HEREDITARY ARTINIAN RINGS OF FINITE REPRESENTATION TYPE

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Recall [11,6] that a hereditary finite dimensional algebra is of finite type if and only if a corresponding diagram is a disjoint union of the Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 occouring in Lie theory. Here, we will consider the general case of a hereditary artinian ring A and associate to it a diagram $\Gamma(A)$. It turns out that A is of finite type if and only if $\Gamma(A)$ is the disjoint union of the Coxeter diagrams A_n, B_n $(= C_n)$, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ which classify the irreducible Coxeter groups [3] . However, the existence of rings of type H_3, H_4 and $I_2(p)$ with $p = 5$ or $p \geq 7$ remains open: it depends on rather difficult questions concerning division rings. On the other hand, we define for any Coxeter diagram "branch system" which generalize the root systems of the Dynkin diagrams. The dimension types of a hereditary artinian ring of finite representation type just form such a branch system.

The results of sections 1, 2 and 5 were obtained by P.Dowbor and D.Simson who announced part of them in the papers [9, 10] and at the Ottawa Conference 1979. The results were obtained independently by C.M.Ringel who announced them at the Oberwolfach meeting on division rings in 1978.

1. Bimodules of finite representation type

Let F and G be division rings, and ${}_FM_G$ a bimodule. Denote by $\mathcal{L}({}_FM_G)$ the category of finite dimensional representations of ${}_FM_G$, a representation being of the form $V = (X_F, Y_G, \varphi: X_F \otimes {}_FM_G \longrightarrow Y_G)$ with dimension type $\underline{\dim} V = (\dim X_F, \dim Y_G)$. Given ${}_FM_G$, let $M^R = \mathrm{Hom}_G({}_FM_G, {}_GG_G)$, $M^L = \mathrm{Hom}_F({}_FM_G, {}_FF_F)$, and $M^{R(i+1)} = (M^{Ri})^R$, $M^{L(i+1)} = (M^{Li})^L$, with $M^{Ro} = M = M^{Lo}$. Note that if $\dim M_G$ is finite,

$M^{RL} \cong M$, whereas if $\dim_F M$ is finite, then $M^{LR} \cong M$. We say that M is a bimodule with <u>finite dualisation</u> if all bimodules M^{Ri} and M^{Li} are finite dimensional on either side.

<u>Proposition 1. Assume $_F M_G$ is of finite representation type, with indecomposable representations $P_1, \ldots, P_m$. Let $\dim P_i = (x_i, y_i)$, and assume we have choosen an orderig with $x_i/y_i \leq x_{i+1}/y_{i+1}$. Then this ordering is unique, and there are Auslander-Reiten sequences</u>

$$0 \longrightarrow P_{i-1} \longrightarrow M^{L(i-1)} \otimes P_i \longrightarrow P_{i+1} \longrightarrow 0$$

<u>The bimodule M has finite dualisation; M^{Li} is one-dimensional as a right vector space, for some i, and the bimodules M and M^{Lm} are (semilinear) isomorphic.</u>

Let outline the main steps of the proof:

If $\dim M_G$ is finite, we define a functor $C_1^+ : \mathcal{L}(_F M_G) \longrightarrow \mathcal{L}(M^L)$ as usual [2,7]: given $(X_F \otimes {_F M_G} \xrightarrow{\varphi} Y_G)$, let $A_G = \ker \varphi$. Using $X_F \otimes {_F M_G} \cong \mathrm{Hom}(M^L, X_F)$, we get from the kernel map $A_G \longrightarrow \mathrm{Hom}(M^L, X_F)$ as adjoint a map of the form $A_G \otimes M^L \longrightarrow X_F$. Clearly, under C_1^+, the full subcategory $\mathcal{L}_1(_F M_G)$ of $\mathcal{L}(_F M_G)$ of objects without simple projective direct summands is equivalent to the full subcategory $\mathcal{L}_2(M^L)$ of $\mathcal{L}(M^L)$ of objects without simple injective direct summands. In particular, $\mathcal{L}(M^L)$ and $\mathcal{L}(M)$ have the same representation type.

Similary, if $\dim(M^R)_F$ is finite, define $C_1^- : \mathcal{L}(_F M_G) \longrightarrow \mathcal{L}(M^R)$ mapping $(X_F \otimes {_F M_G} \xrightarrow{\varphi} Y_G)$ onto the cokernel of the adjoint map $X_F \longrightarrow \mathrm{Hom}_G(_F M_G, Y_G) \cong Y_G \otimes M^R$, and thus establishing an equivalance between $\mathcal{L}_2(M)$ and $\mathcal{L}_1(M^R)$.

If $\dim_F M$ or $\dim M_G$ is infinite, then clearly $\mathcal{L}(M)$ cannot be of bounded representation type. Thus, the functors C_i^+ and C_i^- show that M has to be a bimodule with finite dualisation if $\mathcal{L}(M)$ is of finite representation type.

The functors C_1^+, C_1^- have nice properties: If (X_F, Y_G, φ) is indecomposable in $\mathcal{L}(M)$, then its image in $\mathcal{L}(M^R)$ is either zero, in which case $(X_F, Y_G, \varphi)=(F_F, 0, o)$ is simple injective, or else $\underline{\dim}\, C_1^-(X, Y, \varphi) = (y, yb-x)$, where $\underline{\dim}\,(X,Y,\varphi) = (x, y)$ and $b = \dim(M^R)_F$. In particular, there is a unique representation in $\mathcal{L}(M)$ of type $(x,y) \neq (1,0)$ if and only if there is a unique one in $\mathcal{L}(M^R)$ of type $(y, yb-x) \neq (0,1)$.

Let $C_i^+ : \mathcal{L}(M) \longrightarrow \mathcal{L}(M^{Li})$, $C_i^- : \mathcal{L}(M) \longrightarrow \mathcal{L}(M^{Ri})$ be the iterated functors Call (X, Y, φ) in $\mathcal{L}(M)$ <u>preprojective</u> provided $C_i^+(X,Y,\varphi) = 0$ for some i, and <u>preinjective</u> provided $C_i^-(X,Y,\varphi) = 0$ for some i. Obviously, these modules are characterized by their dimension types. Let P_i be the indecomposable module (if it exists) with $C_i^+ P_i = 0$ and $C_{i-1}^+ P_i \neq 0$. Being determined by their dimension types, P_i, P_j can be ismorphic only for $i=j$. Let P_m be the last such module which exists (assuming $\mathcal{L}(M)$ of finite representation type). Then all P_i with $1 \leqslant i \leqslant m$ exist, and P_{m-1}, P_m have to be injective. Since the set of preprojective modules is closed under indecomposable submodules of direct sums, and contains the indecomposable injective modules, it contains all indecomposable modules. Similary, all indecomposable modules are also preinjective. Next, one proves that $\mathrm{Hom}(P_1,P_2) \cong {}_F M_G$, and therefore $\mathrm{Hom}(P_i,P_{i+1}) \cong M^{L(i-1)}$. Also, there is an obvious exact secuence

$$0 \longrightarrow P_{i-1} \longrightarrow \mathrm{Hom}(P_{i-1},P_i)^L \otimes P_i \longrightarrow P_{i+1} \longrightarrow 0 ,$$

and it is left almost split [1] (prove it by induction using the functors C_i^-).

Similary, let I_i be the indecomposable module with $C_i^- I_i = 0$ and $C_{i-1}^- I_i \neq 0$. Then $\mathrm{Hom}(I_2,I_1) \cong M^{R2}$. Since $P_{m-1} = I_2$, $P_m = I_1$, we conclude that $M^{L(m-2)} \cong M^{R2}$, thus $M^{Lm} \cong M$.

Finally, if the right dimension of all M^{Li} would be $\geqslant 2$, then one proves by induction on i, that P_i exists (for any $i \geq 1$) and that $\underline{\dim}\, P_i = (x_i, y_i)$ with $0 \leq x_i < y_i$. This produces a countable number of isomorphism classes of indecomposable modules.

2. Dimension sequences

A sequence $a = (a_1, \ldots, a_m)$ of length $|a| = m \geq 2$ with $a_i \in \mathbb{N}$ is called a dimension sequence provided there exists $x_i, y_i \in \mathbb{N}$ ($1 \leq i \leq m$), with

$$a_i x_i = x_{i-1} + x_{i+1}, \quad a_i y_i = y_{i-1} + y_{i+1} \quad (1 \leq i < m),$$

$x_0 = -1$, $x_1 = 0$, $x_m = 1$, and $y_0 = 0$, $y_1 = 1$, $y_m = 0$. The set of vectors $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, with $1 \leq i \leq m$, is called the branch system defined by a. Note that it is uniquely determined by a, and conversly, it determines a. This generalizes the positive part of the usual rank 2 root systems: for the dimension sequences $(0,0)$, $(1,1,1)$, $(1,2,1,2)$ and $(1,3,1,3,1,3)$ the branch systems are just the positive roots of $A_1 \times A_1$, A_2, B_2 and G_2, respectively.

Proposition 2. *The branch system in $\mathbb{R}^2$ can be constructed as follows: $\{(1,0), (0,1)\}$ is a branch system. If $\mathcal{B}$ is a branch system, and p, q are neighbors in $\mathcal{B}$, then $\mathcal{B} \cup \{p+q\}$ is a branch system.*

Here, in a finite subset $\mathcal{B}$ of $\mathbb{R}^2$ consisting of pairwise linearly intependent elements, we call two elements neighbors in case the lines through these elements are neighbors in the set of all lines through elements of $\mathcal{B}$. G. Bergman has pointed out that the branch systems $\{(x_i, y_i) \mid 1 \leq i \leq m\}$ correspond just to the Farey sequences $\frac{x_i}{y_i}$ (see [16]). In particular, the numbers x_i, y_i always are without common divisor.

The proposition above can be reformulated as follows.

<u>Proposition 2'. The set $\mathcal{D}$ of dimension sequences can be obtained as follows:</u> $(0,0) \in \mathcal{D}$, <u>and if</u> $(a_1,\dots,a_m) \in \mathcal{D}$, <u>then all the sequences</u> $(a_1,\dots,a_{i-1}, a_i+1, 1, a_{i+1}+1, a_{i+2},\dots,a_m)$ <u>for</u> $1 \leqslant i < m$ <u>belong to</u> $\mathcal{D}$.

Note that for any other dimension sequence a, we have $a_i = 1$ for some $i \geqslant 2$. An easy consequence is the following

<u>Corollary. The set $\mathcal{D}$ of dimension sequences is closed under cyclic permutations.</u>

Given a dimension sequence $a = (a_1,\dots,a_m)$, let $a^+ = (a_m, a_1,\dots,a_{m-1})$. Let us give the list of all dimension sequences of length $\leqslant 7$, up to the cyclic permutations and reversion:

$(0,0)$; $(1,1,1)$; $(1,2,1,2)$; $(1,2,2,1,3)$;
$(1,2,2,2,1,4)$, $(1,2,3,1,2,3)$, $(1,3,1,3,1,3)$;
$(1,2,2,2,2,1,5)$, $(1,2,2,3,1,2,4)$, $(1,2,3,2,1,3,3)$,
$(1,4,1,2,3,1,3)$.

3. Coxeter diagrams

Let $\Gamma = \{1,\dots,n\}$ and assume there is given a set map $d: \Gamma \times \Gamma \longrightarrow \mathcal{D}$. Note that d defines on Γ the structure of an oriented graph, if we draw an arrow $i \longrightarrow j$ in case $d(i,j) \neq (0,0)$. With (Γ, d) we also will consider the unoriented graph given by $(\Gamma, |d|)$, where two different points are connected by at most two edges, and any edge has assigned a number $\geqslant 3$. If i is a sink for (the orientation defined by) d, define $(\pi_i d)(i,j) = d(j,i)^+$, $(\pi_i d)(j,i) = (0,0)$, and $(\pi_i d)(j,k) = d(j,k)$ for all $j, k \neq i$. For any sink i, define on $\mathbb{R}^n$ a linear transformation $\sigma_i = \sigma(d)_i$ as follows: if $x = (x_j) \in \mathbb{R}^n$, let
$(\sigma_i x)_j = x_j$ for $j \neq i$, and $(\sigma_i x)_i = -x_i + \sum_j d(i,j)_1 x_j$.
If $i_1,\dots,i_t$ is a $(+)$-admissible sequence (thus i_s is a sink for $\pi_{s-1}\cdots\pi_1 d$, for all $1 \leqslant s \leqslant t$), define

$$\sigma_{i_t \dots i_1} = \sigma(\pi_{t-1} \dots \pi_1 d)_{i_t} \, \sigma_{i_{t-1} \dots i_1} \, .$$

We call $y \in \mathbb{N}^n$ preprojective provided there exists a (+)-admissible sequence $i_1, \dots, i_t$ such that $\sigma_{i_t \dots i_1}(x)$ is one of the canonical base vectors $(0,\dots,1,0,\dots,0)$. We say that (Γ, d) is of finite type provided there are only finitely many preprojective vectors, and then we call the set of preprojective vectors the branch system of (Γ, d).

Theorem 1. (Γ, d) is of finite type if and only if $(\Gamma, |d|)$ is the disjoint union of Coxeter diagrams A_n, B_n $(=C_n)$, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ ($p=5$ or $p \geq 7$). If $(\Gamma, |d|)$ is one of the Coxeter diagrams of rank n and with Coxeter number h, then the branch system for (Γ, d) has precisely $\frac{1}{2}nh$ elements.

Of course, the branch systems for the diagrams A_n, D_n, E_6, E_7, E_8, F_4, G_2 are precisely the positive roots of the corresponding root system; there are two possible branch systems for B_n, namely the positive roots of B_n, or C_n. The branch systems of the type $I_2(p)$, H_3, and H_4 have p, 15, and 60 elements, respectively.

As an example, let us write down the branch system for $\cdot \longrightarrow \cdot \xrightarrow{5} \cdot$, with second dimension sequence $(2,1,3,1,2)$. It contains the following 15 vectors

$$(1\ 0\ 0),\ (0\ 1\ 0),\ (0\ 0\ 1),\ (1\ 1\ 0),\ (0\ 1\ 1),$$
$$(1\ 1\ 1),\ (0\ 1\ 2),\ (0\ 2\ 1),\ (1\ 1\ 2),\ (1\ 2\ 1),$$
$$(1\ 2\ 2),\ (2\ 2\ 1),\ (1\ 3\ 2),\ (1\ 3\ 3),\ (1\ 4\ 2),$$

and is dependent on the orientation and the given dimension sequence.

The proof of the theorem is rather technical. First, one constructs explicity for every pair (Γ, d) with $(\Gamma, |d|)$ a Coxeter diagram the corresponding branch system. For the converse,one only has to consider the pairs (Γ, d) for wich all proper subdiagrams of $(\Gamma, |d|)$ are Coxeter diagrams, and again an explicit calculation shows that in these cases there are infinitely many preprojective vectors.

4. Hereditary artinian rings

Given an artinian ring A, let A^o be its basic ring, $A^o/\mathrm{rad}A^o = \prod_{i=1}^{n} F_i$, and $\mathrm{rad}A^o/(\mathrm{rad}A^o)^2 = \bigoplus_{i,j} {}_iM_j$ as $\prod_{i=1}^{n} F_i$-bimodule. Then $\mathfrak{S} = (F_i, {}_iM_j)$ is called the species of A. Let $d(i,j)$ be the dimension sequence of the bimodule ${}_iM_j$, thus we have associated to A a pair (Γ, d) and we denote by $\Gamma(A)$ the pair $(\Gamma, |d|)$.

Theorem 2. The hereditary artinian ring A is of finite representation type if and only if $\Gamma(A)$ is disjoint union of Coxeter diagrams A_n, B_n $(=C_n)$, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ ($p=5$ or $p \geq 7$). In this case, the dimension types of the indecomposable A-modules form the branch system for (Γ, d).

As in the case of an algebra [6], one sees that for a hereditary artinian ring A of finite representation type, the basic ring A^o always is the tensor ring over its species.

5. Problems on division rings

The main problem which arises and which we are not able to answer is the question about the possible dimension sequences a of bimodules ${}_FM_G$ of finite representation type. We may assume that $a_2 = 1$ (using a cyclic permutation of a, thus replacing M by some M^{Li}). Then there is given a division ring inclusion $G \hookrightarrow F$, and ${}_FM_G = {}_FF_G$. Since $M^L = {}_GF_F$, the dimension sequence of ${}_FF_G$ starts as follows:

$$a_1 = \dim F_G \ , \ a_2 = 1 \ , \ a_3 = \dim {}_GF \ .$$

Thus, immediatly, we are confronted with the question of differrent left or right index of a division subring G of F, (see [4]).

Of particular interest is the question whether there exists a bimodule with the dimension sequence of type $I_2(5)$, say $(2,1,3,1,2)$, since it would give rise to rings of type H_3 and H_4. In fact, it is easy to see that the only dimension sequence starting with $(2,1,3,1,\ldots)$ is $(2,1,3,1,2)$. Thus, we would need $G \subseteq F$ with $\dim F_G = 2$, $\dim {}_GF = 3$, and $\dim_F \mathrm{Hom}({}_GF_F, {}_GG) = a_4 = 1$. Let us point out certain consequences of these conditions. Since the length of $(2,1,3,1,2)$ is 5, it follows from $M^{L5} \cong M$ that F , G have to be isomorphic. Also F contains a division subring H again isomorphic to F with $\dim F_H = 3$, $\dim_H F = 2$, since $({}_FF_G)^{L3}$ has the dimension sequence $(3,1,2,2,1)$.

Bimodules with suitable dimension sequences would produce interesting examples of artinian rings. For example, we may ask whether there exists a bimodule ${}_FM_G$ of finite representation type with $\dim_F M = 2 = \dim M_G$, thus dimension sequence $(2,2,a_3,\ldots,a_m)$. Of course, according to [13] such a bimodule has to be simple as a bimodule (this also follows from proposition 1, since dualisation leads to a bimodule which is one-dimensional on one side). Note that in case of odd m, the division rings F , G have to be isomorphic, so that we also can form the trivial ring extension $R = F \ltimes M$, and this then would be a local ring with $(\mathrm{rad} R)^2 = 0$, left length 3 , right length 3 , and with precisely $m-1$ indecomposable modules. In particular, the dimension sequences of type $I_2(5)$ would give a local ring R with $(\mathrm{rad} R)^2 = 0$, left length 3 , right length 3 , and precisely 4 indecomposable modules.

Similary, a bimodule with dimension sequence $(a_1,\ldots,a_{u+v+1})$ where $a_1 = u$, $a_2 = 1$, $a_{u+1} = 3$, $a_{u+v} = 1$, $a_{u+v+1} = v$, and the remaining $a_i = 2$ would be of local-colocal representation type (any indecomposable modules has a unique minimal submodule or unique maximal submodule). In contrast, the finite dimensional algebras with this property all have been classified in [15].

This shows very clearly the dependence of the representation theory of artinian rings on questions concerning division rings, a fact which to have been exhibited for the first time in [14].

In particular, under the assumption that there are no pairs of division rings $F \supsetneq G$ with $\dim_G F = 2$, and $\dim F_G = 2$, but finite, the Coxeter diagrams H_3, H_4, and $I_2(p)$ cannot be realized as $\Gamma(A)$ for any artinian ring A, so they could be excluded from theorem 2.

Of course, in case we assume that the division rings F are finitely generated over their centers, then we exclude immediately the cases H_3, H_4, and $I_2(p)$, thus one has the following (see also [8]):

<u>Theorem 3.</u> <u>The hereditary artinian ring</u> A <u>with</u> $A/\mathrm{rad}A$ <u>finitely generated over its center is of finite representation type if and only if</u> $\Gamma(A)$ <u>is the disjoint union of Coxeter diagrams</u> A_n, B_n $(=C_n)$, D_n, E_6, E_7, E_8, F_4, G_2.

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