E(N) is also τ-torsion by the stability of τ and we have $\mathrm{Hom}_R(M',E(N)) \neq 0$, which implies that $\mathrm{Hom}_R(M,E(N)) \neq 0$. This contradicts the fact that M is τ-cotorsionfree and so we must have that M' is also τ-cotorsionfree. □

If the class of all τ-cotorsionfree left R-modules is closed under taking submodules then every τ-cotorsionfree left R-module is also τ-torsionfree. In general, this need not be so. However, for any $\tau \in$ R-tors we note that if M is a τ-cotorsionfree left R-module then any τ-torsion submodule of M is small in M. (For a proof see Goel [77].)

If $\tau \in$ R-tors then any left R-module M has a unique maximal τ-cotorsionfree submodule, namely $C_\tau(M) = \Sigma\{{}_RM' \subseteq M \mid M' \text{ is } \tau\text{-cotorsionfree}\}$. In particular, $C_\tau(R)$ is the unique maximal τ-cotorsionfree left ideal of R and so it must, in fact, be a (two-sided) ideal of R. One easily verifies that $C_\tau(\_)$ is an idempotent subfunctor of the identity endofunctor on R-mod. Also, we note that $C_\tau(M)$ is a submodule of every τ-dense submodule of M.

(2.2) PROPOSITION: <u>If</u> $\tau \in$ R-jans <u>then</u> $C_\tau(R) = L(\tau)$.

PROOF: If $\tau \in$ R-jans then $\mathrm{Hom}_R(L(\tau),M) = 0$ for every τ-torsion left R-module M and so $L(\tau)$ is a τ-cotorsionfree left ideal of R, proving that $L(\tau) \subseteq C_\tau(R)$. Since $L(\tau)$ is τ-dense in R, we have the reverse containment as well. □

If $\tau \in$ R-tors then we say that a left R-module M is τ-<u>surtorsion</u> if and only if $C_\tau(M) = 0$ or, equivalently, if and only if $\mathrm{Hom}_R(N,M) = 0$ for every τ-cotorsionfree left R-module N. Surely every τ-torsion left R-module is τ-surtorsion. Moreover, the class of all τ-surtorsion left R-modules is closed under taking submodules, direct sums, and extensions. It is closed under taking homomorphic images if and only if there exists a torsion theory $\tau^d \in$ R-tors satisfying the condition that a left R-module is τ^d-torsion if and only if it is τ-surtorsion. Beachy [71] has given equivalent conditions for this to happen:

(2.3) PROPOSITION: *The following conditions on* $\tau \in$ R-tors *are equivalent*:

(1) *The class of all* τ-*surtorsion left* R-*modules is closed under taking homomorphic images*.

(2) $C_\tau(M) = C_\tau(R)M$ *for any left* R-*module* M.

(3) *Any* R-*epimorphism* $\alpha: M \to M'$ *restricts to an* R-*epimorphism* $\alpha': C_\tau(M) \to C_\tau(M')$.

Indeed, τ satisfies the equivalent conditions of Proposition 2.3 if and only if $C_\tau(\_)$ is a cotorsion radical in the sense of Beachy [71]. Under these circumstances, $C_\tau(R)$ is an idempotent ideal of R. Indeed, one checks that under these circumstances τ^d is jansian and $C_\tau(R) = L(\tau^d)$. Moreover, the torsion theory τ^d is the unique minimal jansian generalization of τ. Therefore a necessary condition for τ^d to exist is that τ have a unique minimal jansian generalization. Another immediate consequence of Proposition 2.3 is the following.

(2.4) COROLLARY: _If_ $\tau \in$ R-tors _satisfies the equivalent conditions of Proposition_ 2.3 _then a left_ R-_module_ M _is_ τ-_cotorsionfree if and only if_ $[R/C_\tau(R)] \otimes_R M = 0$.

PROOF: We always have $[R/C_\tau(R)] \otimes_R M \cong M/C_\tau(R)M$ and by Proposition 2.3 we see that this is isomorphic to $M/C_\tau(M)$, implying the result we seek. □

Ramamurthi and Rutter [76] have also shown that if $\tau \in$ R-tors satisfies the equivalent conditions of Proposition 2.3 then $C_\tau(\_)$ commutes with direct products if and only if $C_\tau(R)$ is a finitely-generated right ideal of R.

A torsion theory $\tau \in$ R-tors satisfying the condition that the class of all τ-torsionfree left R-modules is closed under taking homomorphic images is said to be _cohereditary_. Rutter [72] has shown that if R is a semiperfect ring then every cohereditary torsion theory on R-mod is jansian. If $\tau \in$ R-tors satisfies the conditions of Proposition 2.3 then τ is cohereditary if and only if every τ-cotorsionfree left R-module is τ-cotorsionfree. (See Golan [75], Proposition

22.12, for the proof of the equivalence of these and other conditions.)

Jansian torsion theories all satisfy the conditions of Proposition 2.3. To see this, it suffices to establish the following result.

(2.5) PROPOSITION: _The following conditions on_ $\tau \in$ R-tors _are equivalent_:

(1) τ _is jansian_.

(2) _A left_ R-_module_ M _is_ τ-_torsion if and only if it is_ τ-_surtorsion_.

(3) $M/C_\tau(M)$ _is_ τ-_torsion for every left_ R-_module_ M.

PROOF: (1) ⇒ (3): If M is a left R-module then $L(\tau)[M/C_\tau(M)] = C_\tau(R)[M/C_\tau(M)] \subseteq C_\tau(M/C_\tau(M)) = 0$ and so $M/C_\tau(M)$ is τ-torsion.

(3) ⇒ (2): We have already noted that every τ-torsion left R-module is τ-surtorsion. The converse follows directly from (3).

(2) ⇒ (1): If $\{M_i \mid i \in \Omega\}$ is a set of τ-torsion left R-modules and if N is a τ-cotorsionfree left R-module then $\mathrm{Hom}_R(N,\Pi M_i) \cong \Pi\mathrm{Hom}_R(N,M_i) = 0$ and so, by (2), ΠM_i is τ-torsion. This proves that τ is jansian. □

In particular, Proposition 2.5 shows that a jansian torsion theory is completely determined by its class of cotorsionfree modules.

As a consequence of results of Goel [77] and Ramamurthi [73a] we can then obtain the following criteria for a jansian torsion theory to be stable.

(2.6) PROPOSITION: The following conditions on $\tau \in$ R-jans are equivalent:

(1) τ is stable.

(2) The class of all τ-cotorsionfree left R-modules is closed under taking submodules.

(3) $C_\tau(N) = N \cap C_\tau(M)$ for every submodule N of a left R-module M.

(4) $C_\tau(I) = I \cap C_\tau(R)$ for every left ideal I of R.

(5) $I = C_\tau(R)I$ for every left ideal I of R contained in $C_\tau(R)$.

Moreover, as Ramamurthi and Rutter [76] have shown, if τ is a stable jansian torsion theory then for any left R-module M we have $Q_{\tau^\perp}(M) = M/C_\tau(M)$.

A jansian torsion theory τ on R-mod is said to be centrally splitting if and only if $R \cong L(\tau) \times T_\tau(R)$ as rings. This condition has been studied by Jans [65], Bernhardt [69, 71, 73], and Golan [75]. From these sources we see that the following conditions on a jansian torsion theory τ are equivalent:

(1) τ is centrally splitting;

(2) $M = C_\tau(M) \oplus T_\tau(M)$ for any left R-module M.

(3) τ is stable and $R/L(\tau)$ has a projective cover in R-mod.

Kurata [72] has shown that if R is a commutative noetherian ring then every jansian torsion theory on R-mod is centrally splitting. Rutter [72] established that if R is a left or right injective cogenerator ring then $\tau \in$ R-tors is centrally splitting if and only if the class of all τ-torsion left R-modules is closed under taking injective hulls of simple modules. Ramamurthi [73a] has proven that if R is a semi-prime right noetherian ring or a quasi-Frobenius ring then every stable jansian torsion theory on R-mod is centrally splitting.

(2.7) PROPOSITION: The following conditions on $\tau \in$ R-jans are equivalent:

(1) τ is centrally splitting.

(2) A left R-module is τ-cotorsionfree if and only if it is τ-torsionfree.

PROOF: (1) ⇒ (2): We have already noted above that (1) implies that τ is stable and so by Proposition 2.6 the class of all τ-cotorsionfree left R-modules is closed under taking submodules. Therefore every τ-cotorsionfree left R-module is τ-torsionfree. Moreover, (1) implies that $R = L(\tau) \oplus T_\tau(R)$ and so if M is a τ-torsionfree left R-module we have

$M = L(\tau)M \oplus T_\tau(R)M = C_\tau(R)M = C_\tau(M)$, proving the reverse containment.

(2) ⇒ (1): Since the class of all τ-torsionfree left R-modules is closed under taking submodules, (2) implies that this is true for the class of all τ-cotorsionfree left R-modules and so $C_\tau(M) \cap T_\tau(M) = 0$ for every left R-module M. On the other hand, (2) implies that τ is cohereditary and so $R = T_\tau(R) + C_\tau(R)$. If m is an element of a left R-module M we then have $m \in T_\tau(R)m + C_\tau(R)m \subseteq T_\tau(M) + C_\tau(M)$ and so $M = T_\tau(M) \oplus C_\tau(M)$. This proves (1). □

Kurata [72] has shown that the set R-jans can be partitioned into the union of three disjoint subsets, which he characterized. From the above discussion we see that Kurata's partition corresponds precisely to the following three cases:

(I) τ-torsionfree ⇔ τ-cotorsionfree;

(II) τ-torsionfree ⇒ τ-cotorsionfree but not conversely;

(III) τ-torsionfree ⇏ τ-cotorsionfree.

<u>3. Torsion theories of the form $\eta(\mathcal{A})$.</u> If $\mathcal{A}$ is a nonempty class of left R-modules let us define the torsion theory $\eta(\mathcal{A})$ to be $\vee\{\tau \in$ R-tors $\mid$ every member of $\mathcal{A}$ is τ-cotorsionfree$\}$. Note that the set over which this join is taken is always nonempty. If M is a left R-module then we will write $\eta(M)$ instead of $\eta(\{M\})$.

(3.1) PROPOSITION: <u>If</u> $0 \to M' \to M \to M'' \to 0$ <u>is an exact sequence in</u> R-mod <u>then</u> $\eta(M' \oplus M'') \leq \eta(M) \leq \eta(M'')$.

PROOF: If τ is any torsion theory on R-mod then the class of τ-cotorsionfree left R-modules is closed under taking extensions and homomorphic images. From this observation both inequalities follow immediately. □

In particular, this implies that if $\{M_i\}$ is a set of left R-modules then $\eta(\oplus M_i) \leq \wedge \eta(M_i)$.

(3.2) PROPOSITION: <u>If</u> N <u>is a small submodule of a left</u> R-<u>module</u> M <u>then</u> $\eta(M) = \eta(M/N)$.

PROOF: If N is a small submodule of a left R-module M then by Proposition 3.1 we know that $\eta(M/N) \geq \eta(M)$. Now let $\tau \in$ R-tors and assume that M/N is τ-cotorsionfree while M is not. Then there exists a τ-torsion left R-module N' and a nonzero R-homomorphism $\alpha: M \to N'$. Since N is small in M, we know that $N + \ker(\alpha) \neq M$ and so there exists an element

$m \in M \setminus N$ satisfying $m\alpha \notin N\alpha$. Therefore α defines a nonzero R-homomorphism $\bar{\alpha}: M/N \to N'/N\alpha$. But $N'/N\alpha$ is a nonzero τ-torsion left R-module, contradicting the fact that M/N is τ-cotorsionfree. Therefore $\eta(M) \geq \eta(M/N)$ and so we have equality. □

(3.3) PROPOSITION: _If_ M _is a simple left_ R-_module then_ $\eta(M) = \chi(M)$.

PROOF: Let $\tau \in$ R-tors. If M is τ-cotorsionfree then M cannot be τ-torsion and so must be τ-torsionfree. This implies that $\tau \leq \chi(M)$ and thus $\eta(M) \leq \chi(M)$. On the other hand, $\mathrm{Hom}_R(M,N) = 0$ for every $\chi(M)$-torsion left R-module N since M is simple and so M is $\chi(M)$ cotorsionfree. This establishes the reverse inequality. □

In particular, Proposition 3.3 implies that for any simple left R-module M the torsion theory $\eta(M)$ is prime and, indeed, is a minimal element of the set R-sp of all prime torsion theories on R-mod.

Following Mares [63], we say that a projective left R-module is _semiperfect_ if and only if each of its homomorphic images has a projective cover.

(3.4) PROPOSITION: _A projective left_ R-_module_ P _is semiperfect if and only if_ $J(P)$ _is small in_ P _and_ $\eta(P) = \wedge\chi(M_i)$, _where the modules_ M_i _are simple left_

R-modules having projective covers.

PROOF: If P is a semiperfect left R-module then Mares [63, Theorem 3.3] has shown that $J(P)$ is small in P and that $P = \oplus P_i$, where the P_i are projective left R-modules which are the projective covers of simple left R-modules M_i. Thus $\chi(M_i) = \eta(M_i) = \eta(P_i) \geq \eta(P)$ for each index i and so $\wedge\chi(M_i) \geq \eta(P)$. To prove the reverse inequality we must show that P is $\wedge\chi(M_i)$-cotorsionfree. Indeed, assume not. Then there exists a nonzero R-homomorphism $\alpha: P \to N$, where N is a left R-module which is $\chi(M_i)$-torsion for each index i. Since $\alpha \neq 0$, its restriction α_h to some summand P_h of P is nonzero. Therefore $\ker(\alpha_h) \subseteq J(P_h)$. Thus we have an induced nonzero R-homomorphism $P_h/\ker(\alpha_h) \to P_h/J(P_h) \cong M_h$. But $P_h/\ker(\alpha_h)$ is isomorphic to a submodule of N and so this map can be extended to a nonzero R-homomorphism from N to $E(M_h)$, contradicting the assumption that N is $\chi(M_h)$-torsion. Thus $\eta(P) = \wedge\chi(M_i)$.

Conversely, assume that $J(P)$ is small in P and that $\eta(P) = \wedge\chi(M_i)$, where the M_i are simple left R-modules having projective covers $P_i \to M_i$. Then $\eta(P) = \wedge\eta(M_i) = \wedge\eta(P_i)$. Set $N = \Sigma\{(\oplus P_i)\alpha \mid \alpha \in \mathrm{Hom}_R(\oplus P_i, P)\}$. If $N \neq P$ then P is not $\xi(P/N)$-cotorsionfree and hence $\xi(P/N) \not\leq \eta(P)$. Therefore there exists an index h satisfying $\xi(P/N) \not\leq \eta(P_h)$. This implies that $\mathrm{Hom}_R(P_h, P/N) \neq 0$, which is a contradiction.

Thus we must have $P = N$. Thus P is a homomorphic image of a direct sum of copies of $\oplus P_i$ and so is isomorphic to a direct summand of a direct sum of copies of $\oplus P_i$. By Mares [63, Theorem 5.2], this suffices to show that P is semiperfect. □

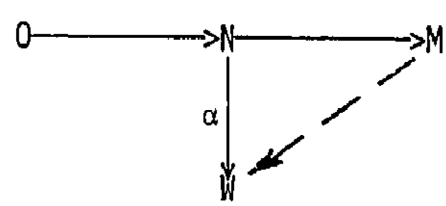
As is to be expected, if P is a pseudoprojective left R-module then the torsion theory $\eta(P)$ defined here coicides with the torsion theory denoted similarly in Section 1. Thus a left R-module M is $\eta(P)$-torsion if and only if $\mathrm{Hom}_R(P,M) = 0$. Since such torsion theories are jansian, it follows that a left R-module M is $\eta(P)$-cotorsionfree if and only if M is a homomorphic image of a direct sum of copies of P. In particular, if P is a pseudoprojective left R-module and if M is an arbitrary left R-module then $C_{\eta(P)}(M) = \Sigma\{P\alpha \mid \alpha \in \mathrm{Hom}_R(P,M)\}$. Moreover, by the fact that jansian torsion theories satisfy the conditions of Proposition 2.3, we see that in fact $C_{\eta(P)}(M) = tr(P)M$, where $tr(P)$ is just the trace of P in R. Thus we see that a left R-module M is $\eta(P)$-torsion if and only if $tr(P)M = 0$ and is $\eta(P)$-cotorsionfree if and only if $tr(P)M = M$.

As a consequence of the above discussion we see that if $\alpha: P \to P'$ is an R-epimorphism between pseudoprojective left R-modules then $\eta(P) \leq \eta(P')$.

4. Relatively projective and injective modules. If $\tau \in$ R-tors then a left R-module M will be said to be τ-projective [resp. τ-injective] if and only if it is projective [resp. injective] relative to every R-epimorphism [resp. R-monomorphism] the kernel [resp. cokernel] of which is τ-torsion. Relative homological properties of modules were first studied by Walker [66]. Rangaswamy [74] has established the following result:

(4.1) PROPOSITION: Let $\tau \in$ R-tors and let M be a τ-projective left R-module. Then the following conditions on a submodule N of M are equivalent:

(1) Any diagram of the form



with W τ-torsion can be completed commutatively.

(2) M/N is τ-projective.

In particular, we note that if $\tau \in$ R-tors and if N is a τ-cotorsionfree submodule of a τ-projective left R-module M then M/N is τ-projective. This result is also due to Bland [74]. We also note that direct sums and direct summands of τ-projective left R-modules are τ-projective.

Another theorem of Rangaswamy [74] characterizes the

τ-projective left R-modules in the case that the torsion theory τ is jansian.

(4.2) PROPOSITION: _If_ $\tau \in$ R-jans _then a left_ R-_module_ M _is_ τ-_projective if and only if_ M _is isomorphic to a direct summand of_ P/N, _where_ P _is a projective left_ R-_module and_ N _is a_ τ-_cotorsionfree submodule of_ P.

As a corollary to this we note that if $\tau \in$ R-jans and if M is any left R-module then there exists an exact sequence $0 \to N' \to N \to M \to 0$ of left R-modules such that N is τ-projective and N' is τ-torsion. Indeed, consider any exact sequence of the form $0 \to L \to P \to M \to 0$ with P projective and set $N = P/C_\tau(L)$ and $N' = L/C_\tau(L)$. The result then follows from Propositions 2.5 and 4.1.

(4.3) PROPOSITION: _If_ $\tau \in$ R-jans _then the following conditions on a submodule_ N _of a_ τ-_projective left_ R-_module_ M _are equivalent_:

(1) M/N _is_ τ-_projective_.

(2) $N/C_\tau(N)$ _is a direct summand of_ $M/C_\tau(N)$.

PROOF: (1) $\Rightarrow$ (2): By Proposition 2.5, $N/C_\tau(N)$ is τ-torsion and so we have an exact sequence of abelian groups:

$$\mathrm{Hom}_R(M/C_\tau(N),N/C_\tau(N)) \to \mathrm{Hom}_R(N/C_\tau(N),N/C_\tau(N)) \to \mathrm{Ext}^1_R(M/N,N/C_\tau(N)) = 0.$$

Therefore the exact sequence of left R-modules

$$0 \to N/C_\tau(N) \to M/C_\tau(N) \to M/N \to 0$$

splits, proving (2).

(2) ⇒ (1): If N' is a τ-torsion left R-module and if $\alpha \in \mathrm{Hom}_R(N,N')$ then $C_\tau(N) \subseteq \ker(\alpha)$ and so α induces an an R-homomorphism $\alpha':N/C_\tau(N) \to N'$. By (2), we can extend this to an R-homomorphism $\beta':M/C_\tau(N) \to N$. If $\nu:M \to M/C_\tau(N)$ is the canonical surjection then $\nu\beta':M \to N'$ extends α. Thus, for any τ-torsion left R-module N' we have an exact sequence of abelian groups

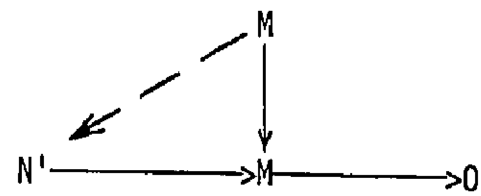
$$\mathrm{Hom}_R(M,N') \overset{\psi}{\to} \mathrm{Hom}_R(N,N') \to \mathrm{Ext}^1_R(M/N,N') \to \mathrm{Ext}^1_R(M,N') = 0,$$

where ψ is an epimorphism. Thus $\mathrm{Ext}^1_R(M/N,N') = 0$ and so M/N is τ-projective. □

(4.4) PROPOSITION: *The following conditions on* $\tau \in R\text{-tors}$ *and on a left* R-*module* M *are equivalent*:

(1) M *is* τ-*cotorsionfree and* τ-*projective*.

(2) *Any diagram of the form*



with $\ker(\alpha)$ *being* τ-*torsion can be completed in a unique manner*.

PROOF: The proof of this proposition is just the dual of the proof of (1) ⇔ (3) of Proposition 5.1 in [Golan, 75]. □

The existence of modules which are τ-cotorsionfree and τ-projective is important in constructing colocalizations. In particular, Sato [76] has established the following result.

(4.5) PROPOSITION: If $\tau \in$ R-jans then $W(\tau) \otimes_R M$ is τ-cotorsionfree and τ-projective for any left R-module M.

Another characterization of such modules for jansian torsion theories is essentially given by Onodera [77]:

(4.6) PROPOSITION: If $\tau \in$ R-jans then the following conditions on a left R-module M are equivalent:

(1) M is τ-cotorsionfree and τ-projective.

(2) There exists an exact sequence of the form $W(\tau)^{(\Lambda)} \to W(\tau)^{(\Omega)} \to M \to 0$.

(3) M is τ-cotorsionfree and for every short exact sequence $0 \to N' \to N \to M \to 0$ we have that N' is τ-cotorsionfree if and only if N is τ-cotorsionfree.

Ohtake [77] has noted that Proposition 4.5 can also be dualized. Namely, we have the following result.

(4.7) PROPOSITION: If $\tau \in$ R-jans then $\mathrm{Hom}_R(W(\tau),M)$ is τ-torsionfree and τ-injective for any left R-module M.

This result has the following consequence.

(4.8) COROLLARY: If $\tau \in$ R-jans then R_τ is isomorphic *to* $\mathrm{Hom}_R(W(\tau),W(\tau))$ *in the category of left* R-*modules and in the category of rings*. *Moreover*, $\mathrm{Hom}_R(W(\tau),\_)$ *is naturally equivalent to the localization functor* $Q_\tau(\_)$.

In Section 2 we considered those torsion theories for which the class of cotorsionfree left R-modules is closed under taking submodules. For such theories, we have a more convenient condition for relative projectivity.

(4.9) PROPOSITION: *If* $\tau \in$ R-tors *satisfies the condition that the class of* τ-*cotorsionfree left* R-*modules is closed under taking submodules then a sufficient condition for a left* R-*module* M *to be* τ-*projective is that* $C_\tau(R)M = M$.

PROOF: Let $\alpha:N \to N''$ be an R-epimorphism the kernel of which is τ-torsion and let $\beta:M \to N''$ be an R-homomorphism. Set $N' = (M\beta)\alpha^{-1}$ and let α' be the restriction of α to N'. Since $C_\tau(R)M = M$ we have $[C_\tau(R)N']\alpha' = C_\tau(R)[N'\alpha'] = C_\tau(R)M\beta = [C_\tau(R)M]\beta = M\beta$. Therefore $N' = C_\tau(R)N' + \ker(\alpha')$. Note that $C_\tau(R)N' \cap \ker(\alpha') \subseteq \ker(\alpha)$ and so $C_\tau(R)N' \cap \ker(\alpha')$ is both τ-cotorsionfree and τ-torsion, implying that it equals 0. Therefore $N' = C_\tau(R)N' \oplus \ker(\alpha')$ and so $C_\tau(R)N' \cong M\beta$. Thus β can be extended to an R-homomorphism from M to N. □

(4.10) PROPOSITION: _If_ $\tau \in$ R-jans _then_ τ _is stable if and only if every_ τ-_cotorsionfree left_ R-_module is_ τ-_projective_.

PROOF: If τ is stable then by Propositions 2.3 and 4.9 it follows that every τ-cotorsionfree left R-module is τ-projective. Conversely, assume that this condition holds. Let M be a τ-cotorsionfree left R-module and let N be a submodule of M. Then we have an exact sequence

$$0 \to N/C_\tau(N) \to M/C_\tau(N) \to M/N \to 0.$$

Since M is τ-cotorsionfree, so is M/N and so, by assumption, it is τ-projective. Therefore this sequence splits, implying that $M/C_\tau(N) \cong M/N \oplus N/C_\tau(N)$. Therefore we have an induced R-epimorphism $M \to M/C_\tau(N) \to N/C_\tau(N)$. Since $N/C_\tau(N)$ is τ-torsion, this implies that $N = C_\tau(N)$ and so N is τ-cotorsionfree. By Proposition 2.6, we have thus shown that τ is stable. □

5. Colocalizations. An R-homomorphism $\alpha:N \to M$ is said to be a colocalization of M at $\tau \in R\text{-tors}$ if and only if

(1) $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are τ-torsion;

(2) N is τ-cotorsionfree and τ-projective.

(5.1) PROPOSITION: *If* $\tau \in R\text{-tors}$ *is jansian then a necessary and sufficient condition for the inclusion map* $\iota:C_\tau(R) \to R$ *to be a colocalization of* R *at* τ *is that* $C_\tau(R)$ *be* τ-*projective*.

PROOF: Since $C_\tau(R/C_\tau(R)) = C_\tau(R)[R/C_\tau(R)] = 0$, it follows that $\operatorname{coker}(\iota)$ is τ-torsion. Moreover, the kernel of ι is surely τ-torsion and its image is surely τ-cotorsionfree. □

(5.2) PROPOSITION: *If* $\tau \in R\text{-tors}$ *and if* $\alpha:N \to M$ *is a colocalization of* M *at* τ *then* $\operatorname{im}(\alpha) = C_\tau(M)$.

PROOF: We know that $\operatorname{im}(\alpha)$ is τ-cotorsionfree and so $\operatorname{im}(\alpha) \subseteq C_\tau(M)$. Furthermore, $C_\tau(M)/\operatorname{im}(\alpha)$ is both τ-cotorsionfree and τ-torsion and so equals 0. Therefore $\operatorname{im}(\alpha) = C_\tau(M)$. □

(5.3) PROPOSITION: *Let* $\tau \in R\text{-tors}$ *and let* $\alpha:N \to M$ *and* $\alpha':N' \to M'$ *be colocalizations of left* R-*modules* M *and* M' *respectively at* τ. *If* $\beta \in \operatorname{Hom}_R(M,M')$ *then there exists a unique* $\beta^* \in \operatorname{Hom}_R(N,N')$ *making the diagram*

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & M \\ {\scriptstyle \beta^*}\downarrow & & \downarrow{\scriptstyle \beta} \\ N' & \xrightarrow{\alpha'} & M' \end{array}$$

commute.

PROOF: By Proposition 5.2 we see that $im(\alpha) = C_\tau(M)$ and $im(\alpha') = C_\tau(M')$ and so we have a diagram of the form

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow{\scriptstyle \alpha\beta} & & \\ N' & \xrightarrow{\alpha'} & C_\tau(M') & \longrightarrow & 0 \end{array}$$

with $ker(\alpha')$ being τ-torsion. The result then follows from Proposition 4.4. □

(5.4) COROLLARY: *If* $\tau \in$ R-tors *and if* $\alpha:N \to M$ *and* $\alpha':N' \to M$ *are colocalizations of a left* R-*module* M *at* τ *then there exists a unique* R-*homomorphism* $\delta:N \to N'$ *satisfying* $\delta\alpha' = \alpha$ *and this* δ *is in fact an isomorphism*.

Thus we see that colocalizations, if they exist, are unique up to isomorphism. The question of the universal existence of colocalizations at a torsion theory τ was solved by Ohtake [77], who proved the following result.

(5.5) PROPOSITION: *A torsion theory* $\tau \in$ R-tors *is jansian if and only if every left* R-*module has a colocalization at* τ.

Thus, combining Propositions 5.5 and 5.3, we see that if $\tau \in$ R-jans then there exists an idempotent right exact endofunctor $K_\tau(\_)$ of R-mod and a natural transformation κ^τ from $K_\tau(\_)$ to the identity endofunctor on R-mod such that for every left R-module M, $\kappa^\tau_M: K_\tau(M) \to M$ is a colocalization of M at τ. Indeed, Sato [76] has shown that we can take $K_\tau(\_)$ to be $W(\tau) \otimes_R \_$, with κ^τ given by $\kappa^\tau_M: \Sigma a_i \otimes b_i \otimes m_i \mapsto \Sigma a_i b_i m_i$. Moreover, we thus see that if $\tau \in$ R-jans then $(K_\tau(\_), Q_\tau(\_))$ is an adjoint pair of endofunctors of R-mod. Note too that by Proposition 1.2 a left R-module M is τ-torsion if and only if $K_\tau(M) = 0$.

(5.6) PROPOSITION: <u>If</u> $\tau \in$ R-jans <u>is stable then</u> $K_\tau(\_)$ <u>is an exact functor</u>.

PROOF: By Proposition 22.10 of Golan [75] we see that if τ is stable then $R/L(\tau)$ is flat as a right R-module and hence $L(\tau)$ is flat as a right R-module. This implies that $W(\tau)$ is flat as a right R-module and so $W(\tau) \otimes_R \_$ is exact. □

Finally, we obtain another characterization of stable jansian torsion theories.

(5.7) PROPOSITION: <u>Let</u> $\tau \in$ R-jans. <u>Then</u> τ <u>is stable if and only if every colocalization of a left</u> R-<u>module at</u> τ <u>is a monomorphism</u>.

PROOF: Let $\tau \in$ R-jans be stable and assume that $\alpha:N \to M$ is a colocalization of M at τ. Then, by definition, N is τ-cotorsionfree and $\ker(\alpha)$ is τ-torsion. On the other hand, by Proposition 2.6 we see that $\ker(\alpha)$ is also τ-cotorsionfree and so it must equal 0. Therefore α is a monomorphism.

Conversely, assume that the colocalization of any left R-module at τ is a monomorphism and let M be a τ-cotorsionfree left R-module. By Proposition 5.5 we know that M has a colocalization $\alpha:N \to M$ at τ which, by hypothesis, is monic. By definition, $\operatorname{coker}(\alpha)$ is τ-torsion and, since the class of all τ-cotorsionfree left R-modules is closed under taking homomorphic images, it is also τ-cotorsionfree. Therefore $\operatorname{coker}(\alpha) = 0$ and so α is an isomorphism. In particular, this implies that every τ-cotorsionfree left R-module is τ-projective. By Proposition 4.10 this shows that τ is stable. □

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NON - SMALL MODULES AND NON - COSMALL MODULES

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0. INTRODUCTION

Recently the author has studied small ring homomorphisms of commutative rings R in [9] and showed that R is not small in any ring extensions of R as an R - module if and only if Krull dimension of R is equal to zero. In this note, we shall consider an analogous situation on R - modules.

Let R be a ring, not necessarily commutative, with identity. A (right) R - module M is called non - small, if M is not a small submodule in its injective envelope E(M), which is equivalent to a fact that M is not a small submodule in any extension module of M (see Proposition 1.1). In the first section, we shall define a subfuctor $Z^{*}(\ )$ of identity in the category of all right R - modules, related to non - small modules and study its elementary properties (cf. [18]).

It is clear that every module containing an injective submodule is always non - small. In the second section, we shall study some rings which satisfy the converse of the above property, namely every non - small module contains a non - zero injective. We shall show that those rings are closely related to QF - 3 rings [19] and give a structure characterization of those artinian rings. In the final section, we shall deal with the dual of non - small modules. M is called a non - cosmall module, following [18], if M is a homomorphic image of a projective module P whose kernel is not essential in P, which is equivalent to a fact that if M is a homomorphic image of a module N, then the kernel is always not essential in N (see Proposition 3.1). We also study some rings with dual property that every non - cosmall module contains a projective submodule as a direct summand. We shall show that they are also closely related to QF - 3 rings.

Throughout every ring R has the identity and every R - module M is a unitary right R - module. E(M), Z(M) and J(M) mean an injective envelope, the singular submodule and the Jacobson radical of M, respectively. Some parts except in the final section overlap with results in [10], however we shall give complete proofs for convenience of the reader.

The author would like to express his thanks to Mr. T. Katayama for informing M. Rayar's paper [18] to the author and also to Prof F. Van Oystaeyen and the staffs at University of Antwerpen for their kind hospitalities during the conference of the ring theory in 1978.

1. FUNCTOR $Z^{\star}$

We know that if Krull dimension of a commutative ring is equal to zero, then R is never small in any ring extension as an R - module. We shall consider an analogous situation on R - modules. First, we take any ring, which is not necessarily commutative.

PROPOSITION 1.1 ([14], theorem 1). Let M be an R - module. Then the following conditions are equivalent.

1) M is not small in any extension module M' of M.

2) M is not small in an injective envelope E(M) of M.

3) There exists an injective module E containing M such that M is not small in E.

PROOF. 1) $\to$ 2) $\leftrightarrow$ 3) are clear. 2) $\to$ 1). We assume $M' \supseteq M$. Then $E(M') = E(M) \oplus E_1$. Hence, M is not small in $E(M')$. Therefore, M is not small in M'.

If M satisfies one of three equivalent conditions in Proposition 1.1, we say M is <u>non - small</u> and other - wise we say M is small [14]. ∎

LEMMA 1.1 Let $0 \to M \to Q$ and $Q' \to M \to 0$ be exact. If M is non - small, then so are Q and Q'.

PROOF. It is clear from the definitions. ∎

We shall define a subfunctor of identity in the category of all right R - modules (cf. the functor Z( )). Let M be an R - module. We put

$$Z^*(M) = \{m \in M, mR \text{ is small}\} \quad [18], \S 2.$$

Since J(M) is the union of all small submodules in M, $Z^*(E) = J(E)$ for any injective E and $Z^*(M) = M \cap J(E(M)) = M \cap J(E')$ for an injective $E' \supseteq M$. It is clear from Lemma 1.1 that $Z^*(\ )$ is a subfunctor of identity. If $M \neq Z^*(M)$, M is non - small, however the converse is not true. If R is a right perfect ring [2], J(M) is a unique maximal small submodule in M and so $M \neq Z^*(M)$ if and only if M is non - small. $Z^*(M) \supseteq J(M)$ and in general $Z^*(M) \neq J(M)$. We can define inductively Z^*_n as follows : $Z^*_n(M)/Z^*_{n-1}(M) = Z^*(M/Z^*_{n-1}(M))$. It is well known $Z_2 = Z_3 = \dots$ for singular submodule Z(M) [7]. We do not know whether $Z^*_2 = Z^*_3 = \dots$ or not.

However we have

PROPOSITION 1.2 We assume R/J(R) is a right artinian. Then $Z^*_2 = Z^*_3$ and $M/Z^*_2(M)$ is semi - simple and injective for every R - module M.

PROOF. Since $Z^*(E) = J(E) = EJ(R)$ for an injective E, $E/Z^*(E)$ is semi -

simple. Let $E/Z^*(E) = \sum_I \oplus S_\alpha \oplus \sum_J \oplus S_\beta$, where the S_α is injective and minimal and the S_β is small and minimal. Hence, $Z^*{}_2(E)/Z^*(E) \approx \sum_J \oplus S_\beta$ and $Z^*{}_2(E) = Z^*{}_3(E)$. We put $\bar{E} = E/Z^*{}_2(E)$. Since $Z^*(\bar{E}) = 0$, $J(E(\bar{E})) \cap \bar{E} = 0$. Hence, $J(E(\bar{E})) = 0$ and $E(\bar{E})$ is also semi - simple. Therefore, $\bar{E} = E(\bar{E})$. Let $E \supseteq M$. Then $E/Z^*(E)$ contains isomorphically $M/Z^*(M)$. Hence, $Z^*{}_2(M)/Z^*(M) = (Z^*{}_2(E)/Z^*(E)) \cap M/Z^*(M)$. Therefore, $M/Z^*{}_2(M)$ is isomorphic to a submodule of $E/Z^*{}_2(E)$, which is semisimple and injective. Thus, $Z^*{}_2(M) = Z^*{}_3(M)$. ∎

From now on, in this section, we shall assume R is (left and) right perfect unless otherwise stated. Then there exists a complete set $\{g_i\}$ of mutually orthogonal primitive idempotents such that $1 = \Sigma\, g_i$. Let $E = E(R)$ and x in $E - J(E)$. Then we obtain an epimorphism $f : R \to xR \subseteq E$. Since xR is non - small by Proposition 1.1, R is non - small by Lemma 1.1. Thus, we shall divide $\{g_i\}$ into two parts $\{g_i\} = \{e_i\}_{i=1}^{r} \cup \{f_j\}_{j=1}^{m}$, where the e_iR is non - small and the f_jR is small. We know $n \geqslant 1$ from the above. We call an idempotent g non - small (resp. small) if gR is non - small (resp. small). If we denote the primitive idempotents by e and f, we mean e is non small and f is small, respectively.

LEMMA 1.2 Let R be right perfect. Then every injective module is a homomorphic image of the form $\Sigma \oplus e_kR$, where the e_k is non - small.

<u>PROOF</u>. Let E be injective and $\varphi : \Sigma \oplus g_iR \to E$ a projective cover of E. Assume $\{g_i\}_{I'}$ are small for $I' \subseteq I$. Then $\varphi(\sum_{I'} \oplus g_iR)$ is a small submodule in E by Lemma 1.1. Since $\sum_I \oplus g_iR$ is a projective cover, $I' = \phi$. ∎

LEMMA 1.3 Let R be as above. If M is not small in $\Sigma \oplus g_iR/g_iA_i$, then there exists π_i such that $\pi_i(M) = g_iR/g_iA_i$, where the A_i is a right ideal and π_i is the projection on $g_iR/.g_iA_i$.

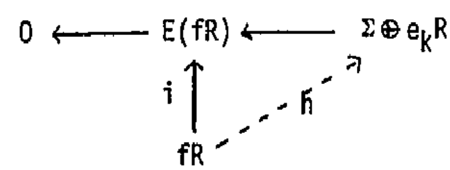
<u>PROOF</u>. Since M $\Sigma \oplus g_jJ(R)/g_jA_j$, $\pi_i(M) \not\subseteq g_iJ(R)/g_iA_i$ for some i. Hence, $\pi_i(M)$=

g_iR/g_iA_i, since g_iR is hollow. We call R a right QF - 3 ring if E(R) is projective as a right R - module [12] and [19]. ■

Theorem 1.3 Let R be right perfect. Then R is right QF - 3 if and only if each e_iR is injective, where the e_i is a non - small primitive idempotent.

<u>PROOF</u>. We assume that R is right QF - 3. Then $E = E(R) = \Sigma\oplus e_kR$ from [2] and [21]. Let e be a non - small primitive idempotent. Then eR is epimorphic to some e_kR by Lemma 1.3. Hence $eR \approx e_kR$ is injective. Conversely, let f be a small idempotent. Then we have an exact sequence $\Sigma\oplus e_kR \rightarrow E(fR) \rightarrow 0$ by Lemma 1.2.

Accordingly, we have a diagram :



where i is the inclusion. Since fR is projective and i is monomorphic, we obtain a monomorphism h of fR to Σ e_kR. Therefore, E(R) is projective. ■

COROLLARY. Let R be a right artinian and QF - 3 ring. Then R is a QF - ring if and only if $Z^*(R)$ $(=l(r(J(R)))) = J(R)$, where r( ) (resp. l( )) means a right (resp. left) annihilator.

<u>PROOF</u>. $Z^*(R) = l(r(J(R)))$ by [18], Proposition 4.8 (see Lemma 2.2 below). ■

2. CONDITION (*)

It is clear from Proposition 1.1 and Lemma 1.1 that every module containing an injective submodule is non - small. We shall study the converse case. For instance, if R is a QF ring, every non - small module contains an injective module (see Proposition 2.6). We shall investigate, in this section, some rings with the property above. Namely, we consider two conditions :

(*) Every non - small module contains a non - zero injective module.

(**) Every indecomposable injective module E is hollow, namely every proper submodule is small in E.

If R is right perfect, (*) is equivalent to " Every finitely generated non - small module contains a non - zero injective module", since $M \neq Z^*(M)$ if M is non - small. The ring Z of integers satisfies the above condition, since every finitely generated module is small by [11], Theorem 2. However, Z does not satisfy (*) by [11], Theorem 9.

Let K be a field and R a K - algebra of finite dimension. Then $\mathrm{Hom}_K(-,K)$ is a dual functor and every indecomposable injective module is of finite length. Hence, the condition (**) is dual to $(**)_1$(resp.$(**)_r$). Every indecomposable projective, left (resp. right) module contains a unique minimal submodule, (QF - 2 [19]).

We shall make use of the notations in § 1.

LEMMA 2.1 We assume R satisfies (*). Then every injective module contains a cyclic injective module and R contains a non - zero injective right ideal.

PROOF. Let E be injective. We consider an exact sequence $\sum_{x\in E} \oplus E_x \to E \to 0$; $E_x = E$ and $\varphi_{E_x} = 1_E$. Then $\varphi(\sum_{x\in E} xR) = E$ and so $\Sigma \oplus xR$ is non - small by Lemma 1.1. Hence, $\Sigma \oplus xR$ contains an injective module F. Therefore, some xR contains an injective submodule isomorphic to a direct summand of F by [22]. If we replace $\Sigma \oplus E_x$ by a free R - module, we obtain the last part.■

PROPOSITION 2.1 Let R be a right noetherian ring satisfying (*). Then R is right artinian.

PROOF. Let E = E(R) and let E be a finite direct sum of indecomposable injective modules E_i. Then E_i is cyclic by Lemma 2.1 and so E is noetherian. Hence, R is right artinian by [20]. ∎

PROPOSITION 2.2 We assume that R contains no infinite set of mutually orthogonal idempotents modulo $Z^*(R)$ and R satisfies (*). Then R is a right QF - 3 ring of finite Goldie dimension.

<u>PROOF</u>. Since R contains a non - zero injective right ideal by Lemma 2.1, we may assume $R = \sum_{i=1} \oplus e_iR \oplus hR$, where the e_iR are indecomposable and injective and hR is small. Let $E_1 = E(hR)$ and $\sum_k \oplus R \overset{\varphi}{\to} E_1 \to 0$ be exact. If $\varphi(\sum_k \oplus hR)$ is not small in E_1, hR contains an injective module by (*) and Lemma 1.1. Hence, $\varphi(\sum_k \oplus (1 - h)R) = E_1$. Thus we have an exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & E_1 & \overset{\varphi}{\longrightarrow} & \sum (1-h)R \\ & & i \uparrow & \nearrow f & \\ & & hR & & \end{array}$$

Since hR is projective and i is monomorphic, f is monomorphic. Therefore, $E_2 = E(R) = \sum \oplus e_{ij}R$, where $e_{ij}R \approx e_iR$ and R is of finite Goldie dimension. ∎

THEOREM 2.3 Let R be perfect. Then (*) holds if and only if there exists n_i for each non - small primitive idempotent e_i such that $e_iR/e_il(J^t)$ is injective for $0 \leq t \leq n_i$ and $e_iR/e_il(J^{n_i+1})$ is small. In this case $e_iR_i/e_il(J^t) \approx e_iR/e_il(J^{t'})$ for $t < t' \leq n_i$ and every submodule of e_iR either contains $e_il(J^{n_i+1})$ or equal to some $e_il(J^t)$, $t \leq n_i + 1$, where $J = J(R)$.

<u>PROOF</u>. We assume (*). Then e_iR is injective and indecomposable from (*). We assume $e_i = e$ and eR/eB is non - small for some right ideal B. Then eR/eB is injective, since eR/eB is indecomposable. Since eR is injective and R is perfect, eR contains a unique minimal submodule el(J) and $eB \supseteq el(J)$ [2]. We have a natural epimorphism $eR/el(J) \to eR/eB$ and eR/eB is injective. Hence, eR/el(J) is injective from (*) and Lemma 1.1. Therefore,

$eR/el(J)$ contains a unique minimal submodule $el(J^2)/el(J)$, which is contained in $eB/el(J)$. Repeating those arguments, we obtain $eR \supset eB \supset el(J^k) \supset \ldots \supset el(J) \supset 0$. First, we shall note $eR/el(J^t) \not\approx eR/el(J^{t'})$ for $k \geq t > t'$, since $l(J^t)$ is a two - sided ideal. Now R/J is artinian and the representative set of minimal modules is finite. Therefore, the above lenght is finite. Accordingly, $eB = el(J^s)$ for some s and $eR/el(J^n)$ is small for some n. Conversely, we assume the n_i in the theorem exists. Let M be a non - small module and $E = E(M)$. Then $M \not\subseteq EJ$. Let m be in $M - EJ$, then mR is non - small by Proposition 1.1. Let $1 = \Sigma\, e_i + \Sigma\, f_j$. Now, $me_iR \approx e_iR/e_iB$ for some right ideal B. Since $e_iR/e_il(J^t)$ is injective for $t \leq n_i$, $el(J^{t+1})$ is a unique minimal submodule of $e_iR/e_il(J^t)$. Hence, either $e_iB = e_il(J^s)$ for some s or $e_iB \supseteq e_il(J^{n_i+1})$. In the latter case, we have an epimorphism $e_iR/e_il(J^{n_i+1}) \to e_iR/e_iB$, which is a contradiction from Lemma 1.1. Hence, me_iR is injective. The last part is clear from the above. ∎

LEMMA 2.2 ([18], Proposition 4.8). Let R be right artinian and M an R - module. Then M is small if and only if $Mr(J) = 0$.

PROOF. See [3], p. 122. ∎

THEOREM 2.4 Let R be right artinian. Then (*) holds if and only if $R/r(J)J^k$ is a direct sum of an injective module and a small projective module for all $k > 0$, where $J = J(R)$.

PROOF. Let $R = \sum_{i=1}^{n} \oplus\, e_iR \oplus \sum_{j=1}^{m} \oplus\, f_jR$ be as in § 1 and $S = r(J)$. Since the f_jR is small, $f_jS = 0$ by Lemma 2.2. Hence, $S = \sum_{i=1}^{n} \oplus\, e_iS$ and $SJ^p = \Sigma \oplus e_iSJ^p$. We assume that $e_i = e$ and eR is injective and $eJ^{q-1} \neq 0$, $eJ^q = 0$. Then eJ^{q-1} is a unique minimal submodule in eR. Hence, $eJ^{q-1} = el(J)$. Similarly, we obtain $eJ^{q-t} = el(J^t)$ if eR/eJ^{q-t+1} is injective. If eR/eJ^s is small and eR/eJ^{s+1} is injective, $eS \subseteq eJ^s$ and hence, $eS = eJ^s$, since eJ^s/eJ^{s+1} is unique minimal. Therefore, if (*) holds, $S = \sum_{i=1}^{n} \oplus e_iS = \sum_{i=1}^{n} \oplus e_iJ^{n_i}$ for some n_i by Theorem 2.3. Hence, $R/SJ^k = \sum_{j=1}^{m} \oplus f_jR \oplus \sum_{i=1}^{n} \oplus e_iR/.e_iJ^{n_i+k}$ and the $e_iR/e_iJ^{n_i+k}$ is injective for $k>0$ by Theorem 2.3. Conversely, we assume the decompositions as in the theorem. We always have $SJ^k = \sum_{i=1}^{n} \oplus e_iSJ^k$. Hence, $R/SJ^k = \sum_{j=1}^{m} \oplus f_jR \oplus \sum_{i=1}^{n} \oplus e_iR/e_iSJ^k$. Therefore, the e_iR/e_iSJ^k is injective for any $k>0$ by Krull - Remak - Schmidt's theorem, since e_iR is non - small. If $e_iSJ^t = 0$ and $e_iSJ^{t-1} \neq 0$, e_iR is injective and e_iSJ^{t-1} is a unique minimal submodule in e_iR and $e_iSJ^{t-1} = el(J)$. Repeating those arguments as in the proof of Theorem 2.3, there exist an integer n_i and a unique series of submodules $e_il(J^t)$ of e_iR such that $e_iR/e_il(J^t)$ is injective and $e_iS = e_il(J^{n_i})$. Therefore, R satisfies (*) by theorem 2.3. ■

LEMMA 2.3 Let R be right perfect. (**) holds if and only if every indecomposable injective module is a homomorphic image of e_iR. (*) implies (**).

PROOF. It is clear from Lemma 1.2. ■

PROPOSITION 2.5 ([10]). Let R be right perfect and (**) holds. Then each e_iR contains a unique minimal submodule if and only if R is right QF -3 (cf. example 2).

PROOF. We assume e_iR contains a unique minimal submodule. $E = E(e_iR)$ is indecomposable and $E/J(E)$ is simple. Hence, e_iR is injective by Lemma 1.3, since some e_jR is projective cover of E and e_iR is projective (see the proof of Theorem 1.3). Therefore, we obtain the proposition by Theorem 1.3. ■

COROLLARY ([19]). Let R be a K - algebra of finite dimension over a field

K. If $(**)_1$ and $(**)_r$ hold (namely R is QF - 2), R is QF - 3.

PROPOSITION 2.6 Let R be a right and left artinian ring. Then the following conditions are equivalent.

1) R is a QF ring.

2) $(*)$ holds and $e_i R f_j = 0$ for every non - small e_i and small f_j.

3) $(*)$ holds and $l(J) \subseteq r(J)$.

4) $e_i R$ is injective and $e_i R/e_i l(J)$ is small whenever $e_i l(J) \neq 0$ for every non - small e_i, (cf. [3], Theorem 2.5).

PROOF. 1) $\to$ 2). Let R be a QF ring. Then $l(J) = r(J)$ and so $l(J)J^k = 0$. Hence, $(*)$ holds by Theorem 2.4. Since $f_j = 0$, $e_i Rf = 0$. 2) $\to$ 1). We assume $(*)$, then R is QF - 3 and $f_j R$ is monomorphic to some $\Sigma \oplus e_{ij} R$ by Theorem 1.3 and its proof, where $e_{ij} R \approx e_i R$. Hence, $e_i R f_j = 0$ implies $f_j = 0$.
3) $\to$ 1). If $l(J) \subseteq r(J)$, $Z^*(R) = J$ by [18], Proposition 4.8. Hence, R is QF by Proposition 2.2 and Corollary to Theorem 1.3.
1) $\to$ 3). It is clear.
1) $\to$ 4). If $e_i R/e_i l(J)$ is non - small, $e_i R/e_i l(J)$ is injective by 2). Hence, since R is a QF ring, $e_i R/e_i l(J)$ is projective. Therefore, $e_i l(J) = 0$.
4) $\to$ 1). Let $\{e_i R\}_1^t$ be a complete set of non - isomorphic right ideals in $\{e_i R\}_1^n$. Then every indecomposable injective modules is the socle of $e_i R$ is not isomorphic to one of $e_j R$ for $i \neq j$. Therefore, $e_i R/e_i J$ is the complete set of non - isomorphic minimal right modules. Let g be a primitive idempotent. Then gR/gJ is isomorphic to one in $\{e_i R/e_i J\}_1^t$. Hence, $gR \approx e_i R$ is injective. ∎

COROLLARY. Let R be a commutative ring. If R is a discrete rank one valuation ring, R satisfies $(**)$ but not $(*)$. If R is artinian, then the following conditions are equivalent.

1) R is a QF ring.

2) (*) holds.

3) (**) holds.

PROOF. The first part is clear from [13], [16], the structure of R and Proposition 2.1. We assume that R is artinian and (**) holds. We may assume R is local. Then a unique indecomposable and injective module is of the form R/A by (**) and [10], where A is an ideal. Hence, $E(R) = \Sigma \oplus R/A$ by [16] and E(R) is faithful. Therefore, A = 0. The remaining parts are clear by Proposition 2.6 and Lemma 2.3. ∎

PROPOSITION 2.7 ([10]). Let R be perfect. When either R is hereditary or $J(R)^2 = 0$, the following conditions are equivalent.

1) (*) holds.

2) (**) holds and each e_iR contains a unique minimal submodule.

3) R is right QF - 3 ring (see example 1)

PROOF. 1)→2)→3) are clear by Lemma 2.3. and Proposition 2.5.
3)→1). Since R is right QF - 3, each e_iR is injective by Theorem 1.3. First, we assume that R is hereditary. Then (*) holds by Theorem 2.3. Next, we assume $J^2 = 0$. Let M be non - small E = E(M). Since $M \subseteq EJ$, there exist m and e_i such that $me_i \in M - EJ$. Hence, me_iR is non - small as in the proof of Theorem 2.3. Now, me_iR is isomorphic either to e_iR or e_iR/e_iJ, since $J^2 = 0$ and e_iR is injective. If $me_iR \approx e_iR/e_iJ$, e_iR/e_iJ is injective, since e_iR/e_iJ is non - small. Therefore, M contains an injective submodule me_iR. ∎

PROPOSITION 2.8 ([10]). Let R be a right artinian and right QF - 3. Then R is hereditary if and only if e_iR/e_iJ^t is injective for every e_i and t.

PROOF. "Only if" part is clear from Theorem 1.3. Conversely, if e_iR/e_iJ^t

is injective for every e_i and t, (*) holds by Theorem 2.3. Hence, every indecomposable injective module is of the form e_iR/e_iJ^t by Lemma 2.3. Let E be injective and M a submodule of E. We shall show E/M is injective. Let S(M) be the socle of M. We define Loewy series $S^i(M)$ as follows : $S^i(M)/S^{i-1}(M) = S(M/S^{i-1}(M))$. We show the above fact by induction on $S^i(M)$. Let $E = E(M)\ \ E_1$ and $E_2 = E(M) = \Sigma \oplus e_{ij}R/e_{ij}{}^{t}ij$. Since $S(M) = S(E_2)$, $E_2/S(E_2) \supseteq M/S(M)$ and $E_2/S(E_2)$ is injective from the assumption. Hence, if $M = S(M)$, E/M is injective. We assume E'/N' is injective for $E' \supset N'$ whenever E' is injective and $S^i(N') = N'$. Let $M = S^{i+1}(M)$. Then E/S(M) is injective and $S^i(M/S(M)) = M/S(M)$. Hence, $E/M \approx (E/S(M))/(M/S(M))$ is injective by the induction. ∎

COROLLARY. Let R be right artinian and basic. Then R is isomorphic to the ring of upper tri - angular matrices over a division ring of degree n if and only if R satisfies the following three conditions,

1) $R = eR \oplus f_2R \oplus \dots \oplus f_nR$,

2) The composition length of eR is equal to n and

3) (*) holds.

<u>PROOF</u>. Conditions 1) 3) and Theorem 2.3 imply that every eR/eJ^t is injective for $t \leq n$. Hence, R is hereditary by Proposition 2.8. Therefore, R is desired ring by [5], Theorem 2. The converse is clear. ∎

We shall study further properties of such a ring in a forthcoming paper.

EXAMPLES ([10]). 1. Let K be a field, M a K - vector space of finite dimension and $M^* = \mathrm{Hom}_K(M,K)$. We put

$$R = \begin{pmatrix} K & M^* & K \\ & K & M \\ & & K \end{pmatrix}$$

Then R is a QF - 3 ring by the natural multiplication $M^{*} \underset{k}{\oplus} M \to K$ (see [6]). If $[M : K] \geq 2$, (**) does not hold, since Re_{22} contains two minimal submodules. We note that R is not hereditary and $J^2 \neq 0$ (see Proposition 2.7).

2. We put

$$R = \begin{pmatrix} K & K & K & K \\ & K & 0 & 0 \\ & & K & 0 \\ & & & K \end{pmatrix}$$

Then (**) holds but R is not QF - 3 and $J^2 = 0$.

3. We put

$$R = \begin{bmatrix} a & b & c \\ o & d & e \\ o & o & a \end{bmatrix} \quad a, b, c, d \text{ and } e \quad K \qquad ([19]).$$

Then $R = eR \bigcirc fR$ and (*) holds. However the composition length of eR = 3 (see Corollary to Proposition 2.8).

4. Let S be the ring of upper tri - angular matrices over K with degree n and R a K - subalgebra of S containing $\{e_{ii}\}_1^n$. We assume R is a two - sided indecomposable ring. Then

R is QF - 3 if and only if (**) holds and $e_{11}R$ contains a unique minimal submodule. R is QF - 3 and hereditary if and only if (*) holds. Let A be a two - sided ideal in S. Then S/A always satisfies (*) (see a forthcoming paper).

3. DUAL CONDITION $(*)^*$

In this section, we shall consider the dual of non - small modules. The following propositions and lemmas are obtained directly from the definition and we shall omit their proofs (cf. [18] pp. 17 - 21).

PROPOSITION 3.1 Let M be an R - module. Then the following conditions are equivalent.

1) For any module T and any epimorphism $f : T \to M$, ker f is always not essential in T.

2) There exist a projective module P and an epimorphism $f : P \to M$ such that ker f is not essential in P.

If M satisfies one of the above conditions, we say M is <u>non - cosmall</u> module, following [18].

LEMMA 3.1 Let $0 \to M \to Q$ and $Q' \to M \to 0$ be exact. If M is non - cosmall, then so are Q and Q'.

PROPOSITION 3.2 ([18], Proposition 2.4). M is non - cosmall if and only if $M \neq Z(M)$.

Every projective module is a non - cosmall module and so every module containing a projective submodule (as a direct summand) is a non - cosmall module. We shall consider the converse.

$(*)^*$ Every non - cosmall module contains a direct summand which is projective.

$(**)^*$ Every indecomposable projective is uniform.

If R is a perfect ring, $(**)^*$ is equivalent to $(**)_r$. If R is a commutative local ring, then $(**)^*$ holds if and only if R is a domain.

LEMMA 3.2 We assume $(*)^*$. Then every indecomposable semi - perfect module i.e. local projective is uniform.

<u>PROOF</u>. Let P be as in the lemma. Then J(P) is a unique maximal submodule of P by [8], [15]. Hence, P/K is indecomposable for any submodule K of P. We assume $K_1 \cap K_2 = 0$. Then P/K_1 is non - cosmall by Proposition 3.1 if $K_2 \neq 0$. Hence, P/K_1 is projective from the above and $(*)^*$. Therefore, $K_1 = 0$. ∎

PROPOSITION 3.3 If R satisfies $(*)^*$, R contains a projective and injective

right ideal.

PROOF. Let $E = E(R)$. Then $E \neq Z(E)$ and so E contains a projective direct summand P. Since P is a summand of a free module and is injective, R contains a direct summand isomorphic to a summand of P by [22]. ∎

PROPOSITION 3.4 1) Let R be a semi - perfect ring with $(**)^*$. Then M is a non - cosmall module if and only if M contains a projective module.

2) We assume $(*)^*$ holds. Then R is right QF - 3 if one of the following is satisfied,

a) Right Goldie dimension of R is finite

b) R is semi - perfect.

PROOF. 1) Let M be non - cosmall. Then M contains a cyclic non - cosmall submodule mR by Proposition 3.2. Let $0 \leftarrow mR \overset{f}{\leftarrow} \Sigma \oplus e_iR$ be a projective cover of mR. Since ker f is non - essential in $\Sigma \oplus e_iR$, $\ker f \cap e_jR$ is not essential in e_jR for some j. Hence, $\ker f \cap e_jR = 0$ and M contains a submodule isomorphic to e_jR.

2) a) Let $\{K_i\}_1^n$ be a set of uniform right ideals in R such that $\sum_1^n \oplus K_i$ is essential in R. Then $E = E(R) = \sum_1^n \oplus E(K_i)$. Let $g : R \to E(R)$ be the inclusion and $Q_j = \ker \pi_j g$, where π_j is the projection of E to $E(K_j)$. Since $E(K_j)$ is uniform, Q_j is irreducible. Furthermore, $0 = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ is irredundant and $E(R/Q_i) = E(K_i)$. If $n \geq 2$, R/Q_i is non - cosmall and so $E(R/Q_i)$ is projective from $(*)^*$. If $n = 1$, R is irreducible and E is indecomposable and non - cosmall. Hence, E is projective.

b) Let $1 = \Sigma e_i + \Sigma f_j$, where $\{e_i, f_i\}$ is a complete set of mutually orthogonal primitive idempotents such that the e_iR is injective (see Proposition 3.3). Since f_jR is uniform by Lemma 3.2, $E_j = E(f_jR)$ is indecomposable and $Z(E_j) \neq E_j$. Hence, E_j is projective. ∎

PROPOSITION 3.5 We assume $(\star)^*$. Then for every uniform projective P, either $Z(P) = 0$ or $Z(P/Z(P)) = P/Z(P)$. Every submodule not contained in $Z(P)$ is projective.

PROOF. Let $P \supseteq T$ and $Z(P) \not\supseteq T$. Then T is non - cosmall by Proposition 3.2 and indecomposable. Hence, T is projective. Let K_i be submodules containing $Z(P)$ properly ($i = 1,2$). Then K_i is projective. Put $K = K_1 \cap K_2$ and consider a natural epimorphism $K_1 \oplus K_2 \to K_1 + K_2 \to 0$, where $\varphi = 1_{K_1} - 1_{K_2}$. Then $\ker \varphi \approx K$. Since $K_1 + K_2$ is projective, so is K. Hence, $K \neq Z(P)$. Therefore, $Z(P)$ is irreducible and so $P/Z(P)$ is indecomposable. Hence, $P/Z(P) = Z(P/Z(P))$ if $Z(P) \neq 0$. ∎

THEOREM 3.6 Let R be semi - perfect. Then $(\star)^*$ holds if and only if there exists sets of primitive idempotents $\{e_i\}$ and of integers $\{n_i\}$ such that

1) the e_iR is injective,

2) $e_iJ^{t_i}$ is projective for $t \leq n_i$ and $e_iJ^{n_i+1}$ is singular and

3) every indecomposable projective is isomorphic to some $e_iJ^{t_i}$.

In this case every submodule e_iB in e_iR either is contained in $e_iJ^{n_i+1}$ or equal to some e_iJ^t, $t \leq n_i + 1$, where $J = J(R)$.

PROOF. We assume $(\star)^*$ holds. Then there exists a complete set of primitive idempotents e_i such that e_iR is injective by Proposition 3.4. Let $e = e_i$ and eK a proper projective submodule of eR. Then $eR \supset eJ \supset eK$. Since eR is uniform and $eK \subset Z(eR)$, eJ is projective by Proposition 3.5. Now, $eJ \approx fR$ by [1] and so eJ^2 is unique maximal submodule of eJ. Therefore, we have a unique chain $eR \supset eJ \supset \dots \supset eJ^t \supset eK$ with eJ^i projective. If $eJ^i \approx eJ^j$, this isomorphism is extended to one of eR. Hence, $i = j$. Thus we can find some m such that eJ^m is projective and eJ^{m+1} is cosmall i.e. singular by Proposition 3.2. If f_jR is not injective, f_jR is contained in some e_iR

by the proof of Proposition 3.4, 2). Hence, $f_jR \cong e_iJ^{t_i}$. Conversely, let M be a non - cosmall module. Then there exist $m \in M$ and a primitive idempotent g such that mgR is cosmall by Proposition 3.2. Since gR is uniform from 1) and 3), $mgR \approx gR$ and $gR \approx e_iJ^t$ by 3). Thus, we have a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & mgR & \longrightarrow & M \\ & & \cong\downarrow & & \swarrow h \\ & & e_iJ^t & & \\ & & \downarrow & & \\ & & e_iR & & \end{array}$$

Then we can find a homomorphism h of M to e_iR, since e_iR is injective. Since $\mathrm{im}\, h \supseteq e_iJ^t$ and $e_iR \supset e_iJ \supset e_iJ^2 \supset \ldots$ is a unique chain as above, $\mathrm{im}\, h = e_iJ^{t_i}$ is projective. Hence, M contains a projective module isomorphic to $e_iJ^{t_i}$ as a direct summand. The last part is clear from the above. ∎

COROLLARY 1. Let R be semi - perfect and hereditary. Then the following conditions are equivalent.

1) $(*)^*$ holds.
2) There exists a set $\{e_i\}$ of primitive idempotents such that the e_iR is injective and f_jR is contained isomorphically in some e_iR for every primitive idempotent f_j.
3) R is Morita equivalent to a direct sum of rings of upper tri - angular matrices over division rings.

PROOF. Since R is hereditary, 1) and 2) are equivalent by the theorem. If $(*)^*$ holds, e_iR is of finite length. Hence, R is right artinian. Therefore, R is QF - 3 artinian and hereditary. Accordingly, we have 3) by [5], Theorem 2. It is clear that 3) implies 2). ∎

COROLLARY 2. Let R be a semi - perfect. Then the following conditions are equivalent.

1) $Z(R) = 0$ and $(*)^*$ holds.

2) R is hereditary and right QF - 3.

PROOF. 1)→2). Since $Z(R) = 0$, every submodule of e_iR is projective by Proposition 3.5. Hence, e_iR is of finite length and so R is right artinian. Furthermore, $J = \Sigma \oplus e_iJ \oplus \Sigma \oplus f_jJ$ is projective from the above. Therefore, R is hereditary.

2)→1). Let e_iR be injective. Then e_iR is uni - serial and of finite length from the proof of Theorem 3.6. Hence, R is right artinian, since R is right QF - 3 and we obtain 1) by [5], Theorem 2. ∎

THEOREM 3.7 Let R be right artinian. Then $(*)^*$ holds if and only if 1) $l(J^k l(J))$ is a directsum of an injective module and a small projective module for all $k>0$ and 2) $(**)_r$ holds.

PROOF. We assume $(*)^*$ holds. Then $(**)_r$ holds by Lemma 3.2. Let $1 = \sum_{i=1}^{n} e_i + \sum_{j=1}^{n} f_j$ be as in § 1 and the e_i (resp. f_j) a non - small (resp. small) primitive idempotent. Then e_iR is injective, eJ^k is projective for $k \leq n_i$ and $e_iJ^{n_i+1} = Z(e_iJ^{n_i+1})$ by Theorem 3.6. Furthermore, since $f_jR \approx e_{\pi(j)}J^{t_j}$, $f_jJ^{n_{\pi(j)}-t_j}$ is projective and $f_jJ^{n_{\pi(j)}-t_j+1} = Z(f_jJ^{n_{\pi(j)}-t_j+1})$. We know by Theorem 3.6 that $e_iR \supset e_iJ \supset \dots \supset e_iJ^{n_i} \supset e_iJ^{n_i+1}$ is a unique series of submodules over $e_iJ^{n_i+1}$ of e_iR. Hence, $e_iZ(R) = Z(e_iR) = e_iJ^{n_i+1}$. Therefore, $Z(R) = \Sigma\, e_iJ^{n_i+1} \oplus \Sigma\, f_jJ^{n_{\pi(j)}-t_j+1}$. Now $l(J^k l(J)) = \{x \in R \mid xJ^k \subseteq Z(R)\}$ and so $l(Jl(J))/Z(R)$ is equal to the socle of $R/Z(R)$ $(\approx \Sigma\, e_iR/e_iJ^{n_i+1} \oplus f_jR/f_jJ^{n_{\pi(j)}-t_j+1})$. Hence, $l(Jl(J)) = \Sigma\, e_iJ^{n_i} \oplus \Sigma\, f_jJ^{n_{\pi(j)}-t_j}$ is projective. If $e_iJ^{n_i} = e_iR$, $e_iJ^{n_i}$ is injective and if $e_iJ^{n_i} \neq e_iR$, $e_iJ^{n_i}$ is small. We can show inductively that $l(J^k l(J)) = \Sigma\, e_iJ^{n_i-k+1} \oplus \Sigma \oplus f_jJ^{n_{\pi(j)}-t_j-k+1}$.

Conversely, we assume 1) and 2). If $J^m = 0$, $R = l(J^m l(J))$ is a directsum of an injective module and a small projective module. Hence, e_iR is injective by Krull - Remak - Schmidt's theorem and so R is right QF - 3 by Theorem 1.3. Since $E = E(R) \approx \Sigma \oplus e_{ij}R$; $e_{ij}R \approx e_iR$ and $(**)_r$ holds, f_jR is monomorphic to some $e_{\pi(j)}R$. Put $e = e_i$ and $e(l(J^n l(J)) = eR$ and $e(l(J^{n-1}l(J)) \neq eR$. Since $e(l(J^{n-1}l(J)) \supseteq eJ$, $eJ = e(l(J^{n-1}l(J))$ is projective by 1) (note that eJ is uniform and $l(J^k l(J))$ is a two - sided ideal). Since eJ has a unique maximal submodule, $el(J^{n-2}l(J)) = eL^2$. Thus, we obtain a unique series of small projective submodules $eJ \supset eJ^2 \supset \ldots \supset eJ^{n-1}$ and $eJ^n = eZ(R)$. Therefore, $f_jR \approx e_{\pi(j)}J^t j$ and $(*)^*$ holds by Theorem 3.6. ∎

REMARK 1. Let Q be a QF ring. Then

$$R = \begin{pmatrix} Q & Q \\ 0 & Q \end{pmatrix}$$

is QF - 3 and satisfies 2) in Theorem 3.7. However $l(Jl(J))$ is always projective and $l(J^2l(J))$ is projective if and only if Q is semisimple. Hence, R satisfies $(*)^*$ if and only if Q is semisimple.

We do not know whether 1) implies 2) in Theorem 3.7.

PROPOSITION 3.8. If R is self injective as a right R - module, $(*)^*$ holds. If R is commutative and noetherian, the converse is true.

PROOF. Let $M \neq Z(M)$ for a right R - module. Then we have m in M such that $mR \neq Z(mR)$. Put $K = \{x \in R \mid mx = 0\}$. We may assume $R = E(K) \oplus E_1$. Since K is not essential in R, $E_1 \neq 0$. Hence, mR contains isomorphically E_1, which is projective and injective. The remaining part is clear by Corollary to Proposition 3.4 and Theorem 3.7. ∎

REMARKS 2. Let R be self injective, even if R is a commutative ring such that R/J(R) is artinian, R does not satisfy (*) in general (see [17],

p. 378 and Lemma 2.1).

3. The ring Z of integers satisfies a condition "Every finitely generated non - cosmall module contains a projective direct summand". However Z does not satisfy $(\star)^*$. If R is a right artinian ring such that every indecomposable injective is finitely generated, the above condition is equivalent to $(\star)^*$ from the proof of Theorem 3.6. We do not know whether conditions $(\star)$ and $(\star)^*$ are right and left symmetric.

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MATRIX VALUATIONS ON RINGS

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0. INTRODUCTION

Given an ideal I in a commutative ring R, there is a well-known lemma in commutative ring theory which says that the radical of I is an intersection of prime ideals P containing I, i.e. $\sqrt{I} = \cap P$, $P \supseteq I$.

One way of generalizing the above result is to develop the notion of a pseudovaluation p on a ring R as it is treated in [2]. There it is shown how to obtain valuations from pseudovaluations on R. As it turns out pseudovaluations on R can be considered analogous to ideals of R and valuations similar to prime ideals. Moreover, given an arbitrary pseudovaluation p and $\{v_i\}_{i\in I}$ a family of valuations on R, then

$$p^* = \inf_{i\in I}\{v_i\},\quad v_i \geqslant p,$$

where p^* is the root of p.

As another generalization of the result quoted above, Cohn in chapter 7 of [3] develops the notion of a matrix ideal of a ring R (not necessarily commutative), and shows how prime matrix ideals can be used to obtain the universal field of fractions of R under certain conditions. Furthermore, it is shown that given a matrix ideal A of R, then the radical of A is an intersection of prime matrix ideals P containing A.

This paper, essentially, deals with a common generalization of those stated above by developing the idea of a matrix valuation and a matrix pseudovaluation on a ring R (not necessarily commutative).

In section 3 we introduce the notion of a matrix valuation on a ring R, and show that any matrix valuation V on R gives rise to a prime matrix ideal of R. Hence any ring R with a matrix valuation V has an epic R - field K associated with V ; we point out that V induces a valuation on K_V.

Section 4 deals with a generalization of the idea of a matrix valuation to that of a matrix pseudovaluation, and presents analogous results to those of Bergmans' [2] for matrix pseudovaluations.

1. PRELIMINARIES

This section recalls some conventions from [3] which we will follow throughout the work. All rings occurring are associative, but not necessarily commutative. Every ring has a unit element, denoted by 1, which is preserved by homomorphisms and inherited by subrings. Given two square matrices A, B over a ring R, the diagonal sum of these matrices is defined

as :
$$A \dot{+} B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

This sum is always defined, and for square matrices of order r, s, the diagonal sum is square of order r + s. We now recall another operation on square matrices over R which is in fact defined only for certain pairs of matrices over R. Let $A = (a_{ij})$, $B = (b_{ij})$ be two n X n matrices over R such that $a_{ij} = b_{ij}$ for all $i = 2, 3, \ldots, n$, $j = 1, 2, \ldots, n$. We shall say that the determinantal sum of A and B with respect to the first row exists ; it is the matrix C whose first row is the sum of the first rows of A and B, and whose other rows agree with those of A and B. The determinantal sum with respect to another row or column, when it exists, defined similarly. We shall write $C = A \nabla B$ for the determinantal sum of A and B. We note that the latter operation is not everywhere defined and to say that C is a determinantal sum of matrices $A_1, A_2, \ldots, A_n$ means that we can replace two of $A_1, \ldots, A_n$ by their determinantal sum with respect to some row or column, and repeat this process on two matrices in the resulting set untill we are left with one matrix, namely C.

DEFINITION. Let R be any ring, A and B be two square matrices over R not necessarily of the same size. We shall say that A and B are <u>stably associated</u> if there exist invertible matrices P, Q such that

$$A \dotplus I = P(B \dotplus I)Q, \quad (1)$$

for unit matrices of suitable size. If P and Q in (1) are products of elementary matrices over R, then A and B are said to be <u>stably E-associated</u>.

An n X n matrix A over a ring R is said to be <u>full</u> if it cannot be written as a product of matrices P, Q, where P is an n X r matrix and Q is r X n, and r X n. Otherwise, A is called <u>non-full</u>.

Now let $\mathcal{P}$ be a set of square matrices over R. Then $\mathcal{P}$ is called a <u>matrix ideal</u> of R if

1. $\mathcal{P}$ includes all non-full matrices,
2. If $A, B \in \mathcal{P}$ and their determinantal sum $C = A \nabla B$ with respect to some row (or column) exists, then $C \in \mathcal{P}$,
3. If $A \in \mathcal{P}$, then $A \dotplus B \in \mathcal{P}$ for all square matrices B over R,
4. $A \dotplus 1 \in \mathcal{P} \rightarrow A \in \mathcal{P}$.

$\mathcal{P}$ is said to be <u>proper</u> if it does not contain the element 1. Furthermore, $\mathcal{P}$ is called a <u>prime matrix ideal</u> if it is a proper matrix ideal with the additional condition

$$A + B \in \mathcal{P} \to A \in \mathcal{P} \text{ or } B \in \mathcal{P}.$$

A set Σ of square matrices over R is said to be multiplicative if $1 \in \Sigma$, and whenever $A, B \in \Sigma$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \Sigma$$

for any matrix C of suitable size over R.

We shall need the following result in section 3, for a proof see chapter 7 of [3].

THEOREM A. Given a ring R with a prime matrix ideal $\mathcal{P}$, there exists an epic R - field K such that $\mathcal{P}$ is the precise class of matrices mapped to singular matrices under the canonical homomorphism $R \to K$.

2. MATRIX VALUATIONS

Let R be any ring and Γ a totally ordered additive abelian group.

DEFINITION. A function v on R with values in $\Gamma \cup \{+\infty\}$ is called a semi - valuation if

i) $v(ab) = v(a) + v(b)$, $a, b \in R$,

ii) $v(a + b) \geq \min\{v(a), v(b)\}$,

iii) $v(0) = +\infty$

We recall that v is a valuation on R if we have $v(a) = +\infty$ if and only if $a = 0$.

OBSERVATION. The set P of all elements $a \in R$ such that $v(a) = +\infty$ is a strong prime ideal (i.e. $\bar{R} = R/P$ is an integral domain), and v induces a valuation on $\bar{R}$.

Now using the operations "diagonal sum" and "determinantal sum" introduced earlier, we generalize the concept of a semi - valuation to the set M(R) of all square matrices over R.

DEFINITION 1. A function V on M(R) with values in $\Gamma \cup \{+\infty\}$ is called a matrix valuation if

i) $V(A \dotplus B) = V(A) + V(B)$, $A, B \in M(R)$

ii) $V(A \quad B) \geq \min \{V(A), V(B)\}$, whenever   is defined for square matrices A, B over R,

iii) V remains unchanged under multiplying any row or column by -1,

iv) $V(1) = 0$ for $1 \in R$,

v) $V(A) = +\infty$ for any non-full matrix A over R.

We now present some consequences of the above axioms in the following

PROPOSITION 2. Given a matrix valuation V on a ring R, we have the following :

V.1. If $V(A) \neq V(B)$, then $V(A \quad B) = \min \{V(A), V(B)\}$ whenever   is defined for square matrices A,B in M(R),

V.2. V is zero on elementary matrices over R. In particular, $V(1) = 0$ for any unit matrix I in M(R)

V.3. If we add to the column (or row) a_i of matrix $A = (a_1, \ldots, a_n)$ a right (or left) multiple $a_j\lambda$, $\lambda \in R$, of another column (or row), V does not change ; i.e. V(A) is unchanged if A is multiplied on the left (or right) by elementary matrices,

V.4. V remains unchanged under any permutation of rows or column of A,

V.5. $V\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = V\begin{bmatrix} A & 0 \\ D & B \end{bmatrix} = V\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = V(A) + V(B)$, where $A, B \in M(R)$, and C, D are matrices of suitable sizes over R,

V.6. If A is stably E - associated to B, then $V(A) = V(B)$,

V.7. $V(AB) = V(A) + V(B)$ for square matrices A, B of the same size in M(R),

V.8. The restriction of V to R is a semi - valuation.

PROOF. The proofs follow almost immediately from definition 1. ∎

The next result points out the interrelationship between matrix valuations and valuations of determinants in the commutative case.

THEOREM 3. Suppose R is a commutative ring with a semi - valuation v. If V is a matrix valuation on R such that V/R = v, then we have $V(A) = v(\det(A))$, $A \in M(R)$.

PROOF. Using proposition 2, one can prove this by induction on the order of square matrices over R.

Thus a matrix valuation on R (in the commutative case) is completely determined by its restriction to R. ∎

LEMMA 4. Given a matrix valuation V on a ring R the set of all square matrices A such that $V(A) = +\infty$ is a prime matrix ideal.

PROOF. Follows from the definition of a prime matrix ideal.

Let P be the prime matrix ideal obtained in lemma 4. By theorem A, there exists an epic R - field K such that P is the precise class of matrices mapped to singular matrices under the canonical homomorphism $R \to K$. We shall call this field, the field associated with V and use the notation K_V. We recall that, by theorem A, each element $x \in K_V$ can be obtained as the first component u_1 of the solution $(u_1, u_2, \ldots, u_n)^t$ of system $Au + a = 0$, where $A = (a_1, a_2, \ldots, a_n)$ lies in the multiplicative set $\Sigma = P^c$ of all square matrices on which V is finite, and a is a column over R. Now define

$$W(u_1, A, a) = V(A_1) - V(A),$$

where $A_1 = (a, a_2, \ldots, a_n)$. It is not hard to show that W is independent of the choice of system, and W is a valuation on K_V. ∎

THEOREM 5. Let R be a ring with a matrix valuation V. Then V induces a valuation on the associated epic R - field K_V.

We now investigate matrix valuations on skew fields. Let K be a skew field, $GL_n(K)$ be the group of all non - singular matrices over K, and $E_n(K)$

the subgroup of $GL_n(K)$ generated by $I + \lambda E_{ij}$ for all $i \neq j$, and $\lambda \in K$, where E_{ij} is the matrix having 1 in the (i, j) - place, 0 elsewhere. In [1] it is shown that $A \in GL_n(K)$ can be written in the form $B.D(\mu)$ where $B \in E_n(K)$ and the matrix $D(\mu)$ differs from the unit matrix only in the element a_{nn} which is $\mu \in K$. The image $\bar{\mu} \in K^{\star ab} \cup \{0\}$ of μ, where $K^{\star ab} = K^{\star}/[K^{\star}, K^{\star}]$; is independent of the choice of decomposition of A into B and $D(\mu)$. $\bar{\mu}$ is known as the Dieudonné determinant of A, and it will be denoted by d(A). We can now state the following

THEOREM 6. Let K be a skew field with a valuation v. Then there exists a unique matrix valuation V on K inducing v, and V is given by

$$V(A) = v(\mu), \ \bar{\mu} = d(A)$$

for each square matrix A over K.

PROOF. Similar to that of theorem 3. ∎

REMARK. Suppose $f : R \to S$ is a ring homomorphism of R into S. Then a matrix valuation V on S determines a matrix valuation W, say, on R by pullback, i.e.

$$W(A) = V(A^f).$$

COROLLARY 1. Given a ring R, let K be an epic R - field with a valuation v. Then v induces a matrix valuation on R.

COROLLARY 2. Given a valuation v on a right (or left) Ore ring R, there is a unique matrix valuation on R inducing v.

4. MATRIX PSEUDOVALUATIONS

Let R be any ring and Γ be the additive ordered semi - group of real numbers with $+\infty$ adjoined. A function p on R with values in Γ is called a pseudovaluation on R if

1) $p(xy) \geqslant p(x) + p(y)$, $x, y \in R$,

2) $p(x - y) \geqslant \min\{p(x), p(y)\}$,

3) $p(1) = 0$, $p(0) = +\infty$

In [2] Bergman shows that given a real - valued pseudovaluation p on a commutative ring R, there exists a valuation $v \geqslant p$ which also satisfies certain upper bounds. In particular, if $p(st) = p(s) + p(t)$ for all $s, t \in S$, where S is a multiplicative semi - group in R, then v can be chosen so that $v(s) = p(s)$ for all $s \in S$. Here in this section we generalize the above result by developing the notion of a matrix pseudovaluation on a ring R (not necessarily commutative), and present analogous results to those of Bergmans' in [2] for matrix pseudovaluations. Let R be a ring and Γ be the additive ordered semi - group of real numbers with $+\infty$ adjoined. Denote by M(R) the set of all square matrices over R.

DEFINITION 1. A function μ on M(R) with values in Γ is said to be a <u>matrix pseudovaluation</u> if the following conditions are satisfied :

i) $\mu(A \dotplus B) \geqslant \mu(A) + \mu(B)$, $A, B \in M(R)$

ii) $\mu(A \quad B) \geqslant \min\{\mu(A), \mu(B)\}$, whenever is defined for square matrices A, B in M(R),

iii) μ remains unchanged under multiplying any row or column by -1,

iv) $\mu(1) = 0$ for $1 \in R$, and $\mu(A \dotplus 1) = \mu(A)$ for any A in M(R),

v) $\mu(A) = +\infty$ for any non - full matrix A over R.

The matrix pseudovaluation μ will be called <u>radical</u> if it satisfies

$$\mu(\overset{n}{\dotplus} A) = n\mu(A),$$

where $A \in M(R)$ and n any positive integer. Thus a matrix valuation V is just a matrix pseudovaluation satisfying the stronger condition

$$V(A \dotplus B) = V(A) + V(B)$$

for all $A, B \in M(R)$.

We now collect some of the consequences of the above axioms in the following

PROPOSITION 2. Let R be any ring with a matrix pseudovaluation μ. Then we have the following :

μ.1. If $\mu(A) \neq \mu(B)$, then $\mu(A \quad B) = \min\{\mu(A), \mu(B)\}$ whenever is defined for matrices A, B over R,

μ.2. $\mu(A)$ is unchanged if A is multiplied on the left (or right) by elementary matrices,

μ.3. $\mu\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \mu\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \mu(A \dotplus B) \geq \mu(A) + \mu(B)$, where A, B $M(R)$, and C, D are matrices of suitable size over R,

μ.4. μ is zero on elementary matrices over R, and $\mu(A \dotplus I) = \mu(A)$ for any unit matrix $I \in M(R)$,

μ.5. If A is stably E - associated to B, then $\mu(A) = \mu(B)$,

μ.6. $\mu(AB) \geq \mu(A) + \mu(B)$ for square matrices A, B of the same size over R,

μ.7. The restriction of μ to R is a pseudovaluation.

PROOF. Similar to that of proposition 3.2. ∎

The following lemma shows how matrix ideals of R and matrix pseudovaluations on R are related.

LEMMA 3. Let R be any ring with a matrix pseudovaluation μ. Then the set P of all square matrices A over R such that $\mu(A) = +\infty$ is a proper matrix ideal.

PROOF. Follows from the definition of a matrix ideal. ∎

One can apply similar method of proofs as used in [2] to prove the following results :

LEMMA 4. Let μ be a matrix pseudovaluation on a ring R. Then the function $\mu^*(A) = \lim_{n\to\infty} \frac{1}{n}\mu(\overset{n}{\dotplus} A)$ is defined for all $A \in M(R)$ and it is a radical matrix pseudovaluation $\geq \mu$.

Furthermore, if $A \in M(R)$ and Σ a multiplicative set of square matrices containing A, then

$$\sup_{X \in \Sigma} \{\mu^*(A \dotplus X) - \mu^*(X)\} \leq \sup_{X \in \Sigma} \{\mu(A \dotplus X) - \mu(X)\}$$

where the supremum in this relation is taken over all $X \in \Sigma$ such that $\mu(X) < +\infty$. Σ will always be non - empty since it contains 1.

The above process of obtaining μ^* from μ will be called "taking the root of μ" and we shall write $\mu^* = \sqrt{\mu}$. We note that μ is radical if and only if $\mu^* = \mu$.

Before stating the next lemma, we need to give the following

DEFINITION. $A \in M(R)$ is called <u>regular</u> under a matrix pseudovaluation μ, or μ is regular at A, if for all B M(R)

$$\mu(A \dotplus B) = \mu(A) + \mu(B).$$

LEMMA 5. Suppose μ is a radical matrix pseudovaluation on R and A M(R) with $\mu(A) \leq +\infty$. Then the function

$$\mu_A(B) = \lim_{n \to \infty} \{\mu[B \dotplus (\dotplus^n A)] - n\mu(A)\}$$

is defined for all B M(R), and it is a radical matrix pseudovaluation $\geq \mu$, which is regular at A.

Furthermore, for any $B \in M(R)$ and any multiplicative set Σ of square matrices containing A, we have

$$\sup_{X \in \Sigma} [\mu_A(B \dotplus X) - \mu_A(X)] \leq \sup_{X \in \Sigma} [\mu(B \dotplus X) - \mu(X)]$$

where the supremum is taken over all $X \in \Sigma$ such that $\mu(X) < +\infty$. Σ is non - empty since $1 \in \Sigma$.

We call the above process of finding $\mu_A(B)$ "regularization at A".

LEMMA 6. Let μ be a matrix pseudovaluation on a ring R. Then there exists a matrix valuation V on R satisfying

$$\mu(A) \leqslant V(A) \leqslant \sup_{X \in M(R)} [\mu(A \dotplus X) - \mu(X)], \; A \in M(R)$$

where the supremum is taken over all $X \in M(R)$ such that $\mu(X) < +\infty$.

<u>PROOF</u>. Let M be the set of all radical matrix pseudovaluations μ' on R satisfying

$$\mu(A) \leqslant \mu'(A),$$

$$\sup_{X \in M(R)} [\mu'(A \dotplus X) - \mu'(X)] \leqslant \sup_{X \in M(R)} [\mu(A \dotplus X) - \mu(X)]$$

for all $A \in M(R)$. Lemma 4 says that M is non - empty, i.e. $\mu^* \in M$. Define a partial ordering on M as follows :

$\mu_1 \leqslant \mu_2$ if $\mu_1(A) \leqslant \mu_2(A)$ for all $A \in M(R)$ and $\mu_1, \mu_2 \in M$.

It is not hard to see that M is inductive under the above ordering and thus by Zorn's lemma M contains a maximal element V, say. Now lemma 5 ensures that V cannot be regularized any further and thus V is the desired matrix valuation on R.

Now let the set M(R) of all square matrices over R be totally ordered in any way and fix $A \in M(R)$. Then one can use the same method of proof as used in [2] to prove ∎

LEMMA 7. Let μ be a matrix pseudovaluation on a ring R. Then there exists a matrix valuation V on R satisfying

$$\mu(A) \leqslant V(A) \leqslant \sup_{X \in \Sigma_A} \{\mu(A \dotplus X) - \mu(X)\}, \; A \in M(R),$$

where Σ_A denotes the multiplicative set of square matrices generated by matrices $\leqslant A$ under the ordering of M(R) and the supremum is taken over all $X \in \Sigma_A$ such that $\mu(X) < +\infty$.

THEOREM 8. Suppose μ is a matrix pseudovaluation on a ring R, and let Σ be a multiplicative set of square matrices over R such that

$$\mu(X \dotplus Y) = \mu(X) + \mu(Y)$$

for all X, $Y \in \Sigma$. Then there exists a matrix valuation $V \geqslant \mu$ on R such that $V = \mu$ on Σ.

PROOF. Take a total ordering of M(R) such that Σ becomes an initial segment under the ordering of M(R). So, by lemma 7, there exists a matrix valuation V on R satisfying

$$\mu(A) \leqslant V(A) \leqslant \sup_{X \in \Sigma} \{\mu(A \dotplus X) - \mu(X)\},\ A \in M(R). \tag{1}$$

Now if $A \in \Sigma$ we know that $\mu(A \dotplus X) = \mu(A) + \mu(X)$, $X \in \Sigma$. Thus the first and the last terms of (1) are equal, i.e. $\mu(A) = V(A)$ for all $A \in \Sigma$. ∎

COROLLARY 9. Given a matrix pseudovaluation μ and $\{V_i\}_{i \in I}$ a family of matrix valuation on a ring R, then

$$\mu(A) = \inf_{i \in I} \{V_i(A)\}$$

if and only if μ is radical.

Furthermore, if μ is an arbitrary matrix pseudovaluation on R, then

$$\mu^{\star} = \sqrt{\mu} = \inf_{i \in I} \{V_i\}$$

where V_i ranges over all matrix valuations $\geqslant \mu$.

PROOF. Follows from theorem 8 and lemma 4. ∎

ACKNOWLEDGMENTS

Most of the results appeared in this note are extracted from the author's thesis [4], the author would like to thank P. M. Cohn for his advice.

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Modules with Baer, CS or Left Utumi Endomorphism rings.

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Let ${}_RM$ be a nonsingular left R-module, where R is an associative ring with 1, and $B = \mathrm{Hom}_R(M,M)$ be the ring of R-endomorphisms of M; let E(M) be the injective hull of M and $A = \mathrm{Hom}_R(E(M),E(M))$ be the ring of R-endomorphisms of E(M). We are interested in questions like the following: what properties of M will make B a Baer ring? A Baer (a Baer *) ring is a ring in which every right - and left - annihilator ideal is generated by an idempotent (a projection).

There is interest in finding out when the matrix ring $M_n(R)$ is a Baer or Baer * ring, for example, see [3], [5], [7]. $M_n(R)$ may be considered as the endomorphism ring of a free R-module with finite basis, so that the question we ask is a generalization to nonsingular modules of the problem of matrix rings of Baer or Baer * rings.

When M is nonsingular, the ring B may be embedded in the ring A, which is a (von Neumann) regular, left self-injective ring. It is known that the maximal left quotient (MLQ) ring of a left nonsingular ring is regular and left self-injective. Hence the embedding of B in A leads naturally to the follwing questions; what properties of M will make B left nonsingular and A the MLQ ring of B?

There is considerable interest in some similar questions when B is a Baer *-ring. For suitable Baer *-rings C, there is a complete *-regular ring D, called the regular ring of C such that C is a subring of D, the involution extends to D and all the projections of D lie in C. Handelman, for example, determines necessary and sufficient conditions for the maximal ring of quotients of a Baer *-ring C to be the regular ring of C([3]), and Pyle determines conditions on C which make its involution extendible to its maximal ring of quotients in such a way that the maximal ring of quotients can be identified with the regular ring of C([4]). For example, Pyle finds that, for a Baer * ring C, a necessary and sufficient condition for the involution to be extendible to the MLQ ring of C is that C satisfy Utumi's condition. This result motivates another question we ask here, namely; what properties of M will make B a left Utumi ring? A ring C is said to satisfy Utumi's condition (on the left) or to be a <u>left Utumi</u> ring if C is left nonsingular and any left ideal of C with zero right annihilator is essential in C.

Before stating our answers to the questions raised above, we make the following definitions: we will call a module $_RM$ <u>retractable</u> (<u>e-retractable</u>) if $\mathrm{Hom}_R(M,U)\neq 0$ for every nonzero submodule (complement) U in M. A submodule U of M will be called <u>a-closed</u> if $U = \ell_M(H)$ for some subset H of B, where $\ell_M(H) = \{m \in M: mH = 0\}$.

{Recall that a submodule U is a complement, or essentially closed, in M, if U has no proper essential extension in M}.

Notation: $r_B(U) = \{b \in B: Ub = 0\}$.

Our results are as follows :

Theorem A: Let ${}_RM$ be nonsingular and retractable. Then B is left nonsingular, ${}_BB$ is essential in ${}_BA$ and A is the MLQ ring of B.

Theorem B: Let ${}_RM$ be nonsingular and e-retractable. Then B is a Baer ring if and only if every a-closed submodule of M is a direct summand in M.

Theorem C: Let ${}_RM$ be nonsingular and retractable. Then B is a left Utumi ring if and only if, for each submodule U of M, $r_B(U) = 0$ implies U is essential in M. When these equivalent conditions hold, B is a Baer ring if and only if every complement in M is a direct summand in M.

Examples of retractable modules are: any generator, in particular any free module, any semisimple module, and any torsionless module over a semiprime ring. Examples of e-retractable modules are given by any of the above; in addition, any injective module and any CS-module (i.e. a module in which every complement is a direct summand) is e-retractable. In connection with Theorem A, we can give an example to show that, even for a nonsingular, e-retractable, projective M, A may not be the MLQ ring of B.

A left CS ring is a ring in which every complement left ideal is a direct summand of the ring. The class of left nonsingular, left CS rings is a subclass of the class of Baer rings, which is a subclass of the class of left Rickart rings (also known as left p.p. rings). A ring R is a left Rickart ring if the left annihilator of each element of R is generated by an idempotent. Baer rings and left Rickart rings are always left nonsingular.

There are several results in the literature characterizing certain classes of rings in terms of the endomorphism rings of their free or projective modules. For example, $\mathrm{Hom}_R(F,F)$ is a left Rickart ring for every free (projective) left R-module F if and only if R is left hereditary, i.e. if and only if every left ideal of R is projective ([6], Theorem 1, or [2], Theorem 2.3); $\mathrm{Hom}_R(F,F)$ is a Baer ring for every free (projective) left R-module F if and only if R is semiprimary hereditary, if and only if every torsionless R-module is projective([6], Theorem 2). It is thus natural to ask which left nonsingular rings R have the property that $\mathrm{Hom}_R(F,F)$ is left nonsingular, left CS for every free left R-module F.

Now, a ring R is left Rickart if and only if every principal left ideal of R is projective, a ring R is Baer if and only if every cyclic torsionless left R-module is projective, and a ring R is left nonsingular, left CS if and only if every cyclic nonsingular left R-module is projective. (One sees easily that these characterizations are the natural ones to expect if one recalls that Baer rings are concerned with left annihilators being direct summands while left CS rings are concerned with left complements being direct summands, and if one notes that a cyclic left R-module R/I is torsionless iff I is a left annihilator while, for a nonsingular ring R, a cyclic left R-module is nonsingular iff I is a left complement).

Hence, the following result (obtained in collaboration with A.W. Chatters) is just the result one would expect:

Theorem D: Let R be a left nonsingular ring. Then $\mathrm{Hom}_R(F,F)$ is a left CS ring for every free left R-module F if and only if every nonsingular left R-module is projective.

Actually, the rings R which have the property that every nonsingular left R-module is projective are precisely the Artinian hereditary serial rings (theorem 2.15 in K.R. Goodearl's "Singular torsion and the splitting properties", Memoirs of the Amer. Math. Soc., 124(1971).

Finally, a small result relating Baer, left Utumi and CS rings: Proposition: R is left nonsingular, left CS iff R is Baer and left Utumi.

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REMARKS ON LOCALIZATION AND DUALITY

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This talk will consist of three parts : a survey of some of the work done jointly with Basil Rattray, a contribution to co-localization and equivalence for additive categories at non-small projectives, and an indication of some possible further developments.

1. Introduction and survey

Given two categories $\mathcal{A}$ and $\mathcal{B}$ and a pair of adjoint functors $U : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ with adjunctions $\eta : id \longrightarrow UF$ and $\varepsilon : FU \longrightarrow id$, there is always induced an equivalence between the full subcategories

$$\mathrm{Fix}\,(FU,\varepsilon) = \{\, A \in \mathcal{A} \mid \varepsilon(A) \text{ is iso} \,\}$$

and

$$\mathrm{Fix}\,(UF,\eta) = \{\, B \in \mathcal{B} \mid \eta(B) \text{ is iso} \,\} .$$

Moreover, as first observed by John Isbell, (UF,η) is an idempotent triple, that is, ηUF is an isomorphism, if and only if (FU,ε) is an idempotent cotriple, that is, εFU is an isomorphism.

In this case Fix(FU,ε) is a coreflective subcategory of $\mathcal{A}$ and Fix(UF,η) is a reflective subcategory of $\mathcal{B}$. (See [3] for a proof.)

Next, an example. Let $\mathcal{A}^{op}$ be the category of topological spaces and $\mathcal{B}$ the category of rings (in deference to the present conference). Let $U = \mathcal{A}(-,\underline{2})$, where $\underline{2}$ is the discrete two-element space, and $F = \mathcal{B}(-,\underline{Z}/(2))$, where $\underline{Z}/(2)$ is the two-element ring. Then Fix(FU,ε) is the opposite of the category of Boolean spaces and consists of all spaces A which are <u>presented</u> by $\underline{2}$, that is, for which there is a coequalizer diagram $\underline{2}^Y \rightrightarrows \underline{2}^X \rightarrow A$, and Fix(UF,η) is the category of Boolean rings and consists of all rings B <u>cogenerated</u> by $\underline{Z}/(2)$, that is, for which there is a monomorphism $B \to (\underline{Z}/(2))^X$.

To explain the title of this lecture, the reflector FU , which assigns to each space A a Boolean space FU(A) , is an example of localization, while the equivalence between the opposite of the category of Boolean spaces and the category of Boolean rings is the well-known Stone duality. (See [4] for more on this subject.)

The famous Gelfand duality between compact Hausdorff spaces and commutative C*-algebras may be treated in a similar fashion [5] . Then $\mathcal{A}$ is as above and $\mathcal{B}$ is the category of commutative Banach algebras. The localization functor is here the Stone-Čech compactification [2] .

Note that in the first example $\mathcal{B}$ was an algebraic category, while in the second example $\mathcal{B}$ is at least a full subcategory of an equational category in the sense of Linton.

Now ring theorists are not usually interested in the category of Banach algebras nor, for that matter, in the category of rings!

Let us look at the situation where $\mathcal{A}$ is a cocomplete additive category. (By "additive" is meant what other people call "pre-additive", and "cocomplete" means that $\mathcal{A}$ has coproducts and co-equalizers, in the additive case, cokernels.) Let P be a given object of $\mathcal{A}$ with endomorphism ring E and $\mathcal{B} = \mathrm{Mod}\,E$. It is of course well-known that the functor $U = \mathcal{A}(P,-)$ possesses a left adjoint F. It may not be so well-known that there is an explicit construction for F; to wit, $\gamma(B) : |B|P \longrightarrow F(B)$ is the joint cokernel of all finitary morphisms $h : P \to |B|P$ for which

$$\sum_{b\in|B|} b\,(p_b h) = 0 ,$$

$p_b : |B|P \to P$ being the canonical projection corresponding to $b \in |B|$. (Here XP denotes the coproduct of copies of P, one for each element of X, and $|B|$ is the underlying set of the E-module B. A morphism $P \to XP$ is called <u>finitary</u> if it factors through a finite subcoproduct.)

The adjunction $\eta(B) : B \longrightarrow UF(B)$ is given by

$$\eta(B)(b) = \gamma(B)\, i_b .$$

$i_b : P \longrightarrow |B|P$ being the canonical injection.

While the details will be found elsewhere [6], it may be instructive to show why $\eta(B)(be) = \eta(B)(b)e$, that is, $\gamma(B)\, i_{be} = \gamma(B)\, i_b e$, for all b in B and all e in E. Indeed, this follows from

$$\sum_{b'\in|B|} b'\, p_{b'}\,(i_{be} - i_b e) = be - be = 0 ,$$

in view of the definition of $\gamma(B)$.

To introduce the other adjunction ε, we first define $\lambda(A) : |U(A)|\; P \longrightarrow A$ by

$$\lambda(A)\, i_f = f ,$$

for all $f \in |U(A)|$, and note that for any finitary $h : P \to |U(A)| P$,

$$\lambda(A)\, h = \lambda(A) \sum_{f : P \to A} i_f\, p_f\, h = \sum_{f : P \to A} f\,(p_f\, h) ,$$

where $p_f h = 0$ for all but a finite number of $f \in |U(A)|$. Then $\varepsilon(A) : FU(A) \to A$ is the unique morphism for which

$$\varepsilon(A)\, \gamma\, U(A) = \lambda(A) .$$

The matter becomes particularly managable when, for any set X, all morphisms $P \to XP$ are finitary. In that case P has been called <u>weakly small</u> in [6].

If P is weakly small, εFU and therefore ηUF are isomorphisms if and only if P is $\gamma U(A)$-projective for all A in $\mathcal{A}$. (If $e : A' \to A$, P is called e-<u>projective</u> if for every $f : P \to A$ there exists $f' : P \to A'$ such that $ef' = f$.) Moreover, $Fix(FU,\varepsilon)$ is then the subcategory of all objects A of $\mathcal{A}$ presented by P, that is, for which there is a cokernel diagram $YP \to XP \to A$. In this situation FU has been called a <u>colocalization</u> functor [2] . Note that $\gamma U(A)$ is then the cokernel of all $h : P \to |U(A)|P$ for which $\lambda(A)h = 0$ and thus coincides with the morphism called $\kappa(A)$ in [2] .

These matters are discussed in [6] and will be generalized later. For the moment we shall require a lemma, which is easier to prove than to cite :

LEMMA 1. P is in $Fix(FU,\varepsilon)$.

<u>Proof</u>: We know from category theory that $U\, \varepsilon(P) \eta\, U(P) = 1$, hence $\varepsilon(P)(\eta(E)(e)) = e$. In particular,

$$\varepsilon(P)\ (\eta(E)(1)) = 1 .$$

Also $\varepsilon(P)\gamma(E)i_e = \varepsilon(P)(\eta(E)(e)) = e$,

hence $\eta(E)(1)\varepsilon(P)\gamma(E)i_e = \eta(E)(1)e = \eta(E)(e) = \gamma(E)i_e$, and therefore

$$(\eta(E)(1))\varepsilon(P) = 1 .$$

If we know that P is $\gamma(B)$ - projective for all B in Mod E , $\mathrm{Fix}(UF,\eta)$ will be subobject-closed; in fact, if $\mathcal{A}$ has a cogenerator C , $\mathrm{Fix}(UF,\eta)$ will consist of all E-modules cogenerated by U(C) . The following, while essentially contained in [6], is not explicitly stated there.

P is called <u>projective</u> if it is e-projective for all regular epimorphisms (that is, cokernels) e .

PROPOSITION 1. Let $\mathcal{A}$ be a cocomplete additive category, P a weakly small object of $\mathcal{A}$, E, U and F as above. Then the following statements are equivalent :

(1) (UF,η) is idempotent and E is projective in $\mathrm{Fix}(UF,\eta)$.

(2) (FU,ε) is idempotent and P is projective in $\mathrm{Fix}(FU,\varepsilon)$.

(3) P is projective in some full coreflective subcategory of $\mathcal{A}$.

(4) P is $\gamma(B)$ - projective for each B .

(5) $\eta(B)$ is a surjective epimorphism for each B .

<u>Proof</u>. (1) ⟹ (2). By Isbell's theorem, (FU,ε) is also idempotent. By Lemma 1, P is in $\mathrm{Fix}(FU,\varepsilon)$. Since U(P) = E , it corresponds to E under the equivalence, hence it is also projective.

(2) ⟹ (3). $\mathrm{Fix}(FU,\varepsilon)$ is a full coreflective subcategory of Mod E .

(3) ⟹ (4). Since a full coreflective subcategory is closed under coproducts and cokernels, and since F(B) is constructed

by means of coproducts and cokernels, $F(B)$ is in the given subcategory. Moreover, $\gamma(B)$ is a regular epimorphism in the subcategory, hence P is $\gamma(B)$-projective.

(4) $\Rightarrow$ (5). Let $f \in |UF(B)|$, that is, $f : P \to F(B)$. In view of (4), we can find $h : P \to |B|P$ so that $\gamma(B)h = f$. Since P is weakly small,

$$h = \sum_{b\in|B|} i_b\, p_b\, h ,$$

where $p_b h = 0$ for all but a finite number of $b \in |B|$. Therefore,

$$f = \gamma(B)h = \sum_{b\in|B|} \eta(B)(b)\, p_b\, h = \eta(B)\Big(\sum_{b\in|B|} p_b\, h\Big) .$$

(5) $\Rightarrow$ (6). Since $U\varepsilon(A)\eta U(A) = 1$, it follows from (5) that $\eta U(A)$ is an isomorphism for each A, hence that (UF,η) is idempotent. To see that E is projective in $\mathrm{Fix}(UF,\eta)$, it suffices to verify that every regular epi in $\mathrm{Fix}(UF,\eta)$ is one in $\mathrm{Mod}\,E$, that is, a surjection.

Let $B_1 \to B_2$ be a regular epi in $\mathrm{Fix}(UF,\eta)$, hence the cokernel of $B_0 \to B_1$ in $\mathrm{Fix}(UF,\eta)$. In view of the way cokernels are constructed in reflective subcategories, $B_1 \to B_2$ is isomorphic with $B_1 \to B \to UF(B)$, where $B_1 \to B$ is the cokernel in $\mathrm{Mod}\,E$ and $\eta(B) : B \to UF(B)$. Since both of these are surjections, so is their composition.

As an application of the above methods I want to mention the main result of [6].

THEOREM. Let I be a quasi-injective right R-module with the discrete topology and E its endomorphism ring. Then the functor $U : (\mathrm{Cont}\,R)^{op} \to \mathrm{Mod}\,E$ gives rise to a duality between the category of continuous right R-modules copresented by I and the category of abstract left E-modules cogenerated by ${}_E I$.

Here Cont R consists of all continuous right R-modules, that is, topological R-modules over the discrete ring R , and all continuous R-homomorphisms. I is small, because it is discrete, and the quasi-injectivity was used in [1] to show via Harada's Lemma that I is m - injective for any regular monomorphism $m : A \to I^X$ in Cont R. The functor FU is an example of localization in Cont R, not in Mod R.

Several classical duality theorems are subsumed under the following corollary to the above theorem.

COROLLARY. Let I be an Artinian quasi-injective cogenerator of Mod R and E its endomorphism ring. Then there is a duality between the category of continuous pro-Artinian right R-modules (with the inverse limit topology) and E Mod.

Another illustration of our methods is afforded by the following example: If P is a finitely generated projective right R-module with endomorphism ring E , the associated functor $U : \mathrm{Mod}\,R \to \mathrm{Mod}\,E$ induces an equivalence between the category of right R-modules presented by P and Mod E. Of course, here $F \cong (-) \otimes_E P$.

Again FU is an example of colocalization in Mod R. It is a pity that P has to be assumed to be finitely generated, in view of the fact that McMaster [7] has discussed colocalization for an arbitrary projective and even shown that it coincides with FU . To adapt the present methods to McMaster's results, a new idea is required. We shall make a little detour and introduce topological considerations.

2. Colocalization and equivalence at non-small projectives

We shall consider a second object P' in our category $\mathcal{A}$. We shall stipulate that P' determines E', U', F', η' and ε' in the same way that P gave rise to E etc. As we shall see later, the assumption on P' made in Proposition 2 will be satisfied, for example, if $\mathcal{A}$ is Abelian and P' is a small generator of $\mathcal{A}$.

PROPOSITION 2. Let $\mathcal{A}$ be a cocomplete additive category, P and P' objects of $\mathcal{A}$ with endomorphism rings E and E' and associated functors $U : \mathcal{A} \longrightarrow \text{Mod}\,E$ and $U' : \mathcal{A} \longrightarrow \text{Mod}\,E'$. Then each object of $\text{Mod}\,E$ of the form $U(A)$ is a topological E-module, a fundamental system of open neighborhoods of zero consisting of all subgroups $V_A(g_1) \cap \ldots \cap V_A(g_n)$ of $U(A)$, where $V_A(g_i) = \{ f : P \to A \mid f g_i = 0 \}$ is associated with $g_i : P' \longrightarrow P$. Moreover, $\eta\, U(A)$ is continuous, $U(A)$ is Hausdorff if P' generates P, and $U(A)$ is complete if $P \in \text{Fix}(F'U', \varepsilon')$.

<u>Proof</u>. Clearly $U(A)$ is a topological group. But also, for each $e \in E$, the mapping $f \longmapsto fe$ is continuous, since

$$e^{-1}(V_A(g_1) \cap \ldots \cap V_A(g_n)) = V_A(e\, g_1) \cap \ldots \cap V_A(e\, g_n)\,.$$

$U(A)$ is Hausdorff, since the intersection of fundamental open neighborhoods of zero is $\bigcap_{g : P' \to P} V_A(g) = 0$, provided P' generates P.

Next, we shall show that $U(A)$ is complete, if $P \in \text{Fix}(F'U', \varepsilon')$. Let $\{ f_x \mid x \in X \}$ be a Cauchy net in $U(A)$, where $(X, \leq)$ is an upward directed set. We shall write $V_A(g) = V_A(g_1) \cap \ldots \cap V_A(g_n)$, when $g = [g_1, \ldots, g_n] : nP' \longrightarrow P$.

Then, for all $g : nP' \to P$, there exists $x(g) \in X$ such that, for all $x \geq x(g)$, $f_x - f_{x(g)} \in V(g)$. We seek a limit f of this net.

Consider the mapping $\varphi : \mathcal{A}(P',P) \longrightarrow \mathcal{A}(P',A)$ defined by $\varphi(g) = f_{x(g)}\, g$. This is easily seen to be an E'-homomorphism. For example, $f_x - f_{x(ge')} \in V_A(ge')$, for all $x \geq x(ge')$,

$f_x - f_{x(g)} \in V_A(g)$, for all $x \geq x(g)$, hence,

for all $x \geq$ both $x(ge')$ and $x(g)$, $f_{x(ge')}\, ge' = f_x\, ge' = f_{x(g)}\, ge'$, and therefore $\varphi(ge') = \varphi(g)e'$.

Suppose for the moment we can find $f : P \longrightarrow A$ such that $\varphi = \mathcal{A}(P',f)$. Then $f_x g = f_{x(g)}\, g = \varphi(g) = fg$ for all $x \geq x(g)$, and so $f_x - f \in V_A(g)$ for all $x \geq x(g)$. Thus f is the limit of the Cauchy net.

It remains to show that $\varphi = \mathcal{A}(P',f)$. Since $\varepsilon'(P)$ is an isomorphism, we can put $f = \varepsilon'(A)\, F'(\varphi)\, (\varepsilon'(P))^{-1}$, then the following square commutes:

$$\begin{array}{ccc} F'U'(P) & \xrightarrow{\;F'(\varphi)\;} & F'U'(A) \\ {\scriptstyle \varepsilon'(P)}\downarrow & & \downarrow{\scriptstyle \varepsilon'(A)} \\ P & \xrightarrow[\;f\;]{} & A \end{array}$$

But this means that $f\varepsilon'(P) : F'U'(P) \to A$ corresponds to $\varphi : U'(P) \longrightarrow U'(A)$ under the adjunction, hence $\varphi = U'(f)$.

Finally, to show that $\eta U(A)$ is continuous, take any $g : nP' \longrightarrow P$, then

$$\begin{aligned} V_A(g) &= \{ f : P \to A \mid fg = 0 \} \\ &= \{ f : P \to A \mid \sum_{f' : P \to A} f' p_{f'} i_f\, g = 0 \} \end{aligned}$$

$$\subseteq \{f : P \to A \mid \gamma U(A)\, i_f\, g = 0\}$$
$$= \{f : P \to A \mid \eta U(A)(f) g = 0\}$$
$$= (\eta U(A))^{-1}\, V_{FU(A)}(g)\ ,$$

which is therefore an open subset of $U(A)$.

The proof is now complete.

The following result shows that the assumption on P' in Proposition 2 will be satisfied if $\mathcal{A}$ is Abelian and P' is a weakly small generator of $\mathcal{A}$. In fact, in this case, $\mathrm{Fix}(U'F', \varepsilon') = \mathcal{A}$. As there is no point in carrying the prime, P should be read as P' when applying Proposition 3 to Proposition 2.

PROPOSITION 3. Let $\mathcal{A}$ be a cocomplete Abelian category, P a weakly small generator with associated functor $U : \mathcal{A} \longrightarrow \mathrm{Mod}\, E$. Then $\mathrm{Fix}(FU, \varepsilon) = \mathcal{A}$, that is, U is full.

Proof. Let A be any object of $\mathcal{A}$. Since $U\varepsilon(A)\, \eta U(A) = 1$, $U\varepsilon(A)$ is epi. Since U is faithful, $\varepsilon(A)$ is epi.

Recall that $\varepsilon(A)\, \gamma U(A) = \lambda(A)$, where $\lambda(A)\, i_f = f$ for all $f : P \to A$. Since $\mathcal{A}$ is Abelian and $\gamma U(A)$ is epi, it will follow that $\varepsilon(A)$ is iso if we show that $\gamma U(A) \ker \lambda(A) = 0$.

Let $k : K \longrightarrow |U(A)|P$ be the kernel of $\lambda(A)$ and let $g : P \to K$. Since P is weakly small, kg is finitary, hence

$$kg = \sum_{f : P \to A} i_f\, p_f\, k g\ ,$$

where $p_f kg = 0$ for all but a finite number of $f \in |U(A)|$. Now

$$0 = \lambda(A) k g = \sum_{f : P \to A} f(p_f k) g\ ,$$

therefore $\gamma U(A) k g = 0$, by definition of γ .

Since this is true for all $g : P \to K$ and P is a generator, $\gamma U(A)k = 0$, as was to be shown.

This proof is reminiscent of the Gabriel-Popescu theorem, where the assumption that P is weakly small is replaced by the assumption that $\mathcal{A}$ has exact direct limits.

Our next proposition and the lemma leading up to it will depend on the following :

ASSUMPTION A. $\mathcal{A}$ is a cocomplete additive category, P and P' are objects of $\mathcal{A}$ with endomorphism rings E and E' and associated functors $U : \mathcal{A} \longrightarrow \mathrm{Mod}\, E$ and $U' : \mathcal{A}' \longrightarrow \mathrm{Mod}\, E'$. Furthermore, every morphism $P' \longrightarrow XP$ is finitary and P' generates P.

Clearly, the last condition is satisfied when P' is a small generator or when $P' = P$ is weakly small.

LEMMA 2. Under Assumption A, the following are equivalent :

(1) the image of $\eta(B)$ is dense,

(2) P is <u>approximately</u> $\gamma(B)$-projective,

that is, for every $f : P \to F(B)$ and every $g : nP' \to P$, there exists $h : P \longrightarrow |B|P$ such that $\gamma(B)h - f \in V_{F(B)}(g)$.

<u>Proof</u>. Assume (1), then, for each $f : P \to F(B)$ and $g : nP' \longrightarrow P$, we can find $b \in |B|$ so that $f - \eta(B)(b) \in V_{F(B)}(g)$, hence

$$\gamma(B)\, i_b\, g = \eta(B)(b) g = f g ,$$

and so we have (2) with $h = i_b$.

Assume (2), and let $f : P \longrightarrow F(B)$, $g : nP' \longrightarrow P$. Find h so that $\gamma(B) h g = f g$. Now $h g : nP' \longrightarrow |B|P$ is finitary, by Assumption A, hence

$$h\,g = \sum_{b\in F} i_b\,p_b\,h\,g \quad ,$$

for some finite subset F of $|B|$ depending on g.

Then

$$f\,g = \gamma(B)h\,g = \sum_{b\in F} \eta(B)(b)p_b\,h\,g = \eta(B)(b_g)g \quad ,$$

where $b_g = \sum_{b\in F} b(p_b\,h)$. Thus $f - \eta(B)(b_g) \in V_{F(B)}(g)$, and so (1) holds.

PROPOSITION 4. Under assumption A, the following are equivalent :

(1) $\eta U(A)$ is surjective ;

(2) P is $\gamma U(A)$ - projective.

<u>Proof</u>. The implication (1) $\Longrightarrow$ (2) is proved as for Lemma 2.

Assume (2). Then, by Lemma 2, $\eta U(A)$ has a dense image. Thus, given $f : P \longrightarrow FU(A)$ and $g : nP' \longrightarrow P$, we can find $f_g : P \longrightarrow A$ such that $\eta U(A)(f_g)g = f\,g$. Now

$$f_g\,g = (U\varepsilon(A)\,\eta U(A))(f_g)g = \varepsilon(A)\,\eta U(A)(f_g)g = \varepsilon(A)fg \ .$$

But this means that the net $\{ f_g \mid g : nP' \longrightarrow P \}$, where $g' \geq g$ means $V_A(g') \subseteq V_A(g)$, has limit $\varepsilon(A)f$.

Now consider the net $\{ \eta U(A)(f_g) \mid g : nP' \longrightarrow P \}$. By density, it has limit f. But, by continuity of $\eta U(A)$, it has limit $\eta U(A)(\varepsilon(A) f)$. Since the topology on $UFU(A)$ is Hausdorff, $f = \eta U(A)(\varepsilon(A)f)$, and so (1) holds.

In view of Proposition 3, the following is of interest, which is also implicit in [6] and could have been treated in Part 1.

PROPOSITION 5. Let $\mathcal{A}$ be a cocomplete additive category, P an object with endomorphism ring E and associated functor $U : \mathcal{A} \to \mathrm{Mod}\,E$, and assume that $\eta U(A)$ is surjective for all A in $\mathcal{A}$. Then

(1) Fix(FU,ε) is a coreflective subcategory of $\mathcal{A}$ consisting of all objects presented by P ;

(2) Fix(UF,η) is a reflective subcategory of Mod E. If $\mathcal{A}$ is copresented by C , this subcategory consists of all E-modules copresented by U(C) .

Proof. Since $\eta U(A)$ is always mono, it follows from the hypothesis that it is an isomorphism. Therefore, (UF,ε) and (FU,η) are idempotent, and so Fix(FU,ε) is a coreflective, Fix(UF,η) a reflective subcategory.

(1) Each object of Fix(FU,ε) has the form F(B) and, according to its construction, is the joint cokernel of a certain collection Y of morphisms $P \to XP$, where $X = |B|$, hence the cokernel of a single morphism $YP \to XP$.

Conversely, Fix(FU,ε) is a full coreflective subcategory of $\mathcal{A}$, hence closed under coproducts and cokernels. By Lemma 1, P is in Fix(FU,ε) , hence so is every object presented by P .

(2) By assumption, for each object A of $\mathcal{A}$ there is a kernel diagram $A \to C^X \to C^Y$. Since U preserves kernels and products, we have a kernel diagram $U(A) \to U(C)^X \to U(C)^Y$ in Mod E. Now each object of Fix(UF,η) has the form U(A) , hence is copresented by U(C) .

Conversely, since $\eta U(C)$ is an isomorphism, U(C) is in Fix(UF,η) . Moreover, being a full reflective subcategory, the latter is closed under products and kernels, hence it contains every object copresented by U(C) .

In view of Proposition 4, the hypotheses of Proposition 5 are satisfied if Assumption A holds and if P is $\gamma U(A)$ - projective for all A in $\mathcal{A}$.

P was called <u>weakly projective</u> in [6] if P is e-projective for every regular epimorphism $e: XP \to A$. This implies, in particular, that P is $\gamma U(A)$ - projective for every A in $\mathcal{A}$.

Putting all this together, we obtain the following consequence of propositions 4 and 5.

PROPOSITION 6. Let $\mathcal{A}$ be a cocomplete additive category with a small generator, and let P be a weakly projective object of $\mathcal{A}$ with endomorphism ring E and associated functor $U: \mathcal{A} \to \mathrm{Mod}\, E$. Then the conclusions (1) and (2) of Proposition 5 hold.

We are finally able to deal with McMaster's colocalization.

PROPOSITION 7. If P is a weakly projective right R-module with endomorphism ring E, the R-modules presented by P form a full coreflective subcategory of Mod R which is equivalent to a full reflective subcategory of Mod E consisting of all E-modules copresented by $\mathrm{Hom}_R(P, \underline{Q}/\underline{Z})$.

<u>Proof</u>. In Proposition 6 take $\mathcal{A} = \mathrm{Mod}\, R$, $P' = R$ and $C = \mathrm{Hom}_{\underline{Z}}(R, \underline{Q}/\underline{Z})$. Then calculate

$$\begin{aligned} U(C) &= \mathrm{Hom}_R(P, \mathrm{Hom}_{\underline{Z}}(R, \underline{Q}/\underline{Z})) \\ &\cong \mathrm{Hom}_R(P \otimes_R R, \underline{Q}/\underline{Z}) \\ &\cong \mathrm{Hom}_R(P, \underline{Q}/\underline{Z}) . \end{aligned}$$

3. Additional remarks

Let us explore some possible further develpments. If we look at Proposition 2, we wonder why some objects of $\mathrm{Mod}\,E$ should be topologized, while others, namely those not in the image of U, have no obvious topology. One could remedy the situation by regarding U as a functor from $\mathcal{A}$ to $\mathrm{Cont}\,E$ instead of $\mathrm{Mod}\,E$. Unfortunately,. it is easily seen that this functor $\mathcal{A} \to \mathrm{Cont}\,E$ does not preserve infinite products, hence cannot have a left adjoint. What is needed is really a different kind of category from $\mathrm{Cont}\,R$.

Given any bimodule ${}_{E'}G_E$, we shall construct a new category $(\mathrm{Mod}\,E)_G$. Its objects are pairs (B,V), where $B \in \mathrm{Mod}\,E$ and V assigns to each $g \in G$ an additive subgroup $V(g)$ of B (not in general an E-submodule) satisfying certain conditions (see below). Its morphisms $\varphi : (B,V) \longrightarrow (B',V')$ are E-homomorphisms $\varphi : B \longrightarrow B'$ such that, for all $b \in B$ and $g \in G$,

$$b \in V(g) \implies \varphi(b) \in V'(g) .$$

The conditions to be satisfied by V are the following :

(1) For all $b \in B$, $e \in E$, $g \in G$,

$$be \in V(g) \iff b \in V(eg) .$$

(2) For all $b \in B$, $e' \in E'$, $g \in G$,

$$b \in V(g) \implies b \in V(ge') .$$

(3) $$\bigcap_{g \in G} V(g) = 0 .$$

$(\mathrm{Mod}\,E)_G$ is an additive category with kernels and products. V may be used to define a topology on each object, as on $U(A)$ before. This topology is Hausdorff, and all morphisms are continuous.

If $\mathcal{A}$ is an additive category satisfying the assumptions of Proposition 2, we let $G = \mathcal{A}(P',P)$ and obtain a functor $U : \mathcal{A} \longrightarrow (\mathrm{Mod}\, E)_G$, where $U(A) = (\mathcal{A}(P,A), V_A)$ and $U(f) = \mathcal{A}(P,f)$.

In order to construct a left adjoint F to U, we shall assume further that every morphism $f : P' \to XP$ is finitary. We define $\gamma(B) : |B|P \to F(B)$ as the joint cokernel of all morphisms $P' \xrightarrow{g} P \xrightarrow{h} |B|P$ such that $\sum_{b \in |B|} b(p_b h) \in V(g)$, in the sense that there is a finite subset F_g of $|B|$ such that, for all finite subsets F containing F_g, $\sum_{b \in F} b(p_b h) \in V(g)$.

As before, we define $\eta(B)(b) = \gamma(B) i_b$ for all $b \in |B|$. It is not difficult to see that F is then left adjoint to U with adjunction η. If we postulate some kind of projectivity for P, it again follows that $\eta(B)$ is dense and consequently an epimorphism in $(\mathrm{Mod}\, E)_G$. However, there is no reason for $\eta(B)$ to be a surjection, unless $B = U(A)$. Thus we are far removed from an algebraic kind of category, in which all epimorphisms have to be surjections.

If we insist on having an algebraic type category in place of $(\mathrm{Mod}\, E)_G$, we can produce one; but it won't be something that is easily recognized by a ring theorist. There are in fact two methods for doing this.

According to the first method, we look at the full subcategory of $\mathcal{A}$ consisting of all XP, where X ranges over all sets, and regard it as an equational theory in the sense of Lawvere-Linton. We then construct an equational category whose objects are product preserving functors from the opposite of this subcategory into the category of sets. We shall not explore this method further here.

We shall briefly sketch the second method. Given a cocomplete category $\mathcal{A}$ and an object P of $\mathcal{A}$, one first forms the functor $U' : \mathcal{A} \longrightarrow$ Sets , such that $U'(A) = \mathcal{A}(P,A)$, and its left adjoint F' , such that $F'(X) = XP$, with adjunctions η' and ε' . One then forms the category Alg P of algebras over the triple $(U'F', \eta', U'\varepsilon'F')$, as in any recent book on category theory, an algebra being a pair (X,ξ) , where X is a set and $\xi : U'F'(X) \to X$ satisfies certain conditions.

There is a well-known comparison functor $U : \mathcal{A} \to$ Alg P , such that $U(A) = (U'(A), U'\varepsilon'(A))$, and this has a left adjoint F with adjunctions η and ε . F is constructed with the help of $\gamma(X,\xi) : F'(X) \to F(X,\xi)$, the coequalizer of $F'(\xi)$ and $\varepsilon' F'(X)$, and η is defined by $\eta(X,\xi)(x) = U'\gamma(X,\xi)(i_x)$.

It is now easy to show that $\eta(X,\xi)$ is surjective if and only if P is $\gamma(X,\xi)$-projective. Moreover, the analogues of Proposition 1 and Proposition 5 hold in this general context, the proofs being almost identical to those given above. The only problem that remains is to identify Alg P in any given situation as a familar category.

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POSTSCRIPT

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MODULES Σ - INJECTIFS

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0. INTRODUCTION

C. Faith a defini dans [7] la notion de module Σ - injectif : un module Q est dit Σ - injectif si $Q^{(I)}$ est injectif pour tout ensemble I.

Soient R un anneau commutatif noethérien et $A = R[X]_{\alpha\in\Lambda}$ l'anneau des polynômes dans les indéterminés $(X_\alpha)_{\alpha\in\Lambda}$ (Λ est un ensemble arbitraire). Soit

$0 \to A \to Q_0 \to Q_1 \to \dots \to Q_n \to \dots$

la resolution injective minimale de A. Dans [2] I. Beck pose le problème suivant : les modules $Q_n (n \geq 0)$ sont - ils Σ - injectifs ? Il montre dans [2] que l'affirmation est vraie si R est un anneau de Cohen-Macaulay. Dans ce travail nous montrerons que l'affirmation est vraie dans le cas où R est de dimension de Krull finie (théorème 3.1). (Le problème rest ouvert pour le cas où R est de dimension de Krull infinie).

Par le théorème 1.4 nous donerons de même une réponse négative à une question posée par J.E. Roos dans [10].

Définitions, notations et résultats préliminaires

Tous les anneaux considérés dans ce travail sont commutatifs et unitaires. Tous les modules sont unitaires. Si R est un anneau, nous noterons par Mod R la catégorie des R - modules. Si M est un R - module, par E(M) nous designons l'enveloppe injective de M. Par Spec R on désigne l'ensemble des idéaux premiers de R. Si $p \in$ Spec R alors ht(p) est l'hauteur de l'idéal p, qui est un nombre naturel ou ∞. La dimension de Krull de l'anneau R est noté par dim R. On sait que dim R = sup(ht(p)). Si M est un R - module alors Ass M est l'ensemble des idéaux $p \in$ Spec R premiers associés à M, c'est-à-dire Ass M = $\{p \in \text{Spec } R \mid \exists\, x \in M,\ x=0$ tel que $p = \text{Ann } x\}$.

Une topologie additive sur R est d'après Stenström [11] un ensemble non vide F d'idéaux de R, vérifiant les conditions suivantes :

1) Si $I \in F$ et $a \in R$, alors $(I : a) \in F$.
2) Si I et J sont deux idéaux de R tels que $J \in F$ et $(I : a) \in F$ pour tout $a \in J$, alors $I \in F$.

Pour la topologie additive F on peut considérer les deux classes de R - modules :

$T_F = \{M \in \text{Mod } R \mid \forall\, x \in M, \text{ Ann } x \in F\}$

$F_F = \{M \in \text{Mod } R \mid x \in M \text{ et Ann } x \in F \rightarrow x = 0\}$

Une module $M \in T_F$(resp. $M \in F_F$) est nomé F - torsionné (resp. F - sans torsion). Le couple (T_F, F_F) est une théorie de torsion héréditaire pour Mod R [11].

Si $M \in$ Mod R nous notons : $t(M) = \{x \in M \mid \text{Ann } x \in F\}$. L'application $M \rightarrow t(M)$ est un foncteur t : Mod R $\rightarrow$ Mod R qui s'appelle le radical associé à la topologie additive F. Nous désignerons par $C_F(R)$ l'ensemble :

$$C_F(R) = \{I \text{ idéal de } R \mid R/I \text{ est } F \text{ - sans torsion}\}.$$

L'ensemble $C_F(R)$ est un treillis modulaire complet. L'etude de ce treillis a été fait dans [1], [9].

Si $C_F(R)$ est un treillis noethérien, alors l'anneau R est dit F - noethérien. Un idéal I de R est dit F - de type fini s'il existe un idéal de type fini $J \subset I$ tel que I/J est F-torsionné. Leséquivalences suivantes sont vraies : R est F - noethérien $\Leftrightarrow$ tout idéal de R est F - de type fini $\Leftrightarrow$ tout idéal premier de R est de F - de type fini (voir [5], [9]). Soit $X \subset \operatorname{Spec} R$; l'ensemble $F_X = \{I \subset R / (R/I)_p = 0 \ \forall\, p \in X\}$ est une topologie additive sur R. Il est clair que $T_{F_X} = \{M \in \operatorname{Mod} R \mid M_p = 0 \ \forall\, p \in X\}$.

Si $X_n = \{p \in \operatorname{Spec} R \mid ht(p) \leqslant n\}$ nous noterons par F_n (resp. (T_n, F_n)) la topologie additive F_{X_n} (resp. le couple (T_{F_n}, F_{F_n})). Si R est F_n - noetherien, nous dirons plus bref que R est n - noetherien. Si F est une topologie additive la sous - catégorie localisante T_F ([6], ch. V) est dit stabile par rapport aux enveloppes injectives si pourtout $M \in T_F$ il résulte que $E(M) \in T_F$.

1. ANNEAUX F - NOETHERIENS

THEOREME 1.1 Soient R un anneau, $(R_\alpha)_{\alpha \in \Lambda}$ une famille filtrante croissante de sous - anneau noethériens de R tel que $R = \bigcup_{\alpha \in \Lambda} R_\alpha$. Supposons que pour tout $\alpha \in \Lambda$ et pour tout idéal premier $p \in \operatorname{Spec} R_\alpha$, l'idéal pR est premier dans R. Alors l'anneau R est n - noethérien pour tout nombre naturel n et les sous - catégories localisantes $T_n (n \geqslant 0)$ sont stable, par rapport aux enveloppes injectives.

<u>DEMONSTRATION</u>. Soit $p \in \operatorname{Spec} R$ avec $ht(p) < \infty$. Nous notons $\bar{p}_\alpha = (p \cap R_\alpha)R$. Alors $\bar{p}_\alpha \in \operatorname{Spec} R$ et $p = \bigcup_{\alpha \in \Lambda} \bar{p}_\alpha$. Parce que $ht(p) < \infty$, il existe $\alpha \in \Lambda$ tel que $p = \bar{p}_\alpha$. Comme R_α est noethérien alors $p \cap R_\alpha$ est un idéal de type fini et donc $\bar{p}_\alpha$ est de type fini et par conséquent p est de type fini. Donc nous avons montré que $ht(p) < \infty \rightarrow p$ de type fini.

Nous démonstrons maintenant que R est n - noethérien. Il suffit de montrer que tout idéal premier p est F_n - de type fini. On peut supposer que $ht(p) = \infty$. Comme $p = \bigcup_{\alpha \in \Lambda} \bar{p}_\alpha$, il existe un $\alpha \in \Lambda$ tel que $ht(\bar{p}_\alpha) = n$.

Mais $\bar{p}_\alpha = (p \cap R_\alpha)R$ est de type fini. Soit $q \in \operatorname{Spec} R$ avec $ht(q) \leqslant n$. On voit que $\bar{p}_\alpha \not\subset q$ et donc il existe $s \in \bar{p}_\alpha$, $s \notin q$. Si $\hat{x} \in p/\bar{p}$, $x \in p$ alors $s\hat{x} = \hat{sx} = 0$ et donc $(p/\bar{p}_\alpha)_q = 0$. Par conséquent $p/\bar{p}_\alpha$ est F_n - torsionné et donc p est F_n - de type fini.

La dernière partie de la théorème se déduit du corollaire 4.2 [9]. ∎

COROLLAIRE 1.2 Soient R un anneau noethérien et $(X_\alpha)_{\alpha \in \Lambda}$ une famille d'indéterminés. Alors l'anneau des polynômes $R[X_\alpha]_{\alpha \in \Lambda}$ et l'anneau des series formelles $R[[X_\alpha]]_{\alpha \in \Lambda}$ sont n - noethériens.

De même les sous - catégories localisante $\mathcal{T}_n (n \geqslant 0)$ sont stable par rapport aux enveloppes injectives.

DEMONSTRATION. Nous pouvons écrire

$R[X_\alpha]_{\alpha \in \Lambda} = \bigcup_{F \subset \Lambda} R[X_\alpha]_{\alpha \in F}$ et $R[[X_\alpha]]_{\alpha \in \Lambda} = \bigcup_{F \subset \Lambda} R[[X_\alpha]]_{\alpha \in F}$

où F est un ensemble fini arbitraire de Λ. On voit facilement que nous sommes dans les conditions du théorème 1.1. ∎

COROLLAIRE 1.3 Nous sommes dans les hypothèses du théorème 1.1. Notons par $F_w = \bigcap_{n \geqslant 0} F_n$, qui est une topologie additive sur R. Alors pour tout modules $M = 0$, F - sans torsion, nous avons Ass $M \neq \emptyset$. En particulier si Q est un module injectif F - sans torsion, il existe une famille d'idéaux premiers $(p_i)_{i \in I}$ avec $ht(p_i) < \infty$ tel que Q est une extension essentielle de la somme directe $\bigoplus_{i \in I} E(R/p_i)$.

DEMONSTRATION. Soit Q un module injectif F - sans torsion. Comme $Q \neq 0$ il existe un nombre naturel n pour lequel $Q \notin \mathcal{T}_n$. $\mathcal{T}_n$ étant stable par rapport aux enveloppes injectives alors $t_n(Q)$ est injectif (t_n est le radical associé à la topologie F_n). Donc $Q \approx t_n(Q) \oplus Q/t_n(Q)$ où $Q/t_n(Q) \neq Q$ et est F_n - sans torsion. R étant F_n - noethérien, d'après le lemme 2.2 [9] on déduit que Ass $Q/t_n(Q) \neq \emptyset$ et donc Ass $Q \neq \emptyset$. Maintenant, si M est un module F - sans torsion alors E(M) est F - sans torsion. Puisque Ass M = Ass E(M) alors Ass M $\neq \emptyset$.

Pour la dernière partie voir le lemme 6.5 [9].

Soit R un anneau noethérien et $A = R[X_\alpha]_{\alpha \in \Lambda}$. Nous désignons par T_W la sous - catégorie localisante associée à la topologie additive (sur l'anneau A). Soit Mod A/T_W la catégorie quotient et T_W : Mod $A \to$ Mod A/T_W le foncteur canonique ([6], ch. 3). Il est bien connu que Mod A/T_W est une catégorie de Grothendieck, c'est - à - dire une catégorie abelienne avec generateur et limites inductives exactes. ∎

THEOREME 1.4 Soient R un anneau et $A = R[X_\alpha]_{\alpha \in \Lambda}$ où Λ est un ensemble infini. Alors :

1) La catégorie Mod A/T_W ne contient pas d'objets simples (en partculier elle est une catégorie sans la dimension de Krull au sens de Gabriel ([6] ch. 4)).
2) Tout objet injectif de Mod A/T_W est une somme direct d'injectifs indécomposables.
3) Toute somme direct (limite inductive filtrante) d'injectifs est un injectif.
4) Tout sous - catégorie localisante de Mod A/T_W est stable par rapport aux enveloppes injectives.

DEMONSTRATION. 1) En effet si S est un objet simple de Mod A/T_W alors d'après le lemme 3.5 [1] il existe un idéal premier $p \notin F_W$ tel que $S \approx T_W(A/p)$. De plus p est un élément maximal dans $C_{F_W}(A)$. En particulier $ht(p) < \infty$. Comme Λ est infini il existe toujours un idéal premier q tel que $p \subsetneq q$ et $ht(q) < \infty$. Comme $q \in C_{F_W}(A)$, nous obtenons une contradiction.

2) Soit $\bar{Q}$ un objet injectif de Mod A/T_W. Alors $\bar{Q} = T_W(Q)$ où Q est un A - module injectif et F - sans torsion. D'après le corollaire 1.3,

Q est une extension essentielle de $\bigoplus_{i\in I} E(A/p_i)$ où p_i sont des idéaux premiers avec $ht(p_i)<\infty$. Nous notons $Q = \bigoplus_{i\in I} E(A/p_i)$. Comme T_n est stable par rapport aux enveloppes injective, alors $T_n(Q)$ est une extension essentielle de $T_n(Q')$ (T_n est le foncteur canonique T_n : Mod $A \to$ Mod A/T_n), A étant F_n - noethérien, d'aprés le théorème 1.6 [9] $T_n(Q) = \bigoplus_{i\in I} T_n(E(A/p_i))$ est un objet injectif. Donc $T_n(Q) = T_n(Q')$ et par suite $Q/Q'\in T_n$. Comme n est arbitraire alors $Q/Q'\in T_w$ et donc $\tilde{Q} = T_w(Q) = T_w(Q') = \bigoplus_{i\in I} T_w(E(A/p_i))$ où $T_w(E(A/p_i))$ sont des objets injectifs indécomposables.

De la même façon on preuve l'affirmation 3).

4) Soit A une sous - catégorie localisante de Mod A/T_w. Alors $T_w^{-1}(A)$ est une sous - catégorie localisante de Mod A et $T_w \subset T_w^{-1}(A)$. D'après la proposition 4.1 on peut écrire $T_w^{-1}(A) = \bigcap_{p\in F} T_p$ où $T_p = \{M\in \text{Mod } A \; M_p = 0\}$ et F l'ensemble des idéaux premiers p pour lequels $A/p \notin T_w^{-1}(A)$. On observe que pour tout $p\in F$, $ht(p)<\infty$. Ensuite on applique la proposition 4.1. ∎

REMARQUE. La catégorie Mod A/T_w n'est pas localement noethérienne [10]. De cette façon nous donons un reponse négative à un problème posé par J.E. Roos dans ([10], pag. 201) au sens suivante ; si dans une catégorie de Grothendieck C, tout objet injectif est une somme directe d'injectifs indécomposable, il ne résulte pas que C est localement noethérienne.

2. LA DIMENSION DOMINANTE

Soient R un anneau commutatif arbitraire et M un R - module. Soit F une topologie additive sur R. Nous dirons que M a la dimension F - dominante $\geq$ n, s'il existe une résolution injective de M dans la quelle les premiers n composantes sont F - sans torsion (voir [4]). Notons la dimension F - dominante par F - $d_R(M)$; elle est un nombre naturel ou ∞.

Les resultats suivants sont bien connus [4] :

a) $F - d_R(M) \geqslant n \Leftrightarrow$ les premiers n composantes de la résolution injective minimale de M sont F - sans torsion.

Soit t : Mod R→Mod R le radical associé à la topologie additive F ; t est un foncteur exact à gauche. Désignons par $R^i t$ $(i \geqslant 0)$ les foncteurs dérivés de t. Alors :

b) $F - d_R(M) \geqslant n + 1 \Leftrightarrow (R^i t)(M) = 0$ pour tout $i \leqslant n$.

Si M est un R - module, nous noterons par $Z(M) = \{a \in R \mid \exists x \in M\ x=0,\ ax=0\}$

Une suite finie d'éléments $a_1, a_2, \ldots, a_{n+1} \in R$ est une M - suite (de longueur n) si $a_1 \notin Z(M), \ldots, a_{i+s} \notin Z(M/a_1M + \ldots + a_iM)$ pour tout $1 \leqslant i \leqslant n-1$. Si I est un idéal qui contient une M - suite de longueur n nous écrivons $G(I, M) \geqslant n$ [8]. Pour l'idéal I de R nous écrivons plus simple $G(I) = G(I, R)$.

THEOREME 2.1 Soient F une topologie additive sur l'anneau R et M un R - module de type fini. Considerons les affirmations suivantes :

1) $F - d_R(M) \geqslant n$

2) $G(I, M) \geqslant n$ pour tout $I \in F$

Alors 2)→1) est toujours verifiée. Si de plus R est F - noethérien alors elle est verifiée de même l'implication 1)→2).

<u>DEMONSTRATION</u>. 2)→1). Par récurrence finie nous vérifions que $G(I, M) \geqslant n \to Ext^i(R/I, M) = 0$ pour tout $i < n$.

En effet si n = 1 la conclusion est immédiatement. Soit $a_1, a_2, \ldots, a_{n+i} \in I$ une M - suite de longueur n + 1. De la suite exacte

$$0 \to M \xrightarrow{a_1} M \to M/a_1M \to 0$$

nous trouvons la suite exacte :

$$\to Ext_R^{n-1}(R/I, M) \to Ext^{n-1}(R/I, M/a_1M) \to Ext^n(R/I, M) \xrightarrow{a_1} Ext^n(R/I, M) \to$$

Nous avons $Ext^{n-1}(R/I, M/a_1M) = 0$ par l'hypothèse de récurence, puisque $G(I, M/a_1M) \geqslant n$. Comme $a_1 \in I$ le morphisme

$$Ext^n(R/I, M) \xrightarrow{a_1} Ext^n(R/I, M)$$

est égal à zéro.

Alors $Ext^n(R/I, M) = 0$.

Comme un module M est F - sans torsion $\leftrightarrow Hom_R(R/I, M) = 0$ pour tout $I \in F$, on voit facilement que 2) → 1).

Supposons maintenant que R est F - noethérien et nous prouvons que 1) → 2). Si $F - d_R(M) \geq 1$ alors M est F - sans torsion. M étant de type fini alors M est F - noethérien (corollaire 1.3 [9]). En vertu du théorème 2.3, Ass M est fini et $Z(M) = \bigcup_{p \in Ass\, M} p$. Soit $I \in F$. Puisque $p \notin F$ pour tout $p \in Ass\, M$ alors $I \not\subset p$ et donc $I \not\subset \bigcup_{p \in Ass\, M} p$. Il existe, donc, un élément $a \in I$ tel que $a \notin Z(M)$. Par conséquence $G(I, M) \geq 1$. En suite nous procédons par récurrence sur $F - d_R(M)$. Supposons que $F - d_R(M) \geq n$ $(n \neq 0)$. Il existe $a_1 \in I$ avec $a_1 \notin Z(M)$. De la suite exacte

$$0 \to M \xrightarrow{a_1} M \to M/a_1M \to 0$$

nous obtenons la suite exacte

$$\to (R^{n-2}t)(M) \to (R^{n-2}t)(M/a_1M) \to (R^{n-1}t)\ M \to \dots$$

d'où nous obtenons que $(R^{n-2}t)(M/a_1M) = 0$ et donc $F - d_R(M/a_1M) \geq n-1$. Par récurrence nous avons $G(I, M/a_1M) \geq n-1$ d'où il resulte que $G(I, M) \geq n$ pour tout $I \in F$. ∎

3. APLICATIONS POUR LES ANNEAUX DES POLYNOMES

Soit R un anneau noethérien commutative et $(X_\alpha)_{\alpha \in \Lambda}$ une famille arbitraire d'indéterminés. Considérons l'anneau $A = R[X_\alpha]_{\alpha \in \Lambda}$. Nous prouvons :

THEOREME 3.1 Supposons que dim $R < \infty$. Soit

$$0 \to A \to Q_0 \to Q_1 \to \dots \to Q_n \to \dots$$

la resolution injective minimale de A. Alors pour tout $i \geq 0$, Q_i sont Σ - injectifs (ou au sens de [2], A est un anneau Σ_∞ - noethérien).

Pour la démonstration, nous utilisons le lemme suivant :

LEMME 3.2 Soit R un anneau noethérien avec dim $R < \infty$. Soient $R[X_1, X_2, \dots, X_n]$ l'anneau des polynômes en n indéterminés et $p \subset R[X_1, \dots, X_n]$ un idéal premier tel que $ht(p) > \dim R$. Alors

$$ht(p) - \dim R \leqslant G(p)$$

<u>DEMONSTRATION</u>. Posons $\dim R = r$ et $ht(p) = r + s$, $s \geqslant 1$. Il est clair que $s \leqslant n$. Procédons par récurrence sur s. Si $s = 1$ alors $ht(p) > \dim R$ et d'après le lemme 3 ([3], pag. 16), p contient un polynôme unitaire f en X_n (faisant une abstraction d'un changement de variable). On voit facilement que f est un élément régulier et donc $G(p) \geqslant 1$.

Supposons l'affirmation vraie pour $s - 1$ $(s > 1)$. Comme $ht(p) > \dim R$ d'après le lemme 3 ([3], pag. 16) il existe un polynôme unitaire $f \in p$ en l'indéterminé X_n.

Posons $q = R[X_1, \ldots, X_{n-1}] \cap p$ et $q^* = qR[X_1, \ldots, X_n]$. On voit que $f \notin q^*$ et on déduit alors que $ht(q) = r + s - 1$ (voir le théorème 39 [8]). Par récurrence q contient une $R[X_1, \ldots, X_{n-1}]$ - suite, $f_1, f_2, \ldots, f_{s-1}$. Mais $f_1, f_2, \ldots, f_{s-1}$ est une $R[X_1, \ldots, X_n]$ - suite. Soit maintenant l'egalité $hf = g_1f_1 + g_2f_2 + \ldots + g_{-1}f_{s-1}$ où $g_1, \ldots, g_{s-1}, h \in R[X_1, \ldots, X_n]$.

Ecrivons

$$f = X_n^k + t_1X_n^{k-1} + \ldots + t_k \text{ et } h = h_0X_n^m + h_1X_n^{m-1} + \ldots + h_m$$

où $t_1, \ldots, t_k, h_0, h_1, \ldots, h_m \in R[X_1, \ldots, X_{n-1}]$.

De l'égalité ci - dessus on obtient que $h_0 = g_1f_1 + g_2f_2 + \ldots + g_{s-1} f_{s-1}$ où $g_1, \ldots, g_{s-1} \in R[X_1, \ldots, X_{n-1}]$. Donc $h_0 \in < f_1, \ldots, f_{s-1} >$ où nous avons noté par $< f_1, \ldots, f_{s-1} >$ l'idéal engendré, dans l'anneau $R[X_1, \ldots, X_{n-1}]$ par les éléments $f_1, \ldots, f_{s-1}$.

En suite, de l'égalité $h_1 + h_0t_1 = g_1f_1 + \ldots + g_{s-1}f_{s-1}$ où $g_1, \ldots, g_{s-1} \in R[X_1, \ldots, X_{n-1}]$ on déduit que $h_1 \in < f_1, \ldots, f_{s-1} >$. Par récurrence nous avons $h_0, h_1, \ldots, h_m \in < f_1, \ldots, f_{s-1} >$ d'où il résulte que $h \in f_1R[X_1, \ldots, X_n] + \ldots + f_{s-1}R[X_1, \ldots, X_n]$.

En conclusion $f_1, \ldots, f_{s-1}$, f est une $R[X_1, \ldots, X_n]$ - suite contienue dans p. ∎

Démonstration du théorème 3.1 Soit $r = \dim R < \infty$. Considérons la topologie additive sur A : $F_{n+r} = \{I \subset A \mid I \not\subset p \ \forall\ p \in \mathrm{Spec}\, A,\ ht(p) \leq n + r\}$. Soient $I \in F_{n+r}$ et $t = G(I)$.

Ainsi que dans le théorème 5.5 [2], on montre qu'il existe un idéal premier $p \supset I$ tel que $t = G(I) = G(p)$. Puisque $p \in F_{n+r}$ alors $ht(p) \geq n + r + 1$. Alors il existe des indéterminées $X_{\alpha_1}, \ldots, X_{\alpha_R}$ tel que $ht(p \cap R[X_{\alpha_1}, \ldots, X_{\alpha_R}]) \geq n + r + 1$.

En vertu du lemme 3.2 nous obtenons que $G(p \cap R[X_{\alpha_1}, \ldots, X_{\alpha_k}]) \geq n+1$ d'où il résulte que $G(p) \geq n+1$. D'après le théorème 2.1, $Q_0, Q_1, \ldots, Q_n$ sont des modules F_{n+r}- sans torsion. Comme A est F_{n+r} - noethérien, en vertu du théorème 1.6 [9] il resulte que tous Q_i $(0 \leq i \leq n)$ sont Σ - injectifs.

Quand R est un anneau noethérien arbitraire nous avons le résultat partial. ∎

THEOREME 3.2 Soit R un anneau noethérien arbitraire. Avec les notations du théorème 3.1 tous Q_i $(i \geq 0)$ sont F_w - sans torsion où $F_w = \bigcap_{n \geq 0} F_n$.

DEMONSTRATION. Il est bien connu que Ass A = $\{pA \mid p \in \mathrm{Ass}\, R\}$. Soit $I \in F_w$. Alors $I \not\subset p$ pour tout $p \in \mathrm{Spec}\, A$ avec $ht(p) < \infty$. D'autre part si $p \in \mathrm{Ass}\, R$ alors $ht(pA) < \infty$ et donc il existe $a_1 \in I$, $a_1 \notin \bigcup_{q \in \mathrm{Ass}\, A} q$. Pour l'élément a_1 il existe des indéterminés $X_{\alpha_1}, \ldots, X_{\alpha_r}$ tel que $a_1 \in R[X_{\alpha_1}, \ldots, X_{\alpha_r}]$, posons

$$R' = \frac{R[X_{\alpha_1}, \ldots, X_{\alpha_r}]}{a_1 R[X_{\alpha_1}, \ldots, X_{\alpha_r}]}, \quad \Lambda' = \Lambda - \{\alpha_1, \ldots, \alpha_r\} \quad \text{et}$$

$A' = A/a_1A$. On voit que $A' = A/a_1A = R'[X_{\alpha'}]_{\alpha' \in \Lambda'}$.

Si on note $I' = I/a_1A$, l'idéal I' est de hauteur infini. Ainsi que ci - dessus en remplacant R par R', il existe un élément $a_2 \in I$ tel que

$\hat{a}_2 \in I'$ et $a_2 \in \bigcup_{q' \in \mathrm{Ass}\, A} q'$. Dans cette manière on obtient la suite a_1, a_2, ..., a_n, ... d'élément qui appartiennent à I et qui forment une A - suite.

Donc $G(I) \geqslant n$, pour tout nombre naturel n. En vertu du théorème 2.1,

$F_w - d_R(A) \geqslant n$ pour tout $n \geqslant 0$. ∎

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ON THE SPECTRA OF LEFT STABLE RINGS

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0. INTRODUCTION. Let R be a left stable ring, R-sp = {the collection of prime torsion theories on R-mod} and Sp(R-mod) = {the collection of isomorphism classes of indecomposable injective R-modules}. For a large class of rings (e.g. D-rings, see J. Golan [1]) the assignment $\chi:F \to \chi(F)$ is a bijection of Sp(R-mod) onto R-sp, thus any topology introduced for one of the spaces can be carried over to the other. The space R-sp (or Sp(R-mod)) with an appropriate topology is called the spectrum of the ring R. For an R-module M, $\chi(M)$ ($\xi(R)$) denotes the unique largest (smallest) torsion theory relative to which M is torsion free (torsion). The notation $F \in$Sp(R-mod) is used to denote both an isomorphism class of indecomposable injective modules and one of the representative elements of the class. A two-sided ideal I is associated to an R-module M, I=ass(M), if there exists a non-zero submodule N of M such that

I is the annihilator of all non-zero submodule of N. It follows that for any $F \in Sp(R\text{-mod})$ ass(F) is a prime ideal and for an element $\pi \in R\text{-sp}$ we can define ass(π) by the equality ass(π)=ass(F) where $\pi = \chi(F)$. The assignment $\theta : \pi \to \mathrm{ass}(\pi)$ maps R-sp to Spec(R) = {the collection of prime ideals of R}. Given the Zariski topology on Spec(R) and the basic order topology on R-sp, θ is a continuous map whenever R is left stable, left noetherian ring. This result implies that the presheaf constructed on Spec(R) by F. van Oystaeyen, [8], is in fact a sheaf for the above class of rings. (It has already been shown in [8] that this is true for prime left noetherian rings.)

A torsion theory τ is called <u>basic</u> if $\tau = \xi(R/L)$ for some left ideal L of R. The set {pgen(τ) | τ is a basic torsion theory}, where pgen(τ) = $\{\pi \in R\text{-sp} \mid \tau \leq \pi\}$, forms a base of open sets for a topology on R-sp which is called the <u>basic order topology</u>.

In the papers [3] and [4] the author showed that the left stable rings are characterized by the fact that the order relations in R-sp are in complete agreement with the existence of non-zero homomorphisms among the elements of Sp(R-mod). Namely, Hom(F,G) $\neq 0$ if and only if $\chi(G) \leq \chi(F)$ for any pair $F, G \in Sp(R\text{-mod})$. Since the basic order topology reflects the order relations in R-sp one can expect that the above fact has its consequences on the structure of the spectrum

R-sp. In this note we are going to study a few of these implications.

All rings R considered here will have identity and each R-module is a unitary left R-module. The usual notation E(M) denotes the injective hull of an R-module M and the notation $E_R(M)$ is used if the ring R has to be emphasized. For the unexplained concepts, results, terminology and notation on torsion theories we refer to the book [1] of J. Golan, and only the most important concepts will be defined in due course.

1. THE CONSEQUENCES OF THE DESCENDING CHAIN CONDITION ON R-SP

Let R be a semi - noetherian ring, then any descending chain of prime torsion theories terminates in finite steps. (See J. Golan [1].) In this section we discuss a few consequences of this result, one of which is the analogue of Theorem 7.6 of R.Gordon and J.C. Robson [2].

THEOREM 1. Let R be left stable semi-noetherian ring. Then any descending chain of torsion theories in the form $\xi(F_1 \otimes \dots \otimes F_n)$, $F_i \in Sp(R\text{-mod})$, terminates in finite steps.

PROOF. First we show that for any two $F,G \in Sp(R\text{-mod})$, a proper inequality $\chi(F) \lessdot \chi(G)$ is equivalent to the proper inequality $\xi(F) < \xi(G)$. The stability of R implies that given $\chi(F) \prec \chi(G)$, then $\xi(F) \leq \chi(G)$, thus $\xi(F) = \xi(G)$ cannot happen. If F is $\xi(G)$ - torsion free, we have $\xi(G) \leq \chi(F) \prec \chi(G)$ which is a contradiction, hence the only alternative is $\xi(F) < \xi(G)$. The converse is similar.

We are going to show that the existence of an infinite sequence of strictly descending torsion theories

$$\sigma_0 > \sigma_1 > \ldots > \sigma_n > \ldots$$

of the form $\sigma_i = \xi(F_1^i \otimes \ldots \otimes F_{n_i}^i)$ implies the existence of a sequence

$$\xi(G_0) > \xi(G_1) > \ldots > \xi(G_k) > \ldots$$

where $G_i \in Sp(R\text{-mod})$ $(i=0,1,2\ldots)$ which, in turn, gives a strictly decreasing sequence of prime torsion theories in contradiction with Proposition 20.13 of J. Golan [1].

Let $S_i^1 = \{\xi(F) \mid F \in Sp(R\text{-mod})$ and $\xi(F) \leq \sigma_i\}$ and let S_i be the set of maximal elements of S_i^1. Then S_i is a finite set. Indeed, if $\xi(F) \leq \sigma_i$, then F is σ_i-torsion, hence there exists an index k, $1 \leq k \leq n_i$ such that F is $\xi(F_k^i)$-torsion, otherwise F would be $\xi(F_k^i)$-torsion free for each k, thus σ_i-torsion free because $\sigma_i = \xi(F_1^i \otimes \ldots \otimes F_{n_i}^i) = \xi(F_1^i) \vee \ldots \vee \xi(F_{n_i}^i)$. This means that $\xi(F) \leq \xi(F_k^i) \leq \sigma_i$, thus every maximal element in S_i^1 equals to some of the $\xi(F_k^i)$ $1 \leq k \leq n_i$, also $\sigma_i = \vee\{\xi(F) \mid \xi(F) \in S_i\}$. We are going to construct the following infinite graph G. Let the set $V = \cup\{S_i \mid i=0,1,\ldots\}$ be the vertices of

G which is an infinite set. The vertices $\xi(F)$ and $\xi(G)$ are connected by an edge if $\xi(F)\in S_i$, $\xi(G)\in S_{i+1}$ and $\xi(F)>\xi(G)$. We claim that if $\xi(F)\in S_i$ and there is an edge going out of the vertex $\xi(F)$, then $\xi(F)\notin S_j$ with $j>i$. Let $\xi(G)\in S_{i+1}$ and $\xi(F)>\xi(G)$. Assume $\xi(F)\in S_j$, $j>i$, then the relations $\sigma_{i+1}\geq\sigma_j\geq\xi(F)>\xi(G)$ contradict with the definition of $\xi(G)$, thus our claim is proved. This insures that there are at most finite many edges going out of any given vertex. Since the sets S_i, $i=0,1,2,\ldots$, are finite sets, for a given $\xi(F)\in S_i$, the number of edges in any path leading from some vertex in S_o to $\xi(F)$ is bounded. This implies that there exists a longest path leading from some element of S_o to $\xi(F)$. If $\xi(F)\in S_o$ we say the height of $\xi(F)$ is 0, otherwise the height of an element $\xi(F)\in V$ is the number of edges in the longest path. Let define $V_k=\{\xi(F) \mid \text{height of } \xi(F) \text{ is } k\}$. Then $V=\cup\{V_k|k=0,1,\ldots\}$ and V_k is a finite set for each $k=0,1,\ldots$. This follows by induction from the fact that V_o is finite and from the above remark about the number of edges that going out from a fixed vertex. Since V is infinite there must exist paths with arbitrary length. An application of the König Graph Theorem insures the existence of an infinite properly decreasing sequence of torsion theories $\xi(G_o) > \xi(G_1)>\ldots>\xi(G_n)>\ldots$as we claimed.

If R is a left stable left noetherian ring and L is a left ideal, then the basic torsion theory $\xi(R/L) = \xi(E(R/L)) = \xi(F_1 \otimes \ldots \otimes F_n)$. Thus Theorem 1 implies the following result:

COROLLARY 1. Let R be a left stable, left noetherian ring. Then any descending chain of basic torsion theories terminates in finite steps.

REMARK. Let L be a left ideal of the left stable, left noetherian ring R. A torsion theory of the form $\chi(R/L)$ is called cobasic torsion theory. By the above method we can prove that the ascending chain condition holds for cobasic torsion theories if and only if it holds for prime torsion theories.

The torsion theory $\xi(F)$, $F \in Sp(R\text{-mod})$ coincide with the one that is called <u>coprime</u> by J. Raynaud, [6], if R is left stable, left noetherian ring, because $\xi(F)$ is the unique minimal element of the set $\{\tau \in R\text{-tors} \mid \chi(F) \notin pgen(\tau)\}$.

Let $P, Q \in Spec(R)$. It is easy to see that $\xi(R/P) \wedge \xi(R/Q) = \xi(R/(P+Q)) = \xi[E(R/(P+Q))] = \xi(F_1) \vee \ldots \vee \xi(F_n)$ with $F_i \in Sp(R\text{-mod})$, $i = 1, 2, \ldots, n$, hence $\xi(R/P) \wedge \xi(R/Q)$ is a finite join of coprimes. If this is true for every pair of coprime torsion theories $\xi(F)$ and $\xi(G)$, $F, G \in Sp(R\text{-mod})$, then we say R satisfies <u>Property F</u>. If

If R is a left stable, left noetherian ring, Property F has an interesting consequence. Let I and J be left ideals of R. Then $\xi(R/I)\wedge\xi(R/J)=[\xi(F_1)\vee\ldots\vee\xi(F_m)]\wedge[\xi(G_1)\vee\ldots\vee\xi(G_n)]=\vee\{\xi(F_i)\wedge\xi(G_j);\ i=1,\ldots,m;\ j=1,\ldots,n\}$ where $E(R/I) = F_1\oplus\ldots\oplus F_m$, $E(R/J)=G_1\oplus\ldots\oplus G_n$ direct sums of indecomposable injective modules. If R has Property F, then we can continue the change and obtain the decomposition $\xi(R/I)\wedge\xi(R/J)=\xi(H_1)\vee\ldots\vee\xi(H_k)=\xi(H_1\oplus\ldots\oplus H_k)$ where we have dropped all the torsion theories which would have made the join redundant. Let $H_i = E(R/L_i)$, L_i irreducible left ideal, we have that $E(R/L_i)\oplus\ldots\oplus E(R/L_k) = E(R/(L_1\cap\ldots\cap L_k))$, thus, with the notation $L=L_1\cap\ldots\cap L_k$, the equality $\xi(R/I)\wedge\xi(R/J) = \xi(R/L)$ shows that the meet of finite many basic torsion theories is a basic torsion theory.

Consider R-sp with the basic order topology and let $U_1\subset U_2\subset\ldots\subset U_n\subset\ldots$ be a strictly increasing sequence of open sets. We can assume that U_i is a finite union of basic open sets of the form pgen $\xi(R/L)$, and then the equalities pgen$\xi(R/L_1)\cup\ldots\cup$ pgen$\xi(R/L_n)=$ pgen$[\xi(R/L_1)\wedge\ldots\wedge\xi(R/L_n)]$ and $\wedge$ pgen $(\tau) = \tau$ change the above sequence to strictly decreasing sequence of basic torsion theories by the result of the preceding paragraph. Corollary 1 contradicts this possibility, thus we have the following theorem.

THEOREM 2. Let R be a left stable, left noetherian ring. Property F implies that R-sp with the basic order topology is a noetherian space.

REMARK. All the examples of left stable, left noetherian rings the author knows about, have Property F.

2. REDUCTION THEOREMS. Let R be a left noetherian, left stable ring. We are going to show that the study of the topological space R-sp can be reduced to the case when R is a semi-prime (or even a prime) ring. In the beginning we point to the importance of the prime torsion theories $\chi(R/P)$, $P \in Spec(R)$. Let start with the known fact that for left noetherian rings R (or left D-rings in general) the map $P \to \chi(R/P)$ is an order reversing injection, thus $P \subseteq Q$ if and only if $\chi(R/P) \geq \chi(R/Q)$. If R is left stable as well $Hom(E(R/P), E(R/Q)) \neq 0$ is equivalent to $\chi(R/P) \geq \chi(R/Q)$, and consequently, to $P \subseteq Q$. This can be extended to include the other indecomposable injective modules as well and it shows that the torsion theories $\chi(R/P)$, $P \in Spec(R)$, are the "local maximums".

PROOF. Let R be a left stable, left noetherian ring. If $F \in Sp(R\text{-mod})$ and $P = ass(F)$, then $\chi(F) \leq \chi(R/P)$.

PROOF. Let $P=ass(F)$ be the annihilator of the submodule N of F and let L be a maximal element of the set $\{ann(a)|a\in N\}$. Then L is either a maximal element in $\{ann(b)|b\in F\}$ or there exists a maximal element I in the latter set that contains L. It follows that I is a critical ideal, $F=E(R/I)$ and $P\subseteq I$, hence the epimorphism $R/P\to R/I\to 0$ insures the relation $Hom(E(R/P),F)\neq 0$, so we can conclude that $\chi(F)\leq\chi(R/P)$ by the stability of the ring R.

PROPOSITION 2. With the assumption of Proposition1. let $F,G\in Sp(R\text{-mod})$, $P=ass(F)$ and $Q=ass(G)$. If $Hom(F,G)\neq 0$, then $P\subseteq Q$.

PROOF. Let $\varphi\in Hom(F,G)$ and $a\in F$ with $\varphi(a)\neq 0$. By Theorem 4.4 of B. Stenström [7] $P^n a=0$ for some natural number n, hence $P^n(Ra)=P^n a=0$ as well. Thus $P^n\varphi(Ra)\subseteq\varphi(P^n Ra)=0$, hence P^n annihilates a nonzero submodule $\varphi(Ra)$ of G. This implies that $P^n\subseteq Q$, and consequently $P\subseteq Q$.

THEOREM 3. Let R be a left stable, left noetherian ring, N the prime radical of R and let $\tilde{R}=R/N$. Consider R-sp and $\tilde{R}$-sp with the respective basic order topologies. Then $\tilde{R}$ is a left stable, left noetherian ring and there exists a bijection $\Phi:R\text{-sp}\to\tilde{R}\text{-sp}$ such Φ is a homeomorphism.

PROOF. Let φ be the epimorphism $\varphi:R\to R/N=\tilde{R}$ and $\varphi^*:\tilde{R}\text{-mod}\to R\text{-mod}$ the restriction functor. If $F\in Sp(R\text{-mod})$ and $P=ass(F)$ then there exists a nonzero submodule M of F such that $PM=0$, hence $NM=0$ which implies that $\tilde{F} = \{a\in F \mid Na=0\}\neq 0$. We claim that $\tilde{F}$ is an indecomposable injective $\tilde{R}$-module and the map $\Psi:F\mapsto\tilde{F}$ gives a bijection $\Psi:Sp(R\text{-mod})\to Sp(\tilde{R}\text{-mod})$. If $F\in Sp(R\text{-mod})$, then $\tilde{F}\in Sp(\tilde{R}\text{-mod})$ and given $H,H_1,H_2\in Sp(\tilde{R}\text{-mod})$ it follows that $\widetilde{E_R(\varphi^*H)}\cong H$ and $E_R(\varphi^*H_1)\cong E_R(\varphi^*H_2)$ if and only if $H_1\cong H_2$. It is also clear that the inverse map Ψ^{-1} is given by $\Psi^{-1}:H\mapsto E_R(\varphi^*H)$ and $E_R(\varphi^*\tilde{F})\cong F$ for $F\in Sp(R\text{-mod})$. Given $F,G\in Sp(R\text{-mod})$, our next claim is that $Hom_R(F,G)\neq 0$ if and only if $Hom_{\tilde{R}}(\tilde{F},\tilde{G})\neq 0$. Since $E_R(\varphi^*\tilde{F})\cong F$, any map $0\neq f\in Hom_{\tilde{R}}(\tilde{F},\tilde{G})$ can be extended to a non-zero map in $Hom_R(F,G)$. On the other hand, let $Hom_R(F,G)\neq 0$. Since $\tilde{F}\neq 0$ and F is $\chi(G)$-torsion free which implies that $\cap\{\ker f \mid f\in Hom_R(F,G)\}=0$, there must exist an element $f\in Hom_R(F,G)$ such that $f(\tilde{F})\neq 0$. But $f(\tilde{F})\subseteq\tilde{G}$, thus the restriction of f to $\tilde{F}$ gives a non-zero element in $Hom_{\tilde{R}}(\tilde{F},\tilde{G})$.

Consider a torsion theory $\tau\in R\text{-tors}$. The collection of $\tilde{R}$-modules $\{M\in R\text{-mod} \mid \varphi^*M \text{ is } \tau\text{-torsion}\}$ gives the torsion class of a torsion theory in $\tilde{R}$-mod which will be denoted $\varphi^{\#}\tau$. (See J. Golan [1] p. 85.) Since both R and $\tilde{R}$ are left noetherian rings the assignment $\chi:F\to\chi(F)$ is a bijection for both rings. Thus the restriction of $\varphi^{\#}$ to R-sp gives a bijection $\Phi:R\text{-sp}\to\tilde{R}\text{-sp}$. Also if

$M \in \tilde{R}$-mod, then M is $\chi(\tilde{F})$-torsion if and only if $\varphi^* M$ is $\chi(F)$-torsion, hence we also have $\varphi^{\#}\chi(F)=\chi(\tilde{F})$.

Now let H_1, $H_2 \in Sp(\tilde{R}\text{-mod})$ and $Hom_{\tilde{R}}(H_1,H_2)\neq 0$. Then $H_i=\tilde{F}_i$ for some $F_i \in Sp(R\text{-mod})$ (i=1,2) and $Hom_R(F_1,F_2)\neq 0$. Since R is stable, $\chi(F_2)\leq\chi(F_1)$ follows and Proposition 9.1 of [1] implies that $\chi(H_2)=\chi(\tilde{F}_2)=\varphi^{\#}\chi(F_2)\leq\varphi^{\#}\chi(F_1)= \chi(\tilde{F}_1)=\chi(H_1)$, hence $\tilde{R}$ is stable by Theorem 2 of [3].

Finally, we are going to show that the map Φ is a homeomorphism if we use the basic order topologies in both R-sp and $\tilde{R}$-sp. Any open set of R-sp is the union of basic open sets, pgen$\xi(R/I)$, I is a left ideal of R. Since R is left noetherian, left stable ring, $\xi(R/I)=\xi(E(R/I))=\xi(F_1 \oplus \ldots \oplus F_n)=\xi(F_1)\vee\ldots\vee\xi(F_n)$, hence pgen $\xi(R/I)$=pgen $\xi(F_1)\cap\ldots\cap$ pgen $\xi(F_n)$. This shows that the set $\{\text{pgen}\,\xi(F) \mid F\in Sp(R\text{-mod})\}$ is a subbase of the basic order topology of R-sp.

The proof of the theorem will be complete if we show that for any $F\in Sp(R\text{-mod})$ $\Phi\,\text{pgen}\,\xi(F)=\text{pgen}\,\xi(\tilde{F})$. Let $\chi(G)\in\text{pgen}\,\xi(F)$. Then F is $\chi(G)$-torsion, hence $Hom_R(F,G)=0$ which in turn implies that $Hom_{\tilde{R}}(\tilde{F},\tilde{G})=0$ that is $\tilde{F}$ is $\chi(\tilde{G})$-torsion and consequently $\chi(\tilde{G})\in\text{pgen}\,\xi(\tilde{F})$. The procedure can be reversed which establishes our claim.

Given a torsion theory τ, pspcl(τ) = $\{\pi\in R\text{-sp} \mid \pi\leq\tau\}$. If the left stable, left noetherian ring is prime, then $\chi(R)$ is the unique maximal (prime) torsion theory and for every torsion theory τ, R is τ-torsion free. In general, let $P_1,\ldots,P_n$ be the minimal primes of R, then $\chi(R/P_1),\ldots,\chi(R/P_n)$ are the maximal prime torsion theories of R-tors. The following result can be established by repeating the steps of the proof of Theorem 3. It shows that the study of the spectrum of left stable, left noetherian rings can be reduced to examine the spectrum of prime rings since R-sp=pspcl$\chi(R/P_1)\cup\ldots\cup$pspcl$\chi(R/P_n)$ and pspcl$\chi(R/P_i)$ is homeomorphic to R/P_i-sp $(i=1,\ldots,n)$.

THEOREM 4. Let R be a left stable, left noetherian ring, P prime ideal of R, $F\in$Sp(R-mod) and let $\tilde{F}=\{a\in F\mid Pa=0\}$ be considered as an R/P-module. Then the map $\Phi:\chi(F)\to\chi(\tilde{F})$ is a bijection Φ:pspcl$\chi(R/P)\to R/P$-sp and becomes a homeomorphism if we consider the basic order topology in R/P-sp and the relativization of the basic order topology of R-sp to the closed set pspcl$\chi(R/P)$.

REMARK. Recall the notation ass(π)=ass(F), where $\pi\in$R-sp, $\pi=\chi(F)$, $F\in$Sp(R-mod). Given $P\in$Spec(R) and consider the subset $(R\text{-sp})_P=\{\pi\in R\text{-sp}\mid \text{ass}(\pi)=P\}$ of R-sp with the relative topology. Since $(R\text{-sp})_P$ is homeomorphic to $(R/P\text{-sp})_0$, 0 is the zero (prime) ideal of of R/P, it is enough to consider a prime ring R and the set $(R\text{-sp})_0=\{\pi\in R\text{-sp}\mid \text{ass}(\pi)=0\}$. If $\pi=\chi(F)$, $F\in$Sp(R-mod), then $\pi\in(R\text{-sp})_0$ if and only if no left ideal L of R with $F\cong E(R/L)$ contains an ideal. Let I be an ideal of R and

$I \subseteq L$. Then $I(R/L)=0$, thus $\mathrm{ass}(\pi)\neq 0$. On the other hand, if $\mathrm{ass}(\pi)=P\neq 0$, then a left ideal L can be found such that $P\subseteq L$ and $F\cong E(R/L)$. Indeed, let $PN=0$ for some nonzero submodule N of F and let $0\neq a\in N$. Then $P\subseteq(0:a)=\{r\in R \mid ra=0\}$ and $F\cong E(R/(0:a))$. The set $(R\text{-sp})_P$ resembles the spectrum of a simple ring. Therefore, the study of left stable, left noetherian simple rings seems to be interesting in order to learn more about the spectrum of left stable rings.

3. THE MAP R - SP → SPEC(R). The aim of this section is to show that the map $\theta:\pi\to\mathrm{ass}(\pi)$ is a continuous map from R-sp with the basic order topology onto Spec(R) with the Zariski topology. This, in turn, implies that the presheaf constructed by F. van Oystaeyen on Spec(R) (see [8]) is a sheaf for left stable, left noetherian rings.

THEOREM 5. Let R be left stable, left noetherian ring. Consider the Zariski topology in Spec(R) and the basic order topology in R-sp. The map $\theta:\pi\to\mathrm{ass}(\pi)$ is a continuous map from R-sp onto Spec(R).

PROOF. Let C be a closed set in Spec(R). Then $C =\{P\in \mathrm{Spec}(R) \mid \mathrm{rad}I\subseteq P\}$, where radI is the prime radical of an ideal I, also radI is a finite intersection of prime ideals of R minimal over I, $\mathrm{rad}I=P_1\cap\ldots\cap P_n$. The inverse image of C, $\theta^{-1}C = \{\pi\in R\text{-sp} \mid \mathrm{ass}(\pi)\in C\}$ and if

$P=\mathrm{ass}(\pi)$, $P\in C$ means $P\supseteq P_k$ for same $1\le k\le n$. The stability of R and Proposition 1 imply the inequalities $\pi\le\chi(R/P)\le\chi(R/P_k)$. Consequently $\theta^{-1}C =$ $\mathrm{pspcl}\chi(R/P_1)\cup\ldots\cup\mathrm{pspcl}\chi(R/P_n)$ which is the union of finite many closed sets in R-sp, hence it is closed itself. This proves the continuity of θ.

Theorem 5 has an interesting consequence. Let O be any open set in Spec(R). Then $O = \mathrm{Spec}(R)\setminus C$, where C is a closed set, thus it has the form $C = \{P\in\mathrm{Spec}(R) \mid \mathrm{rad}I\subseteq P\}$ for some ideal I of R. It follows from the above discussion that $\theta^{-1}C =$ $\{\pi\in R\text{-sp} \mid \mathrm{ass}(\pi)\in C\} = \mathrm{pspcl}\chi(R/P_1)\cup\ldots\cup\mathrm{pspcl}\chi(R/P_n)$ where $\mathrm{rad}\, I = P_1\cap\ldots\cap P_n$. If P is a prime ideal of R, then R/P is either torsion or torsion free with respect to any torsion theory, hence the equalities $\mathrm{pspcl}\chi(R/P_i) =$ $= \mathrm{supp}(R/P_i)$ $(i=1,\ldots,n)$ follow. Since $R\text{-sp}\setminus\mathrm{supp}(R/P)=$ $\mathrm{pgen}\xi(R/P)$, we conclude that the inverse image of the open set O has the following form: $\theta^{-1}O =$ $\mathrm{pgen}\xi(R/P_1)\cap\ldots\cap\mathrm{pgen}\xi(R/P_n) = \mathrm{pgen}\xi(R/P_1)\vee\ldots\vee\xi(R/P_n) =$ $\mathrm{pgen}\xi(R/(P_1\cap\ldots\cap P_n)) = \mathrm{pgen}\xi(R/\mathrm{rad}I) = \mathrm{pgen}\xi(R/I)$. As a consequence, we have $\wedge\,\theta^{-1}O = \xi(R/I)$ which is the torsion theory one uses to construct the ring of quotients in the construction of the presheaf on Spec(R). (See F. van Oystaeyen [8].) The construction of the presheaf on R-sp uses the same torsion theory since the assignment is: $U \to Q_{\Lambda U}(R)$ for an open set U of R-sp. Consequently, for any open set $\mathcal{O}$ of Spec(R), $\mathcal{O} = \{P \in Spec(R) | P \not\supseteq radI\}$, the assigned ring of quotients $Q_I(R) = Q_{\xi(R/I)}(R) = Q_{\Lambda\theta^{-1}\mathcal{O}}(R)$ is the same object which is assigned to the inverse image $\theta^{-1}\mathcal{O}$ of $\mathcal{O}$ in the construction of the presheaf on R-sp. By Theorem 1 of [5] the presheaf constructed on R-sp is a sheaf if R is left stable, left noetherian ring, thus we have the following result.

COROLLARY 2. Let R be a left stable, left noetherian ring. The assignment $Q_I(R) = Q_{\xi(R/I)}(R)$ to the open set $\mathcal{O} = \{P \in Spec(R) \mid P \not\supseteq radI\}$ for an ideal I of R is a sheaf on Spec(R).

Therefore, the collection of left stable, left noetherian rings is another class of rings, besides the class of prime noetherian rings (see [8]), for which the presheaf of F. van Oystaeyen on Spec(R) is in fact a sheaf.

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ON GALOIS EXTENSIONS OVER COMMUTATIVE RINGS

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0. INTRODUCTION

Let B be a commutative ring extension of a subring A with an automorphism group G (= {σ}) of order 2 such that (1) 2 is a unit in B, (2) $j^2 = -1$, $jb = \sigma(b)j$ for σ in G and b in B, and (3) the set of elements B^G in B fixed by σ is A ($B^G = A$). S. Parimala and R. Sridharan ([6]) showed that B is Galois extension over A if and only if $B \otimes_A B[j]$ is isomorphic with the matrix ring of order 2 over B, $M_2(B)$ ([6], Proposition 1.1), where the Galois extension is in the sense of Chase-Harrison-Rosenberg ([2]). We shall generalize the above characterization to cyclic Galois extensions (G is cyclic) from the point of splitting rings for Azumaya algebras. Let G be an automorphism group of B such that (1) G is cyclic generated by σ of order n invertible in B, (2) $j^k b = \sigma^k(b)j^k$, $j^n = -1$, and (3) $B^G = A$. If B is Galois over A then $B \otimes_A B[j] \cong M_n(B)$. The converse holds when n is prime. Moreover, we shall discuss a non - cyclic case : Let G be the automorphism group of B such that G is a non - cyclic group of order 4 invertible in B, G = (α)(β), that i,j, and k are the usual quaternions with $ib = \alpha(b)i$, $jb = \beta(b)j$, $kb = \alpha\beta(b)k$, and that $B^G = A$. Assume each maximal ideal of B is G - invariant. If B is Galois over A then $B \otimes_A B[i, j, k] \cong M_4(B)$. The converse holds when none of the following algebras is commutative : $B/M \otimes_A B[i]$, $B/M \otimes_A B[j]$ and $B/M \otimes_A B[k]$, for each maximal ideal M of B.

1. BASIC DEFINITIONS

Let B be a commutative ring, and A a subring of B with the same identity 1. Then B is called a Galois extension over A ([2]) with a finite automorphism group G (Galois group) if (1) there exist elements in B, $\{a_i, b_i / i = 1, 2, \ldots, n$ for some integer $n\}$ such that $\Sigma a_i b_i = 1$ and $\Sigma a_i \sigma(b_i) = 0$ whenever $\sigma \neq 1$ in G, and (2) the set of elements in B fixed under each element in G is A ($B^G = A$). For characterizations of Galois extensions, see [2] or [3]. Let S be a ring, and R a subring with the same identity 1 (not necessarily commutative). Then S is called a separable extension of R if there exist elements in S, $\{c_i, d_i / i = 1, 2, \ldots, n$ for some integer $n\}$ such that (1) $a(\Sigma c_i \otimes d_i) = (\Sigma c_i \otimes d_i)a$ for each a in S, where $\otimes$ is over R, and (2) $\Sigma c_i d_i = 1$ ([5], Section 2, Definition 2). S is called an Azumaya R - algebra if it is separable over R and its center is R ([1] and [3]). A commutative ring extension B over R is called a splitting ring for the Azumaya R - algebra S if $B \otimes_R S \cong \mathrm{Hom}_B(P,P)$ where P is a progenerator B - module ([3], P. 63). We shall employ the following facts :

PROPOSITION 1. ([3], Theorem 5.5, P. 64) Let S be an Azumaya R - algebra. If B is a maximal commutative subalgebra in S (that is, $S^B = B$, the commutant of B in S is B) and if it is separable over R, then it is a splitting ring for S.

PROPOSITION 2. ([3], Proposition 1.2, P.81) Let B be a commutative ring extension of A. Then, B is Galois over A with the Galois group G if and only if (1) $B^G = A$, and (2) for each $\sigma \neq 1$ and maximal ideal M of B, there exists an element b in B such that $(b - \sigma(b)) \notin M$.

As a consequence of Proposition B, the ideal generated by $\{b - \sigma(b)\}$ for all b in B is B for any $\sigma \neq 1$ in G.

2. MAIN THEOREMS

This section will include a generalization of a theorem of Parimala and Sridharan ([6], Proposition 1.1). Let B be a commutative ring with 1, G an automorphism group generated by σ of order n invertible in B, and $A = B^G$. We define an algebra over A, B[j], such that (1) B[j] is a free B - module with a basis $\{1, j, \ldots, j^{n-1}\}$ (2) $j^n = -1$, $j^k b = \sigma^k(b)j^k$ for all b in B and each positive integer k, and (3) multiplication is distributive over addition.

LEMMA 2.1 If B is a Galois extension over A, B[j] is an Azumaya A - algebra such that B is maximal commutative subalgebra of B[j].

<u>PROOF</u>. We first claim that the center of B[j] is A. Let $\Sigma_{k=0}^{n-1}(b_k j^k)$ for b_k in B be an element in the center. Then $j(\Sigma b_k j^k) = (\Sigma b_k j^k)j$, that is, $\Sigma\sigma(b_k)j^{k+1} = \Sigma b_k j^{k+1}$. Since $\{1, j, \ldots, j^{n-1}\}$ form a basis over B, $\sigma(b_k) = b_k$ for $k = 0, 1, \ldots, n-1$. The automorphism group G is cyclic generated by σ such that $B^G = A$, so b_k are in A. Also, $a(\Sigma b_k j^k) = (\Sigma b_k j^k)a$ for each a in B, so $\Sigma a b_k j^k = \Sigma b_k \sigma^k(a) j^k$. Hence $b_k(a - \sigma^k(a)) = 0$. But B is Galois over A, so Proposition 2 in Section 1 implies that $b_k = 0$ for each $k \neq 0$.

This proves that the center of B[j] is A.

Next we claim that B[j] is a separable extension over B. In fact, the element $x = (1/n)(1 \otimes 1 - \Sigma_{i=1}^{n-1}(j^i \otimes j^{n-i}))$ satisfies the equations : $xu = ux$ for all u in B[j] ...(1), and $(1/n)(1 - \Sigma j^i j^{n-i}) = 1$ ... (2). For any b in

B, $xb = (1/n)(1\otimes b - \Sigma(j^i \otimes j^{n-i}b)) = (1/n)(1\otimes b - \Sigma(j^i \otimes_\sigma{}^{n-i}(b)j^{n-i})) = (1/n)(1\otimes b - \Sigma(j^i\sigma^{n-i}(b)\otimes j^{n-i})) = (1/n)(1\otimes b - \Sigma(bj^i \otimes j^{n-i}))$ for the tensor product is over B and $\sigma^n = 1$ in G. $bx = (1/n)(b\otimes 1 - \Sigma(bj^i \otimes j^{n-i}))$, so $xb = bx$ for each b in B. $j^n = -1$, so $xj = jx$. Thus $xu = ux$ for all u in B[j]. The second equation is clear. Moreover, by hypothesis, B is Galois over A, so it is separable over A ([3], Proposition 1.2, P. 81). Thus B[j] is separable over A by the transitivity of separable extensions ([5], Proposition 2.5). Therefore B[j] is Azumaya over A.

Further, we claim that B is a maximal commutative subalgebra of B[j] by showing that the commutant of B in B[j] is B. Let $\Sigma b_k j^k$ for b_k in B be an element in B[j] such that $a(\Sigma b_k j^k) = (\Sigma b_k j^k)a$ for each a in B. Then, $\Sigma a b_k j^k = \Sigma b_k \sigma^k(a) j^k$, and so $b_k(a-\sigma^k(a)) = 0$ for each k. Thus Proposition B in Section 2 implies that $b_k = 0$ for each $k \neq 0$. Thus $(B[j])^B = B$.

THEOREM 2.2 If B is Galois over A with a cyclic Galois group G generated by σ of order n invertible in B. Then $B\otimes_A B[j]$ given in Lemma 3.1 is isomorphic with the matrix ring $M_n(B)$ of order n over B.

PROOF. By Proposition A and Lemma 2.1, $B\otimes_A (B[j])^0 \cong \mathrm{Hom}_B(B[j], B[j])$ where $(B[j])^0$ is the opposite algebra of B[j] ([3], Theorem 5.5, P. 64). Since B[j] is a free B - module of rank n, $\mathrm{Hom}_B(B[j], B[j]) \cong M_n(B)$, a matrix algebra over B of order n. But then, taking opposite algebras on both sides, we have $B\otimes_A B[j] \cong (M_n(B))^0 \cong M_n(B)$, where the second isomorphism is the transposition map of matrices. ∎

To show the converse of Theorem 2.2, we start with a lemma.

LEMMA 2.3 Let B be a commutative ring with 1, and G the automorphism group (= {σ}) of order n invertible in B such that $B^G = A$. If $B\otimes_A B[j] \cong M_n(B)$, B[j] is an Azumaya A - algebra.

PROOF. Since B is a commutative A - algebra and $M_n(B)$ is an Azumaya B - algebra, $B\otimes_A B[j]$ ($\cong M_n(B)$) is Azumaya over B ; and so it suffices to show that A is an A - direct summand of B by Corollary 1.10 in [3], P.45. In fact, let Tr be the trace map such that $Tr(b) = \Sigma\, \sigma^k(b)$ for $k = 0,1, \ldots, n-1$. Since $B^G = A$, Tr (b) is in A for all b in B and (1/n) is also in A. Clearly, the imbedding map Im : $A \to B$ has an inverse map (1/n)(Tr). Both Im and (1/n)(Tr) are A - module homomorphisms, so A is an A - direct summand of B. Thus B[j] is an Azumaya A - algebra. ∎

LEMMA 2.4 Let B be a commutative ring with 1, and the automorphism group (= {σ}) of order n invertible in B such that $B^G = A$. Assume each maximal ideal of B is G-invariant. If $B\otimes_A B[j]\cong M_n(B)$ and if B is not Galois over A, then there exist a maximal ideal M of B and an integer $k\geq 1$ such that $B/M\otimes_A B[j^k]$ is a commutative subalgebra of $B/M\otimes_A B[j]$.

PROOF. Since each maximal ideal M of B is G-invariant, jM = Mj ; and so MB[j] is an ideal of B[j]. But B[j] is Azumaya over A by Lemma 3.3, so MB[j] = mB[j] for some ideal m of A by a well known fact for Azumaya algebras. Noting that $\{1, j, \ldots, j^{n-1}\}$ is a basis over B, we have M = mB. Now B is not Galois over A, so there exist a maximal ideal M of B and an automorphism σ^k for some k such that $(b - \sigma^k(b))\in M$ for all b in M ([3], Proposition 1.2, P. 80). Hence $\sigma^k(b) = b + c$ for some c in M. This will imply that $(B/M)\otimes_A B[j^k]$ is a commutative subalgebra in $(B/M)\otimes_A B[j]$. In fact, $\bar{1}\otimes j^k b = \bar{1}\otimes\sigma^k(b)j^k = \bar{1}\otimes(b+c)j^k = \bar{1}\otimes bj^k + \bar{1}\otimes cj^k$. Since M = mB (note that $m = M\cap A$), $c = \Sigma\, c_i c_i'$ for some c_i in m and c_i' in B. Thus $\bar{1}\otimes cj^k = \Sigma(\bar{c}_i\otimes c_i' j^k) = \bar{0}$ in $B/M\otimes_A B[j]$. Therefore, $\bar{1}\otimes j^k b = \bar{1}\otimes bj^k$ for all b in B. This implies that $(B/M)\otimes_A B[j^k]$ is commutative. ∎

Now we show the converse of Theorem 3.2 when the order of σ is a prime integer.

THEOREM 2.5 Let B be a commutative ring with 1 and G the automorphism group (= {σ}) of prime order n invertible in B such that $B^G = A$. If $B \otimes_A B[j] \cong M_n(B)$ then B is Galois over A.

PROOF. Assume that B is not Galois over A, there exists a maximal ideal M of B such that MB[j] is an ideal of B[j]. In fact, Proposition 1.2 in [3] implies the existence of a maximal ideal M of B and an integer q such that b - $\sigma^q(b)$ are in M for all b in B. Hence $\sigma^q(b) = b + c$ for some c in M, and so $\sigma^q(b)$ is in M whenever b is in M. By hypothesis, n is prime, so σ^q generates G. But then $\sigma^{mq}(b)$ is in M for all integers m, and hence M is G - invariant. Thus MB[j] is an ideal of B[j]. Now, Lemma 3.4 (which holds when this particular M is G - invariant) implies that $B/M \otimes_A B[j^q]$ is a commutative subalgebra of $B/M \otimes_A B[j]$. Since n is prime such that $j^n = -1$, $B[j^q] = B[j]$, and so $B/M \otimes_A B[j]$ (= $B/M \otimes_A B[j^q]$) is commutative. On the other hand, $B \otimes_A B[j] \cong M_n(B)$ by hypothesis, so $B/M \otimes_A B[j] \cong M_n(B/M)$ which is an Azumaya algebra over B/M. Thus $B/M \otimes_A B[j^q]$ is never commutative, a contradiction. Therefore B is Galois over A. ∎

The algebra given in Theorem 2.2 is derived from a cyclic Galois extension B over A. Now we give an algebra derived from a non - cyclic Galois extension. Our result is another generalization of the theorem of Parimala and Sridharan. Let B be a commutative ring with 1 and with a non - cyclic automorphism group of order 4 invertible in B, where the group G = (α)(β) such that $\alpha^2 = \beta^2 = 1$, and $B^G = A$.
We define an A - algebra B[i, j, k], where i, j, and k are the usual quaternions such that (1) ib = α(b)i, jb = β(b)j and kb = (αβ)(b)k, (2) B[i, j, k] is a free B - module with a basis {1, i, j, k}, and (3) multiplication is distributive over addition.

THEOREM 2.6 If B is Galois over A with the above Galois group. Then (1) B[i, j, k] is an Azumaya A - algebra. (2) B is a maximal commutative subalgebra of B[i, j, k]. (3) $B \otimes_A B[i, j, k] \cong M_4(B)$, a 4 by 4 matrix algebra over B.

PROOF. Considering B as a subring of B[i, j, k], we claim that B[i, j, k] is a separable ring extension of B. Let $\varepsilon = (1/4)(1\otimes 1 - i\otimes i - j\otimes j - k\otimes k)$ (for 4 is a unit in B). Then $i\varepsilon = (1/4)(i\otimes 1 + 1\otimes i - k\otimes j + j\otimes k)$, and $\varepsilon i = (1/4)(1\otimes i + i\otimes 1 + j\otimes k - k\otimes j)$, where $\otimes$ is over B. Hence $i\varepsilon = \varepsilon i$. Similarly, $j\varepsilon = \varepsilon j$ and $k\varepsilon = \varepsilon k$. Also, for each b in B, $b\varepsilon = (1/4)(b\otimes 1 - bi\otimes i - bj\otimes j - bk\otimes k)$ and $\varepsilon b = (1/4)(1\otimes b - i\otimes ib - j\otimes jb - k\otimes kb)$. Noting that $1\otimes b = b\otimes 1$ (for $\otimes$ is over B), $i\otimes ib = i\otimes\alpha(b)i = i\alpha(b)\otimes i = \alpha^2(b)i\otimes i = bi\otimes i$, $j\otimes jb = bj\otimes j$ and $k\otimes kb = bk\otimes k$, we have that $\varepsilon x = x\varepsilon$ for all x in B[i, j, k]. Also, $(1/4)(1 - i^2 - j^2 - k^2) = 1$. Hence B[i, j, k] is separable over B. But B is Galois over A, so it is separable over A. Thus B[i, j, k] is separable over A by the transitivity of separable ring extensions.

Next, we show that the center of B[i, j, k] is A. Let $x = a_1 + a_2 i + a_3 j + a_4 k$ be an element in the center. Then $bx = xb$ for each b in B. This implies that $a_2(b - \alpha(b)) = 0$, $a_3(b - \beta(b)) = 0$ and $a_4(b - \alpha\beta(b)) = 0$. But B is Galois over A, so $a_2 = a_3 = a_4 = 0$ by Proposition B in Section 2. Thus $a_1 = x$. Also, $a_1 i = ia_1$, $a_1 j = ja_1$ and $a_1 k = ka_1$, so $a_1 = \alpha(a_1)$, $a_1 = \beta(a_1)$ and $a_1 = \alpha\beta(a_1)$. Since $B^G = A$, a_1 is in A. Thus x is in A. Clearly, A is contained in the center, so A is the center.

For part (2), we claim that the commutant of B in B[i, j, k] is B. Let x be an element in the commutant. The proof of part (1) implies that x is in B. Clearly, B is contained in the commutant.

Part (1) and part (2) imply that $B\otimes_A(B[i, j, k])^0 \cong Hom_B(B[i, j, k], B[i, j, k])$ ([3], Theorem 5.5, P. 64), where $(B[i, j, k])^0$ is the opposite algebra of $B[i, j, k]$. Since $B[i, j, k]$ is a free B - module of rank 4, taking opposite algebras on both sides, the proof is completed. ∎

As given in cyclic Galois extensions, we can get a similar fact to Lemma 2.4 with a slight modification of the proof of Lemma 2.4.

THEOREM 2.7 Let B be a commutative ring extension of A with a non - cyclic automorphism group of order 4 (= $(\alpha)(\beta)$) invertible in B such that $B^G = A$. Assume each maximal ideal of B is G-invariant. If $B\otimes_A B[i, j, k] \cong M_4(B)$ and if B is not Galois over A, then there exists a maximal ideal M of B such that one of the following algebras is commutative : $B/M\otimes_A B[i]$, $B/M\otimes_A B[j]$ and $B/M\otimes_A B[k]$.

ACKNOWLEDGMENTS

This work was supported by the Board for Research and Productive Creativity of Bradley University, and I would like to thank Professor Dr. F. Van Oystaeyen for his valuable comments.

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A NONCOMMUTATIVE THEORY FOR PRIMES

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0. INTRODUCTION

There have been some attempts to generalize the theory of valuations (primes) in fields to the case of rings. In [2] D.K. Harrison expounded a theory which enabled him to obtain some number theoretic results for commutative rings. The main objects in this theory are not the so-called "primes" but the valuation pairs introduced by Manis in [3], [4]. The relations between primes and valuation pairs are studied in [2]. Harrison primes in noncommutative rings were studied only in some special cases ; e.g. in [6] primes in matrix rings over locally finite fields are characterized and in [12] some results in finite dimensional algebras over fields are obtained, but only for primes containing a basis for the algebra (i.e. "spanning" primes).

Using a weaker definition of primes, Connell constructed a functor (cfr. [1]), which is a transformation of Spec. Van Oystaeyen studied related primes in the noncommutative case (cfr. [5], [10], [11]) he also obtained some numbertheoretical properties for so-called (semi) restricted primes. However one cannot expect to obtain a valution theory for noncommutative rings in this way since (as mentioned in [8]) the extension theorem for primes does not hold.

In this paper we study, more general primes in noncommutative rings (these include <u>all</u> other given definitions !).
For these primes, the "extension theorem" does hold ; all results of papers cited become special cases of our theory here. We characterize all primes in central simple algebras over arbitrary fields and the result on matrix rings over locally finite fields, cf. [6], is a trivial consequence.

Many thanks to Freddy Van Oystaeyen for some useful suggestions.

1. SOME GENERALITIES ON PRIMES

Let R be an arbitrary ring with unit. We are interested in couples (P,R') satisfying :

DEFINITION 1.1 i) R' is a subring of R

ii) P is a prime ideal in R'

iii) If $xR'y \subset P$ with $x,y \in R$ then $x \in P$ or $y \in P$

(P,R') with these properties is called a <u>prime</u> in R. P is called the <u>kernel</u> and R' the <u>domain</u> of the prime.
We are only interested in nontrivial primes, i.e. $P \neq R'$. This definition generalizes the primes studied in [8], the couples studied there will now be referred to as being complete primes. It is obvious that in case R is commutative both definitions are the same. Some of the properties for completely primes are now being restated for general primes ; the proofs are easy adaptions of the former ones, so we refer to [8] for most of them.
LEMMA 1.2 Let P be an additive subgroup of R_+ which is multiplicatively closed, define

$$R^P = \{r \in P \mid rP \subset P \text{ and } Pr \subset P\}$$

then R^P is a subring of R and P is an ideal in R^P.

PROOF. straightforward. ∎

Recall that a subset S of a ring R is called an m-system iff for $s_1, s_2 \in S$ there is an $x \in R$ such that $s_1 x s_2 \in S$. A prime ideal in a ring is then exactly an ideal which is the complement of an m-system.

We call a subset S of a ring R an m-system for T, with $T \subset R$, iff for $s_1, s_2 \in S$ there is an $x \in T$ such that $s_1 x s_2 \in S$.

PROPOSITION 1.3 If (P,R') is a prime in R then $R \setminus P$ is an m-system for R'. Conversely : If P is an additive subgroup of R_+ which is multiplicatively closed and such that $R \setminus P$ is an m-system for R^P then (P, R^P) is a prime in R.

PROOF. The fact that $R \setminus P$ is an m-system for R' follows from the third condition in Definition 1.1.

In view of Lemma 1.2 the converse is also obvious because $R^P \setminus P$ is an m-system (for R^P) since $R \setminus P$ is an m-system for R^P and this again is equivalent to condition iii) in Definition 1.1. ∎

REMARK 1.4 Let (P,R') be a prime in R. Then also (P, R^P) is a prime in R and $R' \subset R^P$. So if P is a kernel of prime in R then R^P is the maximal domain for it. If no domain is mentioned for a prime P of R then it is understood that (P, R^P) is considered.

PROPOSITION 1.5 Let P be a prime in R then there is a prime ideal P^0 of R contained in P. P^0 is the maximal ideal of R in P.

PROOF. One easily verifies that $P^0 = \{x \in R \mid R\, x\, R \subset P\}$ is the desired ideal, cfr. Proposition 1.4 in [8]. ∎

Notation and terminology.

Denote by Prim R the set of all kernels of primes in R, i.e. :

Prim R = {R | P is a prime in R}. Clearly Spec R $\subset$ Prim R and if we endow Prim R with the (Zariski) topology defined in [8], then it follows from 1.5 that Spec R is dense in Prim R. However in the noncommutative case Prim R is not necessarily a functor from Rings $\to$ Top.

Let R be a subring of a ring A then we say that a prime (Q,A') in A is lying over a prime (P,R') in R iff $Q \cap R' = P$ and $A' \cap R \supset R'$. If $\pi = (P,R')$ is a fixed prime in R we define :

$\pi\text{-Prim}_R(A)$ = {Q | Q is a π-prime in A} = {Q | Q is a prime in A lying over π}.

Clearly if P is a prime in R and Q a prime in A, then Q lies over P whenever $Q \cap R = P$ and $R^P \subset A^Q$, i.e. Q is a left and right R^P - module, in this case we have $A^Q \cap R = R^P$.

COROLLARY 1.6 Let $R \subset A$, (P,R') a fixed prime in R then $Q \subset A$ is a P-prime in A iff:

1) Q is a left and right R' - mdule

2) $Q \cap R = P$

3) $A \setminus Q$ is an m-system for A^Q.

PROOF. This follows easily from the definitions an Proposition 1.3. ∎

DEFINITION 1.7 A prime (P,R') is called special iff for all $x \in R \setminus R'$ there is a $\lambda \in P$ such that $\lambda x \in R'$. A prime (P,R') is called semi-restricted iff for all $x \in R \setminus R'$ there is a $\lambda \in P$ such that $\lambda x \in R' \setminus P$. If $R \subset A$, $\pi = (P,R')$ a fixed prime in R then a π-prime Q in A is called π-special, and π-semirestricted iff one may take λ in P.

Note that the correct terminology would be left-special, left-semi-restricted butsince we do not use these conditions on the right the introduced terminology will do. Prime ideals in a ring are obviously semirestricted primes in it.

2. EXISTENCE AND EXTENSION OF PRIMES

In [8] completely primes in algebras over rings were studied. It is mentioned there that in the noncommutative case the existence of completely primes is not guaranteed (cfr. p. 19).

The main example is the following.

Take $A = M_n(\Delta)$, Δ a skewfield, $n \neq 1$. Suppose (P,A') is a completely prime in A, let e_{ij} be matrix units in A. Then for $i \neq j$, $e_{ij}e_{ij} = 0 \in P$ so $e_{ij} \in P$ and $e_{ii} = e_{ij} \cdot e_{ji}$ for $j \neq i$ so $e_{ii} \in P$ but since $1 = \sum_i e_{ii}$, we have $1 \in P$ which means $P = A'$.

In this section we shall prove that with our defintions, extension of primes from the groundring to the algebra considered is always possible. As an example, all primes in a matrix algebra over a field will be constructed.

Throughout we consider algebras A over a groundring R with unit, for simplicity sake we assume $R \subset A$.

If S, R are subsets of A then

$S<T>$ stands for $\{x \in A \mid x = \Sigma s_i t_i,\ s_i \in S,\ t_i \in \bar{T}\}$

where $\bar{T}$ is the multiplicative closed set generated by T.

THEOREM 2.1 Let A be an R-algebra, $\pi = (p,R')$ a fixed semirestricted prime in R. Let $B \subset A$ and $M \subset B$ satisfy the following properties ;

i) $Bp \subset p<B>$ and $BR' \subset R'<B>$

ii) $p<B> \cap R \subset p$

iii) M is a m-system for B

iv) $R' \setminus p \subset M$

v) $M \cap p<B> = \phi$

Then there is a π-prime P in A such that $P \cap M = \phi$, $BP \subset P$, $PB \subset P$ and $R \cap P = p$.

PROOF. Let $S = \{Q \mid Q$ a left and right R' submodule of A which is multiplicatively closed, such that $M \cap Q = \phi$, $BQ \subset Q$, $QB \subset Q$ and $R \cap Q = p\}$.
A straightforward computation (cfr. [8], Theorem 3.7) shows that
$P_0 = \{x \in A \mid xR'<B> \subset p<B>\}$ is an element of S.
In view of Corollary 1.6 we only have to prove that $A \setminus P$ is an m-system for A^p.

Let $x, y \in A \setminus P$ and suppose $xA^p y \subset P$. Define :
$P_{x,i} = \{z \mid z$ is a finite sum of elements of the form $\alpha_1 x \alpha_2 x \ldots \alpha_{i-1} x \alpha_i, \alpha_j \in A^p\}$ and $P_{x,0} = P$. Let $P_x = \sum_{i=0}^{\infty} P_{x,i}$, then P_x is clearly a multiplicatively closed left and right R' - module containing P and x. Define P_y in the same way.

Since $B \subset A^p$ we have $BP_x \subset P_x$ and $P_x B \subset P_x$, also $BP_y \subset P_y$ and $P_y B \subset P_y$. But P was maximal in S so $P_x(P_y)$ satisfies $P_x \cap M \neq \phi$ or $P_x \cap R \neq p$ ($P_y \cap M \neq \phi$ or $P_y \cap R \neq p$).

If $P_x \cap R \neq p$ take $r \in P_x \cap (R \setminus p)$, i.e. there is a $\lambda \neq 0 \in R'$ such that $\lambda r \in R' \setminus p \subset M$ but also $\lambda r \in P_x$; so $P_x \cap M \neq \phi$. Analoguously we deduce $P_y \cap M \neq \phi$.
We now show that $P_x \cap M \neq \phi$ and $P_y \cap M \neq \phi$ leads to a contradiction. Choose $m_x \in P_x \cap M$ and $m_y \in P_y \cap M$, i.e. $m_x = f_0(x) + f_1(x) + \ldots + f_n(x)$ with f_i $f_i(x) \in P_{x,i}$, $m_y = g_0(y) + g_1(y) + \ldots + g_t(y)$ with $g_j(y) \in P_{y,j}$, so that n and t are minimal. Since $xA^p y \subset P$ we have $P_{x,i} P_{y,j} \subset P_{x,i-j}$ for all $i \geq j \geq 1$.

Let $n \geq t$ and let s B be such that $m_y s m_x \in M$ then $(m_y - g_0(y)) s m_x = h \in \sum_{k=0}^{n-t} P_{x,k}$, so $m_y s m_x = g_0(y) s m_x + h$ and $g_0(y) s m_x \in P$ (since $M \subset B \subset A^p$), contradicting the minimality of n. ∎

COROLLARY 2.2 Take A, R, π as in the theorem, then : $\pi\text{-Prim}_R A \neq \phi$.

PROOF. Take $B = M = R' \setminus p$, then i) to v) are trivially fulfilled. ∎

REMARK 2.3 Let us discuss the consequences of the theorem in some special cases :

1) If R is a subring of the center of A then the first condition may be omitted.
2) If R is a field, R' has to be a valuation ring with maximal ideal p (cfr. [8], p. 8) and this is always semirestricted.
3) If we only want to prove the existence of a prime in A then the conditions i) to v) may be replaced by $1 \in M$ and $0 \notin M$ (cfr. [8], theorem 2.2).

Characterisation of primes in matrixrings over fields.

Let A be a matrixring over a field K, then A is isomorphic to an endomorphism ring of a finite dimensional K-vectorspace V. We will describe all primes lying over a fixed valuation ring O_K in K. In case K is the center of A all primes restrict to a valuation ring of K, so we will have a characterisation of the primes in A. Since the commutativity of K shall not be used in the proofs, primes in a matrixring over a skewfield D which restrict to a valuation ring in D, i.e. primes in simple Artinian algebras, may be characterized in the same way.

PROPOSITION 2.4 Let $A = \mathrm{End}_K V$ and (M_K, O_K) a valuation in K. If L,W are O_K - submodules of V, $W \subsetneq L$ and $M_K L \subset W$ then $P = \{\alpha \in A \mid \alpha(L) \subset W\}$ is a prime in A lying over M_K.

PROOF. Clearly P is an O_K - submodule of A which is multiplicatively closed. Suppose $k \in P \cap K$, $k \notin M_K$ then $k^{-1} \in O_K$ and we have $L = k^{-1}kL \subset k^{-1}W \subset W$, a contradiction. So $P \cap K = M_K (M_K \subset P$ by hypothesis). Using Corollary 1.6 it remains to prove that $A \setminus P$ is an m-system for A^P. Take $\alpha, \beta \in A \setminus P$, then there are elements x and y in L such that $\alpha(x) \notin W$ and $\beta(y) \notin W$. Consider the following homomorphism ; $\pi : V \to V : k\alpha(x) \to ky$ for all $k \in K$

$z \to 0$ for all $x \in V \setminus K\alpha(x)$

if $v \in K\alpha(x) \cap L$, $v = k\alpha(x)$, $k \notin O_K$ then $k^{-1}\, k\alpha(x) \in M_K L \subset W$, a contradiction. So $K\alpha(x) \cap L = O_K\alpha(x)$, this yields $\pi(L) \subset O_K y \subset L$. A similar argument leads to $K\alpha(x) \cap W = M_K\alpha(x)$, which yields $\pi(W) \subset W$. It is now obvious that $\pi P \subset P$, and $P\pi \subset P$, i.e. $\pi \in A^P$. But $\beta\pi\alpha(x) = \beta(y) \notin W$, so $\beta\pi\alpha \notin P$ since $x \in L$. ∎

PROPOSITION 2.5 Let P be as in Proposition 2.6, then $A^P = \{\alpha \in A \mid \alpha(L) \subset L$ and $\alpha(W) \subset W\}$.

PROOF. Suppose $\alpha(W) \not\subset W$ and $\alpha P \subset P$ and $P\alpha \subset P$. There is an $x \in W$ such that $\alpha(x) \notin W$, consider the homomorphism

$\pi : V \to V : ky \to kx$ for all $k \in K$, $y \in L \setminus W$

$z \to 0$ for all $z \in V \setminus Ky$

then $L \cap Ky = O_K y$ (cfr. Proof of Proposition 2.6), so $\pi(L) \subset W$ or $\pi \in P$, but $\alpha\pi(y) = \alpha(x) \notin W$, contradiction.

Also, $\alpha(L) \not\subset L$ leads in a similar way to a contradiction. Conversely, the other inclusion is trivial. ∎

LEMMA 2.6 Let P be a prime in A. Then there is an element $x \in V$ such that $v \notin Pv$.

(Px is the set of all images of x under the action of elements of P).

PROOF. Let $\alpha \in A$ and consider the subspace $\mathrm{Ker}(1+\alpha)$ of V. Choose $\beta \in P$ so that $\dim_K \mathrm{Ker}(1+\beta) \geq \dim_K \mathrm{Ker}(1+\alpha)$, $\forall\, \alpha \in P$. If $\mathrm{Ker}(1+\beta) = V$ then $1 + \beta = 0$ this would yield $1 \in P$ which is impossible. Take $x \in V \setminus \mathrm{Ker}(1+\beta)$ and put $v = (1+\beta)x$. Suppose that $v \in Pv$. Then there is an element $\gamma \in P$ such that $\gamma(v) = v$. Consider $(1-\gamma)(1+\beta)$, the kernel of this morphism contains $\mathrm{Ker}(1+\beta)$ and x, so $\dim_K \mathrm{Ker}(1+\beta-\gamma-\gamma\beta) > \dim_K \mathrm{Ker}(1+\beta)$. This contradicts the fact that $\beta-\gamma-\gamma\beta \in P$ and $\dim_K \mathrm{Ker}(1+\beta) \geq \dim_K \mathrm{Ker}(1+\alpha)$, $\forall\, \alpha \in P$. ∎

LEMMA 2.7 Let (M_K, O_K) be a fixed valuation ring in K, (P, A^P) a prime in A lying over (M_K, O_K). Then P is a maximal ideal in A^P.

PROOF. Consider A^P/P as an O_K/M_K - vectorspace. If $\{v_1, \ldots, v_n\}$ is k - independent, $O_K/M_K = k$, then any set of representatives for $\{v_1, \ldots, v_n\}$ in A^P is K - independent (cfr. Proposition 1.1 in [9]). Therefore $[A^P/P : k] \leq [A : K] < \infty$, so since k is a field A^P/P is simple artinian, this entails that P is a maximal ideal. ∎

If P is a prime in A, and $v \in V$ such that $v \notin Pv$ (cfr. Lemma 2.6) then put

$$L = \{x \in V \mid \alpha(x) \in Pv, \ \forall\ \alpha \in P\}.$$

It is obvious that Pv (=W) and L are O_K - modules such that $Pv \subsetneq L$, $M_K L \subset Pv$.

LEMMA 2.8 Let P,L,W be as above then :

1) $\forall\ \alpha \in A^P : \alpha(L) \subset L$ and $\alpha(W) \subset W$.

2) $\beta(L) \subset M_K L$ implies $\beta \in P$.

PROOF. 1) Suppose $\alpha \in A^P$ then $P\alpha(L) \subset W$ but this yields $\alpha(L) \subset L$ by definition of L. Now $\alpha P \subset P$ yields $\alpha Pv \subset Pv$.

2) Take $\beta \in A$ such that $\beta(L) \subset M_K L$. In view of the first part of this lemma and the K - linearity of the maps we have $A^P \beta A^P L \subset M_K L$. If $A^P \beta A^P \not\subset P$ (otherwise the lemma is proven) then $A^P \beta A^P + P = A^P$ by the maximality of P (Lemma 2.7), so there is a $\gamma \in A^P \beta A^P$ such that $\gamma - 1 \in P$. We then have $\gamma L \subset M_K L$ implying $\gamma(v) = v + \pi(v) \in M_K L \subset W$, with $\pi \in P$. Therefore $\pi(v) \in W$, $\gamma(v) \in W$ yield $v \in W$, contradiction. ∎

We now are able to prove that (with the above notations) $P = \{\alpha \in A \mid \alpha(L) \subset W\}$. Together with Proposition 2.4 this characterizes all prime in A.

PROPOSITION 2.9 Let P,L,W be as in Lemma 2.8. Then $P = \{\alpha \in A \mid \alpha(L) \subset W\}$.

PROOF. It is obvious that $P \subset \{\alpha \in A \mid \alpha(L) \subset W\}$. Take $\xi \in \{\alpha \in A \mid \alpha(L) \subset W\}$.
Consider : $\pi : V \to V : kv \to kv$ for all $k \in K$

$$z \to 0 \quad \text{for all } z \in V \setminus kv.$$

Like in Proposition 2.6 a straightforward computation shows that $Kv \cap L = O_K v$ and $Kv \cap W = M_K v$. Therefore :

$$\pi A^p \xi A^p \pi(L) \subset \pi A^p \xi A^p(L) \subset \pi A^p \xi(L) \subset \pi A^p(W) \subset \pi (W) \subset M_K v \subset M_K 1$$

(Lemma 2.8, 1)). Lemma 2.8, 2) yields : $\pi A^p \xi A^p \pi \subset P$ and since $\pi \notin P$ we must have $\xi A^p \pi \subset P$; for the same reason : $\xi \in P$. ∎

COROLLARY 2.10 The primes in A which restrict to the valuation pair (O,K) in K are given by the sets $\{\alpha \in A \mid \alpha(L) \subset W$ where L and W are K - subspaces of V, $W \subsetneq L\}$

This corollary reestablishes the result of [6]. Here maximal primes as defined by Harrison are considered in matrixrings over locally finite fields only.

3. THE RELATION WITH OTHER THEORIES OF PRIMES

In this section the relationship with the theory of Harrison primes, [2], [3], [4], [6] and [12] is described. We will reestablish the main results for our generalized primes !

DEFINITION 3.1 A subset P of a ring R is called an Harrison prime (H - prime) iff it is maximal with respect to the following properties ; it is closed under addition and multiplication and neither containing -1 nor 1.

LEMMA 3.2 Let P be a H - prime in R it is an additive subgroup of R and $x Py \subset P$ and $xy \subset P$ yields $x \in P$ or $y \in P$.

PROOF. cfr. [12], Lemma 1.1. ∎

PROPOSITION 3.3 If P is a H - prime in R then it is a prime in R.

PROOF. Suppose $x\, R^P y \subset P$, then clearly $x\, P\, y \subset P$ and $xy \subset P$ this yields $x \in P$ or $y \in P$. ∎

REMARK 3.4 1) The proposition may also be derived directly from the Extension Theorem 2.1.

2) The converse is obviously not true, since prime ideals are prime but they are H - primes iff they are maximal

3) From [7] it follows easily that all primes are H - primes in case R is a global ring (i.e. a subring of a global number field). Note that in section 3 of [12] Warner imposes a supplementary condition (i.e. "spanning") on H - primes in algebras which is necessary for studying extensions of H - primes. Our definition of special primes has the advantage that it is not linked to the finite dimensionality of the algebra. Recall :

DEFINITION 3.5 Let A be a finite dimensional algebra over a field K, then a H - prime P is called spanning if it contains a K - basis of A, i.e. $KP = A$.

PROPOSITION 3.6 A a K - algebra, K a field and $[A : K] < \infty$. A prime P in A contains a K - basis iff it is π - special, where $\pi = (M_K, O_K)$ is the underlying valuation in K.

PROOF. Let P be a π - special prime in A and let $\{u_1, \dots, u_n\}$ be a K - basis for A. Then there are elements $\lambda_1, \dots, \lambda_n$ in K such that $\{\lambda_1 u_1, \dots, \lambda_n u_n\} \subset A^P$, multiplying $\lambda_i u_i$ with $\lambda \in M_K \setminus \{0\}$ we get $\{\lambda\lambda_1 u_1, \dots, \lambda\lambda_n u_n\} \subset P$, which is still a K - basis since K is a field. Conversely, let $\{v_1, \dots, v_n\} \subset P$ a K - basis for A. Take $x \in A \setminus A^P$ then $x = \sum_{i=1}^{n} \lambda_i v_i$ and there is a λ_j not in O_K so $\lambda_j^{-1} \in M_K$. Consider

$\lambda_j^{-1}x = \sum_{i=1}^{n} \lambda_j^{-1}\lambda_i v_i$ if one of the $\lambda_j^{-1}\lambda_i$ is not in O_K we repeat the above, i.e. multiply with the inverse, which is in M_K, of that element. This can be done untill every coëfficient is in O_K. So we found an element r in M_K such that $rx \in A^P$. ∎

REMARK 3.7 The above proposition still holds if A is a finitely generated algebra which is a free module over a commutative domain R and if one considers primes in A lying over special primes of K.

The commutative interpretation of Warner's paper [12], is the valuation theory develloped by Manis in [3] and [4].

DEFINITION 3.8 Consider pairs (Q,S) where S is a subring of R and Q a prime ideal in S. These may be partially ordered by defining

$(Q,S) \leq (Q',S')$ iff $S \subset S'$ and $Q' \cap S = Q$.

The maximal pairs with respect to this order are called <u>valuation pairs</u>.

We connect this theory to ours.

In what follows R is a commutative ring.

PROPOSITION 3.9 The valuation pairs of a ring R are exactly the semi-restricted primes in R.

<u>PROOF</u>. Theorem 2.1, and [3], Proposition 1. ∎

REMARK 3.10 From Theorem 2.1 follows now immediately that semirestricted primes extend to semirestricted primes, since every pair (Q,S) is contained in a maximal one.

With every valuation pair in R there is a valuation associated i.e. a map $v : R \to \Gamma$, Γ an ordered group, such that

i) $v(xy) = v(x)v(y) \quad \forall\ x,y \in R$

ii) $v(x+y) \leq \max\{v(x),v(y)\}\ \forall\ x,y \in R$.

This valuation is defined as follows :

$$v(x) = \{z \in R \mid [P : z]_R = [P : x]_R\} \text{ and}$$
$$v(x) \leq v(y) \text{ iff } v(x) \supset v(y)$$
$$v(x)v(y) := v(xy).$$

Note that P^0 as defined in Proposition 1.5 is equal to $v^{-1}(v(0))$.
Let A be a commutative R - algebra then one says that a valuation v on R extends to a valuation w of A iff there is an order preserving homomorphism ϕ of the respective ordered groups such that $w \mid_R = \phi \circ v$. In terms of primes this is equivalent to $P_v^0 \subset P_w \subset P_v$ (cfr. [4], Proposition 4). By this one can recover the extension theorem for valuation from our Theorem 2.1. In terms of primes this theorem says :
PROPOSITION 3.11 (cfr. [4], Proposition 5). Let A be an R - algebra (p,R^p) a semirestricted prime in R then there is a semirestricted prime (P,A^P) in A lying over (p,R^p). (cfr. Remark 3.10). The valuation induced by (p,R^p) extends to the one induced by (P, A^P) iff $P^0 \cap R = p^0$.

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RINGS ISOMORPHIC WITH ALL PROPER FACTOR-RINGS

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1. INTRODUCTION

F.A. Szász has raised the following problem (in a letter) :
Determine the structure of all (associative) rings R such that

(1) $R/I \cong R$

for any ideal $I \neq R$. The class of rings satisfying (1) will be called K.
Clearly all simple (prime) rings are in K.
An example of a non-simple ring which belongs to K is the ring $[Z(p^{\infty})]^{\circ}$, the zero-ring with as additive group the quasicyclic group of type p, p a prime number. In section 5 we will construct a non-simple non-zero-ring in K (Theorem 10).
A hereditary class is a class C of rings such that for any ring A and any ideal I in A one has : $A \in C \rightarrow I \in C$.
K is not a hereditary class, since the ideal $[Z(p^2)]^{\circ}$ of $[Z(p^{\infty})]^{\circ}$ does not belong to K. One has : $[Z(p^2)]^{\circ} / [Z(p)]^{\circ} \cong [Z(p)]^{\circ} \not\cong [Z(p^2)]^{\circ}$. The class K is obviously homomorphically closed.
A class C has the extension property if I R, $I \in C$, $R/I \in C$ imply $R \in C$.

Again, K does not have the extension property since

$[Z(p)]^\circ \in K$, $[Z(p)]^\circ \quad [Z(p^2)]^\circ$, $[Z(p^2)]^\circ / [Z(p)]^\circ \in K$ but $[Z(p^2)]^\circ \notin K$.

Hence the class K is neither a radical class nor a semisimple class. We will show that, aside from one exceptional, case, (the ring $[Z(p^\infty)]^\circ$), the additive group R^+ of every ring $R \in K$ is either a divisible torsion-free group or a reduced p-group for some prime p (Corollary 3). We also show that if $R \in K$ with $R^2 \neq 0$ then R is a prime, hereditary idempotent ring (Theorem 1).

For any ring $R \in K$ the ideals of R form a well-ordered set (Lemma 5). This is a crucial property and it enables us to prove one of our main results : Every ring in K is a strong chain ring (Theorem 6). Here a ring R will be called a strong chain ring if it satisfies the following two conditions :

(2) For some ordinal γ the set of all ideals of R can be written $\{H_\alpha\}_{0 \leq \alpha \leq \gamma}$, where $H_\alpha \subset H_\beta$ if and only if $0 \leq \alpha \leq \beta \leq \gamma$, and

(3) For all $\alpha < \gamma$ we have $H_{\alpha+1}/H_\alpha \cong H$ for some ring H which is either simple or a prime order zero-ring.

From this definition it follows that $H_0 = 0$, $H_1 \cong H$, $H_\gamma = R$ and that for β any limit ordinal $H_\beta = \bigcup_{\alpha < \beta} H_\alpha$.

If $R = H_\gamma \in K$ then one might ask what kind of an ordinal γ must be. In order to state this properly we define :

An ordinal γ is called a prime component ordinal or a prime component if $\beta + \gamma = \gamma$ for all $\beta < \gamma$ (see [6], p. 282).

Another main result is now :

Let $R \in K$ so the set of all ideals of R can be written $\{H_\alpha\}_{0 \leq \alpha \leq \gamma}$ for some ordinal γ. If $0 \leq \gamma$ then $H_0 \in K$ if and only if 0 is a prime component (Theorem 13).

2. PROPERTIES OF THE RINGS IN K.

a) Any ring $R \in K$ is subdirectly irreducible.

PROOF. Let $R \in K$ and suppose $R \cong \sum_{\text{subdirect}} R_i$, where the R_i are subdirectly irreducible. Then R contains a class of ideals $\{B_i\}$ such that $\cap B_i = 0$ and $R/B_i \cong R_i$ for all i. Since $\cap B_i = 0$ at least one of the $B_i \neq R$ and $R/B_i \cong R$ implies $R \cong R_i$, so R is subdirectly irreducible.

b) Any ring $R \in K$ is unequivocal, i.e. for any radical Q, R is either Q-radical or Q-semi-simple.

PROOF. Let $R \in K$ and suppose Q is an arbitrary radical (in the Kurosh-Amitsur sense). Let $Q(R) \neq R$ so R is not a Q-radical ring.
Then $R/Q(R) \cong R$ implies that R is Q-semi-simple. K is a proper subclass of the class of all unequivocal rings since K is, for instance, homomorphically closed but the class of unequivocal rings is not.
Also $[Z(\infty)]^\circ$, the zero-ring on an infinite cyclic additive groups, is unequivocal ([1]), p. 682) but $[Z(\infty)]^\circ \notin K$.

c) If $R \in K$ then either $R^2 = 0$ or $R^2 = R$.

PROOF. Assume that $R^2 \neq 0$. If now $R^2 \neq R$ then $R/R^2 \cong R$ implies that $R^2 = 0$, which is not the case. So $R^2 = R$.

d) A nilpotent ring R is in K if and only if R is a zero-ring, i.e. $R^2 = 0$.

PROOF. Let R be a non-zero nilpotent ring in K. Then $R^2 \neq R$. By (c), $R^2 = 0$.

e) A zero-ring is in K if and only if either $R^+ \cong Z(p^\infty)$(quasi-cyclic group of type p) or $R^+ \cong Z(p)$ (cyclic group of order p), where p is a prime number.

PROOF. From (a) we get that the zero-rings in K are exactly the subdirectly irreducible ones. So a zero-ring $R \in K$ if and only if R^+ is a subdirectly irreducible abelian group. This means : $R \in K$ if and only if $R^+ \cong Z(p^\infty)$ or $R^+ \cong Z(p)$.

REMARK.

This result was also obtained by Szélpál [7] for abelian groups.

f) The only rings with unity in K are simple rings.

PROOF. Let $R \in K$ have a unity. Then R has a maximal ideal M. From $R \cong R/M$ we get that R is simple. This also shows that if there exists a (non-zero) non-simple ring R in K then R cannot have maximal ideals.

g) All non-commutative rings in K are idempotent, i.e. $R^2 = R$.

PROOF. Obvious from (c).

h) If $R \in K$ and $R^2 = R$ then $Ann_r(R) = 0$ and $Ann_l(R) = 0$.

PROOF. $Ann_r(R) = \{x \in R \mid Rx = 0\}$ is an ideal in R. Since $R^2 = R$ we have $R^2 \neq 0$ so $Ann_r(R) \neq R$. Then $R/Ann_r(R) \cong R$.
Let $\overline{R} = R/Ann_r(R)$ and suppose $\overline{x} \in Ann_r(R)$. Now $Rx \subseteq Ann_r(R)$ implies $R^2x = 0$. But $R^2 = R$ so $Rx = 0$, which means $x \in Ann_r(R)$ or $\overline{x} = \overline{0}$. Then $Ann_r(\overline{R}) = \overline{0}$ and since $\overline{R} \cong R$ we get that $Ann_r(R) = 0$. Similarly $Ann_l(R) = 0$.

3. A PARTITION OF THE CLASS K.

Divinsky has studied the structure of unequivocal rings. Theorem 4, p. 680, of his paper [1] reads :

There are four kinds of unequivocal rings :

(i) divisible torsion-free,

(ii) reduced torsion-free,

(iii) divisible p-rings,

(iv) reduced p-rings.

Here a ring R is said to be "reduced" if its additive group R^+ is reduced (no divisible subgroups) and similarly R is divisible, torsion-free or a p-ring if R^+ is divisible, torsion-free or a p-group resp.

We now investigate whether all of these four classes contain rings from K.

We first show

THEOREM 1. If $R \in K$ and $R^2 = R$ then R is a prime hereditarily idempotent ring, i.e. for any ideal I in R we have $I^2 = I$.

PROOF. Let H be the heart of R. We claim that $H^2 \neq 0$ since if $H^2 = 0$ the ring R would have a non-zero nilpotent, hence locally nilpotent, ideal and so being unequivocal, $R \in L$ where L is the Levitzki locally nilpotent radical. Now let $0 \neq x \in H$ then since $Ann_r(R) = 0$ and $Ann_\ell(R) = 0$ we have $Rx \neq 0$ so $RxR \neq 0$. But then $RxR \subseteq H \subseteq RxR$ is an ideal in R so $H = RxR$. This implies that $0 \neq x = \Sigma u_i x v_i$ for some finite set $\{u_i, v_i\}$ and this is impossible in a locally nilpotent ring. Thus $H^2 \neq 0$ and if A,B are non-zero ideals of R then $0 \neq H^2 \subseteq AB$ so R is prime. In particular R has no non-zero nilpotent ideals so if I is an ideal in R with $I \neq R$ then $0 \neq R/I^2 \cong R$ and R/I^2 has the nilpotent ideal I/I^2. Thus $I/I^2 = 0$ that is $I^2 = I$ for all ideals I of R.

PROPOSITION 2. Let $R \in K$, then

A) R is not a reduced torsion-free ring, and

B) if R is a divisible p-ring, then $R^2 = 0$.

PROOF. Let $R \in K$ be a reduced torsion-free ring. Since R^+ is reduced there is a prime number p such that $pR \neq R$. Clearly pR is an ideal in R and as $R \in K$ we get $R/pR \cong R$. However R/pR is a p-ring whereas R is supposed to be torsion-free. This contradiction shows that the class K does not contain any reduced torsion-free ring and (A) is established.

Next suppose that R is a divisible p-ring. We claim that every divisible p-ring is a zero-ring. Take $x,y \in R$. Since R^+ is a p-group, $p^n x = 0$ for some power of p. Now $y = p^n z$ for some $z \in R$, since R^+ is divisible. Hence $xy = x(p^n z) = (p^n x)z = 0$. Requiring that $R \in K$ implies that $R \cong [Z(p^\infty)]^\circ$ by (e).

COROLLARY 3. If $R \in K$ and $R^2 = R$ then either R is a divisible torsion-free ring or R is reduced p-ring.

PROOF. Obvious from Proposition 2.

So the only interesting classes in K are

(i) divisible torsion-free,

(iv) reduced p-rings.

Defining the following subclasses of K :

$K_1 := \{R \in K \mid R^+$ is divisible torsion-free$\}$

$K_2 := \{R \in K \mid R^+$ is a reduced p-group$\}$

$K_3 := \{R \in K \mid R^+$ is a divisible p-group$\}$,

we now get a partition of K :

$K = K_1 \cup K_2 \cup K_3$.

The simple rings in K are contained in K_1 if they have characteristic zero and in K_2 if they have characteristic p. The prime order zero-rings are in K_2, the zero-rings in K are contained in the classes K_2 and K_3. K_3 consists entirely of zero-rings, whereas K_1 contains only prime rings.

COROLLARY 4. The only commutative rings in K are zero-rings or fields.

PROOF. Let R be a commutative ring in K. Then either $R^2 = 0$ or $R^2 = R((c))$. If $R^2 = 0$ then either $R^+ \cong Z(p^\infty)$ or $R^+ \cong Z(p)((e))$. In the first case $R \cong [Z(p^\infty)]^\circ$ ($\in K_3$), in the second case $R \cong [Z(p)]^\circ$ ($\in K_2$). Now assume $R^2 = R$. Then R is a prime hereditarily idempotent ring (Theorem 1) and let H be the heart of R. Since $H^2 = H$ and H is a simple commutative ring it follows that H is a field. If R is not simple $H \neq R$. But then H has a unity implies that H is a proper direct summand of R. This is impossible so R must be simple. Consequently $R = H$ is a field.

4. THE STRUCTURE THEOREM

In order to find a non-simple, non-nilpotent ring in K we first investigate the idealstructure of the rings in K. The crucial point is

LEMMA 5. For any ring $R \in K$ the ideals of R form a well-ordered set.

PROOF. If $R^2 = 0$ we know that either $R^+ \cong Z(p^\infty)$ or $R^+ \cong Z(p)$ and the lemma follows with either $R \cong [Z(p^\infty)^\circ]$ or $R \cong [Z(p)]^\circ$. Otherwise $R^2 = R((c))$ and R is hereditarily idempotent prime ring (Theorem 1). Let H be the heart of R, ther H is a simple ring. First we show that the ideals of R form a totally ordered set. For any ideal $I \neq R$ in R we have $R \cong R/I$, so any ideal $I \neq R$ is a prime ideal. Then suppose I_1 and I_2 are two proper ideals of R. Since $I_1 \cap I_2$ is a prime ideal of R, $I_1 I_2 \subset I_1 \cap I_2$ implies either $I_1 \subseteq I_1 \cap I_2 \subseteq I_2$ or $I_2 \subset I_1 \cap I_2 \subset I_1$. Thus the ideals of R form a totally ordered set. Now to show well-ordering let S be any non-empty subset of the ideals of R with $A = \bigcap_{I \in S} I \neq R$. (If $A = R$ then of course $S = \{R\}$ has a least element R). Then $R/A \cong R$ so there exists $B/A \cong H$. Thus there exists some $I \in S$ such that $B \not\subseteq I$. Since the ideals of R are totally ordered it follows that $I \subset B$. But $A \subseteq I$ and since B/A is simple it follows that $A = I \in S$ and so A is the least element of S.

THEOREM 6. Every ring in K is a strong chain ring.

PROOF. Again if $R \in K$ and $R^2 = 0$, i.e. either $R \cong [Z(p^\infty)]^\circ$ or $R \cong [Z(p)]^\circ$, the ring R satisfies (2) and (3) of the definition of strong chain ring of the introduction. The rings H of (3) is now a prime order zero-ring. Then let $R \in K$ and $R^2 = R$. Since the ideals of R form a well-ordered set (lemma 5) it is clear that there exists an ordinal γ such that the set of all ideals of R can be written $\{H_\alpha\}_{0 \leq \alpha \leq \gamma}$, where $H_\alpha \subseteq H_\beta$ if and only if $0 \leq \alpha \leq \beta \leq \gamma$. So R satisfies (2) Now let $\alpha < \gamma$. Then $H_\alpha \neq H_\gamma = R$ so $R/H_\alpha \cong R$.

Also $H_{\alpha+1}/H_\alpha$ is minimal in R/H_α , hence $H_{\alpha+1}/H_\alpha \cong H$, where H is the heart of R, for all ordinals $\alpha < \gamma$. As R is a prime ring, H is a simple ring. Thus R satisfies (3), hence R is a strong chain ring.

REMARK 1. It now follows that K_1 is contained in the class of all strong chain rings with characteristic 0 and that $K_2 \cup K_3$ is contained in the class of all strong chain rings with characteristic p for some prime p.

REMARK 2. Whether the converse of Theorem 6 is true we do not know. Clearly γ must be a limit ordinal unless $\gamma = 1$ and $R = H_1$. In the final section we will show that γ must be a prime component, (Theorem 13). Also note that if $R \cong [Z(p^\infty)]^\circ$ for some p it follows that $R = H_\omega$ and $\gamma = \omega$.

5. A NON-SIMPLE, NON-NILPOTENT RING IN K

In the next lemma we give sufficient conditions for the ideals of a ring R in order that $R \in K$.

LEMMA 7. Let R be a ring which is a sum of a countably infinite set of ideals H_i, i = 0,1,2, ... with $H_0 = 0$ and where the H_i are the only ideals of R. Also assume that $H = H_1$ is simple and that $H_i \subseteq H_j$ if and only if $0 \le i \le j$.

Let there exist a set of isomorphisms $\phi_i : H_{i+1}/H \to H$ for all $i \ge 0$ which are "compatible", that is $\phi_i\big|_{H_i/H} = \phi_{i-1}$ for all $i \ge 1$.

Then for all n, $R/H_n \cong R$, so $R \in K$.

PROOF. Let $\phi : R \to R/H$ be the natural homomorphism and write $\phi\big|_{H_{i+1}} \cdot \phi_i = \alpha_i$ for all $i \ge 0$. Then α_i maps H_{i+1} onto H_i for all $i \ge 0$. Now the mapping $\alpha = \cup\, \alpha_i : R \to R$ defined by $\alpha(x) = \alpha_i(x)$ if $x \in H_{i+1}$ is well-defined, by compatibility, since $\alpha_{i+1}\big|_{H_i} = \alpha_i$ $(i \ge 0)$.

Moreover α is a homomorphism of R onto R, and α maps H_{i+1} onto H_i. Finally,

α^n maps H_{i+n} onto H_i and Ker $\alpha^n = H_n$.
It follows that $R \cong R/H_n$ for all n. This completes the proof of lemma 7.
Now we show that there exists a non-simple, non-nilpotent ring in K. The example is a primitive ring that is artinian relative to two-sided ideals but not noetherian. Let $W = \bigcup_{i=1}^{\infty} W_i$, where each W_i has ordinal ω, be a basis of a vector space V of countably infinite dimension over a field F. Relative to this basis, matrices of linear transformations of V will consist of doubly infinite arrays of blocks each with (countably) infinite rows and columns. Call a matrix bounded row finite if for some n it has no more than n columns containing non-zero elements. It is clear that K, the ring of all linear transformations of finite rank of V, is isomorphic to H, the ring of all bounded row-finite matrices over F. According to an example, due to T.S. Shores [4], we might relabel W and form the ring H_2 of all

$$4)\qquad \begin{pmatrix} A & & & & 0 \\ & A & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \cdot \end{pmatrix} + H$$

where A is some bounded row-finite matrix repeated in every diagonal ω by ω block. Then $H = H_1$ is isomorphic to the ring of all matrices of this block form. Also H_1 is an ideal in H_2 and $H_2/H_1 \cong H_1$, [4]
Now we can continue the process assuming for induction that we have a sub-ring H_n with $H = H_1 \triangleleft H_2 \triangleleft H_3 \triangleleft \ldots \triangleleft H_n$ and all $H_{k+1}/H \cong H_k$ ($k = 0,1,2, \ldots, n-1$) with $H_0 = 0$. Let B H_n (written relative to the basis W) and again relabel the basis. Form the ring H_{n+1} of all

$$5) \qquad \begin{pmatrix} B & & & & \\ & B & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix} + H.$$

Now $H \triangleleft H_{n+1}$, so any non-zero ideal of H_{n+1} must contain H. Let $\pi : H_{n+1} \to \to H_{n+1}/H$ be the natural map then the non-zero ideals of H_{n+1} are H and ideals of the form $J = \pi^{-1}(J')$, where $J' = J/H$ is an ideal of H_{n+1}/H. But $H_{n+1}/H \cong H_n$ and we can go by induction to H_n for any n. Now we show that each H_n is actually contained, as an ideal, in H_{n+1}. Assume for induction that $H_1 \triangleleft H_2 \ldots \triangleleft H_{n-1} \triangleleft H_n$, where $H_{k+1}/H \cong H_k$ for $k = 0,1,2, \ldots, n-1$. By induction we know that we have in H_n every matrix $Q \in H_{n-1}$ (relative to the W-basis). Using again the construction (5) and reordering we can form the set of all

$$6) \qquad \begin{pmatrix} Q & & & & \\ & Q & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix} + H.$$

Since $H_{n-1} \triangleleft H_n$ it follows that the above set is an ideal of H_{n+1}. But this is exactly the way we form H_n, that is, $H_n \triangleleft H_{n+1}$. The idea here is that, starting with H_{n-1} and H we can build H_n and $H_n/H \cong H_{n-1}$.
Now $\pi^{-1}(H_n/H) = H_n$ gives us H_n as an ideal in H_{n+1}. Thus by induction we get the whole sequence of H_n and we can let $R = \cup H_i$.
Now we claim that R has all the required properties of lemma 7. Indeed the H_i are totally ordered in the strictly ascending chain : $H = H_1 \subset H_2 \subset \ldots \subset \subset H_k \subset H_{k+1} \subset \ldots$ and, by construction, $H_k/H \cong H_{k-1}$ for any $k = 1,2, \ldots$. It is also clear from the construction that the isomorphisms are compatible since

$$\alpha\left(\left|\begin{matrix} B & & & \\ & B & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{matrix}\right| + H\right) = B.$$

The ring $H = H_1$ is a simple ring. In order to show that the H_n are the only non-zero ideals of R, we need

LEMMA 8. Let $R = \cup H_i$ be the above ring. If $A \in H_n$ but $A \notin H_{n-1}$ then for each $k < n$ there exists some $C \in R$ such that $AC \in H_k$, $AC \notin H_{k-1}$.

PROOF. This is (vacuously) true for $n = 0$ and for induction assume it for n. Then suppose $A \in H_{n+1}$ but $A \notin H_n$. Now

$$A = \begin{pmatrix} B & & & & \\ & B & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix} + I,$$

where $I \in H$, $B \in H_n$ but $B \notin H_{n-1}$. By induction, for any $k < n$, we have $C \in R$ such that $BC \in H_k$, $BC \notin H_{k-1}$. We get

$$(A-I)\begin{pmatrix} C & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix} = \begin{pmatrix} BC & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{pmatrix} \in H_{k+1}$$

and suppose it were in H_k. Then

$$\begin{bmatrix} BC & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix} = \begin{bmatrix} D & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix} + I_1$$

where $D \in H_{k-1}$. But I_1 has finite rank so eventually we get $BC = D$ contradicting $BC \notin H_{k-1}$.

Since

$$I \begin{bmatrix} C & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix} \in H = H_1 \text{ we have } A \begin{bmatrix} C & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & \cdot & & \cdot \end{bmatrix} \in H_{k+1}$$

but $\notin H_k$ for all $k + 1 < n + 1$. Thus induction is complete.

THEOREM 9. Let J be a proper ideal of R then $J = H_n$ for some n.

PROOF. Since $H = H_1$ is simple and every non-zero ideal of R contains linear transformations of finite rank it follows that $H_1 \subseteq J$. Now $J \neq R$ so there exists some n such that $H_n \subseteq J$ but $H_{n+1} \not\subseteq J$. Suppose $H_n \neq J$ then there would be some $A \in J$ with $A \in H_m$ but $A \notin H_{m-1}$ for some $m > n$. By lemma 8 we then have $AC \in J$ with $AC \in H_{m-1}$ but $AC \notin H_{m-2}$ so without loss of generality we may assume $A \in J$ with $A \in H_{n+1}$ but $A \notin H_n$. But in $\bar{R} = R/H_n$ we know $\bar{H}_{n+1} = H_{n+1}/H_n$ is simple so for any $B \in H_{n+1}$ we have $\bar{B} = \sum_i \bar{C}_i \bar{A} \bar{D}_i$ for some $\bar{C}_i, \bar{D}_i \in \bar{H}_{n+1}$. Then $\sum_i C_i A D_i = B + W$ for some $W \in H_n \subseteq J$. But also $\sum_i C_i A D_i \in J$ so we would arrive at the contradiction $H_{n+1} \subseteq J$. Thus we conclude that $H_n = J$.

We can now state

THEOREM 10. There exists a non-simple, non-nilpotent ring in K.

PROOF. The above ring $R = \cup H_i$ satisfies all the conditions of Lemma 7 by Theorem 9.

6. PRIME COMPONENTS

The following lemma is probably well-known.

LEMMA 11. If a ring R is hereditarily idempotent then every accessible subring of R is an ideal of R.

PROOF. It suffices to show that if $J \triangleleft I \triangleleft R$ then $J \triangleleft R$. But if J' is the ideal of R generated by J then by the Andrunakievič lemma, $J' = J'^3 \subseteq J \subseteq J'$ so $J' < R$. An ordinal γ is called a "prime component" (see [6], p.282) if $\beta + \gamma = \gamma$ for all $\beta < \gamma$ (or equivalently, if $\beta + \alpha \neq \gamma$ for all $\alpha < \gamma$, $\beta < \gamma$).

We then have

LEMMA 12. Let $R \in K$ so the set of all ideals of R can be written $\{H_\alpha\}_{0 \leq \alpha \leq \gamma}$ for some ordinal γ, and $H_\alpha \subseteq H_\beta$ if and only if $\alpha \leq \beta$. Then γ is a prime component and if we let $\beta < \gamma$ and let f be the isomorphism $f : R/H_\beta \cong R$ then $f(H_{\beta+\alpha}/H_\beta) = H_\alpha$ for all $\alpha \leq \gamma$.

PROOF. Suppose $\beta < \gamma$ then by well-ordering there exists some (in fact unique) $0 \leq \gamma$ such that $\beta + \theta = \gamma$. Now $f(H_{\beta+0}/H_\beta) = 0 = H_0$ and for induction assume that for a given ordinal $\phi \leq \theta$ we have $f(H_{\beta+\alpha}/H_\beta) = H_\alpha$ for all $\alpha < \phi$. If ϕ is a limit ordinal then so is $\beta + \phi$ and $H_{\beta+\phi} = \bigcup_{\alpha<\phi} H_{\beta+\alpha}$.
Thus $f(H_{\beta+\phi}/H_\beta) = \bigcup_{\alpha<\phi} f(H_{\beta+\alpha}/H_\beta) = \bigcup_{\alpha<\phi} H_\alpha = H_\phi$.
Otherwise $\phi = \alpha + 1$ for some α with $f(H_{\beta+\alpha}/H_\beta) = H_\alpha$. But $H_{\beta+\alpha+1}/H_\beta$ is a simple extension of H_α, namely $H_{\alpha+1} = H_\phi$. Thus the induction is complete for all $\phi \leq \theta$ so in particular $f(R/H_\beta) = f(H_{\beta+\theta}/H_\beta) = H_\theta$. But then $H_\theta = R = H_\gamma$ so in fact $\theta = \gamma$, that is γ is a prime component and $f(H_{\beta+\alpha}/H_\beta) = H_\alpha$ for all $\alpha \leq \gamma$.

THEOREM 13. Let $R \in K$ so the set of all ideals of R can be written $\{H_\alpha\}_{0 \leq \alpha \leq \gamma}$ for some ordinal γ. If $\theta \leq \gamma$ then $H_\theta \in K$ if and only if θ is a prime component.

PROOF. Suppose first that θ is a prime component for some $\theta \leq \gamma$. By Lemma 11 a proper ideals of H_θ is H_β for some $\beta < \theta$. Then $\beta + \theta = \theta$ so by Lemma 12 we have the isomorphism

$$f(H_\theta/H_\beta) = f(H_{\beta+\theta}/H_\beta) = H_\theta \text{ and so } H_\theta \in K.$$

Conversely, for any $\theta \leq \gamma$ the set of all ideals of H_θ is $\{H_\beta\}_{0 \leq \beta \leq \theta}$ and if $H_\theta \in K$ then for any $\beta < \theta$ there exists an isomorphism $f'(H_\theta/H_\beta) \cong H_\theta$. But in the proof of lemma 12 it is clear that we can substitute θ for γ and f' for f to obtain the result of the lemma, namely that θ is a prime component.

As a corollary we get

COROLLARY 14. Let $R \in K$ so the set of all ideals of R can be written $\{H_\alpha\}_{0 \leq \alpha \leq \gamma}$. If R is not simple or a prime order zero ring then $\gamma \geq \omega$ and $H_\omega \in K$.

REMARK 3. The existence of a ring $R = H_\gamma$ in K with $\gamma > \omega$ is an open question.

7. APPLICATIONS

In this final section we collect some results of a miscellaneous nature.

1. Let R be in K where $R^2 = R$ and let R_n be a matrix ring over R for some n. It is wellknown that the ideals of R_n are of the form U_n where U is an ideal in R. Suppose $U_n \neq R_n$, then $U \neq R$. Hence $R/U \cong R$ and $R_n/U_n \cong (R/U)_n$ imply $R_n/U_n \cong R_n$, so $R_n \in K$.
2. In answer to a question of R. Gilmer and M. O'Malley [2] a non-commutative ring S is constructed by J. Hausen and J.A. Johnson [3], such that S does not satisfy the a.c.c. on two-sided ideals, but every proper two-sided ideal I of S satisfies the a.c.c. In fact, for every proper two-sided ideal I of S, the lattice of all two-sided ideals of I is finite. S has a countable infinite set $\{J_i\}$, $i = 1,2, \ldots$ of proper

non-zero ideals and S has no other ideals. Also J_1 is simple, $S = \cup J_i$ and $J_1 \subset J_2 \ldots \subset J_n \subset \ldots$ So S satisfies all the conditions of Lemma 7 but not the last one, i.e. J_{i+1}/J_1 is not isomorphic to $J_i (i \geq 0)$. From $J_{i+1}/J_1 \cong J_i$ for some $i \geq 0$ we would get $J_{i+1}/J_i \cong J_1$. But this is impossible, since J_{i+1}/J_i is a simple ring without minimal one-sided ideals [8] , whereas J_1 is a simple ring with minimal one-sided ideals. By Theorem 6 (or Lemma 12) $S \notin K$.

However, the ring R of Theorem 10 provides us with another example of a non-zero ring, which does not satisy the a.c.c. on two-sided ideals, but every proper two-sided ideal H_n of R satisfies the a.c.c. By Lemma 11, H_n has only the set $\{H_0 = 0, H_1, \ldots, H_n\}$ as its ideals and hence satisfies the a.c.c.

3. For a class M of rings write $UM = \{R \mid \text{every } 0 \neq R/I \notin M\}$

It is well-known that if M satisfies the condition : If $R \in M$ and $0 \neq I \triangleleft R$ then some $0 \neq I/J \in M$, then UM is radical in the Kurosh-Amitsur sense and UM is called the upper radical defined by the class M. The condition is trivially satisfied if M is a class of simple rings.

Now let M be a class of simple primitive rings without a unity. Suppose $K \in M$ has minimal one-sided ideals. Then K is isomorphic to a ring of linear transformations of finite rank of a vector space V over a field F. First suppose that dim $V = \aleph_0$. Then, by construction above, we can embed K as an ideal of a ring R such that $R/I \cong R$ for any ideal $I \neq R$. This means : $K \in M$, $K \triangleleft R$, but R has no non-zero image in M. By [5 ; Theorem 1] we conclude that UM is not hereditary. Next suppose that dim $V > \aleph_0$. Take a subspace V_0 of V dimension $\aleph_0$. It can easily be seen that K is isomorphic to the ring of linear transformations of V_0 of finite rank. Again we get that UM is not hereditary.

Thus we conclude :

THEOREM 15. Let M be a class of simple primitive rings such that at least one of the rings in M has no unity. A necessary condition for UM is hereditary is that each of the rings in M without a unity has no minimal one-sided ideals.

The first-named author has constructed a simple primitive ring K without unit, without minimal one-sided ideals and with characteristic 2 such that if M = N ∪ {K}, where N is a class of simple rings with unit, then UM is hereditary if and only if $Z_2 \in N$, (to appear in Periodica Math. Hungarica).

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