THE PREPROJECTIVE ALGEBRA OF A MODULATED GRAPH

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The present paper generalizes a recent result of I.M. Gelfand and V.A. Ponomarev [4] reported at the Conference by V.A. Rojter.

A *modulated graph* $\mathfrak{M} = (F_i, {}_iM_j, \varepsilon_i^j)_{i,j \in I}$ is given by division rings F_i for all $i \in I$, by bimodules ${}_{F_i}({}_iM_j)_{F_j}$ for all $i \neq j$ in I finitely generated on both sides and by non-degenerate bilinear forms $\varepsilon_i^j : {}_iM_j \otimes {}_jM_i \to F_i$; here, I is a finite index set. Note that the forms ε_i^j give rise to canonical elements $c_j^i \in {}_jM_i \otimes {}_iM_j$. Namely, if $x_1, \ldots, x_d$ is a basis of $({}_jM_i)_{F_i}$ and $y_1, \ldots, y_d$ the corresponding dual basis of ${}_{F_i}({}_iM_j)$ with respect to ε_i^j , then $c_j^i = \sum_p x_p \otimes y_p$; see section 1.

Define the ring $\Pi(\mathfrak{M})$ as follows. Let $T(\mathfrak{M})$ be the tensor ring of $\mathfrak{M}$: $T(\mathfrak{M}) = \bigoplus_{t \in \mathbb{N}} T_t$, where $T_o = \prod_i F_i$, $T_1 = \bigoplus_{i,j} {}_iM_j$ and $T_{t+1} = T_1 \otimes_{T_o} T_t$ with the multiplication given by the tensor product. Then, by definition, $\Pi(\mathfrak{M}) = T(\mathfrak{M})/\langle c\rangle$, where $\langle c\rangle$ is the principal ideal of $T(\mathfrak{M})$ generated by the element $c = \sum_{i,j} c_i^j$.

Let Ω be an (admissible) orientation of $\mathfrak{M}$; thus, for every pair i,j with ${}_iM_j \neq 0$, we prescribe an order indicated by an arrow $i \longrightarrow j$, or $i \longleftarrow j$ in such a way that no oriented cycles occur. Let $R(\mathfrak{M},\Omega)$ be the corresponding tensor ring of $(\mathfrak{M},\Omega)$: $R(\mathfrak{M},\Omega) = \bigoplus_{t \in \mathbb{N}} R_t$ with $R_o = \prod_i F_i$, $R_1 = \bigoplus_{i \to j} {}_iM_j$ and $R_{t+1} = R_1 \otimes_{R_o} R_t$. For the representation theory of $R(\mathfrak{M},\Omega)$ we refer to [3].

THEOREM. *For each orientation* Ω *of* $\mathfrak{M}$, $R(\mathfrak{M},\Omega)$ *is a subring of* $\Pi(\mathfrak{M})$ *and, as a (right)* $R(\mathfrak{M},\Omega)$ *-module,* $\Pi(\mathfrak{M})$ *is the direct sum of all indecomposable preprojective* $R(\mathfrak{M},\Omega)$*-modules (each occurring with multiplicity one).*

This theorem suggests to call $\Pi(\mathfrak{M})$ the preprojective algebra of $\mathfrak{M}$. Recall that an indecomposable $R(\mathfrak{M},\Omega)$-module P is preprojective if and only if there is only a finite number of indecomposable modules X with $\mathrm{Hom}(X,P) \neq 0$.

COROLLARY. *The ring* $\Pi(\mathfrak{M})$ *is artinian if and only if the modulated graph is a disjoint union of Dynkin graphs.*

Observe that if $\mathfrak{M}$ is a K-modulation (where K is a commutative field), then $\Pi(\mathfrak{M})$ is a K-algebra. In this case, the corollary may be reformulated as follows: The algebra $\Pi(\mathfrak{M})$ is finite-dimensional if and only if $\mathfrak{M}$ is a disjoint union of Dynkin graphs.

Consider, in particular, the case when $(\mathfrak{M},\Omega)$ is given by a quiver; thus, $F_i = K$ for all i and ${}_iM_j$ is a direct sum of a finite number of copies of ${}_KK_K$. For every arrow x of the quiver, define an "inverse" arrow x^* whose end is the origin of x and whose origin is the end of x . Then $T(\mathfrak{M})$ is the path algebra generated by all arrows x and x^* , and $\Pi(\mathfrak{M})$ is the quotient of $T(\mathfrak{M})$ by the ideal generated by the element $\sum_{\text{all } x} (xx^* + x^*x)$.

COROLLARY. *If* $(\mathfrak{M},\Omega)$ *is given by a quiver, then* $\Pi(\mathfrak{M})$ *is finite-dimensional if and only if the quiver is of finite type.*

For a quiver which is a tree, the last result has been announced by A.V. Rojter [6] in his report on the paper [4]. In contrast to the proofs in [4], our approach avoids use of reflection functors and is based on the explicite description of the category

$P(\mathfrak{M},\Omega)$ of all preprojective $R(\mathfrak{M},\Omega)$-modules. The authors are indebted to P. Gabriel for pointing out that the theorem is, in the case when $(\mathfrak{M},\Omega)$ is given by a quiver, also due to Ch. Riedtmann [7].

1. Preliminaries on dualization

Given a finite-dimensional vector space ${}_FM$, denote by ${}^*M$ its (left) dual space $\mathrm{Hom}({}_FM, {}_FF_F)$. If ${}_FM_G$ is a bimodule and ${}_GX$, ${}_FY$ vector spaces, the adjoint map $\overline{f} : X \to {}^*M \underset{F}{\otimes} Y$ to a map $f : M \underset{G}{\otimes} X \to Y$ is given by $\overline{f}(x) = \sum_{p=1}^{d} \phi_p \otimes f(m_p \otimes x)$, where $x \in X$, $\{m_1, m_2, \ldots, m_d\}$ is a basis of ${}_FM$ and $\{\phi_1, \phi_2, \ldots, \phi_d\}$ is the respective dual basis of $({}^*M)_F$. In particular, if M is an End Y - End X-submodule of the bimodule $\mathrm{Hom}(X,Y)$ and $\chi_M : M \otimes X \to Y$ the evaluation map $\chi_M(m \otimes x) = m(x)$, then $\overline{\chi}_M(x) = \sum_p \phi_p \otimes m_p(x)$. Note that $\overline{\chi}_M$ is a (left) G-homomorphism.

Now, given bimodules ${}_FM_G$, ${}_GN_F$ such that ${}_FM$ and N_F are finite dimensional, let $\varepsilon : M \underset{G}{\otimes} N \to F$ be a non-degenerate bilinear form. Thus, the adjoint $\overline{\varepsilon}$ is an isomorphism $\overline{\varepsilon} : N \to {}^*M$; let $\{n_1, n_2, \ldots, n_d\}$ be a basis of N_F and $\{\phi_1, \phi_2, \ldots, \phi_d\}$ the basis of $({}^*M)_F$ such that $\phi_p = \overline{\varepsilon}(n_p)$ for all $1 \le p \le d$. Furthermore, let $\{m_1, m_2, \ldots, m_d\}$ be the dual basis of ${}_FM$. Thus,

$$\varepsilon(m_p \otimes n_q) = (m_p)[\overline{\varepsilon}(n_q)] = (m_p)\,\phi_q = \delta_{pq} .$$

Define the canonical element c_ε of $N \underset{F}{\otimes} M$ (with respect to ε) by

$$c_\varepsilon = \sum_{p=1}^{d} n_p \otimes m_p .$$

Lemma 1.1. *The element* c_ε *does not depend on the choice of a basis.*

Proof. Let $\{n_1', n_2', \ldots, n_d'\}$ and $\{m_1', m_2', \ldots, m_d'\}$ be another bases of N_F and ${}_FM$, respectively, so that

$$\varepsilon(m_p' \otimes n_q') = \delta_{pq} .$$

Then $n_q' = \sum_j n_j b_{jq}$ and $m_p' = \sum_i a_{pi} m_i$ with b_{jq} and a_{pi} from F.

Since $\delta_{pq} = \varepsilon(m_p' \otimes n_q') = \sum_{i,j} a_{pi}\, \varepsilon(m_i \otimes n_j) b_{jq} = \sum_i a_{pi} b_{iq}$, we have also $\sum_p b_{jp} a_{pi} = \delta_{ji}$.

Thus,

$$\sum_p n'_p \otimes m'_p = \sum_{i,j,p} n_j b_{jp} \otimes a_{pi} m_i$$

$$= \sum_{i,j} n_j \left(\sum_p b_{jp} a_{pi}\right) \otimes m_i = \sum_i n_i \otimes m_i . \bullet$$

If we take, in particular, ${}_G N_F = {}^*({}_F M_G)$ and the evaluation map $\chi : M \otimes_G N \to F$ defined by

$$\chi(m \otimes \phi) = (m)\phi ,$$

we obtain, for every bimodule M , *the canonical element* $c(M) = c_\chi$.

Given a bimodule ${}_F M_G$, define the higher dual spaces ${}^{(t)}{}_F M_G$ inductively by

$${}^{(t+1)}{}_F M_G = {}^*({}^{(t)}{}_F M_G) .$$

Thus, ${}^{(t)}M$ is an F-G-bimodule for t even and a G-F-bimodule for t odd.

Lemma 1.2. *Let* ${}_F M_G$ *and* ${}_G N_F$ *be bimodules and* $\varepsilon : {}_F M \otimes_G N_F \to {}_F F_F$ *and* $\delta : {}_G N \otimes_F M_G \to {}_G G_G$ *non-degenerate bilinear forms. Define the maps* ${}^t\eta$ *inductively as follows:*

$${}^0\eta = 1_M : {}_F M_G \to {}^{(0)}M = M ;$$

$${}^1\eta = \overline{\varepsilon} : {}_G N_F \to {}^{(1)}M = {}^*M ;$$

$${}^{2r}\eta = \overline{\delta[({}^{2r-1}\eta)^{-1} \otimes 1_M]} : {}_F M_G \to {}^{(2r)}M \quad \text{and}$$

$${}^{2r+1}\eta = \overline{\varepsilon[({}^{2r}\eta)^{-1} \otimes 1_N]} : {}_G N_F \to {}^{(2r+1)}M .$$

Then

$$[{}^{2r+1}\eta \otimes {}^{2r+2}\eta]\,(c_\varepsilon) = c({}^{(2r)}M) \quad \textit{and} \quad [{}^{2r}\eta \otimes {}^{2r+1}\eta]\,(c_\delta) = c({}^{(2r+1)}M).$$

Proof. Recall that $c_\varepsilon = \sum_p n_p \otimes m_p$, where $\{m_1, m_2, \ldots, m_d\}$ is a basis of ${}_F M$ and $\{n_1, n_2, \ldots, n_d\}$ the dual basis of N_F with respect to ε . Hence, in order to prove the first equality, it is sufficient to show that, for $m \in M$ and $n \in N$,

$$\delta(n \otimes m) = ({}^{2r+1}\eta(n))[{}^{2r+2}\eta(m)] .$$

But, $({}^{2r+1}\eta(n))[{}^{2r+2}\eta(m)] = ({}^{2r+1}\eta(n))[\overline{\delta[({}^{2r+1}\eta)^{-1} \otimes 1_M](m)}] =$

$$= \delta[({}^{2r+1}\eta)^{-1} \otimes 1_M]\,({}^{2r+1}\eta(n)) = \delta[({}^{2r+1}\eta)^{-1}\,{}^{2r+1}\eta(n) \otimes m] =$$

$$= \delta(n \otimes m)\ .$$

Similarly, since

$$({}^{2r}\eta(m))[{}^{2r+1}\eta(n)] = ({}^{2r}\eta(m))[\overline{\varepsilon[({}^{2r}\eta)^{-1} \otimes 1_N](n)}] =$$

$$= \varepsilon[({}^{2r}\eta)^{-1} \otimes 1_N]({}^{2r}\eta(m)) = \varepsilon[({}^{2r}\eta)^{-1}\,{}^{2r}\eta(m) \otimes n] =$$

$$= \varepsilon(m \otimes n)\ ,$$

we can derive the second equality for $c({}^{(2r+1)}M)$.

2. Irreducible maps

Recall the definition of an irreducible map [2]: a map $f : X \to Y$ is called irreducible if f is neither a split monomorphism nor a split epimorphism and if, for every factorization $f = f'f''$, either f'' is a split monomorphism or f' is a split epimorphism. Also, recall the definition of the radical of a module category. If X and Y are indecomposable modules, let rad (X,Y) be the set of all non-invertible homomorphisms. If $X = \bigoplus_p X_p$ and $Y = \bigoplus_q Y_q$ with indecomposable modules X_p and Y_q , define rad $(X,Y) = \bigoplus_{p,q}$ rad (X_p,Y_q) , using the identification $\mathrm{Hom}(X,Y) = \bigoplus_{p,q} \mathrm{Hom}\,(X_p,Y_q)$. The square $\mathrm{rad}^2(X,Y)$ of the radical is thus the set of all homomorphisms $f : X \to Y$ such that $f = f'f''$, where $f'' \in \mathrm{rad}(X,Z)$ and $f' \in \mathrm{rad}(Z,Y)$ for some module Z . Note that both rad and rad^2 are ideals in our module category; in particular, rad (X,Y) and $\mathrm{rad}^2(X,Y)$ are End Y - End X - submodules of the bimodule ${}_{\mathrm{End}\,Y}\mathrm{Hom}(X,Y)_{\mathrm{End}\,X}$. For indecomposable X and Y, the elements in rad $(X,Y)\setminus\mathrm{rad}^2(X,Y)$ are just the irreducible maps. In this case, we write $\mathrm{Irr}(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$, and call $\mathrm{Irr}(X,Y)$ the *bimodule of irreducible maps* (see [5]). In what follows, our main objective is to select a direct complement of $\mathrm{rad}^2(X,Y)$ in $\mathrm{rad}(X,Y)$ which is an EndY-EndX-submodule, and realize in this way

Irr(X,Y) as a subset of Hom(X,Y) rather than just as a factor group. We shall select such complements inductively, using Auslander-Reiten sequences.

Recall that an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is called an Auslander-Reiten sequence if both maps f and g are irreducible. This implies that both modules X and Z are indecomposable, X is not injective and Z is not projective. Conversely, given an indecomposable non-injective module X, there exists an Auslander-Reiten sequence starting with X, and also dually, given an indecomposable non-projective Z, there is an Auslander-Reiten sequence ending with Z. Moreover, if $0 \to X \xrightarrow{f} Y \to Z \to 0$ is an Auslander-Reiten sequence and $h : X \to X'$ is a map which is not a split monomorphism, then there exists $\alpha : Y \to X'$ such that $h = \alpha f$. (For all these properties, we refer to [2]).

In the sequel, we will consider direct sums of the form $\bigoplus_Y \mathcal{U}(Y)$, where $\mathcal{U}(Y)$ is an abelian group depending on Y, with Y ranging over "all" indecomposable modules. Here, of course, we choose first fixed representatives Y of all isomorphism classes of indecomposable modules and then index the direct sum by these representatives. In fact, all direct sum which will occur in this way will have even only a finite number of non-zero summands.

PROPOSITION 2.1. *Let* X *be an indecomposable non-injective module and* G *be a division ring with*

$$\text{End } X = G \oplus \text{rad End } X .$$

Assume that, for every indecomposable module Y, *there is given a direct complement* $M(X,Y)$ *of* $\text{rad}^2(X,Y)$ *in* ${}_{\text{End } Y}\text{rad}(X,Y)_G$. *Let*

$$0 \longrightarrow X \xrightarrow{(\overline{\chi}_{M(X,Y)})_Y} \bigoplus_Y {}^*M(X,Y) \underset{\text{End } Y}{\otimes} Y \xrightarrow{\pi} Z \longrightarrow 0$$

be exact. Then, this is an Auslander-Reiten sequence. Moreover, G *embeds into the endomorphism ring End* Z *of* Z *as a radical complement, and for every* Y, *there is an embedding* σ *of* ${}^*M(X,Y)$ *onto a complement of* $\text{rad}^2(Y,Z)$ *in* ${}_G\text{rad}(Y,Z)_{\text{End } Y}$ *such that*

$$\chi_{\sigma^* M(X,Y)} = \pi \mid {}^*M(X,Y) \otimes Y .$$

Proof. Let

$$0 \longrightarrow X \xrightarrow{(f'_{Y,p})_{Y,p}} \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y \longrightarrow Z' \longrightarrow 0$$

be an Auslander-Reiten sequence starting with X, where $f'_{Y,p} : X \to Y$ for $1 \leq p \leq d_Y$. Then the residue classes of the elements $f'_{Y,1}$, $f'_{Y,2}, \ldots, f'_{Y,d_Y}$ form a basis of the G-vector space $\mathrm{rad}(X,Y)_G/\mathrm{rad}^2(X,Y)_G$ (see Lemma 2.5 of [5]). Let $f_{Y,1}, f_{Y,2}, \ldots, f_{Y,d_Y}$ be a G-basis of $M(X,Y)$. By the factorization property of Auslander-Reiten sequences, there is a map

$$\alpha : \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y \longrightarrow \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y$$

such that $\alpha \circ (f'_{Y,p})_{Y,p} = (f_{Y,p})_{Y,p}$. It follows that α is an automorphism. For, let $E = \mathrm{End}\,(\bigoplus_Y \bigoplus_{p=1}^{d_Y} Y)$ and consider the residue class $\bar{\alpha}$ of α in $E/\mathrm{rad}\,E$. Also, consider the factor group

$$M = \mathrm{rad}(X, \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y)/\mathrm{rad}^2(X, \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y) ,$$

and let $\bar{f}$ and $\bar{f}'$ be the residue classes of $f = (f_{Y,p})_{Y,p}$ and $f' = (f'_{Y,p})_{Y,p}$, respectively. Then $\mathrm{rad}\,E$ annihilates M, and the equality $\bar{\alpha}\,\bar{f}' = \bar{f}$ shows that $\bar{\alpha}$ induces base changes between the bases $(\bar{f}_{Y,p})_p$ and $(\bar{f}'_{Y,p})_p$ of $\mathrm{Irr}(X,Y)$. This implies that $\bar{\alpha}$ is invertible. Since $\mathrm{rad}\,E$ is nilpotent, α is invertible, as well. Thus, we can form the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f'} & \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \| & & \downarrow \alpha & & \downarrow \beta & & \\
0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_Y \bigoplus_{p=1}^{d_Y} Y & \xrightarrow{\pi} & Z & \longrightarrow & 0 ,
\end{array}$$

where both α and β are isomorphisms. As a consequence, also the lower sequence is an Auslander-Reiten sequence.

Note that we can rewrite $\bigoplus_{p=1}^{d_Y} Y$ as $^{*}M(X,Y) \otimes_{\text{End } Y} Y$, and then $(f_{Y,p})_p$ becomes $\overline{\chi}_{M(X,Y)}$. For, if $\phi_{Y,1}, \phi_{Y,2}, \ldots, \phi_{Y,d_Y}$ is the dual basis of $^{*}M(X,Y)_{\text{End } Y/\text{rad End } Y}$ with respect to the basis $f_{Y,1}, f_{Y,2}, \ldots, f_{Y,d_Y}$ of $\text{End } Y/\text{rad End } Y^{M(X,Y)}$, then we identify

$$^{*}M(X,Y) \otimes_{\text{End} Y} Y = \bigoplus_{p=1}^{d_Y} \phi_{Y,p} \otimes Y \approx \bigoplus_{p=1}^{d_Y} Y ,$$

and

$$\overline{\chi}_{M(X,Y)}(x) = \sum_{p=1}^{d_Y} \phi_{Y,p} \otimes f_{Y,p}(x)$$

is identified with $(f_{Y,p}(x))_p$.

Now, $^{*}M(X,Y)$ is a left G-module, and

$$\overline{\chi}_{M(X,Y)} : X \longrightarrow {}^{*}M(X,Y) \otimes_{\text{End } Y} Y$$

is a G-module homomorphism. Hence, under $(\overline{\chi}_{M(X,Y)})_Y$, the module X becomes a G-submodule of $\bigoplus_Y {}^{*}M(X,Y) \otimes_{\text{End } Y} Y$, and therefore also the factor module Z has a left G-module structure. Thus, G embeds canonically into End Z and in this way, G becomes a radical complement. This follows from the canonical isomorphism

$$\text{End } X/\text{rad End } X \approx \text{End } Z/\text{rad End } Z ,$$

which is always valid for the outer terms of an Auslander-Reiten sequence.

The restriction of π to $^{*}M(X,Y) \otimes Y$ defines a map σ of $^{*}M(X,Y)$ into Hom(Y,Z) which is a G-End Y-homomorphism. If we denote again by $\phi_{Y,1}, \phi_{Y,2}, \ldots, \phi_{Y,d_Y}$ an End Y/rad End Y-basis of $^{*}M(X,Y)$, then $\pi \,|\, {}^{*}M(X,Y) \otimes_{\text{End } Y} Y \longrightarrow Z$ can be identified with

$$(\phi_{Y,p})_p : \bigoplus_{p=1}^{d_Y} Y \approx \bigoplus_{p=1}^{d_Y} \phi_{Y,p} \otimes Y \longrightarrow Z .$$

Again, using Lemma 2.5 of [5], we see that the residue classes of $\phi_{Y,1}, \phi_{Y,2}, \ldots, \phi_{Y,d_Y}$ in $\mathrm{Irr}(Y,Z)$ form an End Y/rad End Y-basis and that $^*M(X,Y)$ is therefore mapped injectively onto a complement of $\mathrm{rad}^2(Y,Z)$ in ${}_G\mathrm{rad}(Y,Z)_{\mathrm{End}\ Y}$. This completes the proof.

Now, assume that X is an indecomposable, non-injective module and that G is a radical complement in End X. If there are given direct complements $M(X,Y)$ of $\mathrm{rad}^2(X,Y)$ in ${}_{\mathrm{End}\ Y}\mathrm{rad}(X,Y)_G$, then the $\sigma^*M(X,Y)$ are direct complements of $\mathrm{rad}^2(Y,Z)$ in ${}_G\mathrm{rad}(Y,Z)_{\mathrm{End}\ Y}$, and the Auslander-Reiten sequence starting with X is of the form

$$0 \longrightarrow X \xrightarrow{(\overline{\chi}_{M(X,Y)})_Y} \oplus\ {}^*M(X,Y) \otimes Y \xrightarrow{(\chi_{\sigma^*M(X,Y)})_Y} Z \longrightarrow 0 .$$

Denote by $c(M(X,Y))$ the canonical element in $^*M(X,Y) \otimes M(X,Y)$. Now $\iota : M(X,Y) \hookrightarrow \mathrm{Hom}(X,Y)$ and $\sigma : {}^*M(X,Y) \hookrightarrow \mathrm{Hom}(Y,Z)$, and thus we have a canonical map

$$^*M(X,Y) \otimes M(X,Y) \longrightarrow \mathrm{Hom}(X,Z) ,$$

namely $\sigma \otimes \iota$ followed by the composition map μ.

PROPOSITION 2.2. *Under the map*

$$\bigoplus_Y {}^*M(X,Y) \otimes M(X,Y) \xrightarrow{\oplus(\sigma\otimes\iota)} \bigoplus_Y \mathrm{Hom}(Y,Z) \otimes \mathrm{Hom}(X,Y) \xrightarrow{(\mu)} \mathrm{Hom}(X,Z) ,$$

the element $\sum_Y c(M(X,Y))$ *goes to zero.*

Observe that, for a fixed module X, there is only a finite number of modules Y such that $M(X,Y) \approx \mathrm{Irr}(X,Y) \neq 0$; therefore, we may form the sum $\sum_Y c(M(X,Y))$.

Proof of Proposition 2.2. First, we are going to show that $c(M(X,Y))$ maps onto $\chi_{\sigma^*M(X,Y)} \circ \overline{\chi}_{M(X,Y)}$. Let $f_1, f_2, \ldots, f_d$ be an End Y/rad End Y-basis of ${}_{\mathrm{End}\ Y/\mathrm{rad\ End}\ Y}M = M(X,Y)$, and $\phi_1, \phi_2, \ldots, \phi_d$ the corresponding dual basis in $^*M_{\mathrm{End}\ Y/\mathrm{rad\ End}\ Y}$. Then, for $x \in X$, we have

$$\overline{\chi}_M(x) = \sum_p \phi_p \otimes f_p(x) ,$$

and for $\phi \in {}^*M,\ y \in Y$,

$$\chi_{\sigma^* M}(\phi \otimes y) = \sigma(\phi)(y) \ .$$

Thus,

$$\chi_{\sigma^* M}\, \overline{\chi}_M(x) = \chi_{\sigma^* M}\, (\sum_p \phi_p \otimes f_p(x)) = \sum_p \sigma(\phi_p)(f_p(x)) \ .$$

This shows that $\chi_{\sigma^* M}\, \overline{\chi}_M$ is equal to $\sum_p \sigma(\phi_p) f_p$, and this is the image of $\sum_p \phi_p \otimes f_p = c(M(X,Y))$ under $\mu(\sigma \otimes 1)$. As a consequence, we conclude that under the map $\bigoplus_Y {}^*M(X,Y) \otimes M(X,Y) \xrightarrow{\oplus(\sigma \otimes 1)} \bigoplus_Y \mathrm{Hom}(Y,Z) \otimes \mathrm{Hom}(X,Y) \xrightarrow{(\mu)} \mathrm{Hom}(X,Z)$, the element $\sum_Y c(M(X,Y))$ goes to $\sum_Y \chi_{\sigma^* M(X,Y)}\, \overline{\chi}_{M(X,Y)}$, which is the composite of the two maps in the corresponding Auslander-Reiten sequence and thus zero. The proof is completed.

Let us point out that, in what follows, we shall not specify any longer the embedding σ of ${}^*M(X,Y)$ into $\mathrm{Hom}(Y,Z)$, but shall simply consider ${}^*M(X,Y)$ to be a subset of $\mathrm{Hom}(Y,Z)$.

REMARK. Let us underline the use of the two distinct tensor products $M(X,Y) \otimes {}^*M(X,Y)$ and ${}^*M(X,Y) \otimes M(X,Y)$. Whereas the first one is used for the ordinary evaluation map

$$\chi : M(X,Y) \otimes {}^*M(X,Y) \longrightarrow \mathrm{End}\, Y/\mathrm{rad}\, \mathrm{End}\, Y$$

given by $\chi(f \otimes \phi) = f(\phi)$, it is the second one which has to be used for the composition map μ . Namely, using the above embedding ${}^*M(X,Y) \hookrightarrow \mathrm{Hom}(Y,Z)$, we can consider

$${}^*M(X,Y) \otimes M(X,Y) \hookrightarrow \mathrm{Hom}(Y,Z) \otimes \mathrm{Hom}(X,Y) \xrightarrow{\mu} \mathrm{Hom}(X,Z) \ ,$$

and $\mu(\phi \otimes f) = \phi \circ f$.

3. The preprojective modules

Now, let us consider the particular case of the irreducible maps between indecomposable preprojective $R(\mathfrak{M},\Omega)$-modules. First, recall the way in which these modules can be inductively obtained from the indecomposable projective ones.

For each $i \in I$, there is an indecomposable projective $R(\mathfrak{M},\Omega)$-module $P(i)$. Indeed, denoting by e_i the primitive idempotent of $R(\mathfrak{M},\Omega)$ corresponding to the identity element of the i^{th}

factor F_i in $R_o = \prod_i F_i$, $P(i) = e_i R(\mathfrak{M},\Omega)$. Note that $P(i)/\text{rad}\, P(i)$ is the simple $R(\mathfrak{M},\Omega)$-module corresponding to the vertex i which defines $P(i)$ uniquely up to an isomorphism. Moreover, note that $\text{End}\, P(i) = F_i$, and thus it is a division ring. The irreducible maps between projective modules are always rather easy to determine. Here, for $R(\mathfrak{M},\Omega)$, there are irreducible maps from $P(j)$ to $P(i)$ if and only if $i \to j$ in Ω . In fact, ${}_iM_j$ can be easily embedded in $\text{Hom}\,(P(j), P(i))$ in such a way that

$$ {}_iM_j \oplus \text{rad}^2(P(j), P(i)) = \text{rad}\,(P(j), P(i)) $$

as F_i-F_j-bimodules. This follows either from the explicit description of the modules $P(i)$ given in [3], or from the fact that $\oplus\, {}_iM_j$ is a direct complement of $\text{rad}^2 R(\mathfrak{M},\Omega)$ in $\text{rad}\, R(\mathfrak{M},\Omega)$. As a result, given two indecomposable projective $R(\mathfrak{M},\Omega)$-modules P and P' , we can always choose a direct complement $M(P,P')$ of $\text{rad}^2(P,P')$ in ${}_{\text{End}\,P'}\,\text{rad}(P,P')_{\text{End}\,P}$, and we can identify these $M(P,P')$ with the given bimodules ${}_iM_j$, where $i \to j$.

Now, the indecomposable preprojective modules can be derived from the projective ones by using powers of the Coxeter functor C^- (as defined in [3]) or of the Auslander-Reiten translation $A^- = \text{Tr}\, D$ ("transpose of dual" of [2], and also [1]). Thus, we denote by $P(i,r)$ the module obtained from $P(i)$ by applying the r^{th} power of one of the mentioned constructions. (It is clear from the uniqueness result in [3] that $C^{-r}\, P(i) \approx A^{-r}\, P(i)$.)

<u>LEMMA 3.1</u>. *Assume that* X *and* Y *are indecomposable modules and that there exists an irreducible map* $X \to Y$. *If one of the modules* X, Y *is preprojective, then both are. Furthermore, if* $X = P(i,r)$ *and* $Y = P(j,s)$ *, then either* $s = r$ *and* $i \leftarrow j$ *, or* $s = r+1$ *and* $i \to j$.

<u>Proof</u>. This lemma is well-known, so let us just outline a proof. Using shifts by powers of the Coxeter functors C^+ and C^- (see [3]) or of the Auslander-Reiten translations $A = D\,\text{Tr}$ and $A^- = \text{Tr}\, D$ (see [2] and [1]), we can assume that X is projective. If Y is not projective, then we get from the Auslander-Reiten sequence ending with Y , an irreducible map from AY to X .

Since X is projective, this map cannot be an epimorphism and thus it has to be a monomorphism. Consequently, AY is projective.

Now, in view of Proposition 2.1, we obtain by induction on the "layer" r of the indecomposable preprojective $R(\mathfrak{M},\Omega)$-modules $P(i,r)$ the following result.

PROPOSITION 3.2. a) *If we choose, for any two indecomposable projective modules* P *and* P', *a direct complement* $M(P,P')$ *of* $\operatorname{rad}^2(P,P')$ *in* ${}_{\operatorname{End} P'}\operatorname{rad}(P,P')_{\operatorname{End} P}$, *then this determines a direct complement* $M(P,P')$ *of* $\operatorname{rad}^2(P,P')$ *in* $\operatorname{rad}(P,P')$ *for any indecomposable preprojective modules* P, P'.

b) *If we identify, for any arrow* $i \to j$ *the bimodule* $M(P(j), P(i))$ *with* ${}_iM_j$, *then this yields an identification of any* $M(P(j,r), P(i,r))$ *with* ${}^{(2r)}{}_iM_j$ *and any* $M(P(i,r), P(j,r+1))$ *with* ${}^{(2r+1)}{}_iM_j$ *for* $i \to j$.

PROPOSITION 3.3. *Every map between two indecomposable preprojective modules is a sum of composites of maps from the various* $M(P,P')$.

Proof. Let Y be an indecomposable preprojective module, say $Y = P(i,r)$. Then the radical of the endomorphism ring E of $\bigoplus_{\substack{j \in I \\ 0 \le s \le r}} P(j,s)$ is generated (by using the addition and multiplication) by an arbitrary complement of $\operatorname{Rad}^2 E$ in $\operatorname{Rad} E$. So we may choose as a complement the direct sum of $M(P(j,s), P(j',s'))$.

4. Abstract definition of the full subcategory of the preprojective modules

First, let us introduce the following notation indicating the operation of the division rings F_i and F_j: For $i \to j$, put

$$ {}^{2r}{}_iM_j = {}^{(2r)}({}_iM_j) \quad \text{and} \quad {}^{2r+1}{}_jM_i = {}^{(2r+1)}({}_iM_j) \ . $$

Now, define the category $\mathcal{P}(\mathfrak{M},\Omega)$ as follows: The objects of $\mathcal{P}(\mathfrak{M},\Omega)$ are pairs (i,r), $i \in I$, $r \ge 0$ with the endomorphism rings F_i. For $i \to j$,

$$ M((j,r),(i,r)) = {}^{2r}{}_iM_j $$

and

$$M((i,r),(j,r+1)) = {}^{2r+1}{}_{j}M_{i} .$$

Denote by $F(\mathfrak{M},\Omega)$ the free category generated by these morphisms using the tensor products over F_i . Furthermore, for every (j,r), take

$$c(j,r) = \sum_{i\to j} c({}^{2r}{}_{i}M_{j}) + \sum_{j\to k} c({}^{2r+1}{}_{k}M_{j}) \in$$

$$\bigoplus_{i\to j} ({}^{2r+1}{}_{j}M_{i} \otimes {}^{2r}{}_{i}M_{j}) \oplus \bigoplus_{j\to k} ({}^{2r+2}{}_{j}M_{k} \otimes {}^{2r+1}{}_{k}M_{j}) ,$$

and denote by J the category ideal generated by all elements $c(j,r)$. The category $P(\mathfrak{M},\Omega)$ is then defined as the factor category of $F(\mathfrak{M},\Omega)$ by the ideal J .

Observe that the definition of $P(\mathfrak{M},\Omega)$ requires only the knowledge of the bimodules ${}_{i}M_{j}$ for $i \to j$ (and neither the corresponding bimodules ${}_{j}M_{i}$, nor the bilinear forms ε_{i}^{j} and ε_{j}^{i}).

PROPOSITION 4.1. *The full subcategory of the preprojective modules of the category of all* $T(\mathfrak{M},\Omega)$*-modules is equivalent to* $P(\mathfrak{M},\Omega)$.

Proof. Using Proposition 3.2, there is a canonical functor Γ from $F(\mathfrak{M},\Omega)$ to the subcategory of preprojective $T(\mathfrak{M},\Omega)$-modules given by the choice of $M(P(i),P(j)) = {}_{j}M_{i}$ for projective modules $P(i),P(j)$ where $j \to i$. Also by Proposition 3.3, Γ is surjective. Moreover, according to Proposition 2.2, the elements $c(j,r)$ are mapped to zero.

Conversely, let a morphism $f : (j,r) \to (j',r')$ from $F(\mathfrak{M},\Omega)$ be mapped under Γ to zero. We are going to show that f must lie in the ideal J . This is clear if $r = r'$; for, then $f = 0$. Thus, assume that $f \neq 0$ and proceed by induction on $r' - r$. Now j and r are fixed; let $\{\dots g_p \dots\}$ be the union of bases of all vector spaces ${}_{F_i}({}^{2r}{}_{i}M_{j})$ for all i with $i \to j$ and ${}_{F_k}({}^{2r+1}{}_{k}M_{j})$ for all k with $j \to k$, and let $\{\dots g'_p \dots\}$ be the union of the corresponding dual bases of $({}^{2r+1}{}_{j}M_{i})_{F_i}$ and $({}^{2r+2}{}_{j}M_{k})_{F_k}$.

Thus, $c(j,r) = \sum_p g'_p \otimes g_p$. Now, $f = \sum_p h_p \otimes g_p$, where h_p is a morphism of $\mathcal{F}(\mathfrak{M},\Omega)$ either from (i,r) or $(k,r+1)$ to (j',r'). Since there is an Auslander-Reiten sequence

$$0 \longrightarrow P(j,r) \xrightarrow{(\Gamma(g_p))_p} Q \xrightarrow{(\Gamma(g'_p))_p} P(j,r+1) \longrightarrow 0$$

and since

$$0 = \Gamma(f) = \sum_p \Gamma(h_p)\,\Gamma(g_p) ,$$

we can factor $(\Gamma(h_p))_p : Q \to P(j',r')$ through $(\Gamma(g'_p))_p$. Hence, there is a homomorphism $\tilde{u} : P(j,r+1) \to P(j',r')$ such that

$$\Gamma(h_p) = \tilde{u}\,\Gamma(g'_p) .$$

And, since Γ is surjective, we can find $u : (j,r+1) \to (j',r')$ in $\mathcal{F}(\mathfrak{M},\Omega)$ such that $\Gamma(u) = \tilde{u}$. Obviously, the elements $h_p - u \otimes g'_p$ lie in the kernel of Γ , and therefore, by induction, they belong to J . Consequently,

$$f = \sum_p h_p \otimes g_p = \sum_p (h_p - u \otimes g'_p) \otimes g_p + \sum_p u \otimes g'_p \otimes g_p$$

also belongs to J ; for, $\sum_p u \otimes g'_p \otimes g_p = u \otimes c(j,r)$.

5. Proof of the theorem

The proof of the theorem consists in identifying the additive structure of $\Pi(\mathfrak{M})$ with a factor of a subcategory of $\mathcal{F}(\mathfrak{M},\Omega)$. Indeed, we may consider both $\mathcal{F}(\mathfrak{M},\Omega)$ and $\mathcal{P}(\mathfrak{M},\Omega)$ defined in section 4 as abelian groups forming the direct sum of all $\mathrm{Hom}((i,r),(j,s))$. Denote by $\Phi(\mathfrak{M},\Omega)$ and $\Pi(\mathfrak{M},\Omega)$ the respective subgroups of all $\mathrm{Hom}((i,0),(j,s))$. Then, both $\Phi(\mathfrak{M},\Omega)$ and $\Pi(\mathfrak{M},\Omega)$ contain a subring $R = \bigoplus_{i,j} \mathrm{Hom}((i,0),(j,0))$ which is obviously isomorphic to $R(\mathfrak{M},\Omega)$. Furthermore, under the composition in $\Pi(\mathfrak{M},\Omega)$, $\Pi(\mathfrak{M},\Omega)$ is a right $R(\mathfrak{M},\Omega)$-module; for, if $f : (i,0) \to (j,s)$ and $a : (k,0) \to (i,0)$ from R , then $fa : (k,0) \to (j,s)$ in $\Pi(\mathfrak{M},\Omega)$.

PROPOSITION 5.1. $\Pi(\mathfrak{M},\Omega)_{R(\mathfrak{M},\Omega)}$ *is isomorphic to the direct sum of all*$^{\gamma}$ *preprojective* $R(\mathfrak{M},\Omega)$*-modules (each occurring with multiplicity one).*

$^{\gamma}$ = indecomposable

Proof. Using the notation of section 3, the indecomposable preprojective R-modules are $P(j,s)$, $j \in I$, $s \geq 0$. In particular, $P(j,0)$ are the indecomposable projective R-modules and thus $R_R = \bigoplus_{i \in I} P(i,0)$. For every R-module X_R ,

$$X_R \approx \operatorname{Hom}({}_R R_R, X_R) = \operatorname{Hom}({}_R[\bigoplus_i P(i,0)], X_R) =$$

$$= [\operatorname{Hom}(\bigoplus_i P(i,0)_R, X_R)]_R = [\bigoplus_i \operatorname{Hom}(P(i,0)_R, X_R)]_R .$$

Hence,

$$P(j,s) = [\bigoplus_i \operatorname{Hom}(P(i,0), P(j,s))]_R$$

and thus under the identification of $P(j,s)$ with (j,s) and $\operatorname{Hom}(P(i,0), P(j,s))$ with the maps in $\Pi(\mathfrak{M},\Omega)$, we get the statement.

Now, define the map $\Delta : \mathcal{T}(\mathfrak{M}) \to \mathcal{F}(\mathfrak{M},\Omega)$ as follows. First, the morphisms in $\mathcal{F}(\mathfrak{M},\Omega)$ can be described in the following way: For an (unoriented path) $w = i_{n+1} - i_n - \dots - i_2 - i_1$ of $\mathfrak{M}$, call the number of arrows $i_{t+1} \leftarrow i_t$, $1 \leq t \leq n$, in Ω the layer $\lambda(w)$ of w . Then, the morphisms in $\mathcal{F}(\mathfrak{M},\Omega)$ are the elements of the tensor products

$${}^{r_n}_{i_{n+1}}M_{i_n} \otimes \dots \otimes {}^{r_2}_{i_3}M_{i_2} \otimes {}^{r_1}_{i_2}M_{i_1} ,$$

where $r_t = 2\lambda(i_t - i_{t-1} - \dots - i_2 - i_1) + \begin{cases} 0 & \text{if } i_{t+1} \to i_t \\ 1 & \text{if } i_{t+1} \leftarrow i_t \end{cases}$.

Now, the map Δ is defined by

$${}_{i_{n+1}}M_{i_n} \otimes \dots \otimes {}_{i_3}M_{i_2} \otimes {}_{i_2}M_{i_1} \xrightarrow{{}^{r_n}\eta \otimes \dots \otimes {}^{r_2}\eta \otimes {}^{r_1}\eta} {}^{r_n}_{i_{n+1}}M_{i_n} \otimes \dots \otimes {}^{r_2}_{i_3}M_{i_2} \otimes {}^{r_1}_{i_2}M_{i_1} ,$$

where ${}^{r}\eta$ are the maps of Lemma 1.2 for $M = {}_iM_j$ and $N = {}_jM_i$.

From the definition of $\Phi(\mathfrak{M},\Omega)$, it is clear that $\Phi(\mathfrak{M},\Omega)$ is just the image of $\mathcal{T}(\mathfrak{M})$ under Δ. Also, Δ is obviously $R(\mathfrak{M},\Omega)$-linear.

LEMMA 5.2. $\Delta(\langle c \rangle) = J \cap \Phi(\mathfrak{M},\Omega)$.

Proof. By definition, $c = \sum_j (\sum_i c_j^i) = \sum_j c(j)$; note that $c(j) = e_j \, c \, e_j$, where e_j is the idempotent of $\mathcal{T}(\mathfrak{M})$ corresponding to the identity of F_j ; thus $\langle c\rangle$ is the ideal generated by all $c(j)$'s. Hence, the statement follows from Lemma 1.2 taking into account that, by definition,

$$\Delta(1 \otimes 1 \otimes \dots \otimes c(j) \otimes \dots \otimes 1) = 1 \otimes 1 \otimes \dots \otimes c({}^{r}M) \otimes \dots \otimes 1 .$$

Now, from Lemma 5.2, it follows that Δ defines an isomorphism of $\Pi(\mathfrak{M}) = T(\mathfrak{M})/\langle c\rangle$ onto $\Pi(\mathfrak{M},\Omega) = \Phi(\mathfrak{M},\Omega)/J \cap \Phi(\mathfrak{M},\Omega)$. This completes the proof of the theorem.

The corollaries follow from the results in [2].

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