Indecomposable representations of orders and Dynkin diagrams

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Let R be a complete valuation ring with quotient field K, and Λ an R-order in a semi-simple K-algebra A. It is known that some problems in the representation theory of Λ can be reduced to similar problems in other categories [2], however no attempt seems to exist to develop a general theory. The present note will survey some of the problems which can be tackled in this way. The procedure always will be twofold: First, there will be a reduction to one of those vector space categories which lately attracted nuch interest (categories of representations of quivers, species, partially ordered sets, subspace categories etc.); the second step is then the construction of lattices from the given data. In this way, one wants to list some, or even all, indecomposable Λ-lattices. In contrast to the usually applied methods, there will be little calculation inside the order or the lattices (the calculations are transferred to the frame of vector spaces). On the other hand, the results will usually not depend on the particular structure of the residue field $\underline{k}$ of R, whereas most of the classical results were formulated only for finite $\underline{k}$.

We denote by $N(\Lambda)$ a full set of non-isomorphic indecomposable Λ-lattices, and by $N_d(\Lambda)$ the subset of those lattices of rank d.

Theorem I: If $N(\Lambda)$ is infinite, then there exists an infinite chain of indecomposable Λ-lattices $M_o \subset M_1 \subset \ldots M_i \subset M_{i+1} \subset \ldots$ such that $M_{i+1}/M_i \simeq M_o$ for all i.

This result is a concequence of the validity of a similar statement for certain vector space categories, which turned out to be true in the proof of the second Brauer-Thrall conjecture [4]. Moreover we can formulate a analogue of this conjecture for orders asserting that there are infinitely many natural numbers d such that $N_d(\Lambda)$ is infinite in case $\underline{\underline{k}}$ is infinite, whereas for finite $\underline{\underline{k}}$, the number of elements of $N_d(\Lambda)$ grows rapidely with d. For a special type of orders, the reduction method gives a particulary satisfying theory: Namely it extends a recent result of Bäckström [1] into several different directions and shows that his rather unnatural seeming conditions have a fairly natural interpretation: He describes the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8. We call Λ a <u>Bäckström order</u> provided there exists a hereditary R-order Γ such that $\Lambda \subseteq \Gamma$ and $\mathrm{rad}\Lambda = \mathrm{rad}\Gamma$. Note, that in case A is separable, and Λ is an arbitrary order in A, there is a hereditary order Γ with $\Lambda \subseteq \Gamma \subseteq A$ such that $\Lambda_o = \Lambda + \mathrm{rad}\Gamma$ is a Bäckström order containing Λ. In this way, results concerning lattices over Bäckström orders are of interest also for general orders. For any Bäckström order Λ, we introduce a valued graph in the following way: Let $\Lambda \subseteq \Gamma$, with Γ hereditary, and $J = \mathrm{rad}\Lambda = \mathrm{rad}\Gamma$. By Morita-equivalence, we may suppose $\Lambda/J = \prod_{i=1}^{s} F_i$, and $\Gamma/J = \prod_{i=s+1}^{t} (F_i)_{n_i}$ where F_i are skewfields $1 \le i \le t$, and $(F)_n$ denotes the full nxn-matrix ring over F. Note that for later purposes we denote the various skew fields by F_i, the index ranging from 1 to t. For $s+1 \le j \le t$, let S_j be a simple Γ/J-module with endomorphism ring F_j, and we denote by ${}_iS_j$ the F_i-F_j-bimodule ${}_iS_j = F_i \otimes_\Lambda S_j$, where $1 \le i \le s$, $s+1 \le j \le t$, and by $d_{ij} = \dim_{F_i}({}_iS_j)$, $d'_{ij} = \dim({}_iS_j)_{F_j}$ the resp. two dimensions of ${}_iS_j$. For $i > s$, and for $j < s$, we put $d_{ij} = d'_{ij} = 0$. In this way we get a valued graph G with t vertices and valuation (d_{ij}, d'_{ij}).

Theorem II: Let Λ be a Bäckström order with valued graph G.

(i) $N(\Lambda)$ is finite if and only if G is a Dynkin diagram, and in this case, the elements of $N(\Lambda)$ correspond bijectively to the non-simple positive roots of G.

(ii) If G is an extended Dynkin-diagram, then $N(\Lambda)$ can be classified.

Bäckström [1] proved part (i) under the assumption that the residue field $\underline{\underline{k}}$ of R is finite and that all $F_i=\underline{\underline{k}}$ (note that in this case only the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 can occur), without, however, constructing the indecomposable representations explicitely. The reduction method, on the other hand, allows an actual classification of all indecomposable representations in both cases (i) and (ii). We should remark that every valued graph occurs as the valued graph of a Bäckström order. For example

$$\Lambda = \begin{pmatrix} R & \pi & R & R & R & R & R \\ \pi & \alpha & R & R & R & R & R \\ \pi & \pi & \alpha' & R & R & R & R \\ \pi & \pi & \pi & \alpha'' & \pi & R & R \\ \pi & \pi & \pi & \pi & \beta & R & R \\ \pi & \pi & \pi & \pi & \pi & \beta' & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & R \end{pmatrix} \qquad \alpha \equiv \alpha' \mp \alpha'' \bmod \pi,\ \beta \equiv \beta' \bmod \pi,$$

where π is the maximal ideal of R,

is a Bäckström order with diagram $E_8 = \cdot - \cdot - \cdot - \cdot - \cdot - \cdot - \cdot$. Since E_8 has 120 positive roots, there are 112 indecomposable Λ-lattices. There is one lattice of highest rank 105, with 6 generators, given as follows:

$$M = \begin{pmatrix} \pi & \pi & R & R & \pi & \pi & \pi & R & R & R & R & R & R & R & R \\ \alpha & \alpha' & \alpha'' & \alpha''' & \pi & \pi & \pi & R & R & R & R & R & R & R & R \\ \pi & \pi & \pi & \pi & \beta & \beta' & \beta'' & R & R & R & R & R & R & R & R \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & \gamma & \gamma' & \gamma'' & \gamma''' & \gamma'''' & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & R & \delta & \epsilon & \iota & \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \delta' & \epsilon' & \iota' \\ \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & \pi & R & R & R \end{pmatrix}$$

subject to the following conditions

$\beta \equiv \alpha + \alpha''' + \gamma''' \bmod \pi$
$\beta' \equiv \alpha' + \alpha''' + \gamma'''' \bmod \pi$
$\beta'' \equiv \alpha'' + \alpha''' + \gamma''' \bmod \pi$
$\alpha' \equiv \gamma \bmod \pi$
$\alpha'' \equiv \gamma' \bmod \pi$
$\alpha''' \equiv \gamma'' \bmod \pi$
$\delta \equiv \delta' \bmod \pi$
$\epsilon \equiv \epsilon' \bmod \pi$
$\iota \equiv \iota' \bmod \pi$

Finally, we should state one of the reduction arguments on which the proof of theorems I and II is based, and which allows to reprove many of the known results on $N(\Lambda)$ for arbitrary Λ (for example [3]).

Assume that the R-order Λ in **A** is contained in the hereditary R-order Γ in **A** and let I be a full twosided Γ-ideal contained in Λ such that $I \subseteq \mathrm{rad}\Gamma \cap \mathrm{rad}\Lambda$. Put $\bar{\Lambda} = \Lambda/I$, and $\bar{\Gamma} = \Gamma/I$. We denote by $\underline{\underline{C}} = \underline{\underline{C}}(\Lambda,\Gamma,I)$ the following category. The objects of $\underline{\underline{C}}$ consist of $\bar{\Lambda}$-monomorphisms $i: X \to Y$, where X is a finitely generated left $\bar{\Lambda}$-module, Y is a finitely generated projective left $\bar{\Gamma}$-module, such that $\bar{\Gamma} \cdot \mathrm{Im}(i) = Y$. Morphisms are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{i'} & Y' \end{array}$$

where α is a $\bar{\Lambda}$-homomorphism, and β a $\bar{\Gamma}$-homomorphism.

Theorem III: The category of left Λ-lattices and the category $\underline{\underline{C}}(\Lambda,\Gamma,I)$ are representation equivalent.

In the case of Bäckström order $\Lambda \subseteq \Gamma$, with $\mathrm{rad}\Lambda = \mathrm{rad}\Gamma$, we apply this theorem using $I = \mathrm{rad}\Lambda$.

References:

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