TAME ALGEBRAS

(ON ALGORITHMS FOR SOLVING VECTOR SPACE PROBLEMS. II)

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Aim of these notes is to report on some of the main algorithms developed recently for solving vectorspace problems, the most impressing ones being due to the Kiev school of Nazarova and Roiter, it presents part of lectures *) held at the workshop on representation theory at Ottawa, 1979. For the presentation of these methods, we have choosen one particular question which can be handled without too many difficulties, namely the determination of all tame one-relation algebras. This question was considered lately by Shkabara [36] and Zavadskij [39], and we would like to outline a proof of their results which illustrates some of the recent techniques: on the one hand, the use of partially ordered sets and their representations, or, more generally, of vectorspace categories and their subspaces, and, on the other hand, that of irreducible maps, or the global Auslander-Reiten quiver. These two techniques usually are considered separately, indicating an affinity either to Kiev or Boston. However, they actually fit together very well, as we would like to demonstrate. Namely, partially ordered sets and vectorspace categories will be derived directly from certain Auslander-Reiten quivers. The pattern which appear in this way seem to be of independent interest; they fall into a small number of similarity classes. Some of them may be indexed by the extended Dynkin diagrams, they represent typical "non-domestic" tame vectorspace problems. This is a futher objective of these lectures: we would like to spread some better understanding of tame situations. Recall that an algebra is called tame provided there are at most one-parameter families of indecomposable modules, so that it is possible to obtain a complete classification of all indecomposable modules. We will call a tame algebra domestic in case there is a finite number of one-parameter families such that all other one-parameter familes are obtained by extending modules from F by themselves (for a precise definition, see 1.4). The non-domestic tame algebras seem to be of particular interest, and as we will see, they have some surprising properties. There are plenty of non-domestic tame algebras, however, those considered in these lectures will be associated to very few similarity classes of pattern.

In classifying tame algebras the usually difficult part is to establish that those algebras which are claimed to be tame actually are tame. There are three steps of insight:

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*) The first part dealt with the Brauer-Thrall conjectures, see [35]. The written texts of these two parts are mutually independent.

(i) the first is to determine just the representation type, without giving a complete list of all the indecomposable modules. In many situations, in most of the non-domestic ones, this is at present the only feasible goal;

(ii) the next step is to give a list of all the indecomposable modules, and perhaps even an algorithm for deciding whether a given module is isomorphic to one of the modules in the list, or an algorithm which decomposes a given module into a direct sum of indecomposable modules from the list. Only for very few vectorspace problems such a complete list of indecomposable modules is known;

(iii) the final step is to describe completely the category of modules, not only the indecomposable modules but also all the maps. As first approximation one would like to know all irreducible maps, thus the Auslander-Reiten quiver of the category.

Note that the Kiev school seems to be concentrated on the first goal, whereas Auslander usually stresses the third aspect of considering maps. In dealing with the tame one-relation algebras, we will try to gather as much information as possible. Now, for the domestic algebras it will be easy to establish not only the representation type, but also to determine all indecomposable modules, thus one should proceed to the third step aiming at a description of the Auslander-Reiten quiver. This is the goal of part 2 and section 3.7 of these lectures: we will discuss four constructions which can be used to built domestic algebras starting with tame quivers: concealments, finite enlargements, glueing of components and certain regular enlargements. In all cases it is not difficult actually to determine the new Auslander-Reiten quiver.

Let us stress again the fact that the determination of the Auslander-Reiten quiver of an algebra is not only of interest in itself, but that it can be used for the consideration of further enlargements. In part 3, this will be done, we will consider regular enlargements not only of tame quivers but also of algebras obtained before, and, in this way, many non-domestic tame algebras will be constructed. In the case of non-domestic algebras, we will content ourselves with establishing the representation type without considering further steps (ii) or (iii).

The problem of determining some classes of tame algebras has attracted a lot of attention, lately. Besides, the work of Shkabara and Zavadskij, we also should mention Marmaridis [24] who obtained part of the result of Shkabara using different techniques, namely those of Loupias [23]. Note that our approach is rather similar. On the other hand, S. Brenner also has considered many non-domestic tame algebras [8], using a generalization of the Bernstein-Gelfand-Ponomarev reflection functors developed in her joint work with Butler [9]. She considers with any algebra the corresponding quadratic form, an aspect which we usually will neglect. Also, there is a recent paper by Donovan-Freislich [14], which reduces the investigation of two non-domestic tame algebras (of type $(\tilde{D}_4, 2 \oplus 2)$, see 3.5) to a corresponding vectorspace problem *).

Finally, let us confess that our interest in the work of Shkabara and Zavadskij was motivated by the fact that the report at the conference was intended to include the theory of differential graded categories due to Kleiner and Roiter. In order to get a better understanding of this method it seemed to be convenient to follow its recent applications, and in particular, to see at what point the previously known methods were not strong enough for solving the problems. The differential graded categories were introduced as generalisation of the method of partially ordered sets, and, in fact, both Shkabara and Zavadskij need a further generalisation, namely differential $\mathbb{Z}$-graded categories. However, it turned out that, starting with most tame one-relation algebras, a rather straight forward reduction immediately leads to a subspace problem of a vectorspace category, and usually even to a partially ordered set. Also, in this way, we see that there is no intrinsic difference between quivers with a commutativity relation (as considered by Shkabara) and quivers with a zero relation (as considered by Zavadskij). In fact, in the same manner one can deal with all one-relation algebras, and even with many quivers with more relations.

These notes are organized as follows: there are reports on the two general techiques which will be used: the vectorspace categories including the additive categories of partially ordered sets (2.4), and

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*) Note that in contrast to a claim in the paper, Donovan-Freislich do not give a classification of the indecomposable modules, they determine only the representation type. There is a list of dimension types of indecomposable modules in the paper, which however is incomplete.

the Auslander-Reiten quivers (2.1). Also, we frequently will need the theory of tame quivers. The tame quivers and their representations have been classified by Donovan-Freislich [12] and Nazarova [26], however, use will be made of the full structure theory of the corresponding module category, as established in the joint work with Dlab [11], and there is a report on it in (2.2) and (3.2). The main notions on quivers without or with relations will be found in (1.1) and (1.2), the definition of the various representation types is in (1.4). The remaining sections are rather self-contained. In order to stress various techniques, we give proofs even of some very elementary facts, as the splitting zero relations in (1.3), and detailed information is included on many examples. In part 1, we present the classification of the tame one-relation algebras (theorem 1) and the minimal wild one-relation algebras (theorem 2). Also it is shown that the listed wild algebras actually are wild and some of the combinatorial arguments are provided which are needed to prove that every one-relation algebra is either a specialization of one of the algebras listed in theorem 1 or specializes to one of the algebras listed in theorem 2. However, the combinatorial arguments from Shkabara and Zavadskij are not repeated. Part 2 deals with some special constructions for obtaining domestic algebras from quivers. Here, our aim is always to determine the full Auslander-Reiten quiver. Part 3 considers regular enlargements and, in particular, we obtain a large amount of non-domestic tame algebras. Theorem 3 seems to be of interest: it classifies the vectorspace categories of the form $\mathrm{Hom}(M_\Gamma, M_\Gamma)$, M_Γ a regular Γ-module, with Γ a tame connected quiver, according to their representation type, and determines corresponding similarity classes. The one-relation algebras are not only our object of investigation; they turn out to provide also a method of proof. Namely, we will show that the non-domestic pattern which we encounter have rather strange properties (see 3.4), using certain one-relation algebras.

We assume throughout the paper that our base field k is algebraically closed *). All modules will be assumed to be finitely generated, and usually will be right modules. An additive category of modules is called finite provided it contains only a finite number of indecompo-

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*) Most of the result can be adapted to the case of an arbitrary (commutative) base field. Note that in contrast to a remark in [14], the extension to skew fields will provide substantial changes, since for a skew field D, the polynomial ring D[T] in one variable may be wild.

sable modules. A full subcategory is called cofinite provided there are only finitely many isomorphism classes of indecomposable objects which are not in the subcategory.

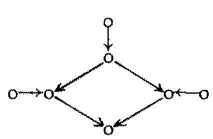
We are endebted to S. Brenner and V. Dlab for several discussions concerning quivers with commutativity relations. Also, I would like to express my thanks to Mrs. Fettköther and Mrs. Oberschelp for their careful and patient typing of this manuscript. Finally, I have to thank D. Happel, and M. Hänsch, A. Höwelmann, H. Klages, N. Kuberski, P.R. Kurth, W. Meier, R. Müller, P. Sudhölter, L. Unger, D. Vossieck for spotting many misprints and inaccura ies in a first draft of these notes.

1. THE TAME ONE-RELATION ALGEBRAS

As we have mentioned in the introduction, we want to report on some of the main techniques of present-day representation theory along the line of giving an outline of the classification of the tame one-relation algebras. Also, we would like, at the same time, to disseminate some feeling for "tame" situations. This last goal will be concentrated on in parts 2 and 3: here, in part 1, we will state the classification of the tame one-relation algebras, and we will deal with the easy part of the proof: that the listed algebras are the only ones which may be tame. Let us start by recalling the basic definitions. In this way, we also will fix the notation which will be used in the sequel.

1.1. Quivers

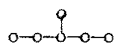
A quiver $\Gamma = (\Gamma_o, \Gamma_1)$ is given by a set Γ_o of "vertices" and a set Γ_1 of "arrows" such that to any arrow, there is assigned its starting point and its endpoint (these are two vertices which may coincide). For example, this is a typical quiver:

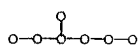


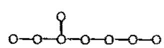
(Note that we usually will draw the vertices of a quiver as small circles, in contrast to the elements of partially ordered sets which we also have to draw rather frequently, and which will be drawn as points). The notion of a quiver and its representations was introduced by Gabriel [17] in order to formulate certain vectorspace problems rather efficiently, and it dominates now a rather large part of the representation theory. If k is a field, a representation of Γ over k is of the form (V_i, φ_α) where, for any vertex $i \in \Gamma_o$, we have the vectorspace V_i, and for any arrow $\underset{i}{\circ}\xrightarrow{\alpha}\underset{j}{\circ}$, we have the linear transformation $\varphi_\alpha : V_i \longrightarrow V_j$. If (V_i, φ_α) and (V_i', φ_α') are representations of Γ over k, then a map $\eta : (V_i, \varphi_\alpha) \rightarrow (V_i', \varphi_\alpha')$ is, by definition, of the form $\eta = (\eta_i)$, where $\eta_i : V_i \rightarrow V_i'$ is a linear transformation such that for any $\underset{i}{\circ}\xrightarrow{\alpha}\underset{j}{\circ}$, we have $\eta_j\varphi_\alpha = \varphi_\alpha'\eta_i$. In this way, the representations of Γ over k form an abelian category, which we will denote by M_Γ or $M_{k\Gamma}$ (it is the module category over the path algebra $k\Gamma$, its definition will be recalled in the next section). In particular, we usually are interested in the indecomposable representations. In [17] Gabriel has shown that a quiver has only a finite number of indecomposable representations if and only if it is the disjoint union of quivers of the form

A_n

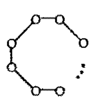
D_n 

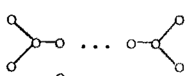
E_6 

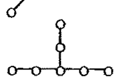
E_7 

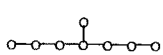
E_8 

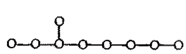
with arbitrary orientations of the edges. Some quivers with infinitely many indecomposable representations, the socalled extended Dynkin diagrams,

$\tilde{A}_n$ 

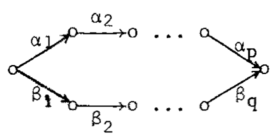
$\tilde{D}_n$ 

$\tilde{E}_6$ 

$\tilde{E}_7$ 

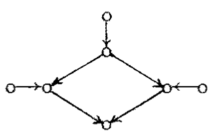
$\tilde{E}_8$ 

have attracted much interest ([12],[26]): they are the only connected quivers of tame representation type, and are even domestic (for the definition of the possible representation types see 1.4). The representation theory for any of these quivers with the exception of $\tilde{A}_n$, does not depend on the orientation of the edges; for $\tilde{A}_n$, it depends on the number of arrows in one direction. The prototype of the quiver with p arrows in clockwise direction and q arrows in counter clockwise direction, where $p+q = n+1$, is

$\tilde{A}_{p,q}$ 

The representation theory of the tame quivers will play a dominant role in the further investigations, it will be recalled in 2.2 and 3.2.

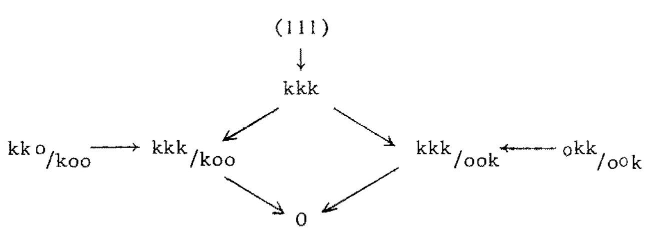
If $V = (V_i, \varphi_\alpha)$ is a representation of Γ where Γ is a quiver with vertices $\{1,\dots,n\}$ its <u>dimension type</u> $\underline{\dim}\, V$ is an element of $\mathbb{Q}^n$, with $(\underline{\dim}\, V)_i = \dim V_i$.
If there exists a unique indecomposable representation V of dimension type $x \in \mathbb{Q}^n$, we often will use x as a symbol for this representation V, or also for its isomorphism class. For example, in dealing with the quiver Γ,



the symbol

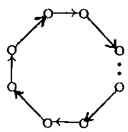
1
3
1 2 2 1
0

stands for the unique indecomposable representation V of this dimension type, namely



1.2. Quivers with relations

We have noted in the last section that for a quiver Γ the category $M_{k\Gamma}$ of all representations of Γ over k is just the module category over the <u>path algebra</u> $k\Gamma$ of Γ over k, defined as follows: If r,s are two vertices, a <u>path</u> from r to s of length p is of the form $(r|\alpha_1,\alpha_2,\dots,\alpha_p|s)$ where the starting point of α_1 is r, the starting point of any other α_i is equal to the end point of the previous α_{i-1}, and finally the end point of α_p is equal to s. Any vertex r gives rise to a path of length 0, namely $(r|r)$. The path algebra $k\Gamma$ has as vectorspace basis the set of all paths, the multiplication being the obvious one: the product of $(r|\alpha_1,\ldots,\alpha_p|s)$ and $(s|\beta_1,\ldots,\beta_q|t)$, in this order, will be $(r|\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q|t)$, and all other products will be zero. Note that $k\Gamma$ is a finite dimensional algebra if and only if Γ has only a finite number of vertices and contains no oriented cycles



If $w = (r|\alpha_1,\ldots,\alpha_p|s)$ is a path in Γ, and $V = (V_i,\varphi_\alpha)$ is a representation, we can evaluate w on V and obtain the linear transformation

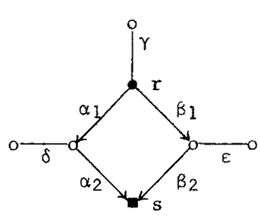
$$w(V) := \varphi_{\alpha_p} \cdot \ldots \cdot \varphi_{\alpha_1} : V_r \longrightarrow V_s.$$

A <u>relation</u> for Γ is a linear combination of paths of length ≥ 2 with same starting point and same end point, not all coefficients being zero. A representation V of Γ is said to <u>satisfy the relation</u> $\rho = \sum_{i=1}^{m} \kappa_i w_i$ or to be a representation of (Γ,ρ) if and only if $\sum_{i=1}^{m} \kappa_i w_i(V) = 0$. In considering relations $\sum_{i=1}^{m} \kappa_i w_i$, we always will assume that all coefficients $\kappa_i \neq 0$. If $m = 1$, then we will speak of a <u>zero relation</u>, and we may assume that the only coefficient is 1. Thus, the representations satisfying a fixed zero relation are those where the evaluation of a certain path is zero.
A relation of the form $w_1 - w_2$ with w_1, w_2 paths, will be called a <u>commutativity relation</u>. In case $w_1 = (r|\alpha_1,\ldots,\alpha_p|s)$, $w_2 = (r|\beta_1,\ldots,\beta_q|s)$, and the $p+q$ vertices which occur as starting points of the α_i or as end points of the β_i are pairwise different, we call $w_1 - w_2$ a <u>strict commutativity relation.</u>

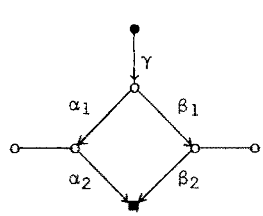
Given a set ρ_i $(i \in I)$ of relations for Γ, the category of representations of Γ over k satisfying ρ_i can be considered as the category M_R of R-modules, where $R = k\Gamma/\langle\rho_i|i \in I\rangle$, with $\langle\rho_i|i \in I\rangle$ the (twosided) ideal generated by the relations ρ_i.

Conversely, in case k is an algebraically closed field, and R is a basic k-algebra (so that R/rad R is a product of copies of k, with rad R the radical of R), then R is of the form $k\Gamma/\langle\rho_i | i \in I\rangle$ for a unique quiver Γ (of course, the generators ρ_i of the ideal $\langle\rho_i | i \in I\rangle$ are not uniquely determined, and we may change them conveniently). The case of one single relation ρ is of particular interest to us, we will call the algebras of the form $k\Gamma/\langle\rho\rangle$ <u>one-relation algebras</u>.

Our present aim is to classify the one-relation algebras according to their representation type. In order to simplify the notation, we will introduce the following convention: Assume Γ is a quiver without oriented cycle and r,s two vertices of Γ. Let Γ_1' be a subset of the set of arrows of Γ, which contains at least one path from r to s. Denote by $\rho = \Sigma\, w_i$ the relation for Γ which is given by the formal sum of all paths from r to s along arrows in Γ_1'. We will see that the representation type of the category M_R with $R = k\Gamma/\langle\rho\rangle$ does not depend on the orientation of the arrows outside Γ_1'. Thus it is sufficient to mark the two vertices r and s (we will draw r as a black circle, and s as a black square), and to note only the orientation of the arrows in Γ_1'. For example



stands for eight different quivers (obtained by adding all possible orientations to the edges $\gamma,\delta,\varepsilon$) together with the relation $\rho = (r|\alpha_1,\alpha_2|s)+(r|\beta_1,\beta_2|s)$, or simply and more suggestively $\alpha_2\alpha_1+\beta_2\beta_1 = 0$, whereas



stands for four quivers with relation $\alpha_2\alpha_1\gamma+\beta_2\beta_1\gamma = 0$. Note that in the first example, the ideal $<\rho>$ of $k\Gamma$ is also generated by a strict commutativity relation, namely $\alpha_2\alpha_1 = \beta_2\beta_1$, whereas in the second example, $<\rho>$ similarly is generated by the commutativity relation $\alpha_2\alpha_1\gamma = \beta_2\beta_1\gamma$, but this is not a strict commutativity relation.

Let Γ be a quiver with relations ρ_i, $i \in I$. There are several ways to obtain from these datas other quivers with relation. If the quiver Γ' with relations ρ_j', $j \in J$, is obtained by a sequence of processes of the following four types, then we will call $(\Gamma',\rho_j')_j$ a <u>specialisation</u> of $(\Gamma,\rho_i)_i$.

1. <u>Adding of relations</u>: let $\Gamma = \Gamma'$, and let $I \subseteq J$, and $\rho_i' = \rho_i$ for $i \in I$. Thus, we add additional relations ρ_j' with $j \in J \setminus I$. In this case, $k\Gamma/<\rho_i|i \in I>$ maps onto $k\Gamma'/<\rho_j'|j \in J>$, thus we have a full exact embedding

$$M_{k\Gamma'/<\rho_j'|j \in J>} \longrightarrow M_{k\Gamma/<\rho_i|i \in I>}$$

2. <u>Deleting of vertices</u>: Let a be some vertex of Γ, and let Γ' be obtained from Γ by deleting the vertex a, and all arows containing a. Also, delete from any relation ρ_i the summands which are multiples of paths going through a, and call the remaining linear combination ρ_i'. Let J be the subset of all $i \in I$ with $\rho_i' \neq o$ in $k\Gamma'$. This defines $(\Gamma',\rho_j')_{j \in J}$. The representations of $(\Gamma',\rho_j')_{j \in J}$ form the full subcategory of all representations V of $(\Gamma,\rho_i)_{i \in I}$ satisfying $V_a = o$. Thus again, we have a full exact embedding

$$M_{k\Gamma'/<\rho_j'|j \in J>} \longrightarrow M_{k\Gamma/<\rho_i|i \in I>}$$

which in this case even gives an extension closed subcategory.

3. <u>Deleting of arrows</u>. Let $\underset{a}{o}\xrightarrow{\beta}\underset{b}{o}$ be an arrow. Let Γ' be obtained from Γ by using the same set of vertices, but deleting the arrow β. If the path $w = (r|\alpha_1,\ldots,\alpha_p|s)$ occurs in the relation ρ_i, and $\beta = \alpha_q$ for some q, then we delete this summand from ρ_i, in this way obtaining a relation ρ_i' (or zero). Let J be the index set of the non-zero ρ_i'. There is a full and exact embedding

$$M_{k\Gamma'/<\rho_j'|j \in J>} \longrightarrow M_{k\Gamma/<\rho_i|i \in I>}$$

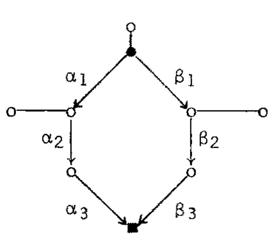
with an extension closed image. The representations of $(\Gamma',\rho'_j)_{j \in J}$ are just those (V_a,φ_α) with $\varphi_\beta = o$.

4. <u>Shrinking of arrows</u>. Let $\underset{a}{o}\xrightarrow{\beta}\underset{b}{o}$ be an arrow. Let Γ' be obtained from Γ by deleting β, and identifying the vertices a and b. If the path $(r|\alpha_1,\ldots,\alpha_p|s)$ occurs in the relation ρ_i, and $\beta = \alpha_q$ for some q, then replace this path by $(r|\alpha_1,\ldots,\alpha_{q-1},\alpha_{q+1},\ldots,\alpha_p|s)$. In this way it may happen that one of the paths, say w, occuring in ρ with a non-zero coefficient, becomes of length 1, say $\rho'_i = \kappa_1 w_1 + \sum_{i=2}^{n} \kappa_i w_i$ where the w_i are pairwise different paths, all $\kappa_i \neq o$, and w_1 is of length 1, given by the arrow γ. Then we delete γ in Γ' and have to replace γ by $-\sum_{i=2}^{n} \kappa_1^{-1}\kappa_i w_i$ in any path occuring in any relation ρ'_j. In this case, we obtain a surjection $k\Gamma/\langle\rho_i | i \in I\rangle \longrightarrow k\Gamma'/\langle\rho'_j | j \in J\rangle$ and this gives rise to a full exact embedding

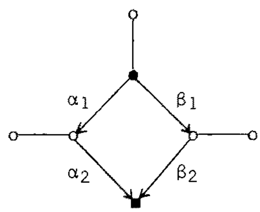
$$M_{k\Gamma'/\langle\rho'_j|j \in J\rangle} \longrightarrow M_{k\Gamma/\langle\rho_i|i \in I\rangle}$$

which again has as image an extension closed subcategory. Note that the representations of $(\Gamma',\rho'_j)_{j \in J}$ are just those representations $V = (V_a,\varphi_\alpha)$ for which φ_β is an identity map.

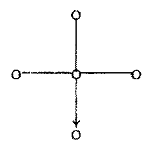
The first three processes of adding relations or deleting points or arrows are rather familiar to anyone, and usually very easy to detect. So let us give just an example for the process of shrinking arrows which is of interest for our study of one-relation algebras: Consider the quiver

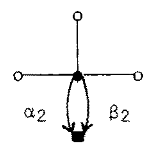


and recall our convention that this means that we are working with the relation $\alpha_3\alpha_2\alpha_1+\beta_3\beta_2\beta_1 = o$. If we shrink both the arrows α_3 and β_3, we obtain the quiver



(with relation $\alpha_2\alpha_1 = \beta_2\beta_1$). A further shrinking of the arrows α_1 and β_1 leads to a quiver



of type $\widetilde{D}_4$. (Here, we first have obtained 

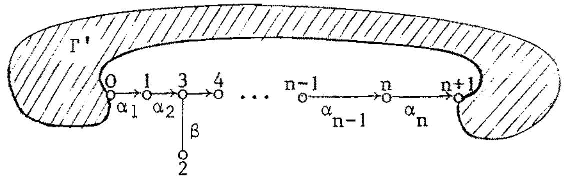
with $\alpha_2 = \beta_2$, so that we can delete β_2 together with the relation).

There is an additional process, namely the <u>dualisation</u>. In this case, we reverse all arrows of Γ and replace any relation ρ by the corresponding linear combination ρ' of the reversed paths. Then $k\Gamma'/\langle\rho_i' | i \in I\rangle$ is just the opposite algebra of $k\Gamma/\langle\rho_i | i \in I\rangle$, and the category $M_{k\Gamma'/\langle\rho_i' | i \in I\rangle}$ is the dual category to $M_{k\Gamma/\langle\rho_i | i \in I\rangle}$.

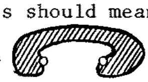
1.3. <u>Splitting zero relations</u>

We want to show that certain zero relations may easily be removed. This process is rather well-known, we will follow the presentation given by Zavadsky [39].

Assume Γ is of the form

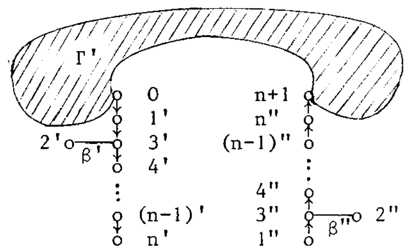


(this should mean that Γ is the union of some subquiver

Γ' =  which is not specified, and the subquiver

o→o→o ... o→o→o with the orientation of one edge not specified, and
  o
that these two subquivers intersect in precisely two vertices named 0 and n+1, but in no arrows). We are interested in the relation $\rho : \alpha_n \cdots \alpha_1 = 0$.

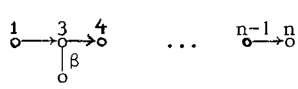
Define Δ to be the quiver of the form



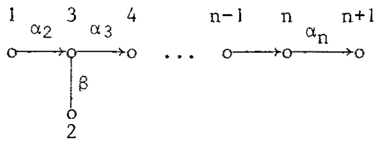
(with the same Γ', and an orientation of the edges β' and β'' corresponding to that of β).
Consider the canonical functor $\Phi : M_{k\Delta} \to M_{k\Gamma}$ which associates to a representation $V = (V_i, \varphi_\alpha)$ of Δ the representation W with $W|_{\Gamma'} = V|_{\Gamma'}$ and $W_i = V_{i'} \oplus V_{i''}$ for $1 \le i \le n$ (and correspondingly forming direct sums of the given linear transformations).

<u>Lemma 1</u>: A representation of $k\Gamma$ is of the form $\Phi(V)$ for some $V \in M_{k\Delta}$ if and only if it satisfies the relation ρ. The functor Φ induces a bijection between the indecomposable representations V in $M_{k\Delta}$ and in $M_{k\Gamma/<\rho>}$ which satisfy (in either category) $V_o \neq 0$ or $V_{n+1} \neq 0$. For any indecomposable representation W in $M_{k\Gamma/<\rho>}$ with $W_o = 0$ and $W_{n+1} = 0$, there are precisely two indecomposable representations V in $M_{k\Delta}$ with $\Phi(V) = W$.

Thus it follows that the functor Φ identifies precisely $n(n-1)$ pairs of indecomposable representations in $M_{k\Delta}$, since $n(n-1)$ is the number of indecomposable representations of



Proof: Let W be indecomposable in $M_{k\Gamma/\langle\rho\rangle}$. Consider its restriction to the subquiver Γ''

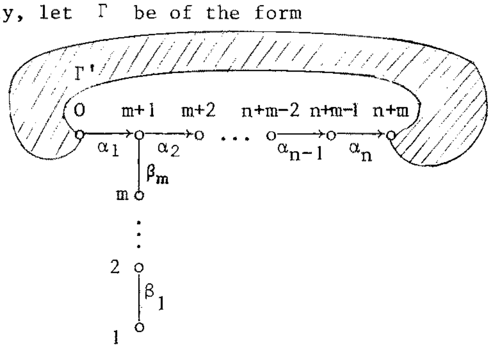


of type D_{n+1}. Let $W|_{\Gamma''} = X \oplus Y$, where Y is a direct sum of indecomposable Γ''-representations with non-zero $(n+1)$-component, and $X_{n+1} = 0$. Case by case inspection of the indecomposable Γ''-representations with non-zero $(n+1)$-component shows that in Y the composition of the maps $\alpha_n \dots \alpha_2$ is a monomorphism. Thus, since $\alpha_n \dots \alpha_1 = 0$, we see that

$$W_o \xrightarrow{\alpha_1} W_1 = X_1 \oplus Y_1$$

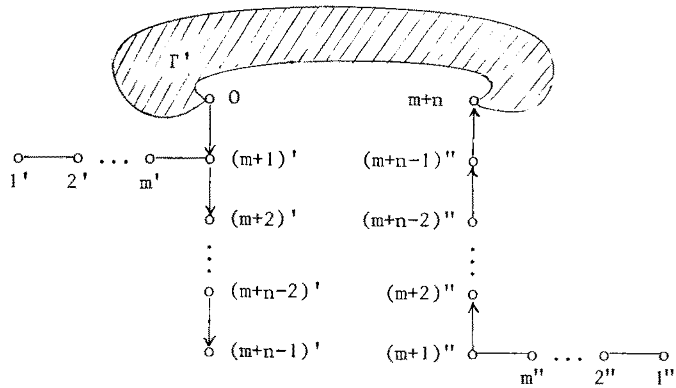
maps into X_1. Let V be the representation of Δ with restrictions: to Γ' being $W|_{\Gamma'}$, to the left arm being X, to the right arm being Y, and using the maps $\alpha_1 : W_o \to X_1$ and $\alpha_n : Y_n \to W_{n+1}$. Then $\Phi(V) = W$, and this is the unique such representation in case $W_{n+1} \neq 0$ or $W_o \neq 0$. If both $W_o = 0 = W_{n+1}$, then $Y = 0$, and there is a second representation $\tilde{V}$ of Δ with $\Phi(\tilde{V}) = W$, namely with X being the restriction of $\tilde{V}$ now to the right arm.

Similarly, let Γ be of the form



again with relation $\rho : \alpha_n \dots \alpha_1 = 0$.

Define Δ to be of the form



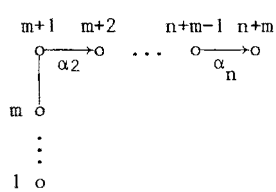
and the functor $\Phi : M_{k\Delta} \to M_{k\Gamma/<\rho>}$ with $\Phi(V)|_{\Gamma'} = V|_{\Gamma'}$ and $\Phi(V)_i = V_{i'} \oplus V_{i''}$ for $1 \leq i \leq n+m-1$.

Again we have

<u>Lemma 2</u>: The functor Φ induces a bijcetion between the indecomposable representations V in $M_{k\Delta}$ and in $M_{k\Gamma/<\rho>}$ which satisfy $V_o \neq 0$ or $V_{n+m} \neq 0$. For any indecomposable representation W in $M_{k\Gamma/<\rho>}$ with $W_o = 0 = W_{n+m}$, there are precisely two indecomposable representations V in $M_{k\Delta}$ with $\Phi(V) = W$.

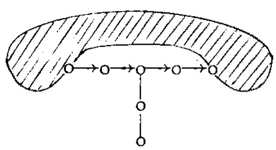
Thus, here the functor Φ identifies precisely $\frac{1}{2}(n+m)(n+m-1)$ pairs of indecomposable representations in $M_{k\Delta}$, and leaves the remaining ones distinct.

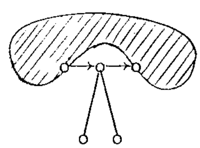
<u>Proof</u>: In this case, the restriction of a representation W of Γ to the subquiver Γ''



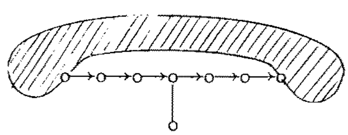
decomposes $W|_{\Gamma''} = X \oplus Y$ where $X_{n+m} = 0$ and $\alpha_n \ldots \alpha_2$ is an monomorphism in Y. Then again $\alpha_1 : W_o \to W_{m+1} = X_{m+1} \oplus Y_{m+1}$ maps into X_{m+1}. Define the representation V of Δ as follows: $V|_{\Gamma'} = W|_{\Gamma'}$, and the restriction to the left arm of Δ being X, to the right arm being Y, and use $\alpha_1 : W_o \to X_{m+1}$, $\alpha_n : Y_{n+m-1} \to W_{n+m}$.

We call relations of the two types above, and the dual ones (with all arows reversed) splitting zero relations. In considering the representation type of a quiver with relation we always may assume that no splitting zero relation occurs. Note that these relations are the only ones which can be separated in such an easy manner. Namely, in all other cases of a single zero-relation $(r|\alpha_1,\ldots,\alpha_p|s)$, there exists an indecomposable representation V satisfying this relation with both $V_r \neq 0$ and $V_s \neq 0$. Clearly, we only have to consider the following cases:

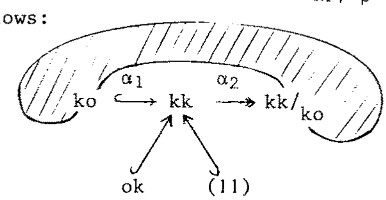


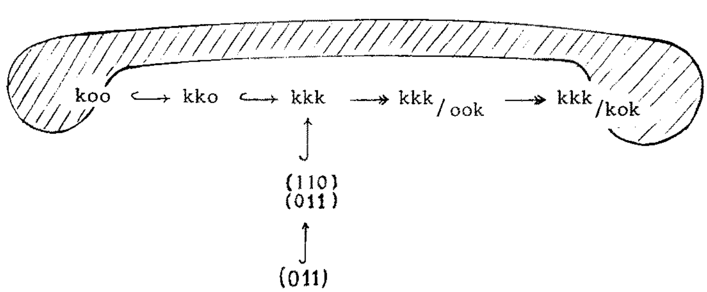


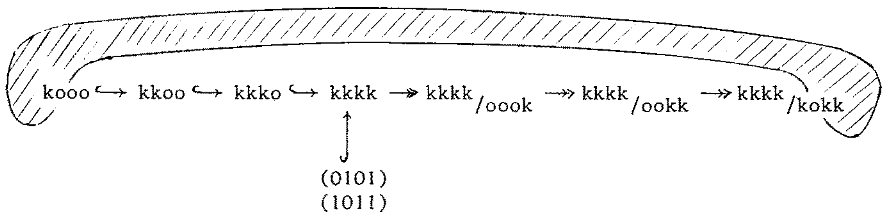
and



Of course, we can choose an arbitrary orientation of the free arms. Examples of representations V in $M_{k\Gamma/<\rho>}$ with $V_r \neq 0$ and $V_s \neq 0$ are as follows:



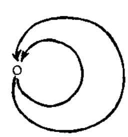




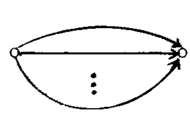
1.4. The representation types

Of course, a finite dimensional algebra R is said to be of finite representation type provided M_R is finite (has only finitely many indecomposable objects).

We will say that R is of wild representation type (or just that R is wild) provided there is an exact embedding of the category of representations of the quiver Ω



into M_R which is a representation equivalence with the corresponding full subcategory of M_R. (Note that we do not assume the embedding to be full). The path algebra $k\Omega$ of Ω is just the free associative k-algebra with two generators, also denoted by $k<X_1,X_2>$. The reason for calling such algebras wild stems from the fact that for any other finite dimensional k-algebra R', there is a full exact embedding $M_{R'} \to M_{k\Omega}$, in particular, there are full exact embeddings $M_{k\Omega_n} \to M_{k\Omega}$ where Ω_n is the n arrow quiver



for any n. For a discussion of categories of wild representation type, see [6, 19].

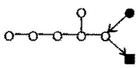
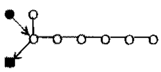
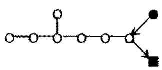
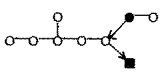
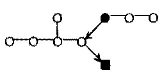
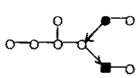
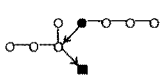
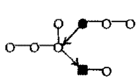
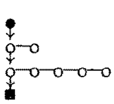
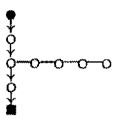
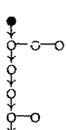
Finally, the algebra R is called to be of tame representation type provided R is not of finite representation type, whereas for any dimension d, there is a finite number of embedding functors $F_i : \mathcal{M}_{k[T]} \to \mathcal{M}_R$ such that all but a finite number of indecomposable R-modules of dimension d are of the form $F_i(M)$, for some i, and some indecomposable k[T]-module M. Note that if for some F_i, almost all $F_i(M)$ are indecomposable and pairwise non-isomorphic, then we will call this set a series of R-modules. In case there exists (independently of d) a finite number of such embedding functors F_i such that, for any dimension d, all but a finite number of indecomposable R-modules of dimension d are of the form $F_i(M)$, then R will be called domestic.

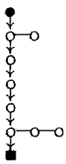
Of particular interest will be embedding functors $F_i : \mathcal{M}_{k[T]} \longrightarrow \mathcal{M}_R$ which are in addition full. As we will see, for the tame one-relation algebras studied in these notes, always such functors will exist. In this case, the irreducible k[T]-modules are mapped under F_i to a one-parameter family of indecomposable R-modules with endomorphism ring k.

Examples of domestic algebras are the path algebras of tame quivers. In case one deals with a connected tame quiver Γ, one only has to delete the images of one full embedding functor $F : \mathcal{M}_{k[T]} \longrightarrow \mathcal{M}_{k\Gamma}$ in order to remain just a finite set of isomorphism classes in any dimension. Also, there are known examples of non-domestic tame algebras, the first one seems to have been the algebras $k[T_1,T_2]/\langle T_1^a,T_2^b\rangle$ with $a \geq 2$, $b \geq 3$, studied by Gelfand and Ponomarev [21]. Further examples have been considered in [29, 33, 13].

1.5. The classification of the tame one-relation algebras

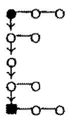
Theorem 1: *Let* Γ *be a connected quiver without oriented cycles, and* ρ *a relation for* Γ *which is not a splitting zero relation. If* (Γ,ρ) *is tame, then it is a specialization of one of the following tame one-relation algebras or their duals.*

1.		$123\overset{2}{4}3^{1}_{1}$	$C\tilde{E}_7$	Z6
2.		${}^{2^{3}}_{2}654321$	$C\tilde{E}_8$	Z8
3.		$24\overset{3}{6}543^{1}_{1}$	$C\tilde{E}_8$	Z9
4.		$24\overset{3}{6}54^{21}_{1}$	$C\tilde{E}_8$	Z10
5.		$24\overset{3}{6}5^{321}_{1}$	$C\tilde{E}_8$	Z11
6.		$24\overset{3}{6}5^{21}_{21}$	$C\tilde{E}_8$	Z12
7.		$24\overset{3}{6}{}^{4321}_{1}$	$C\tilde{E}_8$	Z13
8.		$24\overset{3}{6}{}^{321}_{21}$	$C\tilde{E}_8$	Z14
9.		1 42 54321 2	$C\tilde{E}_8$	Z15
10.		1 3 54321 3 1	$C\tilde{E}_8$	Z16
11.		0 321 4 5 63 42	$F\tilde{E}_8$	Z34

12. 

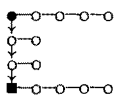
0
21
3
4
5
642
3

$F\tilde{E}_8$ Z37

13. 

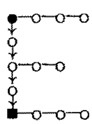
000	321
21	42
3	3
42	21
321	000

$G\tilde{E}_7$-$\tilde{E}_7$ Z33

14. 

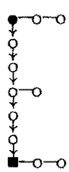
00000	54321
42	63
63	42
54321	00000

$G\tilde{E}_8$-$\tilde{E}_8$ Z31

15. 

0000	4321
3	5
642	642
5	3
4321	0000

$G\tilde{E}_8$-$\tilde{E}_8$ Z32

16. 

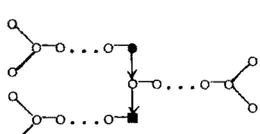
000	321
2	4
4	5
63	63
5	4
4	2
321	000

$G\tilde{E}_8$-$\tilde{E}_8$ Z35

17. 

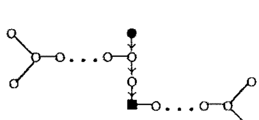
00	42
21	63
3	5
4	4
5	3
63	21
42	00

$G\tilde{E}_8$-$\tilde{E}_8$ Z36

18. 

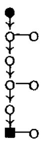
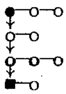
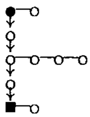
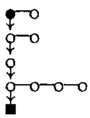
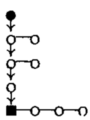
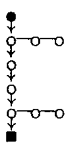
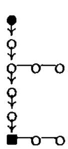
$^{0}_{0}$0...0	$^{1}_{1}$2...2
2...2$^{1}_{1}$	2...2$^{1}_{1}$
$^{1}_{1}$2...2	$^{0}_{0}$0...0

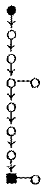
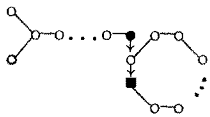
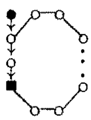
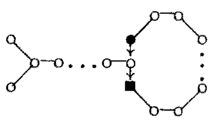
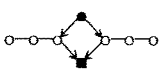
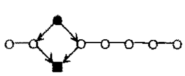
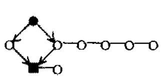
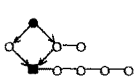
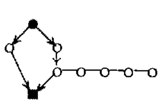
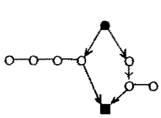
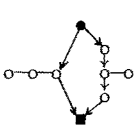
$N\tilde{D}_n$ Z20

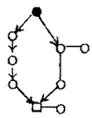
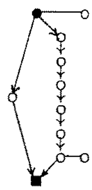
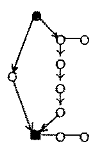
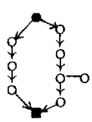
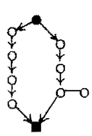
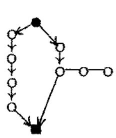
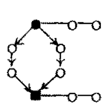
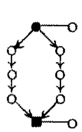
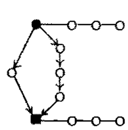
19. 

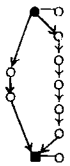
0	1
$^{1}_{1}$2...22	$^{1}_{1}$2...22
2	1
2...2$^{1}_{1}$	0...0$^{0}_{0}$

$N\tilde{D}_n$ Z21

20.		0 21 3 42 3 21	1 21 2 21 1 00		$N\tilde{E}_7$	Z26
21.		000 21 321 21	321 42 321 00		$N\tilde{E}_8$	Z22
22.		00 2 4321 3 21	21 3 4321 2 00		$N\tilde{E}_8$	Z23
23.		00 21 3 4321 2	42 63 5 4321 0	21 42 4 4321 1	$N\tilde{E}_8$	Z24
24.		0 42 63 5 4321	1 21 21 1 0000		$N\tilde{E}_8$	Z25
25.		0 321 4 5 642 3	3 642 5 4 321 0	1 321 3 3 321 1	$N\tilde{E}_8$	Z27
26.		0 3 642 5 4 321	1 2 321 2 1 000		$N\tilde{E}_8$	Z28
27.		0 21 3 4 5 63 4 2	1 21 2 2 2 21 1 0		$N\tilde{E}_8$	Z29

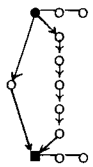
28.		0 1 2 2 4 3 63 42 5 3 4 2 3 1 21 00	$N\tilde{E}_8$	Z30
29.		0 0 11 00...1 1 1 11 ⋮	$R(\tilde{A}_{n,p})$	
30.		0 11 1 1 1 ⋮ 1 11 1	$R(\tilde{A}_n, {}^P_p)$	
31.		1 11 1 1 2...22 ⋮ 1 1 11 1	N	
32.		123^2_2321	$C\tilde{E}_7$	S8
33.		24^3_354321	$C\tilde{E}_8$	S7
34.		2 3 54321 42	$C\tilde{E}_8$	S9
35.		1 3 42 54321	$C\tilde{E}_8$	S10
36.		1 3 2 54321 3	$C\tilde{E}_8$	S5
37.		2 3 1234 3 42	$C\tilde{E}_8$	S6
38.		2 3 123 42 2 3	$C\tilde{E}_8$	S4

39.		1 0 2 21 2 3 3 42	$F\tilde{E}_7$	S16
40.		0 1 0 2 3 2 4 5 4 63	$F\tilde{E}_8$	S11
41.		0 21 3 3 4 5 642	$F\tilde{E}_8$	S12
42.		0 2 1 4 2 63 3 5 4	$F\tilde{E}_8$	S13
43.		1 0 2 2 4 3 63 4 5	$F\tilde{E}_8$	S14
44.		1 0 3 2 642 3 4 5	$F\tilde{E}_8$	S15
45.		0 00 3 21 1 1 2 2 2 2 1 1 3 21 0 00	$G\tilde{E}_6$-$\tilde{E}_6$	S42
46.		0 0 4 2 1 1 3 3 2 2 2 2 3 3 1 1 4 2 0 0	$G\tilde{E}_7$-$\tilde{E}_7$	S40
47.		0 000 4 321 1 3 2 2 2 2 3 1 4 321 0 000	$G\tilde{E}_7$-$\tilde{E}_7$	S39

48. 

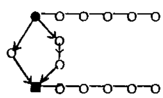
```
  0 0     6 3
   1       5
 2 2     4 4
   3       3
 4 4     2 2
   5       1
  6 3     0 0
```

$G\tilde{E}_8-\tilde{E}_8$ S38

49. 

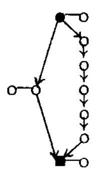
```
  0 00    6 42
   1       5
   2       4
 3 3     3 3
   4       2
   5       1
  6 42    0 00
```

$G\tilde{E}_8-\tilde{E}_8$ S37

50. 

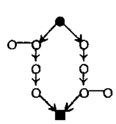
```
 0 00000   6 54321
3 2       3 4
  4         2
  54321    0 00000
```

GE_8-E_8 S41



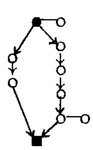
```
   0 0       6 3
    1         5
    2         4
23  3     24  3
    4         2
    5         1
   6 3       0 0
```

$G\tilde{E}_8-G\tilde{E}_8$ S35

52. 

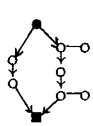
```
   0        5
12 2     36 4
 3 4      4 3
 4 63     2 21
  5         0
```

$G\tilde{E}_8-\tilde{E}_8$ S36

53. 

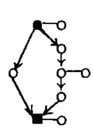
```
  0 0      6 3
1  1     4  5
   2        4
2  3     2  3
   4 2      21
 3         0
```

$G\tilde{E}_7-\tilde{E}_8$ S34

54. 

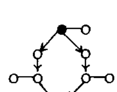
```
 0         0        1
1 21     2 42     1 21
  3         3        2
2 42     1 21     1 21
 3         3        1
```

$N\tilde{E}_7$ S18

55. 

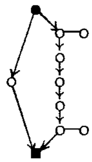
```
  0 0      2 1
   1        2
 1 21     1 21
   2        1
  2 1      0 0
```

$N\tilde{E}_7$ S26

56. 

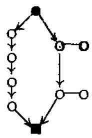
```
  0 0      4 2      2 1
 1 1       3 3      2 2
12 21     12 21    12 21
  2         0        1
```

$N\tilde{E}_7$ S29

57. 

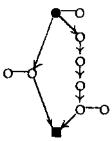
```
 0          4          1          6          2
  21         63         21         105        63
  3          5          2          9          7
2 4        2 4        1 2       4  8       4 8
  5          3          2          7          9
  63         21         21         63         105
 4          0          1          2          6
```

$\widetilde{NE}_8$ S17

58. 

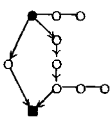
```
 0        5        1
1        4        1
  42       63       21
2        3        1
3        2        1
  63       42       21
4        1        1
 5        0        1
```

$\widetilde{NE}_8$ S19

59. 

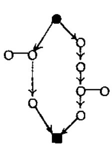
```
  0 0      6 3      2 1
   1        5        2
   2        4        2
12       24       12
   3        3        2
   42       21       21
  3        0        1
```

$\widetilde{NE}_8$ S20?

60. 

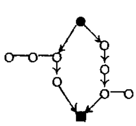
```
 0 00     6 42     3 21
   1        5        3
1  2     3  4     2  3
   321      321      321
 2        0        1
```

$\widetilde{NE}_8$ S21

61. 

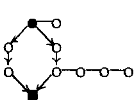
```
  0          2          1
     2          2          2
12         12         12
     4          2          3
     63         21         42
3           1
     5          1          3
  4          0          2
```

$\widetilde{NE}_8$ S22?

62. 

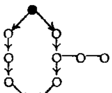
```
    0        5
     2        4
123      246
     4        3
5        3
     63       21
    5        0
```

$\widetilde{NE}_8$ S23

63. 

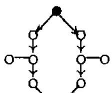
```
 0 0      6 3
1 2      4 5
2 4321   2 4321
 3        0
```

$\widetilde{NE}_8$ S24

64. 

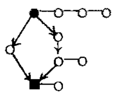
```
 0       4        1
1 3     3 5        2
2 642   2 642    1 321
3 5     1 3        2
 4       0        1
```

$\widetilde{NE}_8$ S25

65. 

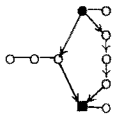
```
 0       2
1 1     2 2
12 21   12 21
2 2     1 1
 2       0
```

$\widetilde{NE}_8$ S27

66. 

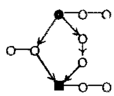
```
 0 000      4 321
1 1        2 3
   21         21
 2 1        0
```

$N\tilde{E}_8$ S28

67. 

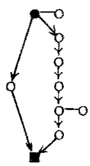
```
    0 0        4 2        2 1
      1          3          2
123  2     123  2     123  2
      3          1          2
    4 2        0 0        2 1
```

$N\tilde{E}_8$ S30

68. 

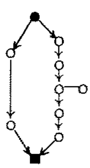
```
  0 00        3 21
12 1       12 2
    2           1
  3 21        0 00
```

$N\tilde{E}_8$ S31

69. 

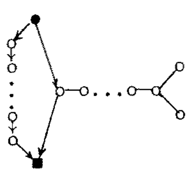
```
 0 0       2 1
   1         2
   2         2
1  3      1  2
   42        22
   3         1
 2         0
```

$N\tilde{E}_8$ S32

70. 

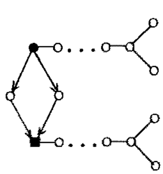
```
   0         3          1
1  2      2  4       1  2
   4         5          3
   63        63         42
2  5      1  4       1  3
   4         2          2
   3         0          1
```

$N\tilde{E}_8$ S33

71. 

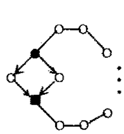
```
 1
1
:        1
:  2...2
:        1
1
 1
```

$C\tilde{D}_n$ S2

72. 

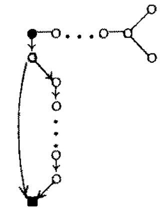
```
        0          1
 0...0     2...2
        0          1
1 1        1 1
        1          0
 2...2     0...0
        1          0
```

$N\tilde{D}_n$ S_3

73. 

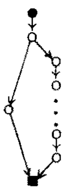
```
  11
 1  1
1 1  :
 1  1.
  11
```

N S1

74. 

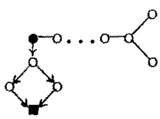
0...0 0 0
1
 1
 :
 1
1

$R\tilde{A}_{1q}$

75. 

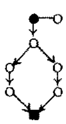
0
1
 1
1 :
 1
1

$R\tilde{A}_{2q}$

76. 

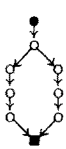
0...0 0 0   2...2 1 1
1   2
1 1   1 1
1

$N\tilde{D}_n$

77. 

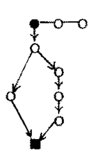
00   21
1   3
1 1   2 2
1 1   1 1
1   0

$N\tilde{E}_6$

78. 

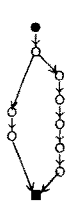
0   2
1   4
1 1   3 3
1 1   2 2
1 1   1 1
1   0

$N\tilde{E}_7$

79. 

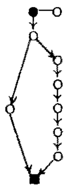
000   321
1   4
 1   3
1 1   2 2
 1   1
1   0

$N\tilde{E}_7$

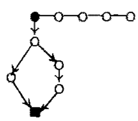
80. 

0   3
1   6
 1   5
1 1   4 4
1 1   2 3
 1   2
 1   1
1   0

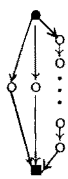
$N\tilde{E}_8$

81. 

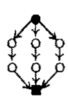
0 0 1 1 1 1 1 1 1 1	4 2 6 5 4 3 3 2 1 0	$\widetilde{NE}_8$

82. 

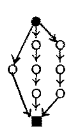
0 0000 1 1 1 1 1	5 4321 6 3 4 2 0	$\widetilde{NE}_8$

83. 

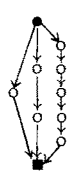
1 1 11 : 1 1	$\widetilde{CD}_n$

84. 

0 111 222 3	3 222 111 0	1 111 111 1	$\widetilde{NE}_6$

85. 

0 11 222 33 4	4 33 222 11 0	1 11 111 11 1	$\widetilde{NE}_7$

86. 

0 1 22 3 3 44 5 6	6 5 44 3 3 22 1 0	1 1 11 1 1 11 1 1	$\widetilde{NE}_8$

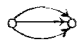
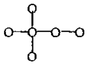
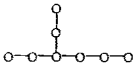
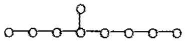
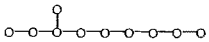
We have first listed the quivers with a zero relation, starting with those without cycles, followed by those with cycles. Next, there are listed those algebras which are given by one commuting cycle with additional arms, and then the remaining quivers with a strict commutativity relation. After this, there are the quivers with a commutativity relation which is not strict, and finally quivers with a relation involving three different paths. For all types, we have grouped together the algebras with similar categories of modules.

Always, we have noted the representation type of the given one-relation algebras. In fact, for the domestic algebras, we will provide rather detailed information on the whole module category, and we distinguish here four different possibilities. First, the symbol C* refers to a concealed quiver of type * in the sense of 2.3 (thus, the first ten algebras all are concealed quivers). We will see that the module category of a concealed quiver is rather similar to that of the corresponding quiver. All other algebras are enlargements of tame quivers, and we denote by F_* a finite enlargement (see 2.6), by R_* a domestic regular enlargement (see 3.7). In these cases C,F,R, there exists precisely one one-parameter family of modules with trivial endomorphism ring. The last domestic case to be considered is the case where two tame connected quivers are glued together, so that there are precisely two one-parameter families of modules with trivial endomorphism ring. This case will be denoted by G*-**, referring to the glueing of a quiver of type * with a quiver of type ** and will be considered in 2.7, these are the domestic cases.
The non-domestic algebras are denoted by N. There are two types of algebras which have to be considered separately in 3.9. All others are non-domestic regular enlargements as considered in part 3, and we have added the similarity type * in writing N_*.

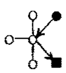
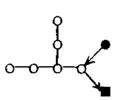
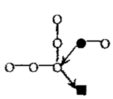
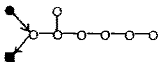
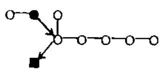
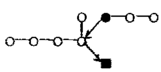
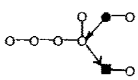
Also, we have listed positive vectors which generate (using non-negative integral linear combinations) the set of all positive vectors on which a corresponding quadratic form takes value zero ([7]). Note that in case the algebra is obtained by glueing two tame quivers together (case G), only the multiples of the given vectors are roots of the quadratic form, whereas for the non-domestic algebras the quadratic form is positiv semi-definite, so that the roots of the quadric form are closed under addition. It is rather easy to see that all listed vectors actually are dimension vectors of modules with trivial endomorphism ring belonging to a series of such modules.

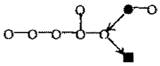
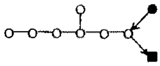
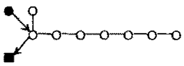
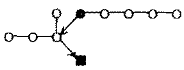
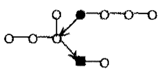
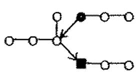
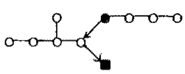
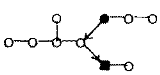
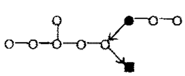
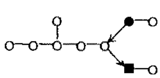
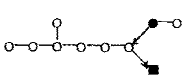
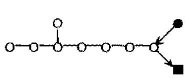
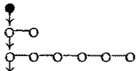
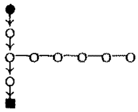
Theorem 2: Let Γ be a connected quiver without oriented cycles, and ρ a relation for Γ. If (Γ,ρ) is not of finite or tame representation type, then it specializes either to a wild quiver with splitting zero relation, or to one of the following wild one-relation algebras or their duals.

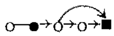
Quivers without relation:

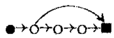
			
			
		$\overset{\approx}{D}_4$	ZII S.TIII
		$\overset{\approx}{E}_6$	ZIII S.TIV
		$\overset{\approx}{E}_7$	ZIV S.TV
		$\overset{\approx}{E}_8$	ZV S.TVI

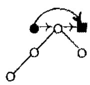
Quivers with one relation:

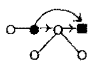
I		$C\overset{\approx}{D}_4$	ZVI
II		$C\overset{\approx}{E}_6$	ZVII
III		$C\overset{\approx}{E}_6$	ZVIII
IV		$C\overset{\approx}{E}_6$	ZIX
V		$C\overset{\approx}{E}_7$	ZX
VI		$C\overset{\approx}{E}_7$	ZXI
VII		$C\overset{\approx}{E}_7$	ZXII
VIII		$C\overset{\approx}{E}_7$	ZXIII

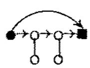
IX		$C\overset{\approx}{E}_7$	ZXIV
X		$C\overset{\approx}{E}_7$	ZXV
XI		$C\overset{\approx}{E}_8$	ZXVI
XII		$C\overset{\approx}{E}_8$	ZXVII
XIII		$C\overset{\approx}{E}_8$	ZXVIII
XIV		$C\overset{\approx}{E}_8$	ZXIX
XV		$C\overset{\approx}{E}_8$	ZXX
XVI		$C\overset{\approx}{E}_8$	ZXXI
XVII		$C\overset{\approx}{E}_8$	ZXXII
XVIII		$C\overset{\approx}{E}_8$	ZXXIII
XIX		$C\overset{\approx}{E}_8$	ZXXIV
XX		$C\overset{\approx}{E}_8$	ZXXV
XXI		$C\overset{\approx}{E}_8$	ZXXVI
XXII		$C\overset{\approx}{E}_8$	ZXXVII

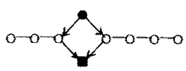
XXIII  $R\tilde{A}_{12}$

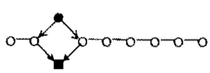
XXIV  $R\tilde{A}_{13}$

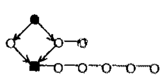
XXV 

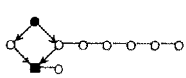
XXVI 

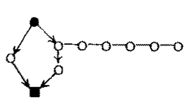
XXVII 

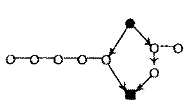
XXVIII  $C\tilde{\tilde{E}}_7$ S.TX

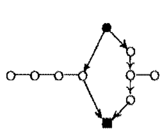
XXIX  $C\tilde{\tilde{E}}_8$ S.TVII

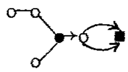
XXX  $C\tilde{\tilde{E}}_8$ S.TIX

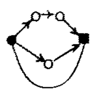
XXXI  $C\tilde{\tilde{E}}_8$ S.TVIII

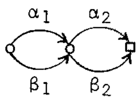
XXXII  $C\tilde{\tilde{E}}_8$ S.TXII

XXXIII  $C\tilde{\tilde{E}}_8$ S.TXI

XXXIV  $C\tilde{\tilde{E}}_8$ S.TXIII

XXXV  $R\tilde{A}_{11}$

XXXVI 

XXXVII Γ = 

$\alpha_2\alpha_1-\beta_2\beta_1$ $R\tilde{A}_{11}$

Four of the wild quivers are denoted by $\tilde{\tilde{D}}_4$, $\tilde{\tilde{E}}_6$, $\tilde{\tilde{E}}_7$, $\tilde{\tilde{E}}_8$, since obviously they relate to the corresponding extended Dynkin diagram in the same way, as these extended Dynkin diagrams relate to the Dynkin diagrams D_4, E_6, E_7, E_8. Most of the wild quivers with relations listed above are concealed quivers in the sense of 2.3, the symbol $C*$ indicates that we deal with a concealment of a quiver of type $*$. Some algebras are regular enlargements of a quiver Γ (see part 3), they are denoted by $R*$, with $*$ denoting the type of Γ. There are four cases which are neither concealed quivers nor regular enlargements of quivers, and which have to be considered separately.

Theorem 1 and 2 essentially are due to Shkabara and Zavadskij who considered the following two important special cases: Zavadskij [40] classified the quivers without cycles (oriented or not) with a single relation (which therefore has to be a zero relation) which are tame. In our lists above, we have added the corresponding numers of Zavadskij's list with the symbol Z. (Since Zavadskij's list contains with any tame algebra also its specialisations, not all numbers from his list appear here). Similarly, Shkabara [36] has classified the quivers with one strict commutativity relation which are tame, we refer to his list by the symbol S. Note however, that Shkabara does not exclude oriented cycles, so his result is more general, but in essence, he does not obtain additional algebras. Also, Marmaridis [24] has considered the algebras which are obtained from a commuting cycle by adding arms, and he determined the representation type in all but three cases (54,55,56).

1.6. Outline of proof

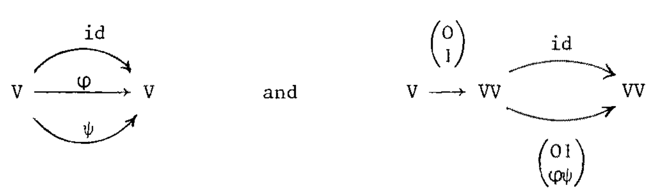
The proof of theorems 1 and 2 uses three different types of arguments.

(a) We have to show that the one-relation algebras listed in theorem 1 are tame. This will be the most interesting part, and we will use most of sections 2 and 3 for this part of the proof. Besides developping some rather general methods we will try to give a good insight into the actual behaviour of the corresponding module categories. In particular, we will explain the different ways of behaviour marked in the list by the letters C,F,G, ...

(b) We have to show that any one relation algebra which is not a specialisation of one of those listed in theorem 1 has a specialisation of one of the forms listed in theorem 2. This is the combinatorial part of the proof, and rather technical. Some of the arguments will be given in the next paragraph, the remaining ones are very similar (but perhaps even more boring): also, they may be found in the literature.

(c) Finally, one has to show that the algebras listed in theorem 2 are wild. This is the easiest part of the investigation, so let us use the remainder of this section to write down some of the embeddings $M_{k\Omega} \to M_R$ which show that the corresponding algebras R are wild.

For the quivers without relations, these embeddings are well-known [12,26]: For example, given a representation (V,φ,ψ) of Ω, define representations



of $\circ \rightrightarrows\!\!\!\!\!\rightarrow \circ$, and $\circ \to \circ \rightrightarrows \circ$, respectively. Note that we will denote the direct sum of two vectorspaces V and W just by VW. Then, given a representation $U \to V \overset{\varphi}{\underset{\psi}{\rightrightarrows}} W$ of $\circ \to \circ \rightrightarrows \circ$, we consider the following representations

$$\begin{array}{ccccccc} & & \Gamma(\varphi) & & & & \\ & & \downarrow & & & & \\ OW & \longrightarrow & VW & \longleftarrow & VO & \longleftarrow & UO \\ & & \uparrow & & & & \\ & & \Gamma(\psi) & & & & \end{array}$$

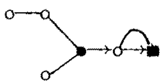
or

$$\begin{array}{ccccccccccc} & & & & O\Gamma(\varphi) & & & & & & \\ & & & & \downarrow & & & & & & \\ & & & & OWV & & & & & & \\ & & & & \downarrow & & & & & & \\ \Gamma(\psi)O & \longrightarrow & VWO & \longrightarrow & VWV & \longleftarrow & VOV & \longleftarrow & \{(vov)\mid v\in V\} & \longleftarrow & \{(uou)\mid u\in U\}, \end{array}$$

where $\Gamma(\varphi)$ denotes the graph of $\varphi : V \to W$, as a subset of VW or WV, whatever is more convenient, and similarly for $\tilde{\tilde{E}}_7$ and $\tilde{\tilde{E}}_8$, using for example the embeddings of the representations of  into the representations of quivers of tpyes $\tilde{E}_7$ and $\tilde{E}_8$, as listed in the tables of [11].

The algebras I - XXII, and XXVIII - XXXIV are concealed quivers, with obvious concealments of quivers of types $\tilde{\tilde{D}}_4$, $\tilde{\tilde{E}}_6$, $\tilde{\tilde{E}}_7$, and $\tilde{\tilde{E}}_8$, see 2.3. For the remaining cases, we have to define again individual embeddings.

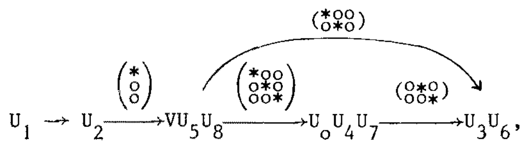
The algebras XXIII, XXIV and XXXV can be treated rather similar. Note that XXXV can be defined also by a zero relation, namely

 .

In all three cases, we define functors from $M_{k\Gamma}$ into M_R, where Γ is a quiver of type $\tilde{\tilde{E}}_8$, which will be a representation equivalence between a cofinite subcategory of $M_{k\Gamma}$ (defined by the requirement that the maps are either monomorphisms or epimorphisms, as indicated) and the image category in M_R. Namely,

$$\begin{array}{c} U_o \\ \uparrow\!\uparrow \\ U_1 \longrightarrow U_2 \longrightarrow V \twoheadrightarrow U_3 \longleftarrow U_4 \longleftarrow U_5 \twoheadrightarrow U_6 \longleftarrow U_7 \longleftarrow U_8 \end{array}$$

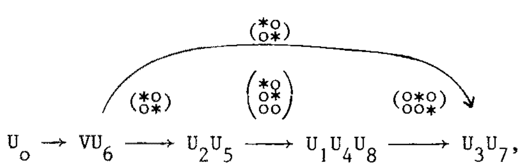
shall go to



similarly, in the case of XXIV,

$$\begin{array}{c} U_o \\ \downarrow \\ U_1 \twoheadleftarrow U_2 \twoheadleftarrow V \twoheadrightarrow U_3 \longleftarrow U_4 \longleftarrow U_5 \longleftarrow U_6 \twoheadrightarrow U_7 \longleftarrow U_8 \end{array}$$

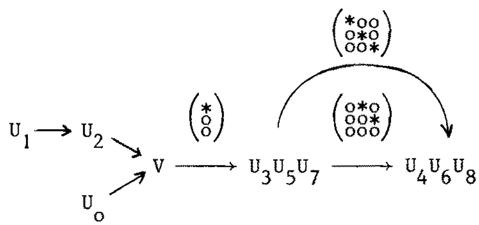
shall go to



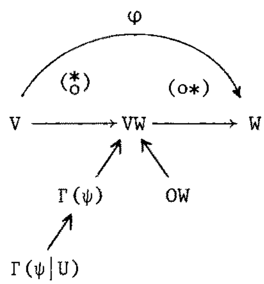
and finally in the case of XXXV,

$$\begin{array}{c} U_o \\ \downarrow \\ U_1 \rightarrow U_2 \rightarrow V \twoheadrightarrow U_3 \leftarrow U_4 \twoheadrightarrow U_5 \leftarrow U_6 \twoheadrightarrow U_7 \leftarrow U_8 \end{array}$$

shall go to

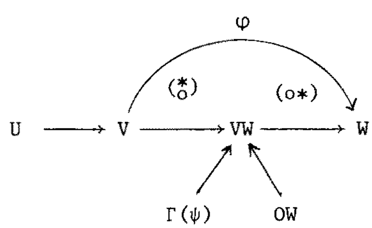


In the cases XXV and XXVI, we define again a functor from $\circ\!\to\!\circ \rightrightarrows \circ$, namely we send $U \to V \overset{\varphi}{\underset{\psi}{\rightrightarrows}} W$ to

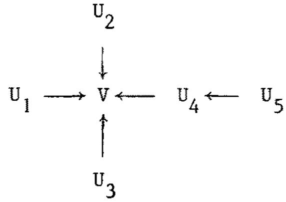


,

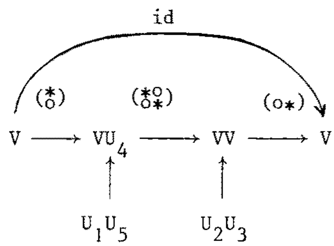
and



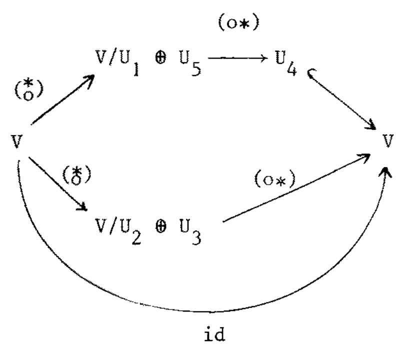
In the cases XXVII and XXXVI, we define a functor from the $\tilde{\tilde{D}}_4$-quiver with subspace-orientation: we send



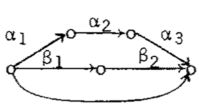
to the representations



and to

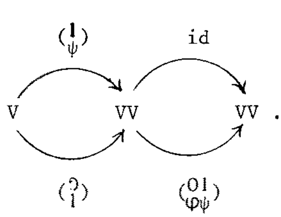


Note that in the last case, we even have obtained a representation of



with $\alpha_3\alpha_2\alpha_1 = \beta_2\beta_1 = 0$.

It remains to consider the case XXXVII. This time, we define directly an embedding of $M_{k\Omega}$ into M_R. Namely, we send (V,φ,ψ) onto

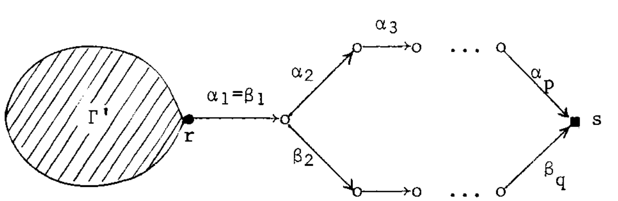


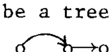
1.7. The combinatorial part of the proof

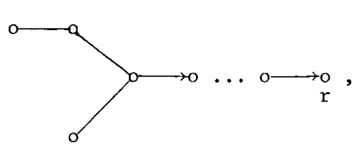
Let (Γ,ρ) be a quiver with one relation ρ which is not a splitting zero condition. We assume that neither (Γ,ρ) nor its dual has a specialisation to one of the quivers with or without relation given in theorem 2. We have to show that then (Γ,ρ) is a specialisation of one of the quivers with relations given in theorem 1.

Let r be the starting point, and s the endpoint of the relation. Since the quivers $\Gamma\setminus\{r\}$ and $\Gamma\setminus\{s\}$ both are specialisations of (Γ,ρ), they have to be disjoint unions of quivers of types A_n, D_n, E_6, E_7, E_8, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. Let $\rho = \sum_{i=1}^{m} \kappa_i w_i$ with pairwise different pathes w_i, and all $\kappa_i \neq 0$.

Consider first the case where two of paths w_i have an arrow in common: say $w_1 = (r|\alpha_1,\dots,\alpha_p|s)$, $w_2 = (r|\beta_1,\dots,\beta_q|s)$, and $\alpha_i = \beta_j$ for some i,j. Since $w_1 \neq w_2$, we can assume $(\alpha_{i+1},\dots,\alpha_p) \neq (\beta_{j+1},\dots,\beta_q)$, otherwise we consider the dual situation. Now $\Gamma\setminus\{r\}$ has a subquiver of type $\tilde{A}_n$, thus this must be a component of $\Gamma\setminus\{r\}$, and therefore $i = j = 1$. Also, w_1 and w_2 are the only possible paths with starting point r and endpoint s, thus $m = 2$. It follows, that (Γ,ρ) is of the form



Note that Γ' has to be a tree, since otherwise we obtain a specialisation of the form  If Γ' contains a subquiver of the form



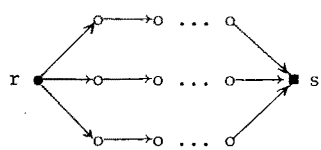
then (Γ,ρ) specialises to XXX. In case $p = 2$ or $q = 2$, we therefore see that (Γ,ρ) is a specialisation of 74. However, if both

$p > 2$, $q > 2$, then we use that the connected subquiver $\Gamma \setminus \{s\}$ can be embedded into one of $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, and then the only possibilities are specialisations of 75-82.

Thus, we can assume that no two paths w_i have an arrow in common. Since there is no specialisation of the form XXXVII, we see that they also cannot have any vertex but starting and end point in common.

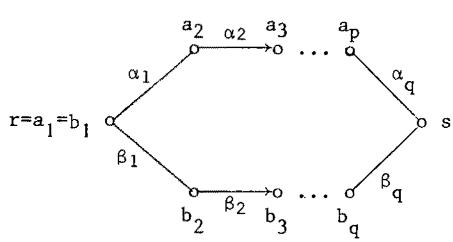
Consider now the case $m \geq 3$.

If we shrink all arrows occuring in w_1 but one, then we can delete the remaining one together with the relation. We obtain a quiver Γ' without a relation containing a subquiver of the form $\tilde{A}_{pq}$, thus this subquiver must be all of Γ', and therefore $m = 3$, and (Γ,ρ) is of the form



Since $\Gamma \setminus \{r\}$ must be embeddable into D_n, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, we see that (Γ,ρ) is a specialisation of one of 83-86.

In case $m = 2$, the relations ρ can be assumed to be a strict commutativity relation, say $\rho = \alpha_p \dots \alpha_1 - \beta_q \dots \beta_1$. Let a_i be the starting point of α_i, and b_i the starting point of β_i, thus we have a subquiver



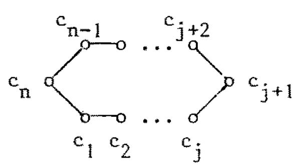
(*)

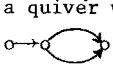
If there is any additional (non-oriented) path joining the a_i, or the b_i, then we obain  as a specialisation. If there is an additional (non-oriented) path joining one of the a_i with one of the

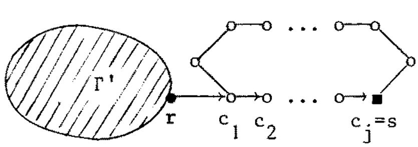
b_j (with $i,j \geq 2$), then we obtain XXXVII as specialisation. Finally, if there is an additional (non-oriented) path joining r with s, then XXXVI shows that $p=q=2$, and then we clearly deal with case 73, since any additional arrow would give  or the dual as specialisation. This shows that we can assume that Γ is obtained from (*) by only adding arms at the various vertices. For a detailed investigation of this situation we refer to Shkabara [36] and Marmaridis [24].

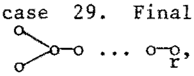
Similarly, in case $m=1$ and Γ does not contain any subquiver of the form $\tilde{A}_{pq}$, we refer to the investigation of Zavadskij [40].

Thus, it remains to deal with the case that ρ is a zero relation and Γ contains a (non-oriented) cycle. Thus, let $\rho = (r|\alpha_1,\ldots,\alpha_p|s)$. Also let

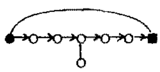


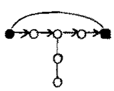
be a cycle in Γ. First, assume that r does not belong to the cycle. Since $\Gamma\setminus\{r\}$ contains a cycle, this cycle must be a component of $\Gamma\setminus\{r\}$, thus r is the neighbor of one of the c_i. Therefore, we may assume $\underset{r}{o}\xrightarrow{\alpha}\underset{c_1}{o}$. Also, $\alpha = \alpha_1$, since otherwise the deletion of α gives a quiver without relation which has a specialisation of the form . Thus (Γ,ρ) is of the form

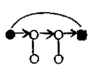


Since there is no specialisation of the form XXIII or XXIV, we see that $j \leq 3$, and $\Gamma' \neq \{r\}$ implies even $j=2$. If $j=3$ and $\Gamma'=\{r\}$, then we deal with the case 29. Finally, XXXV shows that for $j=2$, Γ' is a subquiver of , thus (Γ,ρ) is a specialisation of 31.

Next, assume that r belongs to the cycle, and, by duality, also s belongs to the cycle. Note that the path w has to be part of the cycle since otherwise we will obtain  as a specialization. We want to show that p=2. Since the relation is not a splitting zero relation, the assumption $p \geq 3$ shows that (Γ,ρ) specializes to one of the following quivers with relations:

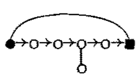
a) 

b) 

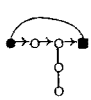
c) 

d) 

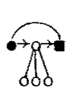
In case a), we obtain as further specialization



which is a splitting zero relation of wild type.
Case b) specializes to



which again is a splitting zero relation of wild type, namely $\tilde{\tilde{E}}_8$. Case c) ist just XXVII and therefore impossible, whereas in case d) the deletion of α_3 gives a quiver to type $\tilde{\tilde{D}}_4$. Now in the case p=2, we cannot have



as specialization (since this further specializes to $\tilde{\tilde{D}}_4$). Also, specializations of the forms XXV, and XXVI being impossible, we see that (Γ,ρ) has to be of the form 31.

2. CONCEALMENTS, FINITE ENLARGEMENTS AND GLUEING

2.1 The Auslander-Reiten quiver

Let X_R, Y_R be modules over some ring R. A homomorphism $f : X \to Y$ is called irreducible in case f is neither a split monomorphism nor a split epimorphism, and if for any factorization $f = f_2 \cdot f_1$, we have that f_1 is a split monomorphism or f_2 is a split epimorphism. Thus the irreducible maps are those maps which have no non-trivial factorizations. In order to be able to speak of the multiplicity of irreducible maps between two modules of finite length, we have to introduce the abelian group $\mathrm{irr}(X,Y)$.

Let X, Y be R-modules of finite length, say $X = \bigoplus_{j=1}^{n} X_j$, $Y = \bigoplus_{i=1}^{m} Y_i$, with X_j, Y_i indecomposable. A homomorphism $f : X \to Y$ is said to belong to the radical $\mathrm{rad}(X,Y)$ provided no component $f_{ij} : X_j \to Y_i$ is an isomorphism, where $f = (f_{ij})$ is the matrix description of f. Let $\mathrm{rad}^2(X,Y) = \{f \mid \exists\, f_1 \in \mathrm{rad}(X,Z),\ f_2 \in \mathrm{rad}(Z,Y)$ with $f = f_2 \cdot f_1\}$. For X,Y indecomposable, a homomorphism $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}(X,Y) \setminus \mathrm{rad}^2(X,Y)$, and, in this situation we call $\mathrm{irr}(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ the group of irreducible maps.

In case R is a finite dimensional algebra over an algebraically closed field k, the group $\mathrm{irr}(X,Y)$ is a finite dimensional vectorspace over k, and we call its dimension the multiplicity of irreducible maps from X to Y. The Auslander-Reiten quiver of R has as vertices the isomorphism classes of indecomposable R-modules, and the number of arrows from $[X]$ to $[Y]$ is precisely $\dim \mathrm{irr}(X,Y)$, where $[X]$ denotes the isomorphism class of the module X. If we choose for every pair X,Y of indecomposable modules, a fixed set of representatives of a basis of $\mathrm{irr}(X,Y)$, then we call these maps fixed irreducible maps.

One can obtain a rather large amount of information concerning the Auslander-Reiten quiver from the Auslander-Reiten sequences. Assume that R is a finite dimensional algebra over some field (or, at least, an artin algebra). Recall that an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \qquad (*)$$

is an Auslander-Reiten sequence if and only if both f and g are irreducible maps. Originally they have been defined by the following universal properties: The sequence (*) is an Auslander-Reiten sequence if and only if both X and Z are indecomposable and (*) satisfies the following equivalent conditions:

(i) Any homomorphism $\alpha : X \to X'$ which is not a split monomorphism, can be extended to Y: there exists $\alpha' : Y \to X'$ with $\alpha = \alpha' \cdot f$.

(ii) Any homomorphism $\beta : Z' \to Z$ which is not a split epimorphism can be lifted to Y: there exists $\beta' : Z' \to Y$ with $\beta = g \cdot \beta'$.

Auslander and Reiten have shown both the existence and unicity of these sequences: For any indecomposable module X which is not injective, there exists an Auslander-Reiten sequence (*), for any indecomposable module Z which is not projective, there exists an Auslander-Reiten sequence (*). If we have two Auslander-Reiten sequences, say (*) and

$$o \to X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to o \qquad (**)$$

then X and X' are isomorphic iff Z and Z' are isomorphic iff the sequences (*) and (**) are equivalent. In particular, the two ends of an Auslander-Reiten sequence (*) determine each other up to isomorphism, and we call $X = A(Z)$ the <u>Auslander-Reiten translate</u> of Z, and also write $Z = A^{-}(X)$. Note that A and A^{-} are only functorial in certain quotient categories, but that there is an explicit construction, namely $A = D\,\mathrm{Tr}$, $A^{-} = \mathrm{Tr}\, D$, where Tr Z denotes the "transpose" of Z (form a minimal projective resolution $Q \xrightarrow{h} P \to Z \to 0$ of Z, and let Tr Z be the cokernel of $\mathrm{Hom}(h,R) : \mathrm{Hom}(P,R) \to \mathrm{Hom}(Q,R)$, whereas D denotes the ordinary duality with respect to the base field. The relation between Auslander-Reiten sequences and irreducible maps is given in the following lemma:

<u>Lemma 1</u>: Let R be an algebra over an algebraically closed field, and let

$$0 \to X \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_t \\ f' \end{pmatrix}} \left(\bigoplus_{i=1}^{t} Y\right) \oplus Y' \xrightarrow{(g_1,\dots,g_t,g')} Z \to 0$$

be an Auslander-Reiten sequence, with Y indecomposable, such that Y' has no direct summand isomorphic to Y. Then $\bar{f}_1,\dots,\bar{f}_t \in \mathrm{rad}(X,Y) / \mathrm{rad}^2(X,Y) = \mathrm{irr}(X,Y)$ is a basis, and $\bar{g}_1,\dots,\bar{g}_t$ is a basis of $\mathrm{irr}(Y,Z)$.

This shows that for X non-injective, the Auslander-Reiten sequence starting with X determines completely the arrows in the Auslander-Reiten quiver with starting point $[X]$. Also, for Z non-projective, the Auslander-Reiten sequence ending with Z determines completely the arrows in the Auslander-Reiten quiver with end point $[Z]$. There is a similar, even easier way to obtain all arrows starting with $[X]$ in case X is indecomposable injective, and all arrows ending in $[Z]$ in case Z is indecomposable projective, as follows:

Lemma 2: Let R be an algebra over an algebraically closed field. Let I be indecomposable injective with socle $\mathrm{soc} I$. Let

$$0 \longrightarrow \mathrm{soc}\, I \rightarrow I \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_t \\ f' \end{pmatrix}} \left(\bigoplus_{i=1}^{t} Y\right) \oplus Y' \longrightarrow 0$$

be exact, with Y indecomposable, such that Y' has no direct summand isomorphic to Y. Then $\bar{f}_1,\dots,\bar{f}_t$ is a basis of $\mathrm{irr}(I,Y)$.

Let P be indecomposable projective with radical $\mathrm{rad}\, P$.

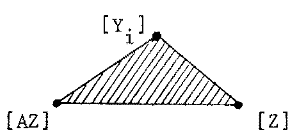
Let

$$0 \longrightarrow \left(\bigoplus_{i=1}^{t} Y\right) \oplus Y' \xrightarrow{(g_1,\dots,g_t,g')} P \rightarrow P/\mathrm{rad}\, P \rightarrow 0$$

be exact, with Y indecomposable, such that Y' has no direct summand isomorphic to Y. Then $\bar{g}_1,\dots,\bar{g}_t$ is a basis of $\mathrm{irr}(Y,P)$.

As a consequence we see that the Auslander-Reiten quiver of a finite dimensional algebra R over an algebraically closed field always is locally finite (any vertex is starting point or end point of at most finitely many arrows), thus its connected components are finite or countable. In fact, in case R is connected and not of finite representation type, then no component is finite (see [2],or also [35]). Another consequence of the previous assertions is that for X indecomposable, $[X]$ is a source if and only if X is simple projective, and a sink if and only if X is simple injective.

Gabriel and Riedtmann have proposed to consider in addition to the Auslander-Reiten quiver also a two-dimensional cell complex, derived from the underlying graph and the action of the Auslander-Reiten translate A, at least when all multiplicities of the irreducible maps are ≤ 1. Namely, the points are the vertices of the Auslander-Reiten quiver, there are two kinds of edges: the ones corresponding to the arrows of the Auslander-Reiten quiver (just forget the orientation), and, in addition, for any point $[Z]$, with Z non-projective, there is an additional edge between AZ and Z. Finally, there are triangles of the form



in case we have the Auslander-Reiten sequence

$$0 \longrightarrow AZ \longrightarrow \bigoplus_{i=1}^{s} Y_i \longrightarrow Z \longrightarrow 0,$$

with all Y_i indecomposable, the boundary edges of such a triangle being the edges corresponding to the arrows $[AZ] \to [Y_i]$ and $[Y_i] \to [Z]$ in the Auslander-Reiten quiver, and the additional edge between $[AZ]$ and $[Z]$. Note that the connected component of the Auslander-Reiten quiver give rise to the (topological) components of this cell complex.

In case some multiplicites are > 1, we proceed similarly, provided the Auslander-Reiten sequences in some component are of the form

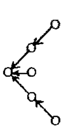
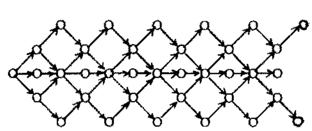
$$0 \longrightarrow X \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_s \end{pmatrix}} \bigoplus_{i=1}^{s} Y_i \xrightarrow{(g_1,\dots,g_s)} Z \longrightarrow 0$$

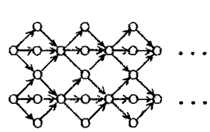
with Y_i indecomposable and fixed irreducible maps f_i, g_i. Here we construct a cell complex only for this particular component. Again, there will be s different triangles for the Auslander-Reiten sequence above, which, as before, have the one boundary edge between $[X]$ and $[Z]$ in common, but which in addition may also have some of the points $[Y_i]$ in common, namely in case Y_i, Y_j are isomorphic for some $i \neq j$.

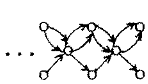
2.2 Quivers without oriented cycles

The representation theory of quivers without oriented cycles will play a dominant role in the further investigations. Here, we present the structure of the components of the Auslander-Reiten quiver which contain projective or injective modules. Thus, let Γ be a (finite) quiver without oriented cycles, and k some commutative field.

If $[a,b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ is some intervall in $\mathbb{Z}$, where $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{\infty\}$, and $a \leq b$, we denote by $[a,b]\Gamma$ the following quiver: its set of vertices is $\Gamma \times [a,b]$, and for any arrow $\overset{i}{\circ} \xleftarrow{\ \alpha\ } \overset{j}{\circ}$, there are arrows $(i,z) \xrightarrow{(\alpha,z)} (j,z)$ for any $a \leq z \leq b$, and arrows $(j,z) \xrightarrow{(\alpha^*,z)} (i,z+1)$, for $a \leq z < b$. Let us give some examples:

 I = [1,6] 

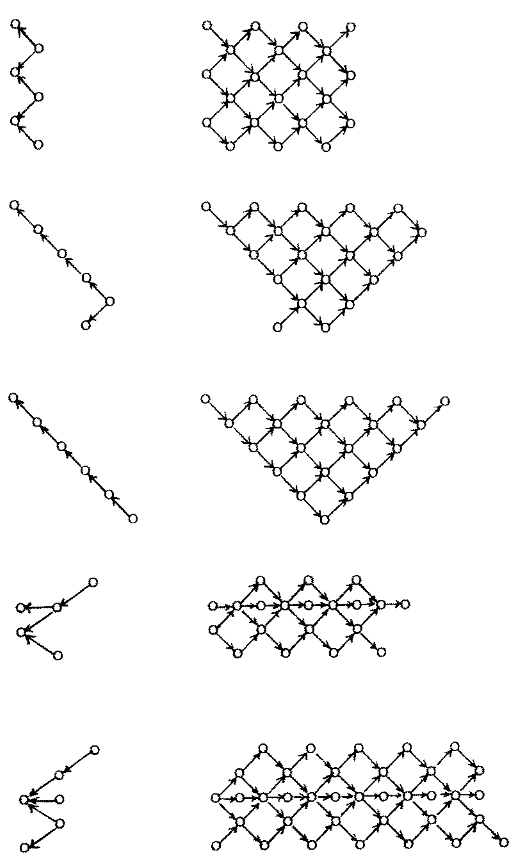
 I = $\mathbb{N}$ 

 I = $\mathbb{N}^-$ 

where $\mathbb{N} = \{1,2,3,\ldots\}$ are the natural numbers, and $\mathbb{N}^- = \{-z \mid z \in \mathbb{N}\}$.

Assume now that Γ is a quiver of type A_n, D_n, E_6, E_7, or E_8. We are going to define a subquiver $A(\Gamma)$ of $\mathbb{N}\Gamma$ as follows. Let σ denote the following permutation of the vertices of Γ: If Γ is of type D_n, with $n \equiv 0(2)$, or of type E_7 or E_8, let σ be the identity. For Γ of type A_n, with n arbitrary, D_n, with $n \equiv 1(2)$, or E_6, let σ be the unique non-trivial automorphism of the underlying graph of Γ (one obtains the underlying graph of Γ by replacing the arrows by edges). For any vertex i of Γ, there is an unoriented path from i to $\sigma(i)$, let a_i be the number of arrows in this path directed towards i, and b_i the number of arrows in this path directed towards $\sigma(i)$. Also, denote by h the "Coxeter number" for Γ, thus $h = n+1$, $2(n-1)$, 12, 18, or 30, for Γ of type A_n, D_n, E_6, E_7, or E_8, respectively. Define now $A(\Gamma)$ as the full subquiver of $\mathbb{N}\Gamma$ of all vertices (i,z) satisfying $1 \le z \le \frac{1}{2}(h+a_i-b_i)$. We give some examples:

Γ $A(\Gamma)$

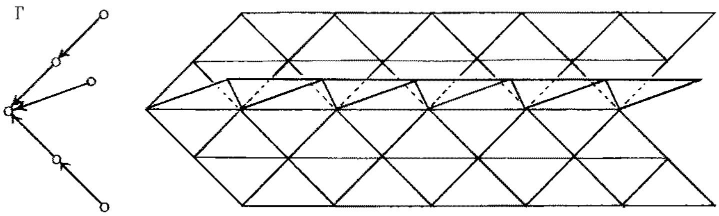


<u>Lemma.</u> For a quiver of type A_n, D_n, E_6, E_7, or E_8, the Auslander-Reiten quiver of $k\Gamma$ is of the form $A(\Gamma)$. If Γ is a connected quiver without oriented cycles, and not of the form A_n, D_n, E_6, E_7, or E_8, then one component of the Auslander-Reiten quiver of $k\Gamma$ contains all indecomposable projective modules and is of the form $\mathbb{N}\Gamma$ (the modules in this component are called preprojective), another component of the Auslander-Reiten quiver contains all indecomposable injective modules and is of the form $\mathbb{N}\Gamma$ (the modules in this component will be called preinjective). In all cases, the Auslander-Reiten translate is $(i,z) \to (i,z-1)$.

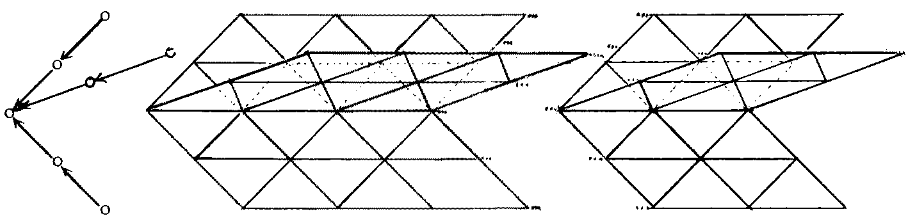
Thus, the modules corresponding to the vertices labelled $(r,1)$ in the Auslander-Reiten quiver of a quiver of finite type (the cases A_n, D_n, E_6, E_7 and E_8), and in the preprojective component of the Auslander-Reiten quiver of a connected quiver of any other type, are just the projective modules. In fact, we can describe this correspondence in more detail as follows: For any vertex r of Γ, denote by $P(r)$ the following representation of Γ: Let $P(r)_s$ be the vectorspace with basis the set of paths $(r|\alpha_1,\ldots,\alpha_p|s)$ from r to s, and for any arrow $\underset{s}{\circ}\xrightarrow{\alpha}\underset{t}{\circ}$ in Γ, let $\varphi_\alpha : P(r)_s \to P(r)_t$ be defined by $(r|\alpha_1,\ldots,\alpha_p|s) \to (r|\alpha_1,\ldots,\alpha_p,\alpha|t)$. This representation $P(r) = (P(r)_s,\varphi_\alpha)$ is indecomposable projective, and it corresponds to the point $(r,1)$. Clearly, if $\underset{i}{\circ}\xrightarrow{\beta}\underset{j}{\circ}$ is an arrow in Γ, then $P(j)$ is isomorphic to a direct summand of the radical $\operatorname{rad} P(i)$ of $P(i)$, an embedding $\mu_\beta : P(j) \to P(i)$ being given by left multiplication with $(i|\beta|j)$. In this way, $\operatorname{rad} P(i) = \bigoplus_{\beta:i\to j} \mu_\beta P(j)$. But this shows that the full subquiver of the Auslander-Reiten quiver generated by the indecomposable projective modules is just the quiver $[1,1]\Gamma$, which is obtained from Γ by reserving the orientation of all edges. In the same way, the full subquiver generated by the indecomposable injective modules is also of this form.

From the knowledge of the indecomposable projectives or the indecomposable injectives, the dimension types of the remaining projective or preinjective modules can be calculated either by the use of the Coxeter transformation, or by using the additivity property of <u>dim</u> on exact sequences, here the Auslander-Reiten sequences.

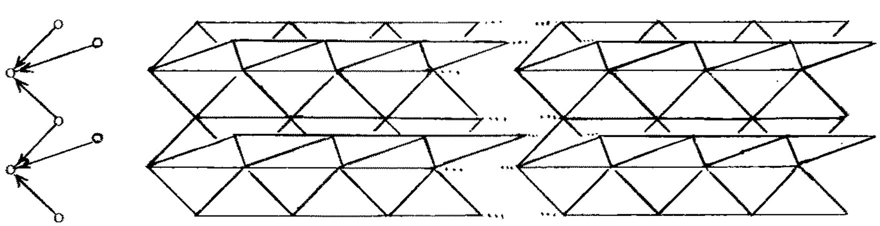
Also, let us sketch the corresponding two-dimensional complexes for some examples. The first example Γ discussed above was a quiver of type E_6:



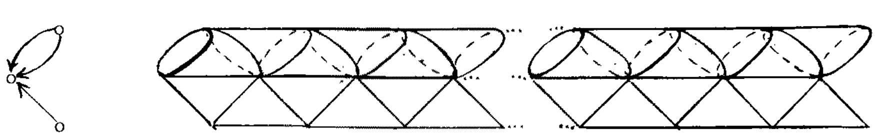
Similarly, for the following quiver Γ of type $\tilde{E}_6$, we give the preprojective and the preinjective components of the Auslander-Reiten quiver:



For the case of the quiver Γ discussed above of type $\tilde{D}_6$, the preprojective and the preinjective components are



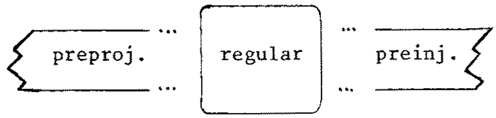
And finally, we give the preprojective and the preinjective component of one wild quiver:



Note that in this case we can define inductively fixed irreducible maps occurring in the Auslander-Reiten sequences, so that we are able to construct the corresponding complex.

It should be noted that for a quiver Γ, the modules of the form $A^{-m}P$ with P indecomposable projective, will always be called preprojective, those of the form $A^{m}I$ with I indecomposable injective will be called preinjective. It follows for Γ connected, that $k\Gamma$ is of finite representation type if and only if all modules are both preprojective and preinjective, if and only if there is at least one module which is both preprojective and preinjective.

The indecomposable representations of a quiver which are neither preprojective nor preinjective, are called regular. Note that there are no homomorphisms from a regular module to a preprojective module, and no homomorphisms from a preinjective module to a preprojective or a regular module. The shape of the category of representations of a connected quiver which is not of finite type therefore can be remembered rather easily as follows:



the possible homomorphisms going from left to right. A survey on results concerning the regular part in the case of Γ being of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, will be given in 3.2.

For a proof of the results above, one can use the following fact: for the path algebra $R = k\Gamma$ of the quiver Γ, the Auslander-Reiten translations A and A^{-} in fact are functors, and they have the following properties:

(1) As always: if X is indecomposable, then $AX = 0$ if and only if X is projective, and if X is not projective, $X \approx A^{-}AX$.

(2) Therefore, by induction: if X is indecomposable, then $A^{m}X = 0$ for some $m \in \mathbb{N}$ if and only if X is preprojective. Thus, if X is not preprojective, then $X \approx A^{-m}A^{m}X$ for all $m \in \mathbb{N}$.

(1*) Dually: if X is indecomposable, then $A^{-}X = 0$ iff X is injective, and if X is not injective, then $X \approx AA^{-}X$.

(2*) if X is indecomposable, then $A^{-m}X = 0$ for some $m \in \mathbb{N}$, iff X is preinjective. Thus, if X is not preinjective, then $X \approx A^{m}A^{-m}X$ for all $m \in \mathbb{N}$.

(3) If X is without projective summand, then $\mathrm{Hom}(X,Y) \approx \mathrm{Hom}(AX,AY)$; and if Y is without injective summand, then $\mathrm{Hom}(X,Y) \approx \mathrm{Hom}(A^{-}X,A^{-}Y)$.

Also,there is a linear transformation $c : \mathbb{Q}^{n} \to \mathbb{Q}^{n}$, called the Coxeter transformations, with

(4) If X is indecomposable and not projective, then $\underline{\dim}\, AX = c\, \underline{\dim}\, X$, and if X is indecomposable and not injective, then $\underline{\dim}\, A^{-}X = c^{-1}\, \underline{\dim}\, X$.

The following lemma can be derived rather easily from (3):

Lemma: If X,Y are indecomposable and either both preprojective or both preinjective, then any homomorphism $f : X \to Y$ is the sum of compositions of fixed irreducible maps.

In particular, the endomorphism ring of any indecomposable preprojective or preinjective module is just k.

There are similar endofunctors, the so-called Coxeter functors introduced by Bernstein-Gelfand and Ponomarev [5], which are equivalent to the Auslander-Reiten translation functors up to categorical equivalence. The precise relation has been determined recently by Gabriel [20]. The Coxeter functors are built up from certain reflection functors which also will be used in the sequel.

For any vertex r of Γ, denote by $\sigma_r\Gamma$ the quiver obtained from Γ by changing the orientation at all arrows involving r. If r is a sink or a source, then there exists a functor $M_{k\Gamma} \to M_{k\sigma_r\Gamma}$, also denoted by σ_r, and called a reflection functor, with the following properties: $\sigma_r(E_\Gamma(r)) = 0$, and σ_r is an equivalence between the full subcategory of $M_{k\Gamma}$ of all representations without direct summand $E_\Gamma(r)$, and the full subcategory $M_{k\sigma_r\Gamma}$ of all representations without direct summand $E_{\sigma_r\Gamma}(r)$. Here, $E_\Gamma(r)$ denotes the simple representation of Γ with $(E_\Gamma(r))_r = k$. Note that for a quiver without cycles, any change of orientation can be achieved by a sequence of σ_r, where we use only sinks (or only sources).

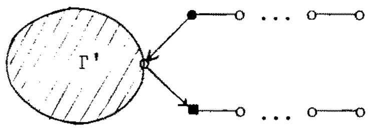
2.3. Concealed quiver algebras

In this section, we will study those one-relation algebras R which are rather similar to the path algebra of a (tame or wild) quiver without relations, and which can be thought of being "disguised" or "concealed" forms of such path algebras. The tame concealed quiver algebras have the property that the deletion of any point immediately leads to an algebra of finite representation type. We later will discuss other classes of one-relation algebras also obtained from quiver algebras by easy modifications, but in contrast to the concealed quiver algebras, these always will be proper enlargements of path algebras of quivers. Note that the representation theory of a concealed algebra turns out to be rather similar to that of the corresponding quiver.

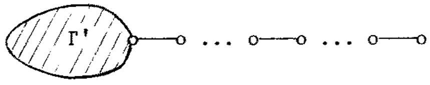
Definition: A finite dimensional k-algebra R will be called a <u>concealed quiver algebra</u> provided there exists a connected quiver Γ of infinite type with the same number of simple modules such that there are cofinite subcategories $\mathcal{U}$ of M_Γ and $\mathcal{V}$ of M_R which are equivalent. In this case, we will say that R is a <u>concealment</u> of the path algebra $k\Gamma$, or just of Γ.

Let us consider some easy examples. First, we show that we can remove completely a zero relation of the following type

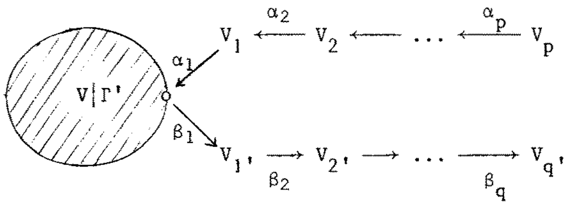
Γ :



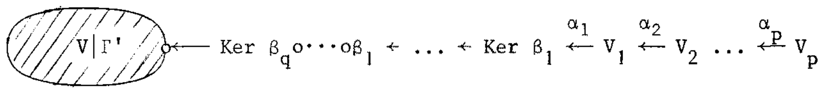
where Γ' is a quiver with or without relations, and obtain the following situation (with the same number of points):



Namely, if there is given a representation V

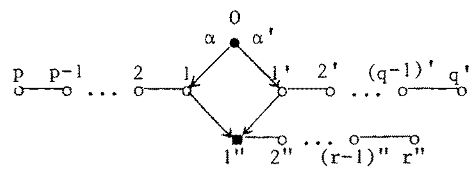


with $\beta_1\alpha_1 = 0$, then we form

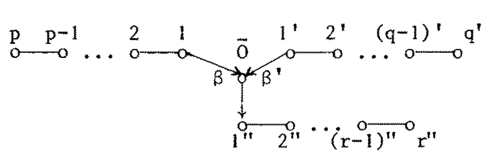


Note that under this functor, we loose precisely $\frac{1}{2}q(q+1)$ indecomposable representations, and gain $\frac{1}{2}q(2p+q+1)$, whereas, on the remaining indecomposables, the functor gives an equivalence of subcategories. This shows that in case Γ' is a quiver, without relation, then Γ is a concealed quiver. This deals with the cases 1-8 of theorem 1, and I-XX of theorem 2.

Next, consider the following type of quivers with commutativity relation:

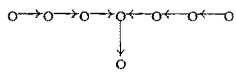


Forming the pushout of the two maps α, α' for a given representation leads to a representation of



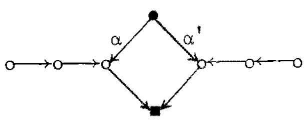
The new quiver has the same number of points and no relation. Note that we loose just one indecomposable representation, namely a simple injective one, but we gain r+1 indecomposable representations (those indecomposable representations W of the new quiver which have $W_1 = W_{1'} = 0$ and $W_{\bar{0}} = k$). Again, we see that the quiver with commutativity relation, was a concealed quiver. In theorem 1, this deals with cases 32-35; in theorem 2, with cases XXVIII-XXXI.

We will consider two algebras in greater detail, namely two concealments of the following quiver Γ of type $\tilde{E}_7$

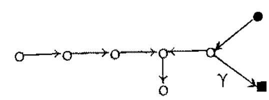


a quiver with a commutative square and one with a zero relation

R_1:

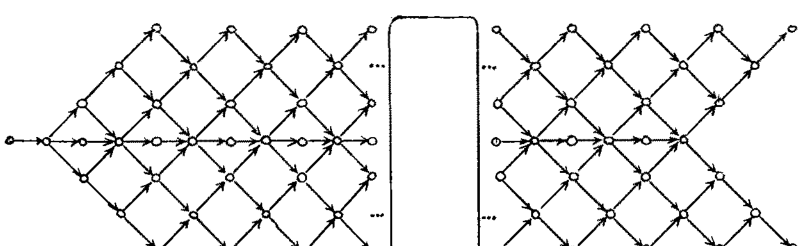


R_2:

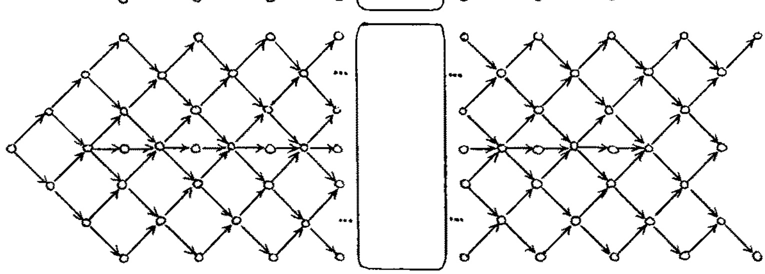


In the case of R_1, forming the pushout of α,α' gives an equivalence between the full subcategory V_1 of M_{R_1} of all representations with $\operatorname{Ker}\alpha \cap \operatorname{Ker}\alpha' = 0$, thus without direct summand of the form $\begin{smallmatrix} & & & 1 & & & \\ o & o & o & & o & o & o \\ & & & o & & & \end{smallmatrix}$, and the full subcategory U_1 of $M_{k\Gamma}$ of all representations without direct summands $\begin{smallmatrix} o & o & o & 1 & o & o & o \\ & & & 1 & & & \end{smallmatrix}$ and $\begin{smallmatrix} o & o & o & o & o & o & o \\ & & & 1 & & & \end{smallmatrix}$. Similarly, in case of R_2, forming the kernel of γ gives an equivalence between the full subcategory V_2 of M_{R_2} of all representations with γ surjective, thus without direct summand of the form $\begin{smallmatrix} o & o & o & o & o & o & o \\ & & & o & & & 1 \end{smallmatrix}$, and the full subcategory U_2 of $M_{k\Gamma}$ of all representations without direct summand $\begin{smallmatrix} o & o & o & o & o & 1 & 1 \\ & & & o & & & \end{smallmatrix}$ and $\begin{smallmatrix} o & o & o & o & o & o & 1 \\ & & & o & & & \end{smallmatrix}$. We compare the three Auslander-Reiten quivers:

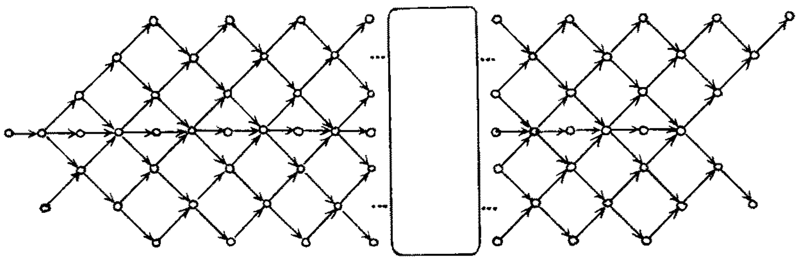
$k\Gamma$



R_1



R_2



As in the case of a quiver, we see that the Auslander-Reiten quiver separates into three different parts: the preprojective component, the regular part (which is the union of many components), and the preinjective component. In general, we have:

Lemma: Let R be a concealed quiver algebra, say a concealment of the quiver Γ. Then M_R has a full subcategory R_R which is equivalent to the category of regular Γ-modules. The remaining indecomposable R-modules form two components of the Auslander-Reiten quivers: one containing the indecomposable projective R-modules (the modules in the corresponding subcategory P_R will be called preprojective), the other containg the indecomposable injective R-modules (the modules in this subcategory I_R will be called preinjective).

Proof: Let U be cofinite in $M_{k\Gamma}$. Γ a quiver, V cofinite in M_R, with an equivalence $\eta : U \to V$. Note that if U, U' are in U, and $f : U \to U'$ is irreducible in $M_{k\Gamma}$, then there is a chain of irreducible maps $\eta(U) \to \ldots \to \eta(U')$ in V, since there can be at most finitely many nontrivial ways of factoring $\eta(f)$ in V; Similarly, for V, V' in V with $g : V \to V'$ irreducible, there is a chain of irreducible maps $\eta^{-1}(V) \to \ldots \to \eta^{-1}(V')$ in U.

We want to show that for almost all indecomposable preprojective representations U in U, with Auslander-Reiten-sequence

$$o \to U \to U' \to A^{-}U \to o,$$

also $U', A^{-}U$ belong to U and

$$o \to \eta(U) \to \eta(U') \to \eta(A^{-}U) \to o$$

is an Auslander-Reiten-sequence in M_R. Of course, we may assume that U has the following property: if U_1, U_2 are indecomposable preprojective, and $f : U_1 \to U_2$ is irreducible, then $U_1 \in U$ implies

$U_2 \in \mathcal{U}$. Then the first condition will be automatically satisfied. Now, there are only finitely many indecomposables $V \in \mathcal{V}$ with $\eta^{-1}(V)$ preprojective, such that there exists a chain of irreducible maps $V \to \dots \to X$ where X is indecomposable and not in $\mathcal{V}$. For, we may assume that we have choosen a minimal chain
$V = V_q \to V_{q-1} \to \dots \to V_1 \to X$ of irreducible maps, then all $V_i \in \mathcal{V}$, and since $\eta^{-1}(V_q)$ is preprojective, also $\eta^{-1}(V_1)$ has to be preprojective. Since there are only finitely many possibilities for V_1 and $\eta^{-1}(V_1)$ is preprojective, there can be only finitely many possibilities for V_q. Now if we choose V with $\eta^{-1}(V)$ preprojective and such that for no chain of irreducible maps $V \to \dots \to X$, the module X lies again in $\mathcal{V}$, then the Auslander-Reiten sequence starting with $\eta^{-1}(V)$ goes under η to an Auslander-Reiten sequence.

Let n be the number of simple R-modules (or simple kΓ-modules). We see that there are at least n pairwise disjoint sequences of the form $A^{-m}V$ with $V \in \mathcal{V}$ and $\eta^{-1}(V)$ indecomposable preprojective, and none of them is periodic. It is also easy to see that for $V \in \mathcal{V}$ with $\eta^{-1}(V)$ indecomposable preprojective, there exists an $m \in \mathbb{N}$ with $A^m V = o$. For, inside $\mathcal{V}$ there are only finitely many indecomposable modules X with $\mathrm{Hom}(X,V) \neq o$, so there are all together only finitely many indecomposable modules X with $\mathrm{Hom}(X,V) \neq o$. Since V is not A-periodic, we must have $A^m V = o$ for some $m \in \mathbb{N}$. Thus, we see that the component of the Auslander-Reiten-quiver containing the indecomposable R-modules V with $\eta^{-1}(V)$ preprojective, contains at least n indecomposable projective modules, thus all of them.

Similarly, we see that the component containing the indecomposable R-modules W with $\eta^{-1}(W)$ preinjective contains all indecomposable injective modules, and also that this component differs from the previous one.

It is now rather easy to see that the regular kΓ-modules may be supposed to be contained in $\mathcal{U}$, and that there are no additional R-modules. This then finishes the proof.

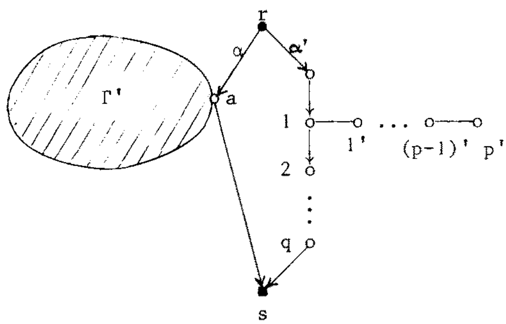
Also, we should remark that it follows from the proof above that for a concealed quiver algebra R, say a concealment of kΓ, with preprojective subcategory $\mathcal{P}_R$, regular subcategory $\mathcal{R}_R$, and preinjective subcategory $\mathcal{I}_R$, and similarly $\mathcal{P}_\Gamma$, $\mathcal{R}_\Gamma$, $\mathcal{I}_\Gamma$, there are full cofinite embeddings

$$P_R \cup R_R \to P_\Gamma \cup R_\Gamma \to P_R \cup R_R$$

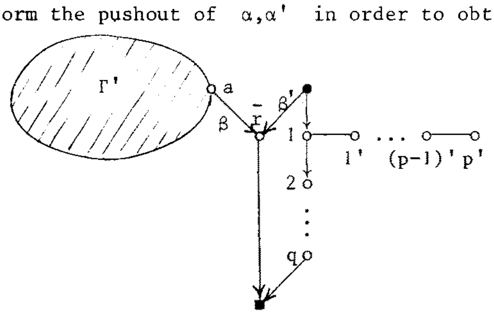
und

$$R_R \cup I_R \to R_\Gamma \cup I_\Gamma \to R_R \cup I_R .$$

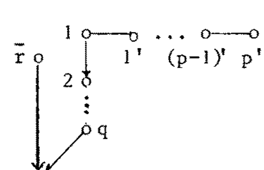
Let us continue to verify that certain algebras are concealed quivers. Consider the following



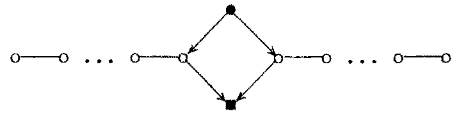
we form the pushout of α,α' in order to obtain representations of



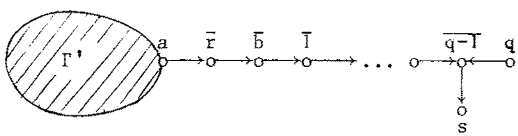
Note that in this way, we loose just one indecomposable representation, and we obtain $p+q+2$ new indecomposable representations, namely those W with support on the subquiver



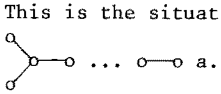
and $W_{\bar{r}} = k$. Let us apply this in different situations. If we start with R_1 with $q = 1$, and Γ' of type A_n, we obtain a quiver of type



thus a concealed quiver, and therefore R itself is a concealed quiver: this deals with the cases 36, 37, XXXIII, and XXXIV. If we start with R_2 with $q = 2$, and again Γ' of type A_n, we obtain the dual of the situation of R_1, thus again R_2 is a concealed quiver, this is 38 and XXXV. Finally, consider the case of R where $p = o$, then we can use induction and obtain after q steps the quiver

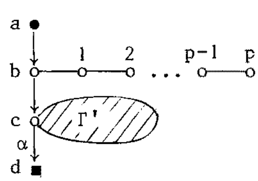


Thus, if Γ' is a quiver without relations, R is a concealed quiver. This is the situation of 71, with Γ' being of type D_n, namely



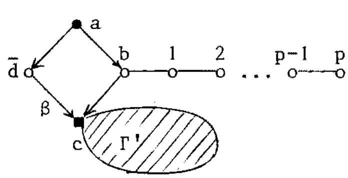
Next, consider

R:



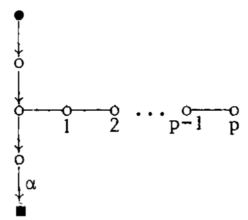
Then, forming the kernel of α, we obtain a representation of

$\overline{R}$:

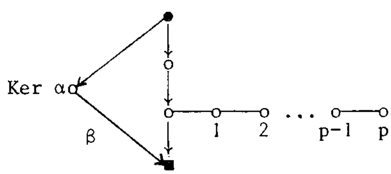


with β a monomorphism (the inclusion map $\ker\alpha \to V_c$). Note that all but p+3 indecomposable representations of $\overline{R}$ are in the image of the functor. If Γ' is of type A_n, then we have seen that $\overline{R}$ is a concealed quiver, thus, in this case, also R is a concealed quiver. This deals with case 9 of theorem 1, and case XXI of theorem 2.

Similarly, we reduce

R: 

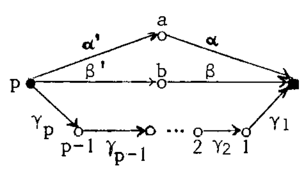
to

$\overline{R}$: 

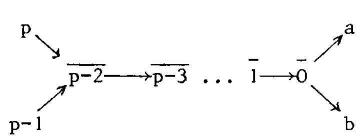
,

forming the kernel of α and obtaining all representations with β a monomorphism. Note that for all but $p+4$ indecomposable representations of $\overline{R}$, the map β really is a monomorphism . This deals with the cases 10 and XXII.

Finally, we have to consider the case 83, namely

R: 

We define a functor from M_R to the representations of the quivers Δ



as follows: if V is a representation of R, let

$$V_{\bar{i}} = \{(x,y,z) \in V_a \oplus V_b \oplus V_{i+1} \mid \alpha(x) + \beta(y) + \gamma_1 \cdots \gamma_i(z) = 0\}$$

with the obvious maps. In this way, one can show that R is a concealment of Δ.

2.4. Vectorspace categories

One of the main working tools in the sequel will be the notion of a vectorspace category, and, as a special case, the additive category of a partially ordered set. These concepts were introduced by Nazarova and Roiter [28,30], and they and their students developped a rather eleborate theory. We will use many of their results. At present, the criteria for partially ordered sets to be of finite or tame representation type seem to be the most powerful tool in representation theory.

By definition, a vectorspace category K is nothing else than a additive k-category together with a faithful functor from K to the category of k-vectorspaces (usually denoted by $|\cdot|$) such that idempotents in K split. If X is an object of K, we will call the vectorspace $|X|$ the "underlying vectorspace" of X, and $|\cdot|$ the forget functor. Recall that the condition that idempotents in K split, means that given an idempotent $e = e^2$ in the endomorphism ring of an object X, there exists a direct decomposition $X = X_1 \oplus X_2$ such that e is the projection onto X_1.

Using the forget functor $|\cdot|$, a vectorspace category K may be considered as a (usually not full) subcategory of the category of k-vectorspaces. Thus, there are given objects which may be thought of consisting of a vectorspace with some additional (not further specified) structure (such as a fixed subspace, a fixed endomorphism etc), and the homomorphisms $K(X,Y)$ from X to Y form a k-linear subspace of the space $\mathrm{Hom}(|X|,|Y|)$ of all linear transformations - the elements of $K(X,Y)$ may be thought of as those linear transformations which preserve the additional structure. Note that we assume that K is additive, which means that for any two objects X, Y in K, there exists the direct sum $X \oplus Y$ in K. Also, in case the underlying vectorspace of X is finite-dimensional (and this usually will be assumed), then our assumption that all idempotents split implies that X can be written as the direct sum of a finite number od indecomposable objects and these have local endomorphism rings. Thus, such a decomposition is unique up to isomorphism.

The usual construction of vectorspace categories will be by the use of functors such as Hom or Ext^i. Namely, let K be any additive k-category with split idempotents, and M an object in K. Then we denote by $\mathrm{Hom}(M,K)$ the vectorspace category whose objects are of the form $\mathrm{Hom}(M,X)$, with X an object in K, and with maps

$$\mathrm{Hom}(M,X) \longrightarrow \mathrm{Hom}(M,Y)$$

being of the form $\mathrm{Hom}(M,f)$, where $f : X \to Y$ is a map in K. Thus we consider something like the image category of the representable functor $\mathrm{Hom}(M,-)$. Of course, we let $|\mathrm{Hom}(M,X)|$ be just the underlying vectorspace of $\mathrm{Hom}(M,X)$. In a similar way, we define vectorspace categories such as $\mathrm{Ext}^i(M,M)$, where M is a module in the module category M (so that Ext^i is defined).

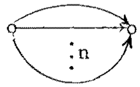
Given a vectorspace category K, its <u>subspace category</u> $U(K)$ is defined as follows: its objects are triples of the form (U,φ,X), where U is a k-space, X is an object of K, and $\varphi : U \to |X|$ is a k-linear transformation. A homomorphism from (U,φ,X) to (U',φ',X') is given by a pair (α,β) where $\alpha : U \to U'$ is k-linear, $\beta : X \to X'$ is a map in K, such that $\beta\varphi = \varphi'\alpha$. Clearly, $U(K)$ is again an additive category with split idempotents, the direct sum of (U,φ,X) and (U',φ',X') being given by $(U \oplus U', \begin{pmatrix}\varphi & 0\\ 0 & \varphi'\end{pmatrix}, X \oplus X')$. Clearly, any object of $U(K)$ is isomorphic to a direct sum of a triple (U,φ,X) with $\varphi : U \to |X|$ being an inclusion map and several copies of $(k,o,0)$ where 0 denotes the zero object of K. This explains the notion subspace category: up to the one indecomposable object $(k,o,0)$ in $U(K)$, we may suppose that we deal with an object of K with a preassigned subspace (note: a subspace, not necessarily a subobject!). If (U,φ,X) belongs to $U(K)$, then $\dim U + \dim |X|$ is called its dimension.

As in the case of algebras, also for vectorspace categories we have to distinguish the different representation types. A vectorspace category K is of <u>finite representation type</u>, if $U(K)$ is finite and K is <u>wild</u> if there is an exact embedding of M_Ω into $U(K)$ which is a representation equivalence with the image category. We have to be slightly more carefully in order to define the notions tame and domestic: In case K has only finitely many indecomposable objects, then K is called <u>tame</u> provided for any dimension d, there is a finite number of embedding functors $F_i : M_{k[T]} \to U(K)$ such that all but a finite number of indecomposable objects of $\mathcal{U}(K)$ of dimension d are of the form $F_i(M)$ for some indecomposable k[T]-module M. In this case, K is called <u>domestic</u> provided we can choose the functors F_i independently from d. In general, a vectorspace category K is called tame or domestic provided any full subcategory with only finitely many indecomposable objects is tame, or domestic, respectively.

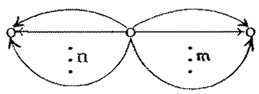
Of great interest is the special case of a vectorspace category K where for any indecomposable object X of K, the endomorphism ring End(X) is a division ring (thus, in case k is algebraically closed, equal to k). In this case, K is called <u>Schurian</u>. In particular, if all indecomposable objects of K are one-dimensional, then K is Schurian.

<u>Lemma</u>: Let k be an algebraically closed field, and K be a Schurian vectorspace category. If K is of finite representation type, then any indecomposable object of K is one-dimensional. If K is not of wild representation type, then any indecomposable object of K is at most two-dimensional. Moreover, in this case, if X, Y are indecomposable in K, and $\dim|X| = 2$, then $\mathrm{Hom}(X,Y) \neq 0$ or $\mathrm{Hom}(Y,X) \neq 0$.

<u>Proof</u>: Denote by add X the additive subcategory generated by the direct summands of X. Now let X be any object of a vectorspace category with End(X) = k, and with $\dim|X| = n$. Then clearly $\mathcal{U}(\text{add } X)$ is isomorphic to the category of representations of the quiver



with n arrows, thus add X is of finite representation type only for n = 1, and is of wild representation type for $n \geq 3$. Similarly, if X,Y are objects with End(X) = End(Y) = k and Hom(X,Y) = 0 = Hom(Y,X) then $\mathcal{U}(\text{add } X \oplus Y)$ is isomorphic to the category of representations of the quiver



where $n = \dim|X|$, $m = \dim|Y|$. This category is wild in case $n+m \geq 3$.

The vectorspace categories with only one-dimensional indecomposable objects have been investigated in great detail by Nazarova and Roiter and their students. Namely, these correspond just to the partially ordered sets. Let S be a partially ordered set. Define a k-category kS as follows: its objects are the elements of S, and for $i,j \in S$, let

$$\mathrm{Hom}(i,j) = \begin{cases} k & \text{iff} \quad i \leq j \\ 0 & \phantom{\text{iff}} \quad i \nleq j \end{cases},$$

with composition of maps the ordinary multiplication in k. In order to deal with an additive category, we just adjoin the finite direct sums, and obtain in this way the category add kS. This is a vectorspace category if we consider k as the underlying vectorspace for any element i of S. We will call add kS <u>the additive category of the partially ordered set</u> S. Note that add kS has only one-dimensional indecomposable objects. Also, conversely, assume $\mathcal{K}$ is a vectorspace category with only one-dimensional indecomposable objects (and such that all objects are finite-dimensional). We claim that $\mathcal{K}$ is of the form add kS. Namely, the indecomposable objects in $\mathcal{K}$ (or better their isomorphism classes) form the elements of S, and for X,Y indecomposable in $\mathcal{K}$, with isomorphism classes $i = [X]$ and $j = [Y]$, we have $i \leq j$ iff $\mathrm{Hom}(X,Y) \neq 0$. Clearly, in this way we obtain a reflexive relation which is both transitive and anti-symmetric due to the fact that the indecomposables are one-dimensional.

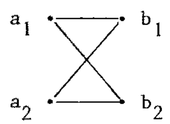
There is another interpretation of the subspace category $\mathcal{U}(\text{add } kS)$ of the additive category of a partially ordered set S. By definition, an S-<u>space</u> (V,V_i) is a vectorspace V together with subspaces V_i, for all $i \in S$, such that $i \leq j$ in S implies $V_i \subseteq V_j$. The S-spaces form a category $\mathcal{S}(S)$, with maps from (U,U_i) to (U',U_i') being given by linear transformations $f : U \to U'$ satisfying $f(U_i) \subseteq U_i'$. This definition is due to P. Gabriel [17], and he showed that the categories $\mathcal{U}(\text{add } kS)$ and $\mathcal{S}(S^{op})$ are representation equivalent where S^{op} is the reversed partially ordering on the underlying set of S. Namely, let (U,φ,X) be an object in $\mathcal{U}(\text{add } kS)$. For any $i \in S$, let X_i be the subobject of X which is the union of all images $i \to X$, and let $U_i = \varphi^{-1}(X_i)$. For $i \leq j$, we have $X_i \supseteq X_j$, thus also $U_i \supseteq U_j$; therefore (U,U_i) is an S^{op}-space which we denote by $\eta(U,\varphi,X)$.

Proposition (Gabriel) : η: (add kS) $\to$ $S(S^{op})$ is a full, dense functor which reflects isomorphisms, thus a representation equivalence.

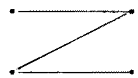
In fact, one shows that the objects in $S(S^{op})$ have projective covers, and that U(add kS) is just the category of projective covers of $S(S^{op})$, see [18] or [15].

In contrast to this subspace interpretation of U(add kS), of dealing with one vectorspace with a set of prescribed subspaces, the definition of U(add kS) may seem to be less intuitive. However, in the sequel, we always will need to work with this definition directly. Namely, we usually will have to consider subspace categories of the form U(Hom(M,M)), and sometimes it will turn out that Hom(M,M) is just the additive category of a partially ordered set. Also, we should remark that the definition of U(add kS) is precisely the base-free translation of a very important class of matrix problems which has been considered for a long time by Nazarova and Roiter. Namely, there are given matrices over k with the columns labelled by elements of a partially ordered set, and the rules of allowed transformations are as follows: we are allowed to use all possible row operations (adding multiples of one row to any other row, and multiplying any row with a non-zero scalar), we also may multiply any row with a non-zero scalar, but we only may add a multiple of a row labelled s to a row labelled t, if $s \leq t$ in S.

In order to visualize partially ordered sets rather easily, we will use the following convention: the elements of S will be represented by points, and the relation $\leq$ is always thought of as going from left to right along the drawn edges. For example,



is the four-element set $\{a_1,a_2,b_1,b_2\}$ with $a_i \leq b_j$ for all i,j. The n-element chain $\{1 < 2 < \ldots < n\}$ will be denoted by (n), the disjoint union of (n_1), ... (n_s) by $(n_1,\ldots,n_s)$. By N we will denote the following partially ordered set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and (N, $n_2,\dots,n_s$) stands for the disjoint union of N and chains ($n_2,\dots,n_s$). (The reason for drawing the relation $\leq$ from left to right, and not as usually from below upwards, comes from the fact that most partially ordered sets considered will be derived from Auslander-Reiten-quivers, the relation $\leq$ being derived from the arrows $\longrightarrow$ which usually are drawn from left to right. Also considering the partially ordered set as a vectorspace category, the relation $\leq$ means nothing else then the existence of a non-trivial map $\longrightarrow$.)

Let us quote now the two main theorems on the representation type of partially ordered sets. By the representation type of the partially ordered set S we will mean that of the vectorspace category add kS.

Kleiner's theorem. The partially ordered set S *is of finite representation type if and only if* S *is finite and does not contain as a full subset one of the sets* (1,1,1,1), (2,2,2), (1,3,3), (1,2,5) or (N,4).

This theorem has been proved in [22] using techniques due to Nazarova and Roiter developped in [28]. A different approach to the main working tool, the "differentiation with respect to maximal elements", has been given by Gabriel in [18], using homological notions. Let us remark that Kleiner has given a complete list of all partially ordered sets of finite representation type which have a faithful representation: this result will not be needed here, for the one-relation algebras,but we will comment on it further in the report on the Brauer-Thrall conjectures [35].

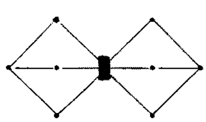
Nazarova's theorem. The partially ordered set S *is of wild representation type if and only if* S *contains as a full subset one of the sets* (1,1,1,1,1), (1,1,1,2), (2,2,3), (1,3,4), 1,2,6) *or* (N,5). *Also, if* S *is not of wild representation type, then* S *is of tame representation type.*

The proof of this theorem is in [27].
Let us also mention that Otraševskaja has given a complete classifica-

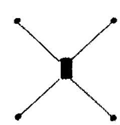
tion of the partially ordered sets with precisely one series of indecomposable representations.

Let us return to the general vectorspace categories which are not necessarily obtained from a partially ordered set. We also will adopt conventions for visualizations, at least for some special cases. The case of a non-Schurian vectorspace category of finite representations type will be considered in part 5. Here, we will deal with a Schurian vectorspace K which is not of wild representation type. The one-dimensional objects of K will again be represented by vertices • , the two-dimensional objects either by black squares ▮, or, if we want to specify two base vectors, by squares ⁞ with two vertices. Edges will represent non-zero homomorphisms, but usually we will have to specify rather carefully the relations.

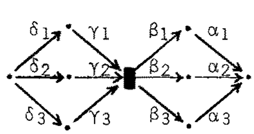
Two special situations will occur more frequently, so that it seems to be helpful to introduce the following convention: We will use the two symbols



and



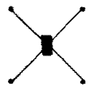
in the following situation: The first refers to a vectorspace category with 8 one-dimensional objects, 1 two-dimensional object with trivial endomorphism ring, and non-zero homomorphisms

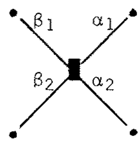


satisfying the following relations:

$$\sum_{i=1}^{3} \alpha_i\beta_i = 0, \quad \sum_{i=1}^{3} \gamma_i\delta_i = 0, \quad \beta_1\gamma_1 = \beta_2\gamma_2 = \beta_3\gamma_3 = 0.$$

This implies that the images of $\gamma_1, \gamma_2, \gamma_3$ are three pairwise different one-dimensional subspaces of the two-dimensional object, and these are also just the kernels of $\beta_1, \beta_2, \beta_3$, respectively. A great number of vectorspace subcategories of this form will be exhibited in section 3.4. In 3.3, one special case will be derived in great detail.

The second symbol  refers to a vectorspace category with 4 one-dimensional objects, 1 two-dimensional object with trivial endomorphism ring, and non-zero homomorphisms



where $\alpha_1\beta_2 = 0 = \alpha_2\beta_1$. Examples of this type will also be found in 3.4.

2.5. <u>Subspace categories arising from simple injective modules</u>

Let R be a finite dimensional k-algebra, and E a simple injective R-module with endomorphism ring k. Let S be the largest factor ring of R without composition factor E, and $P(E)$ a projective cover of E. We denote by $M = \text{rad } P(E)$ the radical of $P(E)$; it is easy to see that M is an S-module.

<u>Lemma 1</u>: The category $\mathcal{M}_R$ of R-modules is equivalent to the category $\mathcal{M}({}_kM_S)$ of representations of the bimodule ${}_kM_S$.

Recall, that for rings S,T and a bimodule ${}_TM_S$, the category $\mathcal{M}({}_TM_S)$ of <u>representations of the bimodule</u> ${}_TM_S$ has as objects the triples $(U_T, X_S, \gamma : U_T \otimes {}_TM_S \to X_S)$, and a map from (U_T, X_S, γ) to (U'_T, X'_S, γ') is given by a pair (α, β) of maps $\alpha : U_T \to U'_T$, $\beta : X_S \to X'_S$ satisfying $\beta\gamma = \gamma'(\alpha \otimes 1_M)$. This category also can be described as the category of triples $(U_T, X_S, \tilde{\gamma} : U_T \to \text{Hom}({}_TM_S, X_S))$, using the adjoint $\tilde{\gamma}$ of γ.

For the proof of lemma 1, we may suppose that R is basic. Let e be an idempotent of R with $P(E) = eR$. Then $(1-e)Re = 0$, since $E = eR/\text{rad}(eR)$ is injective, and $S = (1-e)R(1-e) = (1-e)R$. Also, $M = eR(1-e)$. We can write R in matrix form as follows:

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix} = \begin{pmatrix} k & M \\ 0 & S \end{pmatrix}$$

and the representations of such a matrix ring can be described by triples $(U_k, X_S, U \underset{k}{\otimes} M_S \to X_S)$.

If we start with a quiver with relations $(\Gamma, \rho_i)_{i \in I}$, and $R = k\Gamma/\langle\rho_i | i \in I\rangle$, and we consider the simple R-module $E = E(a)$ corresponding to the vertex a, then E is injective if and only if a is a source. Let $(\Gamma_a, \rho_j')_{j \in J}$ be the quiver with relations obtained from $(\Gamma, \rho_i)_{i \in I}$ by deleting the vertex a. Then Γ_a is the full subquiver of Γ containing all vertices different from a, and, if a is a source, then $\{\rho_j' | j \in J\}$ in fact is a subset of $\{\rho_i | i \in I\}$. Note that in this situation $S = k\Gamma_a/\langle\rho_j' | j \in J\rangle$ is the largest factor ring of R without composition factor E, and $\mathcal{M}_S$ is the full subcategory of $\mathcal{M}_R$ of all R-modules without composition factor E.

Given a module M_S, let $\mathcal{r}(M_S)$ be the full subcategory of $\mathcal{M}_S$ of all objects X_S with $\mathrm{Hom}(M_S, X_S) = 0$. Recall that $\mathrm{Hom}(M_S, \mathcal{M}_S)$ denotes the vectorspace category given by the representable functor $\mathrm{Hom}(M_S, -)$.

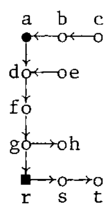
Lemma 2: Let S be a finite dimensional k-algebra. The full subcategory of $\mathcal{M}({}_kM_S)$ of all representations of ${}_kM_S$ without non-zero direct summand of the form $(0,Y,o)$ with Y in $\mathcal{r}(M_S)$, is representation equivalent to $\mathcal{U}(\mathrm{Hom}(M_S, \mathcal{M}_S))$, where (U_k, X_S, γ) goes to $\tilde{\gamma}$.

Proof: Clearly, the objects of the form $(0,Y,o)$ with Y in $\mathcal{r}(M_S)$ generate the kernel of this functor $(U_k, X_S, \gamma) \mapsto \tilde{\gamma}$. On the other hand, the functor is full and dense, and it reflects isomorphisms if we restrict it to the subcategory of representations without non-zero direct summands of the form $(0,Y,o)$, with Y in $\mathcal{r}(M_S)$.

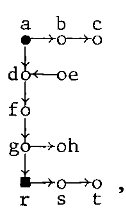
According to the lemma, the representation type of $\mathcal{M}_R$ depends both on $\mathcal{r}(M_S)$ and $\mathcal{U}(\mathrm{Hom}(M_S, \mathcal{M}_S))$. It is easy to construct examples where they have completely different representation types: any one of them may be wild, the other of finite type. However, since most considerations will be done by induction on the number of simple modules, thus the number of vertices of Γ, we then may assume that $\mathcal{r}(M_S)$ is known and will have to concentrate on the subspace category

$U(\mathrm{Hom}(M_S, M_S))$.

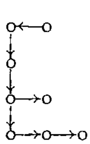
In the case of one-relation algebras $R = k\Gamma/\langle\rho\rangle$, with a being the starting point of the relation ρ, we usually will take this vertex a in order to construct S and $U(\mathrm{Hom}(M_S, M_S))$. This clearly has the advantage that Γ_a is a quiver without relations, thus $S = k\Gamma_a$, which means that we know the S-modules rather well. Of course, in general a does not need to be a source, so that we cannot apply the considerations above, However, we see that most of the algebras listed in theorem 1 of 1.5 have the following property:
Γ decomposes into $\Gamma = \Gamma' \cup \Gamma''$ where Γ', Γ'' are two full subquivers which have only the vertex a in common, such that the paths involved in ρ all lie inside Γ', a is a source in Γ', and Γ'' is a tree. Clearly, in this case we may apply reflection functors to M_Γ with respect to some points lying inside $\Gamma'' \setminus \{a\}$ so that a becomes a source also in Γ''. For example, in the case of



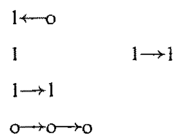
we first apply reflection functors at the points c,b, and c, in order to obtain



then Γ_a has the form $\Gamma_a = \Gamma'_a \dot{\cup} \Gamma''_a$

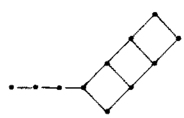
Γ'_a  Γ''_a

and M is the representation of Γ_a with dimension type

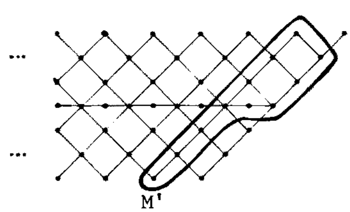


Now Γ'_a is a quiver of type $\tilde{E}_7$, thus tame, and $M' = M|\Gamma'_a$ is preinjective, thus $\mathcal{R}(M)$ contains all preprojective and all regular Γ'_a-modules.

We show that $\mathrm{Hom}(M',\mathcal{M}_{k\Gamma'_a})$ is of the form



Namely, we determine the position of $\begin{smallmatrix} 1 & o & \\ 1 & & \\ 1 & 1 & \\ o & o & o \end{smallmatrix}$ in the preinjective component of $\mathcal{M}_{k\Gamma'_a}$

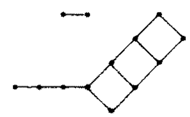


and encircle those indecomposable $k\Gamma'_a$-modules X with $\mathrm{Hom}(M',X) \neq$ o. This is rather easy, since any non-zero map $M' \to X$ has to be a sum of compositions of irreducible maps, thus only the modules to the right of M' may occur. The same calculation shows that $\dim \mathrm{Hom}(M',X) = 1$ in case it is non-zero.

Of course we know that for $M'' = M|\Gamma''_a$, the vectorspace category $\mathrm{Hom}(M'',\mathcal{M}_{k\Gamma''_a})$ is of the form

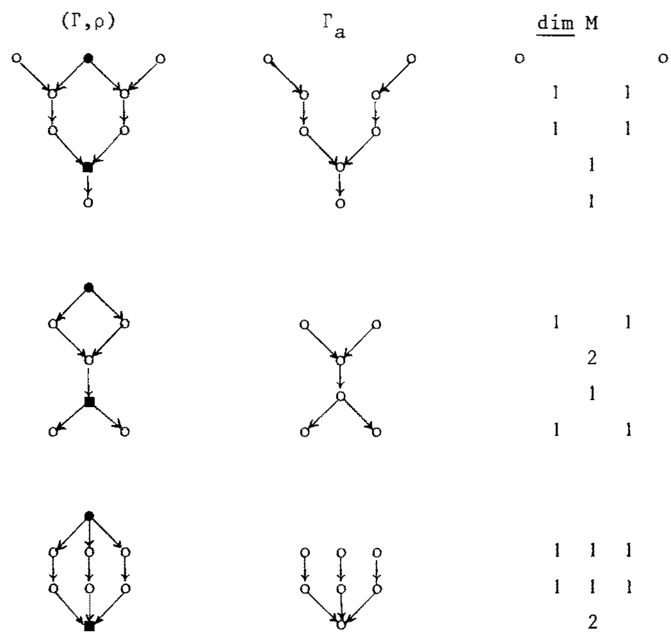
 ,

thus $\mathrm{Hom}(M,\mathcal{M}_{k\Gamma})$ is



which is of tame type according to the theorem of Nazarova, but not of finite representation type, according to the theorem of Kleiner. In fact, $\mathrm{Hom}(M,\mathcal{M}_{k\Gamma})$ contains a full subset of the form (2,2,2).

Let us write down M also in some other cases:



Our further interest will usually be concentrated around the calculation of the vectorspace category $\mathrm{Hom}(M_S,\mathcal{M}_S)$, where $S = k\Delta$, the path algebra of a quiver Δ (without relations).

In order to know whether $\mathrm{Hom}(M_S,\mathcal{M}_S)$ is Schurian or not, we have to consider the endomorphism rings of the objects $\mathrm{Hom}(M_S,X_S)$ in $\mathrm{Hom}(M_S,\mathcal{M}_S)$. But clearly this endomorphism ring is $\mathrm{End}(X_S)/I$ where I is the annihilator of the $\mathrm{End}(X_S)$-module ${}_{\mathrm{End}(X_S)}\mathrm{Hom}(M_S,X_S)$. Thus, if for all indecomposable modules X_S with $\mathrm{Hom}(M_S,X_S) \neq o$, we have $\mathrm{End}(X_S) = k$, then $\mathrm{Hom}(M_S,\mathcal{M}_S)$ is Schurian. In particular, this happens in case M_S is preinjective. For, in this case $\mathrm{Hom}(M_S,X_S) \neq o$

implies for X_S indecomposable that also X_S is preinjective, and therefore $\mathrm{End}(X_S) = k$. Thus we have shown the following:

<u>Lemma 2</u>: If S is the path algebra of a connected quiver, and M_S is preinjective, then $\mathrm{Hom}(M_S, M_S)$ is Schurian.

Also, we note some necessary conditions on M_S for $\mathrm{Hom}(M_S, M_S)$ to be of finite or tame representation type.

<u>Lemma 3</u>: Let S be the path algebra of a connected quiver Δ, and M_S a module.

If $\mathrm{Hom}(M_S, M_S)$ is a finite category (in particular, if $\mathrm{Hom}(M_S, M_S)$ is of finite representation type), then M_S is preinjective.

If $\mathrm{Hom}(M_S, M_S)$ is of tame representation type, then either Δ is of finite representation type, or M_S has no non-zero preprojective direct summand.

<u>Proof</u>: For both assertions, we may suppose that M_S is indecomposable.

If M is not preinjective, then $A^m A^{-m} M \approx M$ for all $m \in \mathbb{N}$. Now $\mathrm{Hom}(A^{-m}M, I) \neq 0$ for some indecomposable injective module I, and therefore also

$$\mathrm{Hom}(M, A^m I) \approx \mathrm{Hom}(A^m A^{-m} M, A^m I) \approx \mathrm{Hom}(A^{-m} M, I) \neq 0.$$

Thus $\mathrm{Hom}(M, M_S)$ has infinitely many indecomposable objects.

Now let M be preprojective, say $M = \mathbf{A}^{-m}P(r)$ for some indecomposable projective module $P(r)$. In case of Δ being not of finite representation type, and I indecomposable injective, we know that $\dim \mathrm{Hom}(P(r), A^n I) = \dim(A^n I)_r$ is unbounded (for $n \to \infty$), thus also the dimension of $\mathrm{Hom}(M, A^n I) = \mathrm{Hom}(A^{-m}P, A^n I) \approx \mathrm{Hom}(P, A^{m+n} I)$ is unbounded. However, for $\mathrm{Hom}(M_S, M_S)$ of tame representation type, we have seen that the objects X_S with $\mathrm{End}(X_S) = k$ satisfy $\dim \mathrm{Hom}(M_S, X_S) \leq 2$.

2.6. Finite enlargements

We consider now algebras S,T together with a bimodule ${}_TM_S$. We always will consider $\mathcal{M}_S$ as a full subcategory of $\mathcal{M}({}_TM_S)$, identifying the S-module X_S with the tripel $(0,X,o)$. We call $\mathcal{M}({}_TM_S)$ a finite enlargement of $\mathcal{M}_S$ in case $\mathcal{M}_S$ is cofinite in $\mathcal{M}({}_TM_S)$. Note that in case S is the path algebra of a connected quiver, this immediately implies that $\mathcal{M}_S$ is preinjective.

Lemma 1: Let ${}_TM_S$ be a bimodule. Let $f : X_S \to Y_S$ be irreducible, and $\mathrm{Hom}(M_S,X_S) = 0$. Then also $(o,f) : (0,X,o) \to (0,Y,o)$ is irreducible.

Proof: Let $(o,f) = (g_1,g_2)(h_1,h_2)$ be a factorisation, with $(h_1,h_2) : (0,X_S,o) \longrightarrow (U_T,Z_S,\gamma : U_T \otimes {}_TM_S \longrightarrow Z_S)$.
Now $f = g_2h_2$ is a factorisation of an irreducible map, thus either h_2 is a split monomorphism or g_2 is a split epimorphism. If g_2 is a split epimorphism, then also (g_1,g_2) is a split epimorphism. So assume $h_2 : X_S \to Z_S$ is a split monomorphism, thus $Z_S = h_2(X_S) \oplus C_S$ for some complement C_S, and $\gamma(U_T \otimes {}_TM_S) \subseteq C_S$, since there is no non-zero homomorphism $M_S \to X_S \approx h_2(X_S)$. As a consequence, (h_1,h_2) is a split monomorphism.

Recall that we identify $\mathcal{M}_S$ with a full subcategory of $\mathcal{M}({}_TM_S)$, namely we identify X_S of $\mathcal{M}_S$ with $(0,X,o)$ in $\mathcal{M}({}_TM_S)$.

Corollary: Let S be the path algebra of a connected quiver, and ${}_TM_S$ be a bimodule with M_S preinjective. Then all components but the preinjective one of the Auslander-Reiten quiver of $\mathcal{M}_S$ remain unchanged in the Auslander-Reiten quiver of $\mathcal{M}({}_TM_S)$.

This follows directly from the previous lemma, since the projective S-modules remain projective in $\mathcal{M}({}_TM_S)$, and the Auslander-Reiten sequences

$$0 \longrightarrow X_S \xrightarrow{f} Y_S \xrightarrow{g} Z_S \longrightarrow o$$

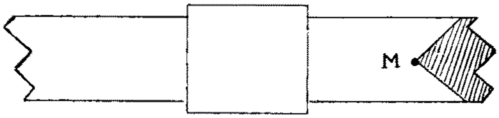
in $\mathcal{M}_S$ which do not involve preinjective modules remain Auslander-Reiten sequences in $\mathcal{M}({}_TM_S)$, since $\mathrm{Hom}(M_S,X_S) = 0 = \mathrm{Hom}(M_S,Y_S)$ and the sequence is characterised by the fact that both f and g are irreducible.

Of course, we similarly see that only that part of the preinjective component of M_S is changed which involves modules X_S with chains of irreducible maps $M' \to \dots \to X$, where M' is an indecomposable direct summand of M_S.

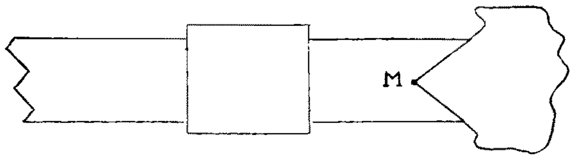
Lemma 2: Let S be the path algebra of a connected quiver, let ${}_TM_S$ be a bimodule with M_S cofinite in $M({}_TM_S)$. If (U_T, X_S, γ) is an indecomposable representation of ${}_TM_S$ with $\gamma \neq o$, then there exists a chain of irreducible maps $M' \to \dots \to (U,X,\gamma)$, where M' is an indecomposable direct summand of M_S.

Proof: Let M' be an indecomposable direct summand of M such that $\gamma | U \otimes M'$ is non-zero. Also, let $\varepsilon : \tilde{U} \to U$ be an epimorphism with $\tilde{U}$ a free module. Then $\gamma(\varepsilon \otimes 1_{M'})$ is a non-zero map from a direct sum of copies of M'_S to X_S, thus we see that $\mathrm{Hom}(M'_S, X_S) \neq o$. Let $o \neq \gamma' : M_S \to X_S$ and consider $(o,\gamma') : (O,M,o) \to (U_T, X_S, \gamma)$. Note that there are only a finite number of indecomposable triples in $M({}_TM_S)$ which have a non-zero map from (O,M,o) : namely, the indecomposable preinjective S-modules Y_S with $\mathrm{Hom}(M_S, Y_S) \neq o$, and some of the remaining indecomposable triples which do not belong to M_S. As a consequence, the process of factorizing (o,γ') non-trivially has to stop after a finite number of steps.

As a consequence, we see that the Auslander-Reiten quiver of $M({}_TM_S)$ is obtained from the Auslander-Reiten quiver M_S by extending the preinjective component in the following way: we know that the Auslander-Reiten quiver of M_S will have the shape

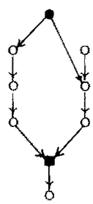


where we have marked the indecomposable modules X_S for which there exists a chain of irreducible maps $M_S \to \dots \to X_S$. The Auslander-Reiten quiver of $M({}_TM_S)$ is then of the following form

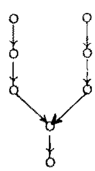


where the additional indecomposable modules (U,X,γ) lie in the region right of M, and most of them (at least those with $\gamma \neq o$) have a chain of irreducible maps $M \to \ldots \to (U,X,\gamma)$.

We consider one example in detail: the one-relation algebra

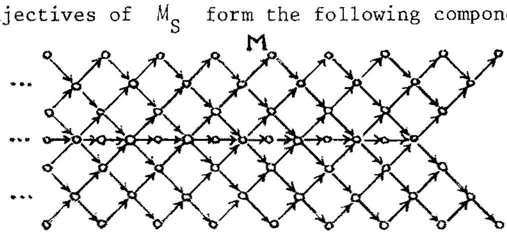


Here, S is the path algebra of

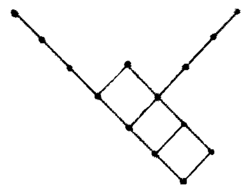


the module M_S is of dimension type $\begin{smallmatrix}1 & & 0\\ 1 & & 1\\ 1 & & 1\\ & 1 & \\ & 1 & \end{smallmatrix}$, and $T = k$.

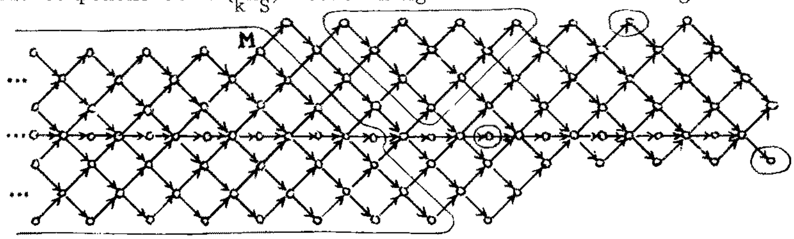
The preinjectives of M_S form the following component:



Note that we can calculate without difficulty that $\mathrm{Hom}(M_S, M_S)$ is the additive category generated by the partially ordered set



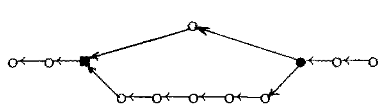
The component of $M({}_kM_S)$ containing M has the following form



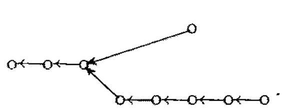
and we have encircled the indecomposable modules belonging to M_S. Note that this component contains all nine indecomposable injective modules, and one indecomposable projective module.

2.7 Gluing of two components

We proceed now to consider bimodules ${}_TM_S$ for which $M({}_TM_S)$ is no longer a finite enlargement of M_S, but where again M_S is a preinjective module over the path algebra of a connected quiver of infinite type. It turns out that in this situation we also have to work with the dual situation of the path algebra kS' of a connected quiver of infite type and a preprojective S'-module $N_{S'}$. Namely, consider the following one-relation algebra R given by



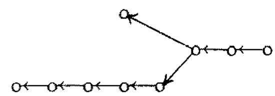
Of course, our previous device was to consider the subquivers

S:  and T:

and the indecomposable ${}_TM_S$ of dimension type

$$\underline{\dim}\ M_S = \begin{pmatrix} & & & & & 1 & & \\ 1 & 1 & 1 & & & & & \\ & & & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \underline{\dim}\ {}_T M = (9\ 0\ 0)$$

so that M_R is reduced to $M({}_T M_S)$. However, we also may consider the subquivers

T': $o \leftarrow o \leftarrow o$ and S': 

and the indecomposable bimodule ${}_{T'}N_{S'}$ of dimension type

$$\underline{\dim}\ {}_{T'}N = (0\ 0\ 9), \quad \underline{\dim}\ N_{S'} = \begin{pmatrix} & & 1 & & & & & \\ & & & & & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & & & \end{pmatrix}.$$

It is easy to see that M_R can be reduced to the category of triples $(X_{S'}, Y_{T'}, \gamma : X_{S'} \longrightarrow Y_{T'} \otimes {}_{T'}N_{S'})$, thus to the category $M(\ ({}_{T'}N_{S'})^*)$. Of course, we always will consider M_S and $M_{S'}$ as full subcategories of M_R. Note that $N_{S'}$ is preprojective and $\mathrm{Hom}(M_R, N_R) \neq o$.

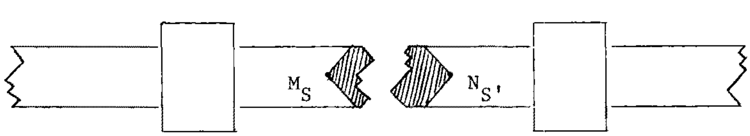
In the following, we will assume that M_R can be reduced both to $M({}_T M_S)$ and $M(\ ({}_{T'}N_{S'})^*)$ where S, S' are path algebras of connected quivers of infinite type, M_S is preinjective, $N_{S'}$ is preprojective with $\mathrm{Hom}(M_R, N_R) \neq o$, and, moreover, $M_S \cup M_{S'}$ is a cofinite subcategory of M_R. Note that these conditions all are satisfied for the one-relation algebras listed in theorem 1 and marked with "G". This letter stands for <u>gluing</u> since we will see that the Auslander-Reiten quiver of M_R is obtained from the Auslander-Reiten quivers M_S and $M_{S'}$ by joining together the preinjective component of M_S and the preprojective component of $M_{S'}$.

Let us first show that there is a chain of irreducible maps $M_R \to \ldots \to N_R$. By assumption, there exists a non-zero homomorphism $\gamma : M_R \to N_R$. Let G be the following set of indecomposable R-modules: it should contain the indecomposable S-modules X with $\mathrm{Hom}(M_S, X_S) \neq o$, the indecomposable S'-modules $Y_{S'}$ with $\mathrm{Hom}(Y_{S'}, N_{S'}) \neq o$, and those indecomposable R-modules which are not in $M_S \cup M_{S'}$. Then G has only finitely many objects, and any non-trivial factorisation of γ will be of the form $M_R \to G_R \to N_R$ with G_R a direct sum of modules in G. As a consequence, the process of factorising γ non-trivially stops after a finite number of steps, thus there exists a chain of irreducible maps $M_R \to \dots \to N_R$.

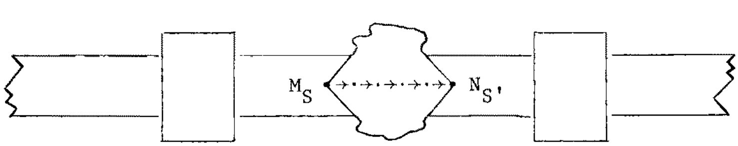
As a consequence, we see that the Auslander-Reiten quiver of $\mathcal{M}_R$ has one component which contains the preinjective S-modules, the preprojective S'-modules and the indecomposable R-modules which are outside of $\mathcal{M}_S \cup \mathcal{M}_{S'}$. All other components lie either in $\mathcal{M}_S$ or in $\mathcal{M}_{S'}$ and are unchanged. For, we know from the Corollary in 2.6, that all components of $\mathcal{M}_S$ but the preinjective one, remain unchanged in $\mathcal{M}_R$. Similarly, all components of $\mathcal{M}_{S'}$ but the preprojective one, remain unchanged in $\mathcal{M}_R$. Again from lemma 1 of 2.6, we know that the chains of irreducible maps $X_S \to \dots \to M_S$ remain chains of irreducible maps in $\mathcal{M}_R$; similarly, the chains of irreducible maps $N_{S'} \to \dots \to Y_{S'}$ remain chains of irreducible maps in $\mathcal{M}_R$. Since there is a chain of irreducible maps $M_R \to \dots \to N_R$, we see that all these modules X_S and $Y_{S'}$ belong to the same component of the Auslander-Reiten quiver of $\mathcal{M}_R$. There are only finitely many additional R-modules (some preinjective S-modules, some preprojective S'-modules, and the indecomposable R-modules outside $\mathcal{M}_S \cup \mathcal{M}_{S'}$). Since there cannot be a finite component, we conclude that these additional R-modules also have to belong to the component containing M_R and N_R. A final application of 2.6 shows that the Auslander-Reiten sequences

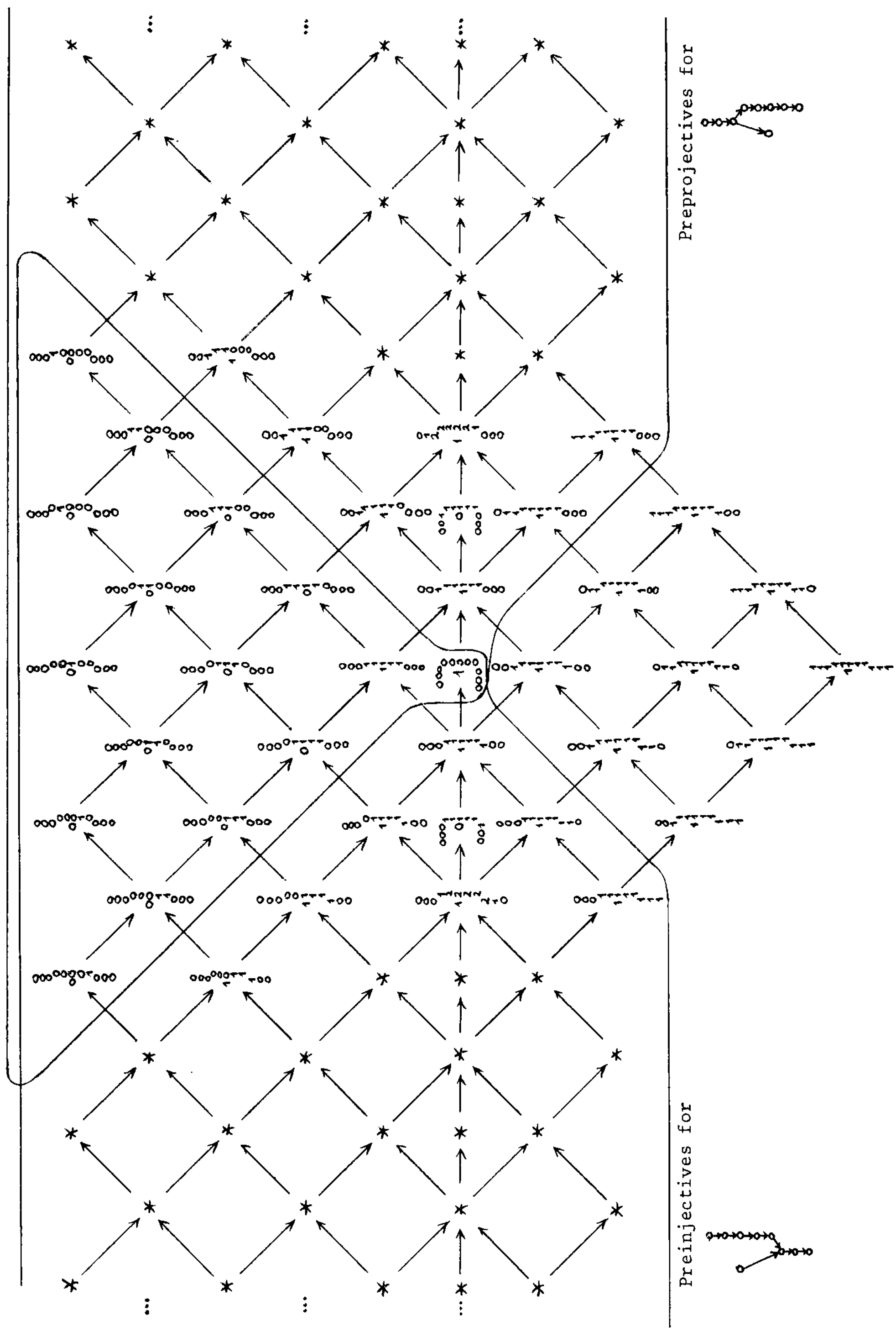
$$o \to X_1 \to X_2 \to X_3 \to o$$

in $\mathcal{M}_S$ with $\mathrm{Hom}(X_1,M) = \mathrm{Hom}(X_2,M) = o$ remain Auslander-Reiten sequences in $\mathcal{M}_R$, and similarly, those in $\mathcal{M}_{S'}$ with $\mathrm{Hom}(X_2,N) = \mathrm{Hom}(X_3,N) = o$ remain Auslander-Reiten sequences in $\mathcal{M}_R$. Thus, the Auslander-Reiten quiver of $\mathcal{M}_R$ is obtained from those of $\mathcal{M}_S$ and $\mathcal{M}_{S'}$



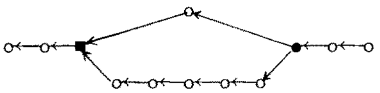
as follows:



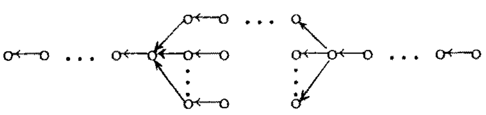

Preprojectives for
Preinjectives for

The new component of M_R containing both modules M_R and N_R will be called the <u>glued component</u>.

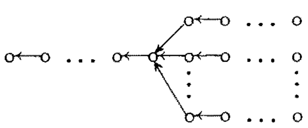
For example in case of the one-relation algebra R given by



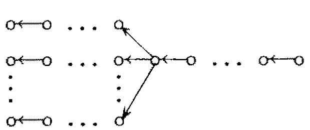
the glued component is exhibited on the previous page. In general, in case we deal with the (usually wild) quiver



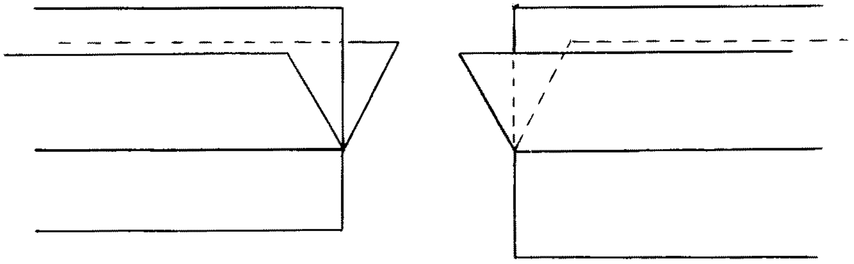
with all possible commutativity relations, let S be the path algebra of



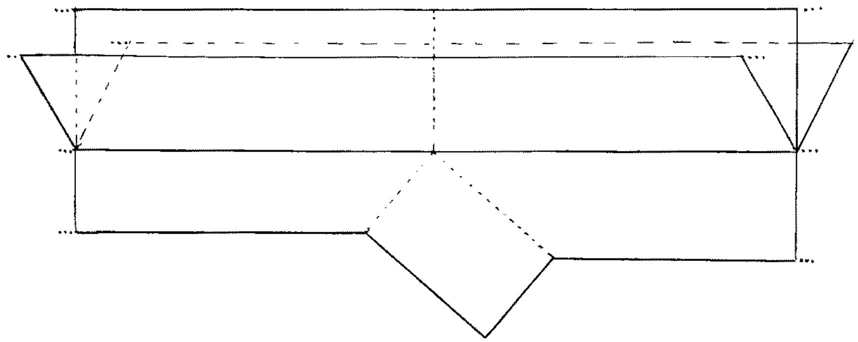
and S' the path algebra of



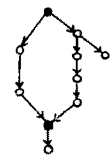
Then the preinjective S-modules and the preprojective S'-modules form components of the following form:



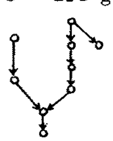
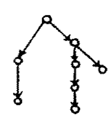
and the glued component of M_R looks as follows:

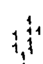
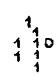


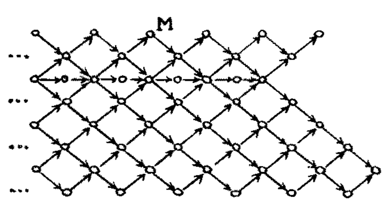
As another example, we exhibit the glued component of the one-relation algebra

R: 

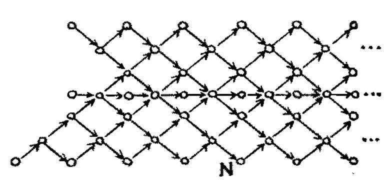
Here, S and S' are given by the following quivers

S:    S': 

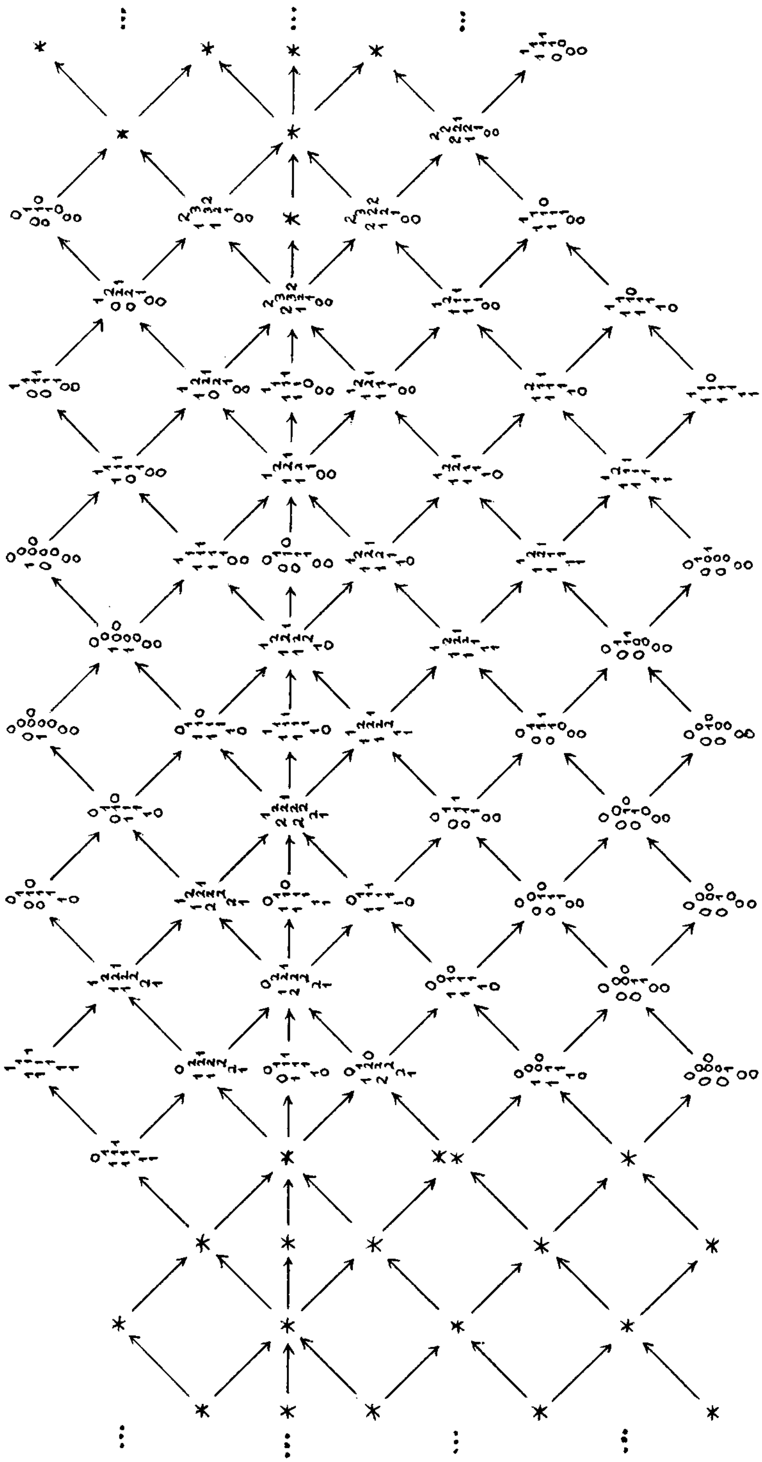
and the dimension type of M_S is , that of $N_{S'}$ is . The preinjective component of M_S is



the preprojective component of $M_{S'}$ is



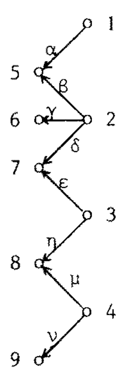
The next page shows the glued component of M_R.

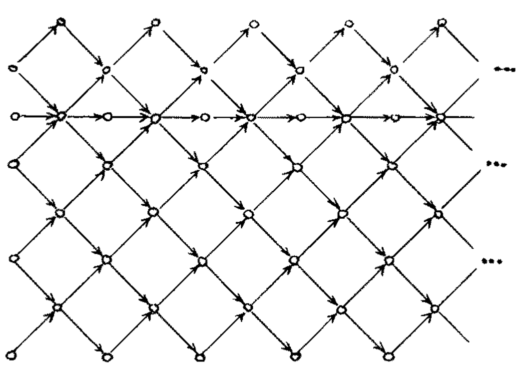


2.8 Glueing à la Müller ($\frac{1}{2}$)

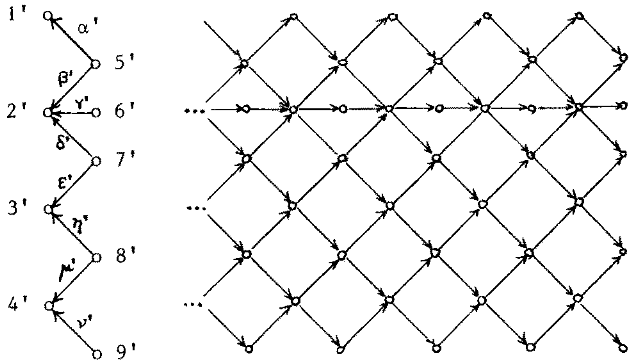
Since we have seen a typical way of glueing together two components of Auslander-Reiten quivers, we want to exhibit some similar processes of glueing, in this and in the next section. In this section, we try to give some insight into a construction which was used very efficiently by W. Müller [25]. Namely, given any quiver without paths of length ≥ 2, he constructed a corresponding quasi-Frobenius algebra with radical cube zero. We will see that his method amounts to a glueing of certain components.

Let Δ be a finite quiver with vertices $r,s,\ldots,$ and arrows $\alpha,\beta,\ldots$ without paths of length ≥ 2. Equivalently, Δ has no oriented cycles and $k\Delta$ has radical square zero. Note that any vertex of Δ is either a sink or a source. For example, we may take the following quiver Δ of type $\tilde{E}_8$, and we outline also its preprojective component:

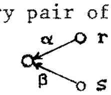




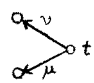
Consider also the quiver Δ' with reversed orientation: denote its vertices by $r',s',\ldots,$ the arrows by $\alpha',\beta',\ldots$. Here we outline the preinjective component



We form now the quiver $\Gamma = \Delta \cup \Delta'$ with identification $r = r'$ for r a sink in Δ (and therefore a source in Δ'), and consider the following relations: For every pair of arrows in Δ with same end point, say

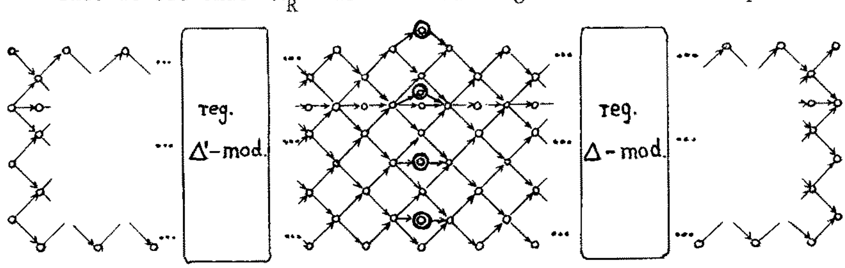


consider $\rho_{\alpha\beta} = \langle r \mid \alpha,\beta' | s \rangle$. For every pair of arrows in Δ with same starting point, say



consider $\rho_{\mu\nu} = \langle t|\mu,\mu'|t'\rangle - \langle t|\nu,\nu'|t'\rangle$. Let I be the ideal generated by all these relations $\rho_{\alpha\beta}, \rho_{\mu\nu}$ and $R = k\Gamma/I$. Then by construction, for r a source in Δ, the indecomposable projective R-module P_r with head $E(r)$ is also injective, and has socle $E(r')$. Thus we see that $M_\Delta \cup M_{\Delta'}$ is cofinite in M_R, the only indecomposable not belonging to $M_\Delta \cup M_\Delta$, being the P_r, for r a source. On the other hand, the only indecomposables in $M_\Delta \cap M_\Delta$, are the simple modules $E(s)$, with s a sink in Δ (and a source in Δ').

Thus we see that M_R has the following Auslander-Reiten quiver



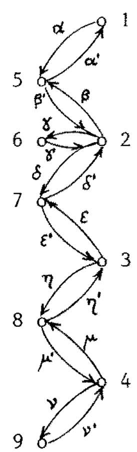
where we have encircled the projective-injective modules. Note that the middle component has the following property: if we omit the projective-injective modules, then the remaining quiver (it is called the stable part) is of the form $\mathbb{Z}\Gamma$. Thus we have seen:

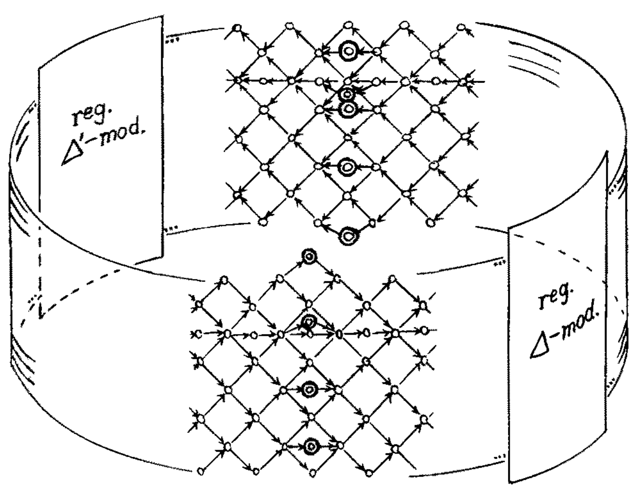
Lemma: Let Γ be any connected quiver which is not of type A_n, D_n, E_6, E_7, or E_8 and which has no paths of length ≥ 2. Then there exists an algebra R and a component of its Auslander-Reiten quiver whose stable part is of the form $\mathbb{Z}\Gamma$.

This seems to be of interest with respect to the recent results of Riedtmann [32].

The glueing process outlined above is one-half of what Müller actually did: he also glued the remaining two ends: the preprojective Δ'-modules and the preinjective Δ-modules along their respective simple modules, using again additional projective-injective modules.

In the example below, we obtain the quiver





with relations:

$\alpha\alpha' = \beta\beta'$, $\delta\delta' = \varepsilon\varepsilon'$, $\eta\eta' = \mu\mu'$;

$\beta'\beta = \gamma'\gamma = \delta'\delta$, $\varepsilon'\varepsilon = \eta'\eta$, $\mu'\mu = \nu'\nu$;

$\varphi\psi = 0 = \psi'\varphi$ for $\varphi \neq \psi \in \{\alpha,\beta,\gamma,\delta,\varepsilon,\mu,\nu\}$

An extension of this method to arbitrary quivers Δ without oriented cycles has been given recently by Tachikawa (see these proceedings [38]).

2.9 Glueing using splitting zero relations

We come back to the consideration of splitting zero relations and want to show the behaviour of the corresponding Auslander-Reiten quiver. We only consider the case of a connected quiver Γ with one splitting zero relation ρ. As we have seen in 1.3, the representations of (Γ,ρ) can be derived from some quiver Δ without relation by identifying a subquiver of Δ either of type A_n or D_n. In case (Γ,ρ) is of tame representation type, there are three cases possible:

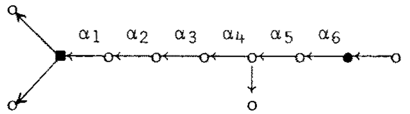
(i) Δ is connected, and therefore tame, thus of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$.

(ii) Δ decomposes into a tame quiver Δ_1 and a quiver Δ_2 of finite type.

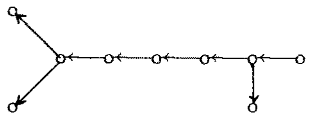
(iii) Δ decomposes into two tame quivers Δ_1 and Δ_2.

The last two cases deal with situations we are already familiar with, namely (ii) provides a finite enlargement of the quiver Δ_1, whereas (iii) is the case of a glueing of the quivers Δ_1 and Δ_2. However, here the glueing is established without the introduction of additional modules. Let us just give two examples:

First, consider the following quiver

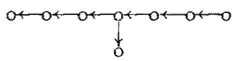


(with relation, as indicated, $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 = 0$). Here, Δ is the disjoint union of the quiver Δ_1 of type $\tilde{D}_8$

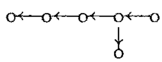


,

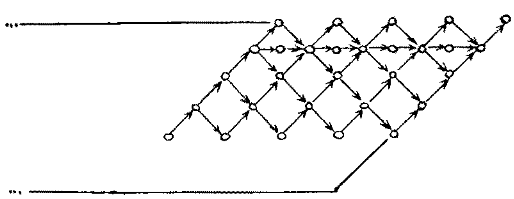
and the quiver Δ_2 of type $\tilde{E}_7$



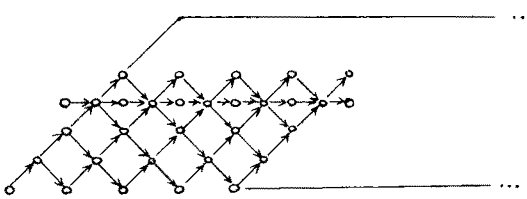
We have to identity the full subcategories $\mathcal{U}$ of $\mathcal{M}_{k\Delta_1}$ and $\mathcal{V}$ of $\mathcal{M}_{k\Delta_2}$ which are of the form $\mathcal{M}_{k\Delta_3}$, with Δ_3 being the following quiver of type D_6:



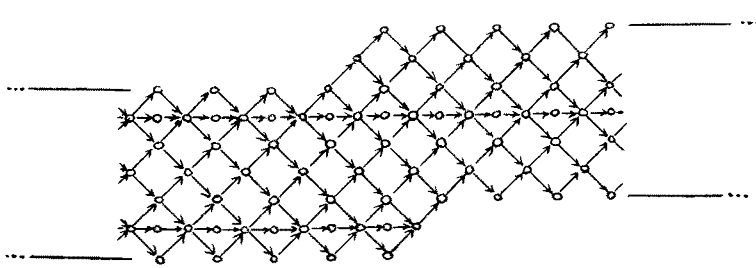
Note, that $\mathcal{U}$ lies in the preinjective component of $\mathcal{M}_{k\Delta_1}$



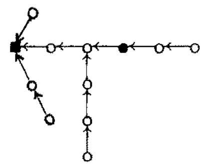
whereas $\mathcal{V}$ lies in the preprojective component of $\mathcal{M}_{k\Delta_2}$



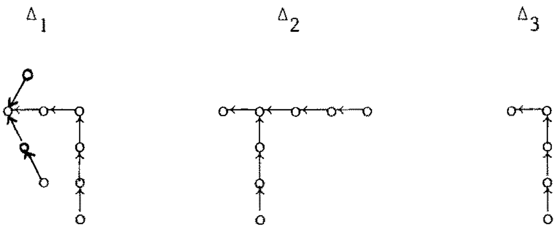
Thus $\mathcal{M}_{k\Gamma}$ has the following component which is obtained from the preprojective component of $\mathcal{M}_{k\Delta_1}$ and the preinjective component of $\mathcal{M}_{k\Delta_2}$ by the identification of the subcategories $\mathcal{U}$ and $\mathcal{V}$.



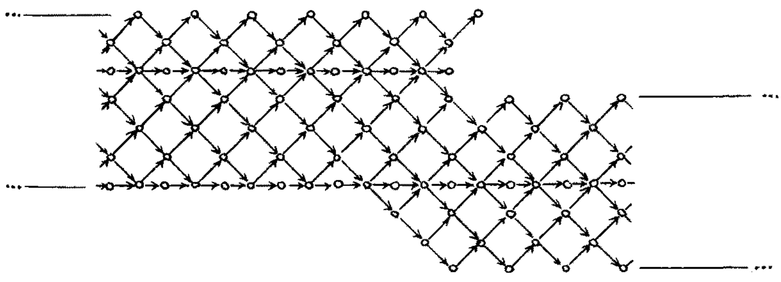
Similarly, the quiver Γ



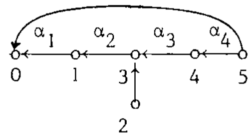
has subquivers Δ_1 of type $\tilde{E}_8$, and Δ_2 of type $\tilde{E}_7$ which intersect in the subquiver Δ_3 of type A_5



and $M_{k\Delta_2}$ and $M_{k\Delta_1}$ are glued along the joint full subcategory $M_{k\Delta_3}$ in order to form $M_{k\Gamma}$. The glued component of $M_{k\Gamma}$ has the following form:

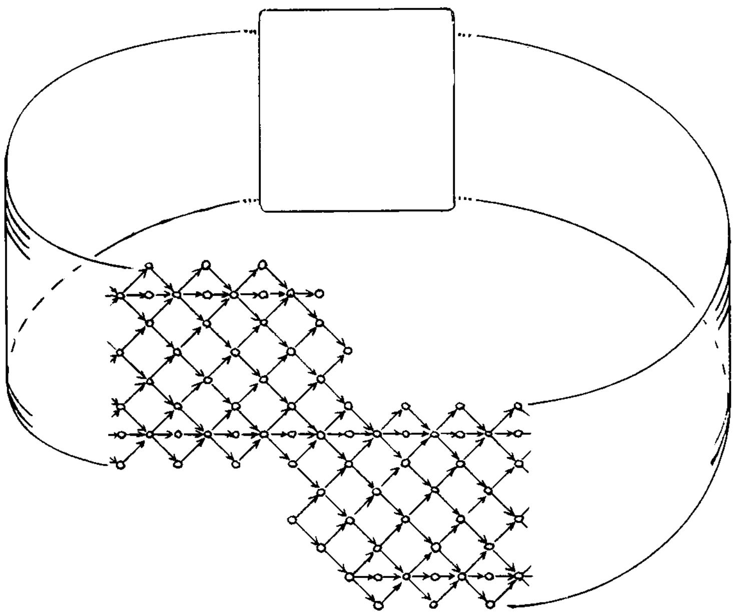


In the first case (i), the Auslander-Reiten quiver of $M_{k\Gamma}$ is obtained from that of $M_{k\Delta}$ correspondingly: this time, the preinjective component of $M_{k\Delta}$ is glued to the preprojective component of $M_{k\Delta}$ itself. For example, let Γ be given by



with $\alpha_1\alpha_2\alpha_3\alpha_4 = o$

then Δ is a quiver of type $\tilde{D}_9$, and the Auslander-Reiten quiver of Γ is as follows:



3. REGULAR ENLARGEMENTS

We are going to study the subspace categories $\mathcal{U}(\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma))$ where Γ is a tame connected quiver and M_Γ is regular. The pattern which occur in this situation seem to be of independent interest. We will encounter a large class of non-domestic tame algebras, which, however, fall into a small number of similarity classes. These similarity classes will be labelled using the extended Dynkin diagrams.

Before we recall some properties of the regular representations of the tame quivers, let us introduce the notion of a pattern.

3.1 Pattern

We want to define an equivalence relation on the class of all vectorspace categories, the equivalence classes being called pattern. By definition, two vectorspace categories $\mathcal{K}$ and $\mathcal{L}$ belong to the same pattern provided there are full cofinite embeddings $\mathcal{K} \to \mathcal{L}$ and $\mathcal{L} \to \mathcal{K}$ of vectorspace categories (note that an embedding $\iota : \mathcal{K} \to \mathcal{K}$ of vectorspace categories is assumed to be compatible with the given forget functors to the category of vectorspaces, thus there should be a canonical isomorphism $|\iota(X)| \approx |X|$ for any object X in $\mathcal{K}$). The pattern of $\mathcal{K}$ will be denoted by $[\mathcal{K}]$.

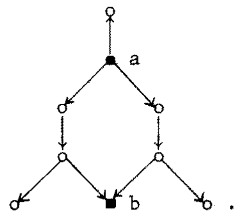
Let us make the following remark: in considering pattern, we only will be interested in vectorspace categories $\mathcal{K}$ which are infinite and for which there exists a cofinite embedding $\varphi : \mathcal{K} \to \mathcal{K}$ of vectorspace categories such that $\bigcap_m \varphi^m(\mathcal{K})$ is finite.

Note that if two vectorspace categories $\mathcal{K}$ and $\mathcal{L}$ belong to the same pattern, then the corresponding subspace categories $\mathcal{U}(\mathcal{K})$ and $\mathcal{U}(\mathcal{L})$ can be derived from each other: namely, there are full embeddings $\mathcal{U}(\mathcal{K}) \to \mathcal{U}(\mathcal{L})$ and $\mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{K})$ which can be controlled rather easily.

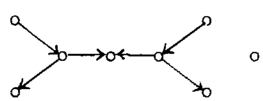
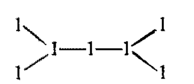
There is an additional situation where two vectorspace categories $\mathcal{K}$ and $\mathcal{L}$ yield rather similar subspace categories $\mathcal{U}(\mathcal{K})$ and $\mathcal{U}(\mathcal{L})$. Namely, recall that we have constructed rather frequently categorical equivalences of the form $\mathcal{M}_R \approx \mathcal{M}({}_kM_S)$, where R and S are finite dimensional algebras with S domestic (usually, S will be the path algebra of a tame quiver). We know that there exists a representation equivalence between a full subcategory $\mathcal{V}$ of $\mathcal{M}({}_kM_S)$ and

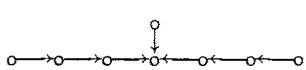
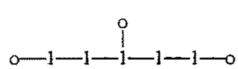
$U(\mathrm{Hom}(M_S, M_S))$, where the complement of V in $M({}_kM_S)$ consists of certain S-modules (those in the right annihilator $r(M_S)$.) In this case, we will say that M_R <u>can be reduced to</u> $U(\mathrm{Hom}(M_S, M_S))$. In case there are vectorspace categories K and L such that M_R can be reduced to $U(K)$, and $M_{R^{op}}$ can be reduced to $U(L)$, we will call the pattern $[K]$ and $[L]$ <u>similar</u>.

For example, let R be the one-relation algebra



Let S be the path algebra of the quiver obtained by deleting a, and T the path algebra obtained from the dual quiver by deleting b. Let M_S and N_T be the corresponding modules with $M_R \approx M({}_kM_S)$, $M_{R^{op}} \approx M({}_kN_T)$. Thus

S:  , $\underline{\dim}\ M_S =$  ,

T:  , $\underline{\dim}\ N_T =$  .

By definition, the pattern $[\mathrm{Hom}(M_S, M_S)]$ and $[\mathrm{Hom}(N_T, M_T)]$ are similar. In the next sections, we will calculate vectorspace categories of the form $\mathrm{Hom}(M_S, M_S)$ rather explicitly, and we will see that similar pattern may appear to look rather differently. However, we stress the fact that for vectorspace categories K, L with similar pattern $[K]$, $[L]$, the subspace categories $U(K)$ and $U(L)$ are not too different.

3.2 THE REGULAR MODULES OF A TAME QUIVER

Let Γ be a tame connected quiver, thus Γ is of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$. Recall that a Γ-module has been called regular, provided it contains no indecomposable direct summand which is preprojective or preinjective. It follows directly from the definition that for X, Y indecomposable, and $f : X \to Y$ irreducible, then X is regular iff Y is regular. Namely, the set of indecomposable regular modules is the union of full components of the Auslander-Reiten quiver.

Proposition. Let Γ be a tame connected quiver. The full subcategory $\mathcal{R}$ of regular $k\Gamma$-modules is an abelian category, and is the direct sum of indecomposable categories $\mathcal{R}_t$ all of which are serial and of global dimension 1. The number n_t of simple objects in $\mathcal{R}_t$ is finite, and is different from 1 for at most three t.

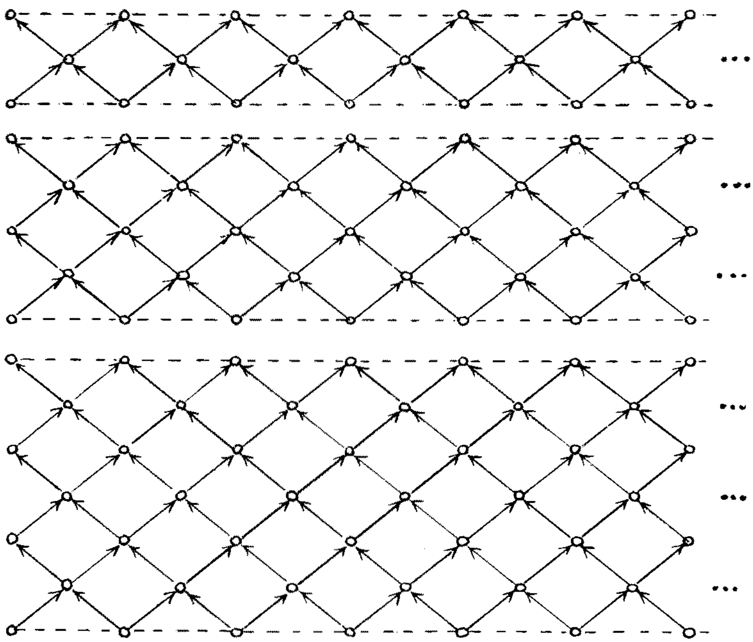
Recall that an abelian category is called serial provided any indecomposable object has a unique composition series. In this case, any indecomposable object is uniquely determined by its socle and its composition length. If we consider a regular representation X of Γ then we call its socle $\mathrm{soc}_{\mathcal{R}}X$ inside the category $\mathcal{R}$ the regular socle, and its length inside $\mathcal{R}$ the regular length, in order to distinguish from the notions of socle and length in $\mathcal{M}_{k\Gamma}$. Similarly, the simple objects in $\mathcal{R}$ will be called the simple regular modules. If U is simple regular, we denote by U(m) the unique regular module with regular socle U and regular length m.

Since $\mathcal{R}$ does not contain projective or injective modules, any indecomposable regular module X has both an Auslander-Reiten sequence ending with, and an Auslander-Reiten sequence starting with X. Since $\mathcal{R}$ is closed under irreducible maps, these Auslander-Reiten sequences lie inside $\mathcal{R}$, and therefore can be calculated inside $\mathcal{R}$: they are of the form

$$o \to U(m) \to U(m+1) \oplus U(m)/\mathrm{soc}_{\mathcal{R}}U(m) \to U(m+1)/\mathrm{soc}_{\mathcal{R}}U(m+1) \to o.$$

Namely, it is easy to check that these sequences have the necessary lifting property inside $\mathcal{R}$. Consequently, $A^{-}U(m) = U(m+1)/\mathrm{soc}_{\mathcal{R}}U(m+1)$ and the irreducible maps are the inclusions $U(m) \longrightarrow U(m+1)$ and the projections $U(m+1) \to U(m+1)/\mathrm{soc}_{\mathcal{R}}U(m+1)$, for all $m \geq 1$. If U(m) belongs to $\mathcal{R}_t$, then clearly $A^{n_t}U(m) \approx U(m)$, and n_t is the smallest number with this property, thus we call n_t the period of U(m). The modules of the form $A^{s}U(m)$ with $o \leq s < n_t$ will be said to form the orbit of U(m).

The indecomposable modules in any $\mathcal{R}_t$ form a complete component of the Auslander-Reiten quiver, and the form of this component only depends on n_t. These are the first cases (the dotted lines have to be identified in order to form a cylinder):



$n_t = 1$

$n_t = 2$

$n_t = 3$

In the next section, we will consider vectorspace categories of the form $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$ in great detail, where Γ is a tame connected quiver, and M_Γ is regular. Let us state some general assertions. Note that if M is an indecomposable regular module in $\mathcal{R}_t$, then the regular length of M is $\leq n_t$ if and only if $\mathrm{End}(M) = k$.

<u>Lemma 1.</u> Let Γ be a tame connected quiver, and M_Γ regular. If $M = \oplus M_i$ with $\mathrm{End}(M_i) = k$, then $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$ is a Schurian vectorspace category.

<u>Proof:</u> For X indecomposable regular, $\mathrm{Hom}(M_i, X)$ is annihilated by the radical of $\mathrm{End}(X)$, thus the same is true for $\mathrm{Hom}(M,X) = \oplus \mathrm{Hom}(M_i, X)$. If X is indecomposable and not regular, then anyway $\mathrm{End}(X) = k$.

<u>Lemma 2.</u> Let Γ be a tame connected quiver, and $\sigma : \mathcal{M}_\Gamma \longrightarrow \mathcal{M}_{\sigma\Gamma}$ a reflection functor. Let M_Γ be a regular Γ-module. Then the vectorspace categories $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$ and $\mathrm{Hom}(\sigma M, \mathcal{M}_{\sigma\Gamma})$ belong to the same

pattern.

Proof: Let $\sigma = \sigma_r$, where we may assume that r is a sink in Γ. Then σ defines a full embedding of the vectorspace category $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ into $\mathrm{Hom}(\sigma M, M_{\sigma\Gamma})$, and the only indecomposable object of $\mathrm{Hom}(\sigma M, M_{\sigma\Gamma})$ not being in the image, is $\mathrm{Hom}(\sigma M, E_{\sigma\Gamma}(r))$.

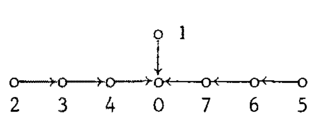
Lemma 3. Let Γ be a tame connected quiver. Let M_Γ regular, and assume $M = \oplus M_i$ with $\mathrm{End}(M_i) = k$ for all i. We denote the vectorspace category $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ by K. Then there exists a cofinite full embedding $\varphi : K \to K$ with $\bigcap_{n \in \mathbb{N}} \varphi^n(K) = 0$.

Proof: If X is simple regular, let $X(m)$ be the indecomposable regular module with regular socle X and regular length m. Let p be the smallest common multiple of the periods of the simple regular Γ-modules. We denote by K' the full subcategory of K of all objects $\mathrm{Hom}(M_\Gamma, Y_\Gamma)$ with Y_Γ preinjective. On K', we define φ to be given by A^p, thus $\varphi(\mathrm{Hom}(M_\Gamma, Y_\Gamma)) = \mathrm{Hom}(M_\Gamma, A^p Y)$, which is canonically isomorphic to $\mathrm{Hom}(A^p M, A^p Y) \approx \mathrm{Hom}(M, Y)$, since $A^p M \approx M$. For X simple regular with $\mathrm{Hom}(M, X(m)) \neq 0$, define $\varphi(\mathrm{Hom}(M, X(m)) = \mathrm{Hom}(M, X(m+1))$. It is easy to see how φ has to be defined on morphisms in order to be functorial, and that it has the desired properties.

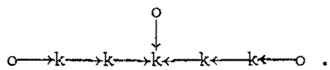
3.3 Calculations of pattern

It is easy to calculate the pattern of $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ for any regular Γ-module M_Γ. Let us show this in great detail in one example.

The case $(\tilde{E}_7, 3)$. We consider a quiver Γ of type $\tilde{E}_7$, and a simple regular Γ-module M_Γ of period 3. Of course, there are many possible Γ (using the different orientations of $\tilde{E}_7$), and for every Γ, there are three different simple regular modules of period 3. Since all the different M_Γ are obtained from any one of them by the use of reflection functors, we may choose an arbitrary one. Thus, let us choose the "subspace orientation", thus Γ is



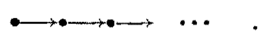
and let M_Γ be the simple regular module



In order to calculate $\mathrm{Hom}(M_\Gamma, M_\Gamma)$, we note that for an indecomposable Γ-module X with $\mathrm{Hom}(M_\Gamma, X_\Gamma) \neq 0$, either X is regular with regular socle M, or else X is preinjective. As before, let $M(n)$ denote the (unique) regular Γ-module with regular socle M and regular length n, and let $\mu_n : M(n) \to M(n+1)$ be the inclusion map. Since $\mathrm{Hom}(M, M(n))$ is one-dimensional over k, and $\mathrm{Hom}(M, \mu_n)$ is an isomorphism of vectorspaces, we see that the chain of inclusions

$$M(1) \xrightarrow{\mu_1} M(2) \xrightarrow{\mu_2} M(3) \longrightarrow \ldots$$

gives raise under $\mathrm{Hom}(M, -)$ to the following vectorspace category



Next, we calculate $\mathrm{Hom}(M,X)$ for X indecomposable preinjective. Now, $X = A^j I(r)$ for some $j \in \mathbb{N}$ and some $0 \leq r \leq 7$. Since we know that

$$\mathrm{Hom}(M,X) = \mathrm{Hom}(M, A^j I(r)) \simeq \mathrm{Hom}(A^{-j}M, I(r)) \simeq (A^{-j}M)_r^*$$

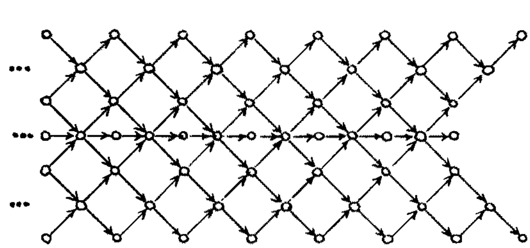
and $A^3 M \approx M$, we only have to determine the following dimension types:

$$\underline{\dim}\ M = \begin{pmatrix} & & & 0 & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

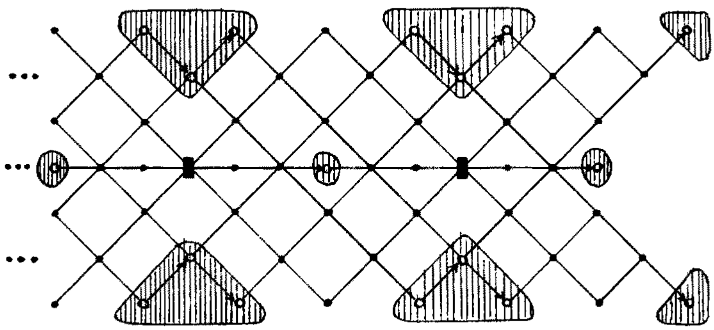
$$\underline{\dim}\ A^{-1}M = \begin{pmatrix} & & & 1 & & & \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$$\underline{\dim}\ A^{-2}M = \begin{pmatrix} & & & 1 & & & \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

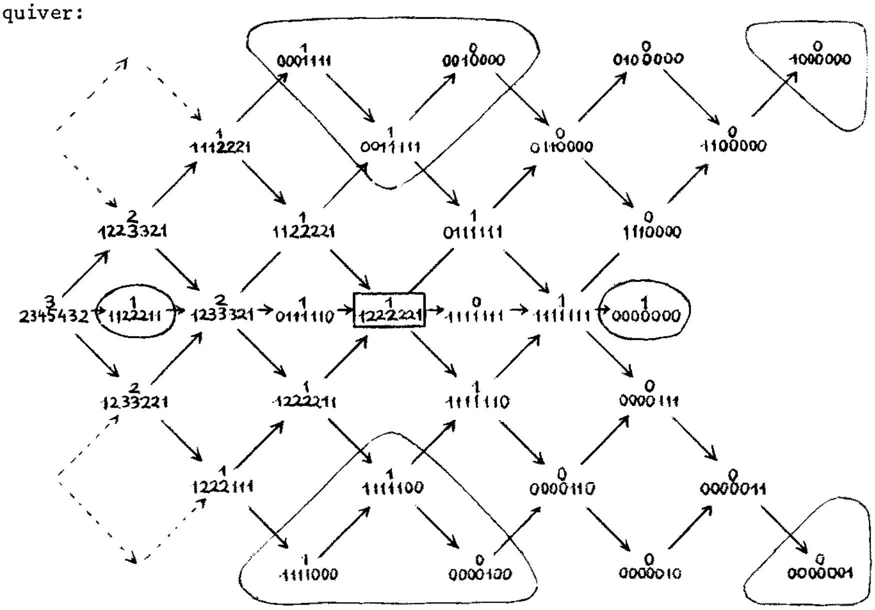
As we know, the preinjective component of the Auslander-Reiten quiver has the following shape:



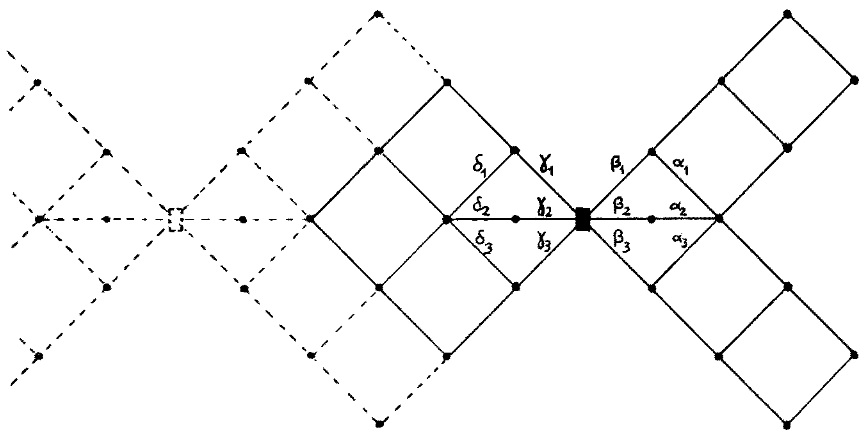
We obtain those indecomposable objects $\mathrm{Hom}(M,X)$ in our vectorspace category Hom(M,$\mathcal{M}$) which come from preinjective Γ-modules X, by deleting the encircled points (for these modules Y, we have Hom(M,Y)=0), and all the remaining points with the exception of the points in squares become one-dimensional in Hom(M,$\mathcal{M}$), whereas the points in squares become two-dimensional.



The fact that M has period 3 implies that the obtained pattern is very regular. Of course, we will use this fact in order to determine the maps Hom(M,f) where $f : X \to Y$ is a map between indecomposable preinjective Γ-modules. In order to do so, we consider the full subcategory $\mathcal{U}$ of all modules $A^j I(r)$ with $0 \leq r \leq 7$, and $0 \leq j \leq 3$. Since in $\mathcal{U}$ any map is a sum of compositions of irreducible maps, we may again work with the corresponding part of the Auslander-Reiten quiver:



It turns out that for any indecomposable X_Γ, Y_Γ in $\mathcal{U}$ with $\mathrm{Hom}(M_\Gamma, X_\Gamma) \neq 0$, $\mathrm{Hom}(M_\Gamma, Y_\Gamma) \neq 0$, and any irreducible map $f : X_\Gamma \to Y_\Gamma$, the induced map $\mathrm{Hom}(M_\Gamma, f) \neq 0$. The maps in $\mathrm{Hom}(M, \mathcal{U})$ are therefore generated by maps corresponding to the edges in



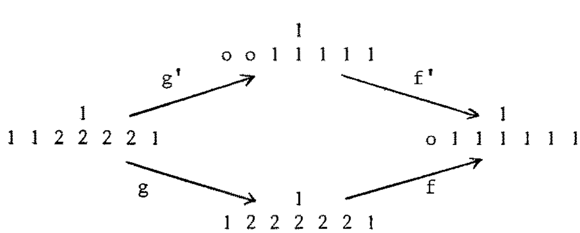
With dotted lines, we have indicated the continuation to the left given by the Auslander-Reiten translation. In a vectorspace category, the composition of two non-zero maps between one-dimensional objects has to be non-zero, so we only have to worry with respect to maps between objects where at least one is not one-dimensional. Here we can use now the relations given by the Auslander-Reiten sequences: they immediately yield

$$\sum_{i=1}^{3} \alpha_i \beta_i = 0, \quad \beta_2 \gamma_2 = 0, \quad \sum_{i=1}^{3} \gamma_i \delta_i = 0 .$$

Also we have in addition

$$\beta_1 \gamma_1 = 0, \quad \text{and} \quad \beta_3 \gamma_3 = 0.$$

Namely, let $\beta_1 = \mathrm{Hom}(M, f)$, $\gamma_1 = \mathrm{Hom}(M, g)$, where f, f', g, g' are irreducible maps



with $fg + f'g' = 0$. Thus $\beta_1\gamma_1 = \mathrm{Hom}(M,fg) = -\mathrm{Hom}(M,f'g')$ factors through $\mathrm{Hom}(M, {}^{\;\;\;\;\;\;\;\;\;1}_{0\;0\;1\;1\;1\;1\;1}) = 0$. And similarly, for $\beta_3\gamma_3$.

Finally, it remains to consider the set of maps between any $\mathrm{Hom}(M,M(n))$ and any $\mathrm{Hom}(M,A^jI(r))$. We claim that this is the full set of linear transformations. This follows easily from the fact that any Γ-homomorphism $M \to A^jI(r)$ can be lifted to $M(n)$

$$\begin{array}{ccc} M & \longrightarrow & A^jI(r) \\ \downarrow & \nearrow & \\ M(n) & & \end{array}$$

(apply first A^{-j} to the inclusion $M \hookrightarrow M(n)$ and use the fact that $I(r)$ is injective).

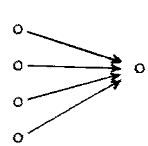
Remark: Let us give some geometrical interpretation to the result of our calculations above. Any Γ-module $V =$

$$\begin{array}{ccccccccccccc} & & & & & & V_1 & & & & & & \\ & & & & & & \downarrow & & & & & & \\ V_2 & \to & V_3 & \to & V_4 & \to & V_o & \leftarrow & V_7 & \leftarrow & V_6 & \leftarrow & V_5 \end{array}$$

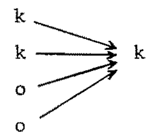
for which all maps are injective, may be considered as a vectorspace V_o with seven prescribed subspaces $V_1,\ldots,V_7$ such that $V_2 \subseteq V_3 \subseteq V_4$ and $V_5 \subseteq V_6 \subseteq V_7$, thus as an S-space for the partially ordered set (1,3,3).

It is easy to see that the elements of $\mathrm{Hom}(M_\Gamma,V_\Gamma)$ just correspond to the elements in the intersection $V_3 \cap V_6$. What we have shown above implies that for indecomposable V, we always will have $\dim V_3 \cap V_6 \leq 2$, and that there is just one infinite sequence of indecomposable S-spaces $V = (V_o,V_1,\ldots,V_7)$ with $\dim V_3 \cap V_6 = 2$.

Similar results can be obtained by the choice of any other regular module of a tame quiver. For example, there is the well-known assertion that for an indecomposable quadrupel $(V_o,V_1,\ldots,V_4)$, the intersection $V_i \cap V_j$ for any two different subspaces V_i, V_j $(1 \leq i,j \leq 4)$ is at most one-dimensional. Here, we have to consider the "four subspace quiver" Δ



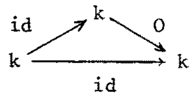
and the simple regular Δ-module



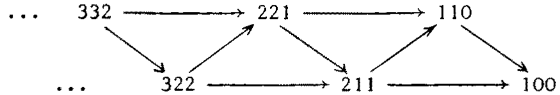
The investigation of $\mathrm{Hom}(M_\Delta, \mathcal{M}_\Delta)$ shows that this vectorspace category is the additive category of a partially ordered set. That is, $\dim \mathrm{Hom}(M_\Delta, V_\Delta) \leq 1$ for any indecomposable V_Δ.

General rules: The previous example is rather typical for the actual work one has to do in calculating the vectorspace categories $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$, where M_Γ is a regular module. In many cases it actually happens that $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$ is the additive category of a partially ordered set. In this case, the procedure is even easier: As above, one first determines $\dim \mathrm{Hom}(M_\Gamma, X_\Gamma)$ for the indecomposable Γ-modules, and here one finds out that one deals with a partially ordered set. Thus, it remains to check whether for X_Γ, Y_Γ indecomposable with $\mathrm{Hom}(M_\Gamma, X_\Gamma) \neq 0$, $\mathrm{Hom}(M_\Gamma, Y_\Gamma) \neq 0$, there is some $f : X_\Gamma \to Y_\Gamma$ with $\mathrm{Hom}(M_\Gamma, f) \neq 0$. Note that there cannot be any additional relations. If X_Γ, Y_Γ both are preinjective, it suffices to consider the irreducible maps f. Note however that it may happen that there is an irreducible map $f : X_\Gamma \to Y_\Gamma$ where $\mathrm{Hom}(M_\Gamma, X_\Gamma) \neq 0 \neq \mathrm{Hom}(M_\Gamma, Y_\Gamma)$, such that $\mathrm{Hom}(M_\Gamma, f) = 0$. For, consider the example of the quiver Δ

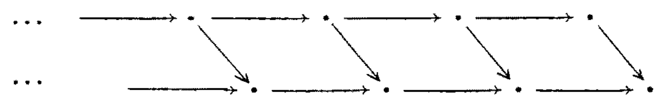
of type $\tilde{A}_{21}$, and the regular module M_Δ



of period 2. The preinjective component of the Auslander-Reiten quiver $\mathcal{M}_\Delta$ is of the form



whereas, however, we obtain under $\mathrm{Hom}(M_\Delta, -)$ the partially ordered set

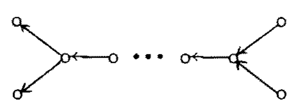


(For example, let $\underline{\dim}\ X = (211)$, $\underline{\dim}\ Y = (110)$. The image of any map $M_\Delta \to X_\Delta$ lies inside the unique submodule of X_Δ of dimension type (101). But this is the kernel of the irreducible map $X_\Delta \to Y_\Delta$).

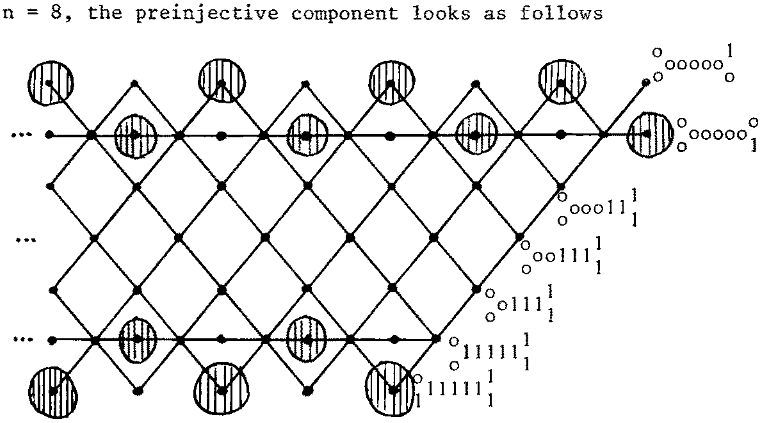
One other calculation has to be done rather carefully. We have seen above that in the case of M_Γ of type $(\tilde{E}_7,3)$, and X_Γ regular, Y_Γ preinjective, one obtained as set of maps $\mathrm{Hom}(M_\Gamma,X_\Gamma) \to \mathrm{Hom}(M_\Gamma,Y_\Gamma)$ in $\mathrm{Hom}(M_\Gamma,M_\Gamma)$ the full set of all linear transformations. However, this is only true in case M_Γ is simple regular. In case M_Γ is not simple regular, the situation may be more complicated as we will see in examples of the next sections.

Finally, let us show the structure of two other classes of pattern of the form $\mathrm{Hom}(M_\Gamma,M_\Gamma)$, with M_Γ regular, which we will need in the later discussion.

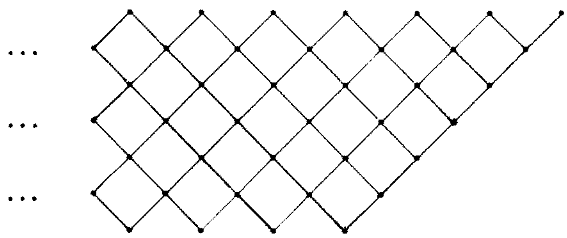
<u>The case</u> $(\tilde{D}_n,2)$. This should indicate that we deal with a quiver of type $\tilde{D}_n$, and a simple regular module of period 2. Let us consider the quiver Γ



and $\underline{\dim}\ M = \begin{smallmatrix}1\\0\end{smallmatrix}\ 1\ 1 \cdots 1\ 1\ \begin{smallmatrix}1\\0\end{smallmatrix}$. The only interesting part of the pattern is that coming from the preinjective component. In case of $n = 8$, the preinjective component looks as follows



and for the encircled modules X, we have Hom(M,X) = 0. For all the other indecomposable preinjective modules Y, the vectorspace Hom(M,Y) is one-dimensional, and no additional irreducibel maps are cancelled, thus we obtain the following partially ordered set

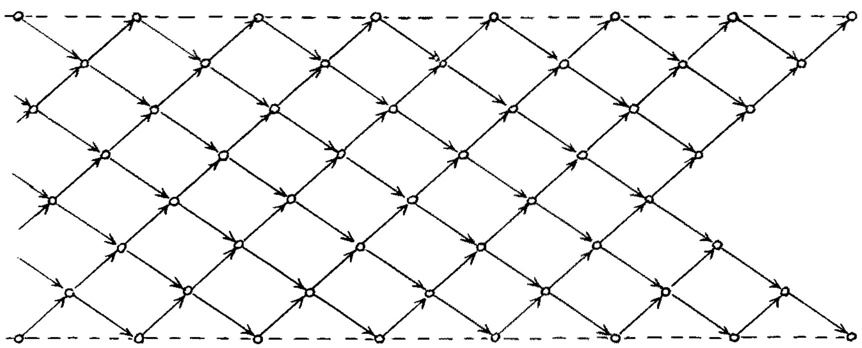


Note that this is precisely the partially ordered set obtained from the quiver $\mathbb{N}^-\Delta_7$ by adding all possible commutativity relations, where Δ_7 is the A_7-quiver $\circ\to\circ\to\circ\to\circ\to\circ\to\circ\to\circ$.

In general, denote by Δ_n the quiver $\circ\leftarrow\circ\leftarrow\circ\ \cdots\ \circ\leftarrow\circ$ of type A_n, with all arrows going in one direction. The preinjective part of the vectorspace category $\mathrm{Hom}(M_\Gamma, M_\Gamma)$, where M_Γ is a simple regular representation of period 2, and Γ of type $\tilde{D}_n$, will always be in the same pattern as $\mathbb{N}^-\Delta_{n-1}$. The cases n = 5,6,7,8 are depicted in 3.5.

The case $(\tilde{A}_{pq}, 1)$. Let Γ be any quiver of type $\tilde{A}_n$, and $M = (M_i, \varphi_\alpha)$ a simple regular module of period 1. Then for all indecomposable preinjective modules X, the vectorspace Hom(M,X) is one-dimensional. Namely, all components M_i are one-dimensional, and $AM \approx M$. Also, if X,Y are indecomposable preinjective, and there exists an irreducible map $f : X \to Y$, then there also is a non-zero homomorphism from Hom(M,X) to Hom(M,Y) in $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ (but not necessarily $\mathrm{Hom}(M,f) \neq 0$). For the proof, we may assume that Y is simple injective, using reflection functors. Then $Y = I(r)$ for a source of Γ, and $X = I(s)$ for some other vertex with an arrow $\underset{r}{\circ}\xrightarrow{\alpha}\underset{s}{\circ}$ which gives rise to the irreducible map $f : I(s) \to I(r)$. Now, if Γ is not of type $\tilde{A}_{1q}$, then $\varphi_\alpha \neq 0$ for all regular modules of period 1, thus $\mathrm{Hom}(M,f) \neq 0$. On the other hand, if Γ is of type $\tilde{A}_{1q}$ and $\varphi_\alpha = 0$ for our M, then $\varphi_{\beta_q} \cdots \varphi_{\beta_1} \neq 0$ in M, where $(r|\beta_1, \ldots, \beta_q|s)$ is the other path from r to s in Γ. This path gives rise to another non-zero homomorphism $g : I(s) \to I(r)$, and $\mathrm{Hom}(M,g) \neq 0$.

It follows that the preinjective part of the vectorspace category $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ ist just the additive category of the partially ordered set obtained from $\mathbb{N}^{-}\Gamma$ by adding all possible commutativity relations. In particular, the vectorspace category $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ only depends on Γ, and not on the isomorphism class of M (note that there is a one-parameter family of such modules). Examples of these pattern will be found in 3.5. This is the case of $\Gamma = \tilde{A}_{34}$ (with identified dotted lines in order to form a cylinder):



3.4. Some non-domestic pattern

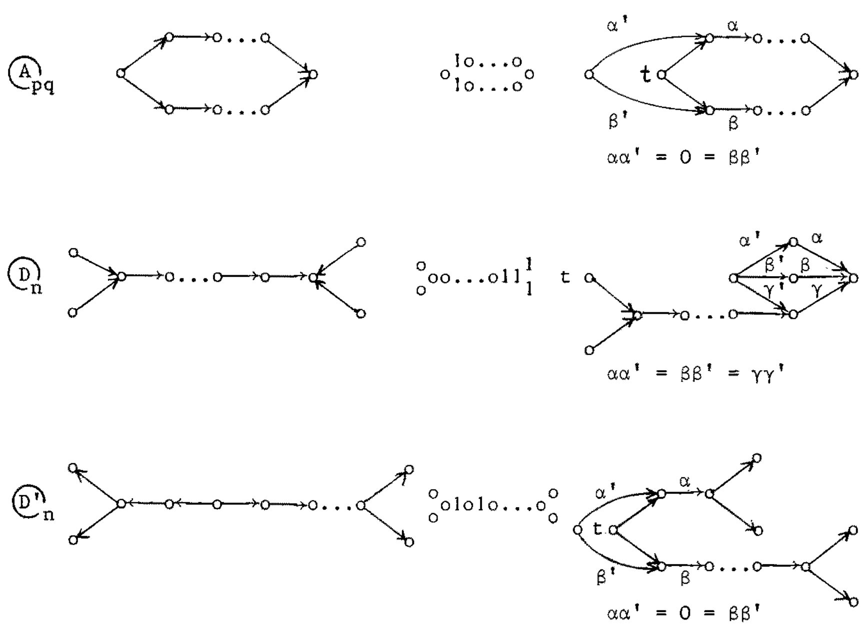
We will encounter, in the sequel, many pattern which are tame, but non-domestic.
In this section, we want to exhibit a small number of such pattern, and it will turn out that any other which we later have to consider is similar to one of those of this section.

The vectorspace categories discussed here are not additive categories of partially ordered sets, so it is difficult to decide whether they are tame or not. Actually, for two of them (the cases $\textcircled{A}_{pq}$ and $\textcircled{D}'_n$), this is known for a long time [29], also that they are non-domestic. For the remaining ones we have to postpone the proof of the tameness to the next section, where we will see that their pattern are similar to pattern of tame partially ordered sets, thus also tame. Actually, all other tame vectorspace categories (but one) which we will have to consider will be additive categories of partially ordered sets.

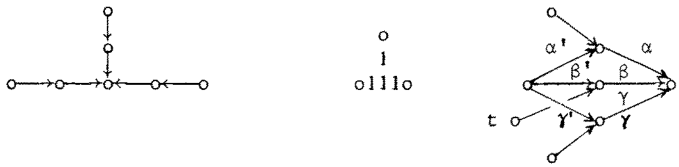
Here, we concentrate on the point that for the non-domestic vectorspace categories K which we will encounter, no domestication will be possible. All have the property that there are either infinitely many pairwise different 2-dimensional objects with endomorphism ring k, or that there are infinitely many pairwise different full embeddings of the additive category of one of the partially ordered sets (1,1,1,1), (2,2,2), (1,3,3), (1,2,5) or (N,4). Of course, these embeddings give rise to infinitely many series which immediately shows that an algebra R which reduces to $U(K)$ cannot be domestic. However, we will see that there are finite full subcategories K_i of K (and, again, infinitely many) such that any K_i itself is non-domestic, and in fact that the whole of $U(K)$ can be rebuilt inside $U(K_i)$: there is a cofinite subcategory of $U(K)$ which is representation equivalent to a subcategory of $U(K_i)$.

We consider the following cases: always, S will be the path algebra of a tame connected quiver, and M_S a regular module which is in two cases ((A_n) and (D'_n)) the direct sum of two simple modules, in all other cases indecomposable. Besides writing down S and M_S, for the benefit of the reader, we also note the quiver with relations with algebra R such that M_R reduces to $M({}_kM_S)$.

case S $\underline{\dim}\, M_S$ R

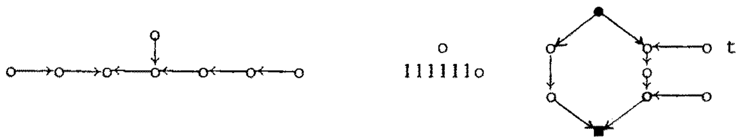


(E_6)

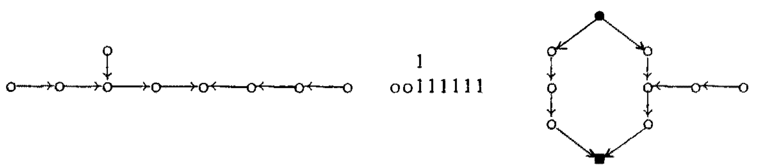


$\alpha\alpha' = \beta\beta' = \gamma\gamma'$

(E_7)

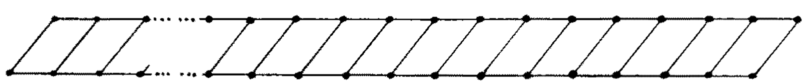


(E_8)

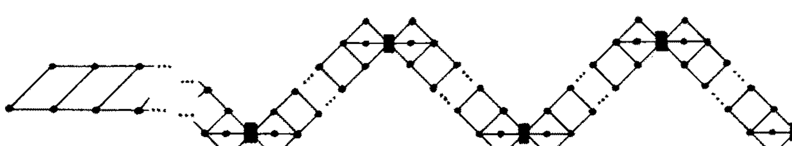


The calculation of the vectorspace category $K = \mathrm{Hom}(M_S, M_S)$ gives the following:

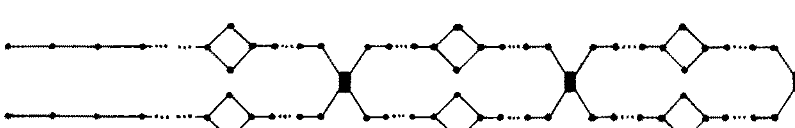
(A_{pq})



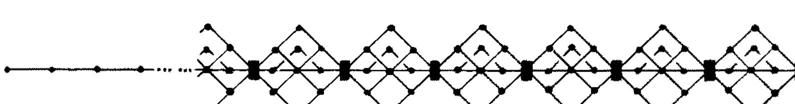
(D_n)

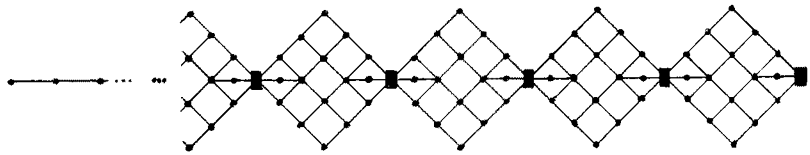


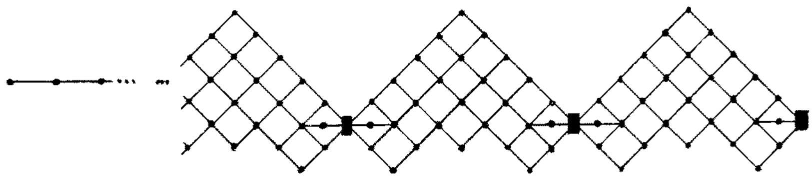
(D'_n)



(E_6)



(E_7) 

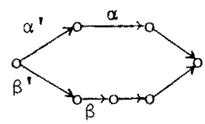
(E_8) 

Note that all of them contain countably many two-dimensional objects with endomorphism ring k, and all but the cases (A_n) and (E_6) contain a countable number of subsets of the form (1,1,1,1). On the other hand, (E_6), (E_7) and (E_8) contain countably many subsets of the form (2,2,2), and (E_7) and (E_8) such of the form (1,3,3). Finally, (E_8) also has subsets of the form (1,2,5) and (N,4). Of course, in this way we obtain many one-parameter families of indecomposable objects in $\mathcal{U}(\mathrm{Hom}(M_S,\mathcal{M}_S))$.

We use now the vertex marked t in the quiver of R, in order to obtain an equivalence between $\mathcal{M}_R$ and the category of representations of a different bimodule ${}_kX_T$. Thus, T is obtained from the given quiver with relation by deletion of the vertex t (note that t is always a source), and X_T is the radical of the projective R-module corresponding to t (note that X_T itself is always projective, and in the cases (D_n), (E_6), (E_7), (E_8) even indecomposable). It turns out that in all cases, T is of finite representation type (we will indicate its complete Auslander-Reiten quiver), so that the subcategory of $\mathcal{M}_R$ which is canonically representation equivalent to $\mathcal{U}(\mathrm{Hom}(X_T,\mathcal{M}_T))$, is cofinite in $\mathcal{M}_R$.

Let us exhibit the Auslander-Reiten quiver of $\mathcal{M}_T$, and indicate the corresponding vectorspace category $\mathcal{L} = \mathrm{Hom}(X_T,\mathcal{M}_T)$.

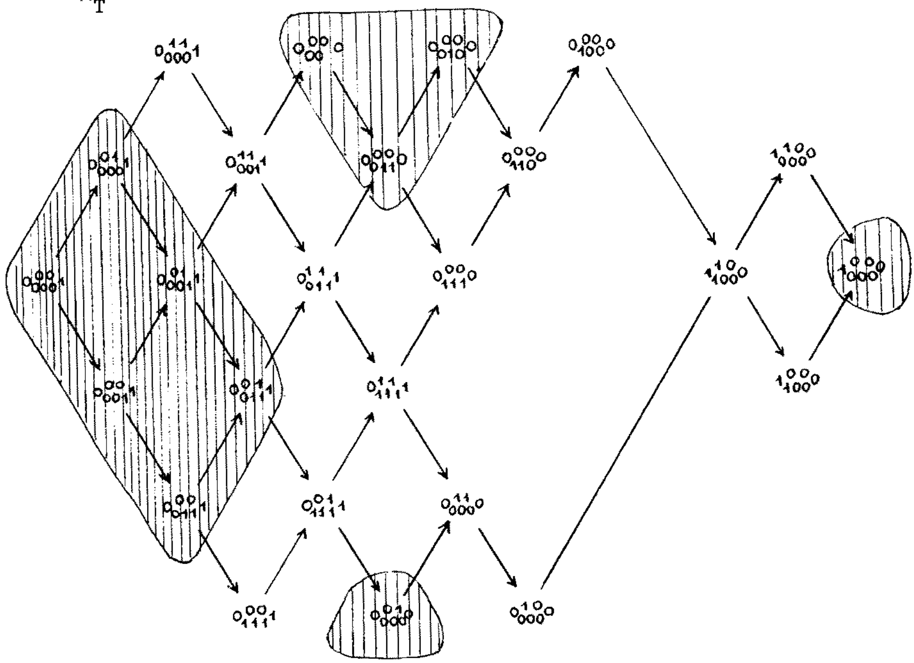
<u>Case</u> (A_{pq}), say $(A_{3,4})$ T:



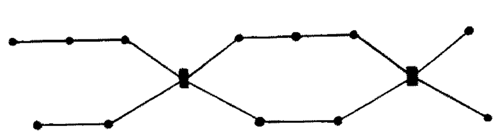
$\dim X_T = o^{11}_{111}2$

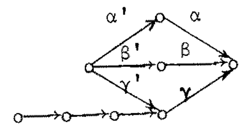
$\alpha\alpha' = 0 = \beta\beta'$

M_T:



$\mathrm{Hom}(X_T, M_T)$:

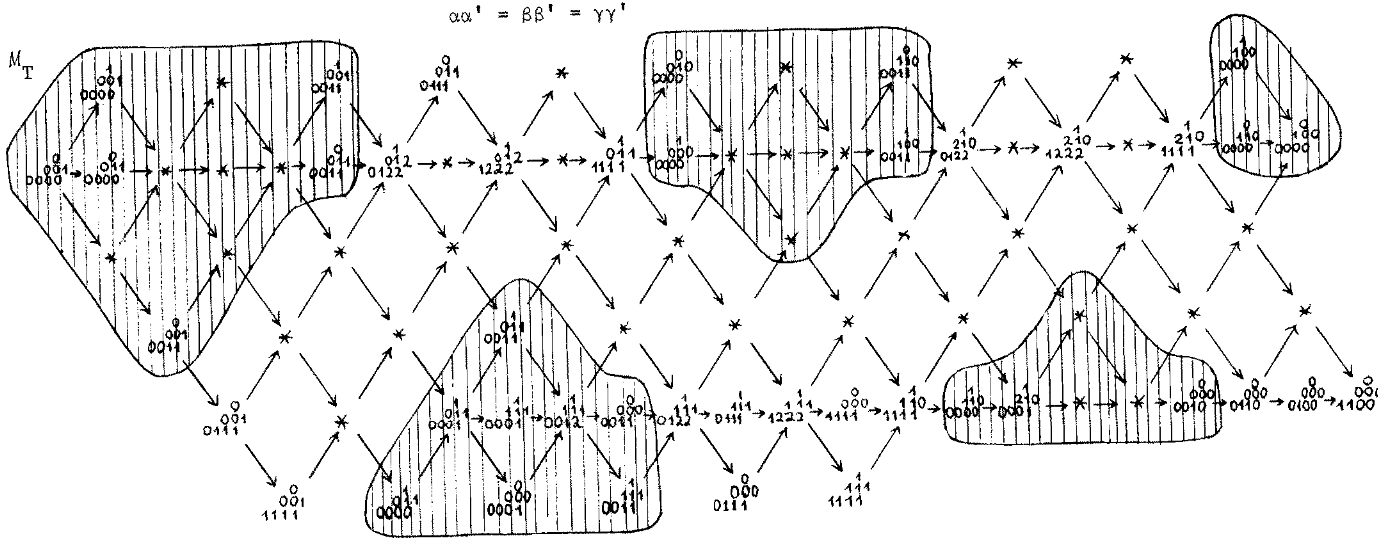


Case (D_n), say (D_7). T: 

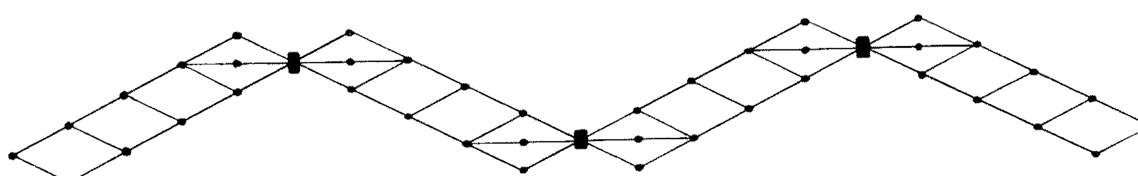
$\alpha\alpha' = \beta\beta' = \gamma\gamma'$

$\underline{\dim}\ X_T = \begin{smallmatrix} & & o \\ & o & o & 1 \\ o & 1 & 1 & 1 \end{smallmatrix}$

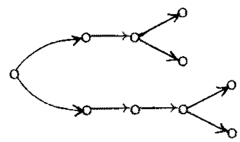
M_T



$\mathrm{Hom}(X_T, M_T)$:

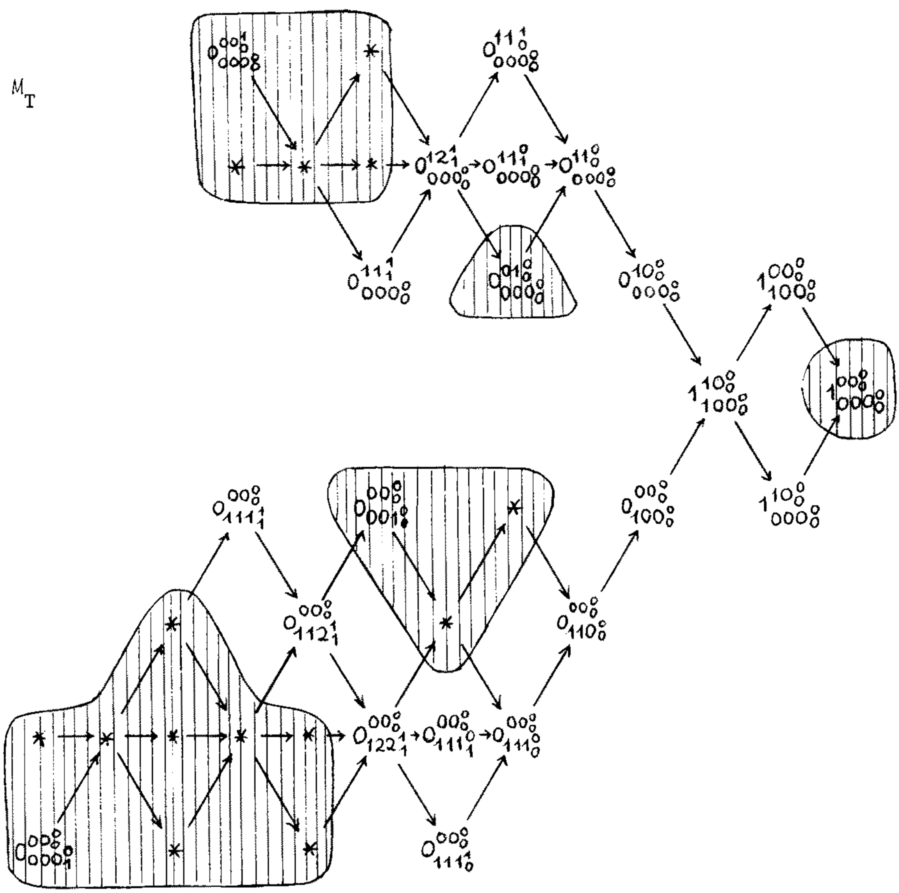


<u>Case</u> (D'_n), say (D'_9) T:

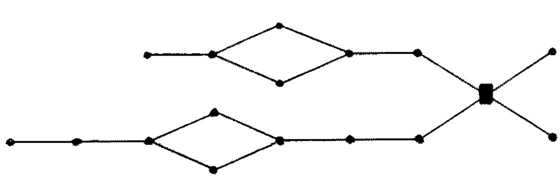


$\underline{\dim}\ X_T = $ o $\begin{smallmatrix} 11^{1}_{1} \\ 111^{1}_{1} \end{smallmatrix}$

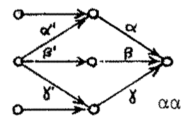
M_T



$\mathrm{Hom}(X_T, M_T)$

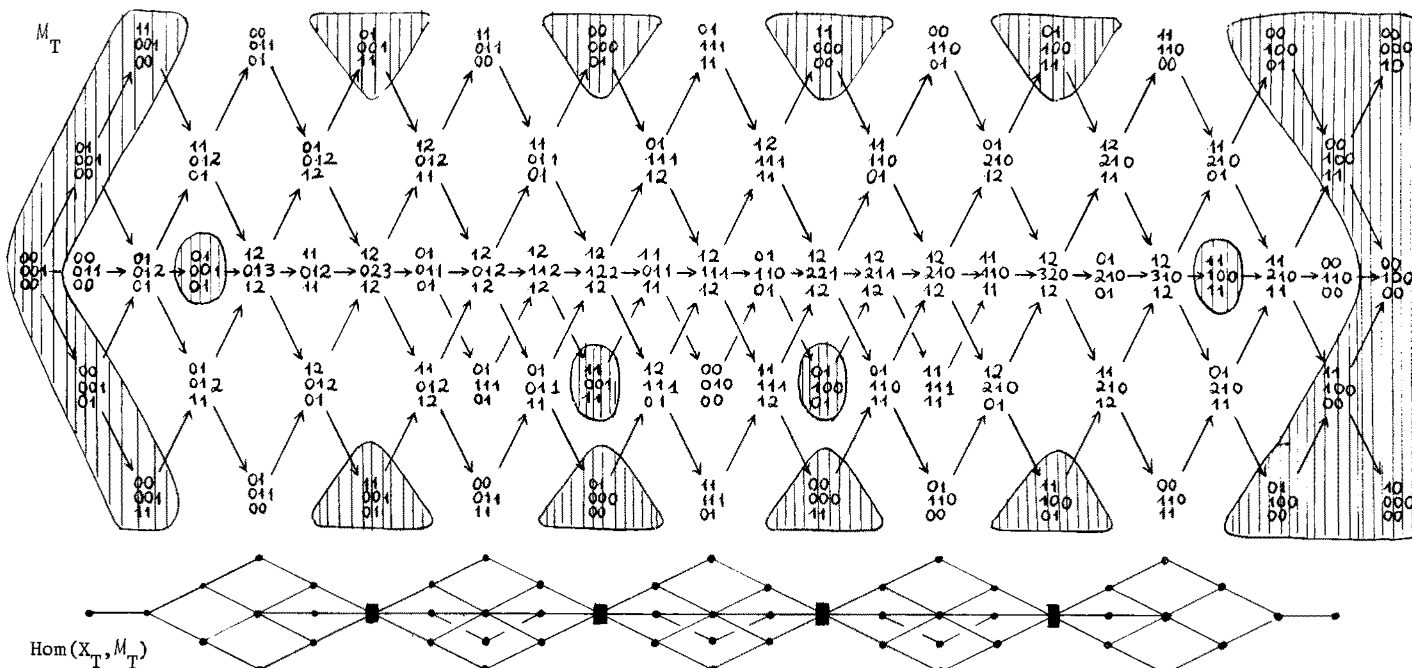


Case $\textcircled{E}_6$

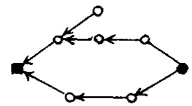
T: 

$\alpha\alpha' = \beta\beta' = \gamma\gamma'$

$\underline{\dim}\ X_T = \begin{matrix} 01 \\ 011 \\ 01 \end{matrix}$

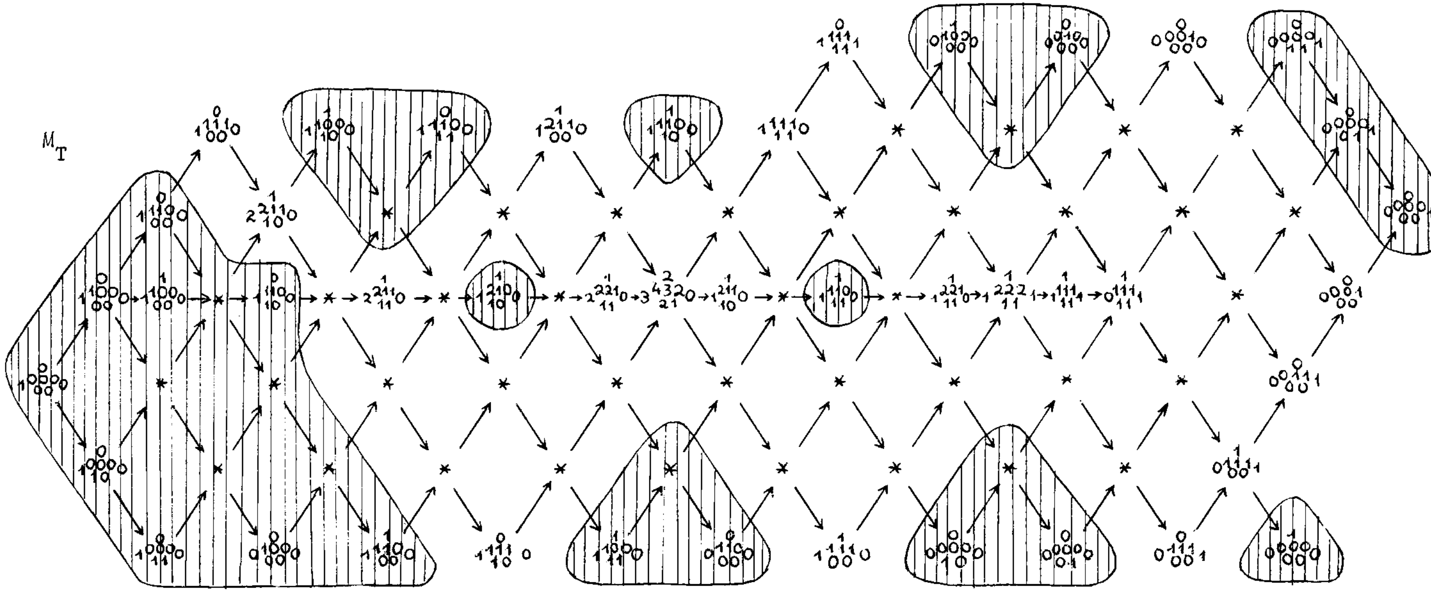


Case $\textcircled{E_7}$

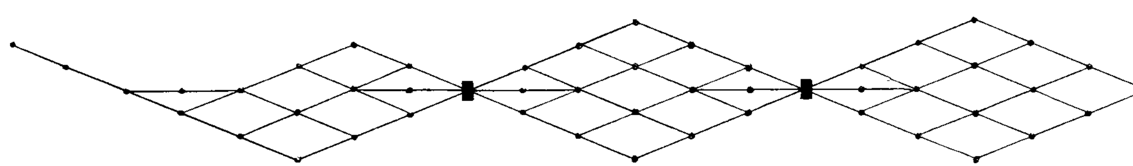
T: 

$\underline{\dim}\, X_T = \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & & \end{smallmatrix}$

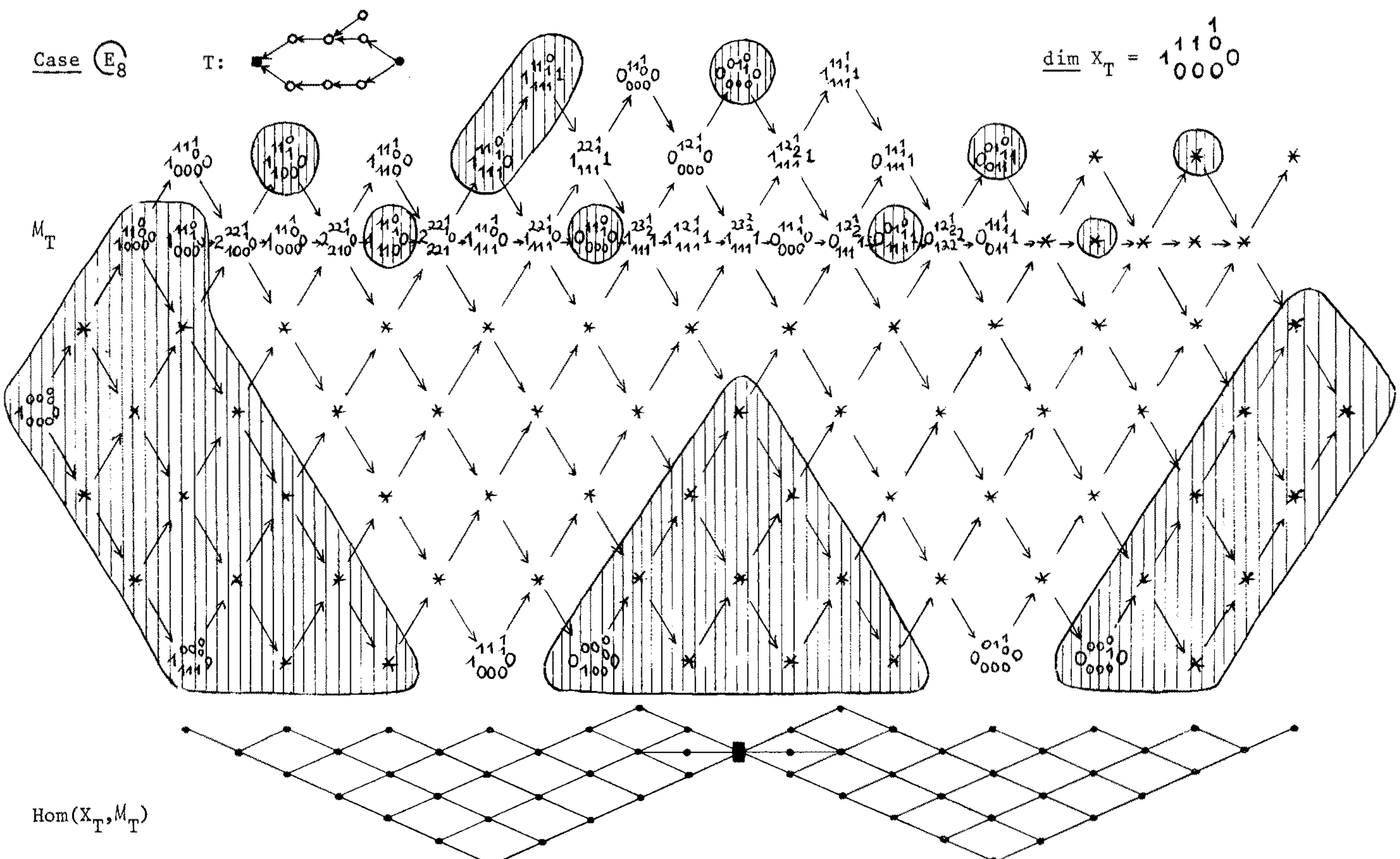
M_T



$\mathrm{Hom}(X_T, M_T)$



Case E_8



In all cases, we see that $L = \mathrm{Hom}(X_T, M_T)$ can be embedded as a full vectorspace subcategory into $K = \mathrm{Hom}(M_S, M_S)$ in a countable number of ways, with pairwise disjoint images. For later references, let us formulate this result as a proposition. Note that in case K is of type * (* = one of $(A)_{pq}, (D)_n, (D')_n, \ldots$), then we call the pattern of K <u>the pattern of type</u> *.

PROPOSITION: <u>For the pattern of type</u> $(A)_{pq}, (D)_n, (D')_n, (E)_6, (E)_7, (E)_8$, <u>there exists a vectorspace category</u> K <u>in the pattern, and a finite vectorspace category</u> L <u>which is equivalent to countably many pairwise disjoint vectorspace subcategories</u> K_i <u>of</u> K, <u>such that a cofinite subcategory of</u> $U(K)$ <u>is representation equivalent to a (codomestic) subcategory of</u> $U(L)$.

Namely, let $K = \mathrm{Hom}(M_S, M_S)$, $L = \mathrm{Hom}(X_T, M_T)$, as above. Let V be the full subcategory of objects of M_R with no non-zero direct summand in $\kappa(M_S)$, and W the full subcategory of objects of M_R with no non-zero direct summand in $\kappa(X_T)$. Then V is codomestic in M_R and representation equivalent to $U(K)$, and W is even cofinite in M_R and representation equivalent to $U(L)$. Thus $V \cap W$, being cofinite in V and codomestic in W, is representation equivalent both to a cofinite subcategory of $U(K)$ and to a codomestic subcategory of $U(L)$.

Note that the proposition above implies, in particular, that the vectorspace categories of type $(A)_{pq}, (D)_n, (D')_n, (E)_6, (E)_7, (E)_8$ cannot be domestic. In fact, it allows for the finite subcategories K_i of K to construct infinitely many pairwise different series.
Perhaps we should stress the following implication of the proposition: nearly everything nice or pathological what happens in $U(K)$ happens not only ones but at least a countable number of times: just consider the various $U(K_i)$. This is the reason for our feeling that these situations should be called non-domestic!

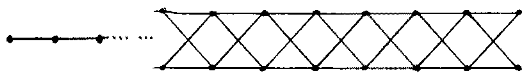
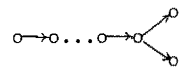
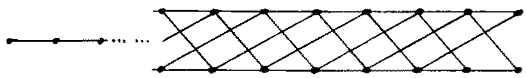
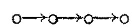
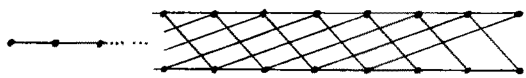
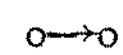
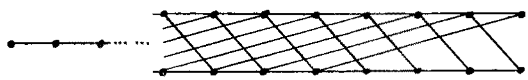
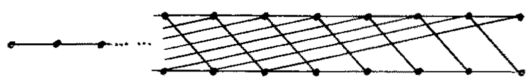
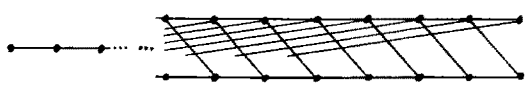
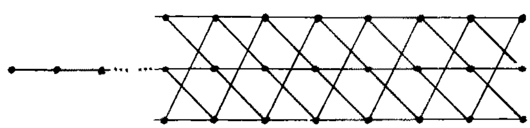
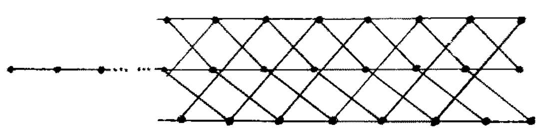
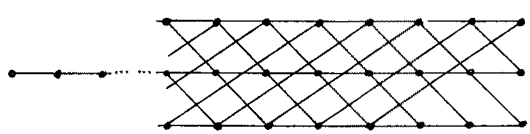
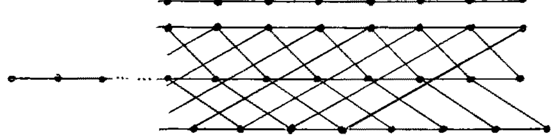
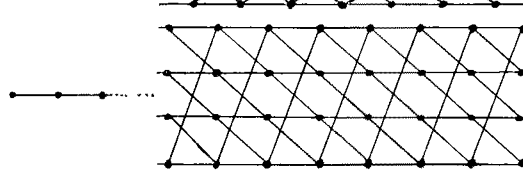
3.5 <u>Regular modules</u> M_T <u>with</u> $\mathrm{Hom}(M_T, M_T)$ <u>tame.</u>

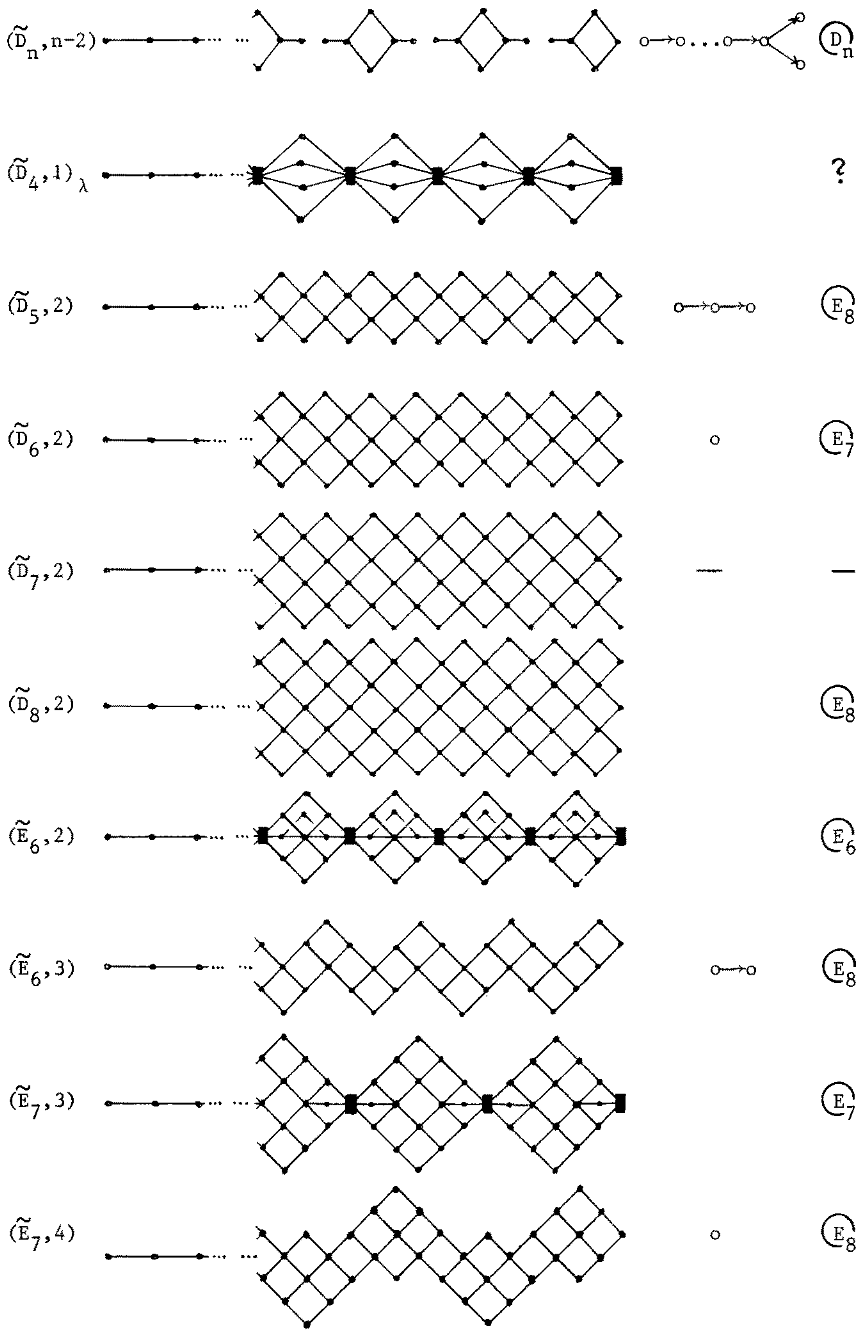
Let Γ be a tame connected quiver. We want to list all regular Γ-modules M_T such that the corresponding vectorspace category $\mathrm{Hom}(M_T, M_T)$ is of tame representation type. We will use the following notation for the <u>type</u> of M_T. Assume that Γ is of type * (one of

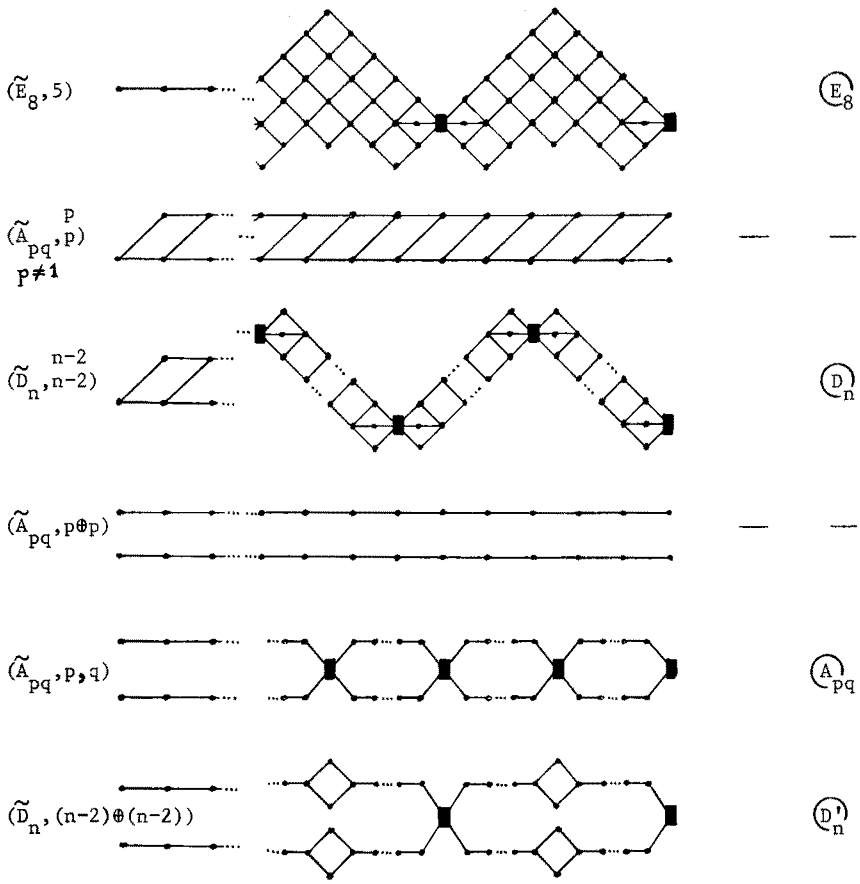
$\tilde{A}_{pq}$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$). If M is simple regular, and of period p, we say that M_Γ is of type (*,p). Note that for $p > 1$ the pattern of $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ will only depend on * and p. This, in fact, is an easy consequence of Lemma 2 in 3.2, since for two quivers Γ and Γ' of the same type *, and M a Γ-module, M' a Γ'-module, both simple regular with same period p, we can identify the underlying graphs of Γ and Γ' in such a way that the module M' can be obtained from M by a sequence of reflection functors σ. Also, we have seen in 3.3 that there is only one pattern of type $(\tilde{A}_{pq},1)$. In case M_Γ is indecomposable regular, of regular length 2, and of period p, then we say that M_Γ is of type $(*,{p \atop p})$. In case of $\Gamma = \tilde{A}_{pq}$, we write $(\tilde{A}_{pq}, p \oplus p)$ for the type of the module $M = M' \oplus M''$ with M', M'' simple regular in the same orbit, but non-isomorphic, whereas $(\tilde{A}_{pq}, p\ ,\ q)$ will be the type of $M = M' \oplus M''$ with M', M'' simple regular in different orbits. Similarly, $(\tilde{D}_n, (n-2) \oplus (n-2))$ will stand for Γ to be of type $\tilde{D}_n$ and a Γ-module $M = M' \oplus M''$ with M', M'' both simple regular in the same orbit, but non-isomorphic.

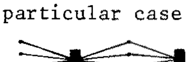
Also, in many cases there exists a Dynkin-diagram Δ and a Δ-module N_Δ, such that the $\Gamma \dot{\cup} \Delta$-module $M \oplus N$ again gives a tame vectorspace category $\mathrm{Hom}(M \oplus N, M_{\Gamma \dot{\cup} \Delta})$ which moreover is non-domestic. In these cases, we also write down Δ, note that Δ is essentially unique. The corresponding Δ-module M' always is indecomposable and has, for the noted orientation of Δ, all components = k. The pair $(\Gamma \dot{\cup} \Delta,\ M \oplus N)$ will be called a <u>completion</u> of (Γ, M), and we denote its type by $\overline{(*,p)}$ in case M_Γ is of type (*,p). In case a completion exists, we write down the similarity class of the pattern of $\overline{(*,p)}$, it is represented by the pattern of one of the vectorspace categories introduced in the last section, and we call it the similarity type of $\overline{(*,p)}$.

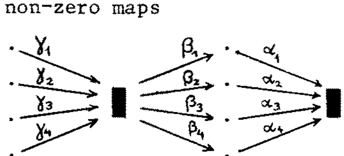
THEOREM 3. <u>Let</u> Γ <u>be a tame connected quiver, and</u> M_Γ <u>a regular representation. If</u> $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ <u>is not wild then</u> (Γ, M) <u>is of one of the following types, and conversely, in all these cases, with the possible exception of</u> $(\tilde{D}_4, 1)$, $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ <u>is tame</u>.

type	pattern of $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$	Δ	Simil. type of $\overline{(*,p)}$
$(\widetilde{A}_{pq},p)$		—	—
$(\widetilde{A}_{22},1)$			D_n
$(\widetilde{A}_{23},1)$			E_8
$(\widetilde{A}_{24},1)$			E_7
$(\widetilde{A}_{25},1)$		—	—
$(\widetilde{A}_{26},1)$		o	E_8
$(\widetilde{A}_{2q},1)$ $q \geq 7$		—	—
$(\widetilde{A}_{33},1)$		o	E_6
$(\widetilde{A}_{34},1)$		—	—
$(\widetilde{A}_{35},1)$		—	—
$(\widetilde{A}_{36},1)$			E_8
$(\widetilde{A}_{44},1)$			E_7

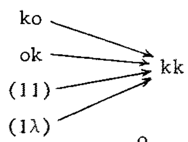

$(\tilde{D}_n, n-2)$
D_n
$(\tilde{D}_4, 1)_\lambda$
?
$(\tilde{D}_5, 2)$
E_8
$(\tilde{D}_6, 2)$
E_7
$(\tilde{D}_7, 2)$
—
—
$(\tilde{D}_8, 2)$
E_8
$(\tilde{E}_6, 2)$
E_6
$(\tilde{E}_6, 3)$
E_8
$(\tilde{E}_7, 3)$
E_7
$(\tilde{E}_7, 4)$
E_8



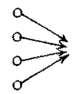
For the conventions for illustrating vectorspace categories, we refer to 2.4. One particular case has to be explained separately, namely $(\tilde{D}_4, 1)$. Here  indicates a vectorspace category with 8 one-dimensional and 2 two-dimensional objects with trivial endomorphism ring, and non-zero maps

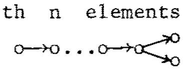


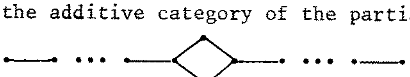
satisfying the relations $\beta_i\gamma_i = 0$ for all i, and $\sum_{i=1}^{3} \alpha_i\beta_i = 0$, $\alpha_1\beta_1+\lambda\alpha_2\beta_2+\alpha_4\beta_4 = 0$. This pattern is obtained for the following simple regular representation

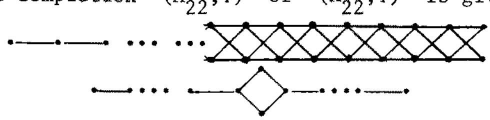
M = 

(with $\lambda \neq 0,1$)

of the $\tilde{D}_4$-quiver Γ =  . We do not know whether it is tame or not.

If Γ, Γ' are quivers, M a representation of Γ, and M' one of Γ', then the vectorspace category $\mathrm{Hom}(M \oplus M', \mathcal{M}_{\Gamma \dot{\cup} \Gamma'})$ clearly is the product of the categories $\mathrm{Hom}(M, \mathcal{M}_\Gamma)$ and $\mathrm{Hom}(M', \mathcal{M}_{\Gamma'})$. This remark will be used in those cases where we had to add to Γ a Dynkin diagram Δ and to the Γ-module M the Δ-module N in order to obtain the non-domestic vectorspace category $\mathrm{Hom}(M \oplus N, \mathcal{M}_{\Gamma \dot{\cup} \Delta})$. For the module N with all components one-dimensional, the vectorspace category $\mathrm{Hom}(N, \mathcal{M}_\Delta)$ has the following form: For the quiver Δ of the form o⟶o...o⟶o with n edges, we obtain the additive category of the chain •—• ••• •—• with n elements (which we also have denoted by (n)); for the quiver  of type D_n, we obtain the additive category of the partially ordered set

 with 2n-2 points. For example, the pattern for the completion $\overline{(\tilde{A}_{22},1)}$ of $(\tilde{A}_{22},1)$ is given by



This is a partially ordered set of width four.

We will call an algebra R a <u>regular enlargement</u> of a tame quiver Γ provided there exists a Γ-module M with $\mathcal{M}_R \simeq \mathcal{M}({}_kM_\Gamma)$ such that for any connected component Γ_i of Γ which is not a Dynkin diagram, $M|\Gamma_i$ is regular. Of course, we always can assume that all $M|\Gamma_i$ are non-zero. In case all but one connected components of Γ are Dynkin diagrams, and Γ is not of type $\tilde{D}_4$, then theorem 3 gives the precise conditions on M for R to be tame. In case at least two components of Γ are tame, then R is tame if and only if Γ has precisely these two components Γ_1, Γ_2, and any $M_i|\Gamma_i$ is of one of the following forms $(\tilde{A}_{pq},p)$, $(\tilde{A}_{22},1)$ and $(\tilde{D}_n,n-2)$. In this case, if at least one $M_i|\Gamma_i$ is of the form $(\tilde{A}_{pq},p)$, then R is domestic, otherwise R is non-domestic. All these assertions follow directly from Theorem 3.

For the proof of Theorem 3, we have to show that the listed cases are tame, and that these are the only ones. The proof that the remaining cases all are wild, will be given in the next section. In the present section, we concentrate on the listed cases, show that they are tame, and deal with the corresponding similarity classes. In fact, we consider first the similarity classes, since it will turn out that nearly all contain pattern of additive categories of partially ordered sets, so that the tameness follows from the theorem of Nazarova.

Calculation of the similarity types

Let $(*,p)$ be a type for which a completion is claimed to exist. We have to exhibit an algebra R such that $\mathcal{M}_R \approx \mathcal{M}({}_kM_\Gamma)$ with M_Γ of type $\overline{(*,p)}$, and $\mathcal{M}_{R^{op}} \approx \mathcal{M}({}_kM'_{\Gamma'})$, where the pattern of $\mathrm{Hom}(M'_{\Gamma'},\mathcal{M}_{\Gamma'})$ is of the indicated type $\textcircled{D}_n$, $\textcircled{E}_6$, $\textcircled{E}_7$, or $\textcircled{E}_8$. It is of interest that for R we always can take a one-relation algebra, thus we just give the corresponding number in Theorem 1.

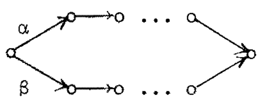
$\overline{(\tilde{A}_{22},1)}$	$\overline{(\tilde{A}_{23},1)}$	$\overline{(\tilde{A}_{24},1)}$	$\overline{(\tilde{A}_{26},1)}$	$\overline{(\tilde{A}_{33},1)}$
76	82	79	81	77

$\overline{(\tilde{A}_{36},1)}$	$\overline{(\tilde{A}_{44},1)}$	$\overline{(\tilde{D}_n,n-2)}$	$\overline{(\tilde{D}_5,2)}$	$\overline{(\tilde{D}_6,2)}$
80	78	18	24	20

$(\tilde{D}_8,2)$	$\overline{(\tilde{E}_6,3)}$	$\overline{(\tilde{E}_7,4)}$
27	60	23

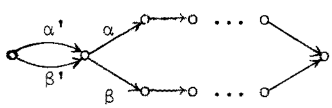
Proof of tameness. If the vectorspace category $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ is the additive category of a partially ordered set, then we just use the theorem of Nazarova in order to verify that $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ is tame. Note that this is very easy to check in our cases due to the fact that the pattern which occur are periodic. Now the similarity of $(\tilde{E}_6,2)$ and $\overline{(\tilde{A}_{33},1)}$, of $(\tilde{E}_7,3)$ and $\overline{(\tilde{A}_{24},1)}$, of $(\tilde{E}_8,2)$ and $\overline{(\tilde{A}_{23},1)}$, and finally of $(\tilde{D}_n, {}^{n-2}_{n-2})$ and $\overline{(\tilde{D}_n, n-1)}$, show that all these pattern are tame.

Thus there only remain two cases, namely the types $(\tilde{A}_{pq}, p\ ,\ q)$ and $(\tilde{D}_n, (n-2) \oplus (n-2))$, which we also have denoted just by $\textcircled{A}_{pq}$ and $\textcircled{D'}_n$, respectively. For these two cases, we do not know any direct reduction to the case of a partially ordered set, using the concept of similarity of pattern. However, fortunately, both pattern are known, for a long time, to be tame. Namely, they give rise to matrix problems solved by Nazarova and Rojter in [26]. The case $\textcircled{A}_{pq}$ later also has been solved, with a different technique, by Donovan and Freislich in [13]. Namely, we consider the following representation $M = M' \oplus M''$ for

$\Gamma =$ 

,

where M' is the simple regular representation for which the map corresponding to α is $k \to o$, and M'' the simple regular representation with $k \to o$ being the map corresponding to β. Then $\mathcal{M}({}_kM_\Gamma) = \mathcal{M}_R$, with R being given by the quiver



with relations $\alpha\alpha' = 0 = \beta\beta'$. The indecomposable R-modules can easily be described by "strings" and "bands".

Remark. The proof above essentially finishes the investigation of the tame one-relation algebras. It remains to consider two special algebras, namely 31 and 73, which will be done in 3.9. These two algebras are not themselves regular enlargements of tame quivers, but specialisations of such enlargements, however not of one-relation algebras.

Note that not all pattern do occur for one relation algebras: the regular enlargements of a tame quiver Γ by a decomposable modules M will always be defined by more than one relation, thus the non-domestic cases $\textcircled{A}_{pq}$ and $\textcircled{D'}_n$ are impossible for tame one-relation algebras. Also, the case $(\tilde{D}_4, 1)$ will never lead to a one-relation algebra. If we restrict to one-relation algebras with zero condition or with strict commutativity condition, as in [39] or [36], then also the pattern (E_6) does not occur.

3.6. Proof of Theorem 3

It remains to show that the listed cases are the only ones which can be tame. We will need the following lemma.

Lemma. Let X, Y be non-isomorphic two-dimensional objects in a vectorspace category K with $\mathrm{End}(X) = k = \mathrm{End}(Y)$. If $\dim \mathrm{Hom}(X,Y) = 1$, and $o \neq f \in \mathrm{Hom}(X,Y)$ is bijective, then K is wild.

Proof: Let $0 \neq f \in \mathrm{Hom}(X,Y)$, and x_1, x_2 a basis of the underlying vectorspace $|X|$ of X. Since f is bijective, $y_1 = f(x_1)$, $y_2 = f(x_2)$ is a basis of Y. Since X, Y are not isomorphic, and $\mathrm{End}(X) = k$, it is easy to see that $\mathrm{Hom}(Y,X) = 0$. Define an embedding $F : M_{k\Omega} \to U(K)$ as follows: If (V,φ,ψ) is a vectorspace with two endomorphisms, let its image under F be given by the object $(U, \mu, V \otimes_k (X \oplus Y))$, where U is the subspace of $V \otimes_k (X \oplus Y)$ generated by the elements $v \otimes (x_1, y_2)$ and $v \otimes (x_2, 0) + \varphi(v) \otimes (0, y_1) + \psi(v) \otimes (0, y_2)$, with μ the inclusion maps. Note that we can identify μ with the map $V \oplus V \to V \oplus V \oplus V \oplus V$ given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \varphi \\ 1 & \psi \end{pmatrix},$$

and then for two objects (V,φ,ψ) and (V',φ',ψ') of $M_{k\Omega}$, a homomorphism from $F(V,\varphi,\psi)$ to $F(V',\varphi',\psi')$ is given by a pair of matrices (A,B), say

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & b_2 & 0 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_3 \end{pmatrix}$$

with $a_{ij}, b_i \in \mathrm{End}(V)$, satisfying

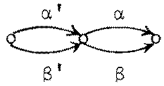
$$B \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \varphi & \psi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \varphi' & \psi' \end{pmatrix} A.$$

But this implies that $a_{11} = a_{22} = b_1 = b_3$, $a_{12} = a_{21} = b_2 = 0$, and $a_{11}\varphi = \varphi' a_{11}$, $a_{11}\psi' = \psi a_{11}$, thus a_{11} is a map in $M_{k\Omega}$ from (V,φ,ψ) to (V',φ',ψ'), and its image under F is just (A,B).

We start with considering one special case, namely we show that $(\tilde{A}_{1,1},{}^1_1)$ is wild. Thus, consider $\Gamma =$  , and M

$$kk \underset{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}}{\overset{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}{\rightrightarrows}} kk .$$

The corresponding algebra R with M_R reducing to $M({}_kM_\Gamma)$ is given by



with relations $\alpha\alpha' = \beta\beta'$, $\beta\alpha' = 0$, independently from λ. (For, we first obtain the relations $\alpha\alpha'+\lambda\alpha\beta'-\beta\beta' = 0$, $\lambda\alpha\alpha'-\beta\alpha' = 0$, but we can replace α,β by linear combinations in order to get the relations above). We consider now representations of an $\tilde{\tilde{E}}_7$-quiver

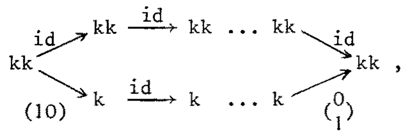
$$\begin{array}{ccccccccccccc} & & & & & & U_0 & & & & & & \\ & & & & & & \downarrow & & & & & & \\ U_1 & \hookrightarrow & U_2 & \twoheadleftarrow & U_3 & \hookrightarrow & V & \twoheadrightarrow & U_4 & \hookleftarrow & U_5 & \twoheadrightarrow & U_6 & \hookleftarrow & U_7 \end{array}$$

with maps being monomorphisms and epimorphisms as indicated, and define a functor into M_R by sending this representation to

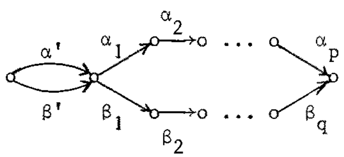
$$U_1U_3U_0 \underset{\begin{pmatrix}* & 0 & 0\\ 0 & * & 0\\ 0 & 0 & *\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}{\overset{\begin{pmatrix}0 & * & 0\\ 0 & 0 & *\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}{\rightrightarrows}} U_2VVU_5U_7 \underset{\begin{pmatrix}0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & * & 0\\ 0 & 0 & 0 & 0 & *\end{pmatrix}}{\overset{\begin{pmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & * & 0 & 0\\ 0 & 0 & 0 & * & 0\end{pmatrix}}{\rightrightarrows}} VU_4U_6 .$$

In this way, we obtain an exact embedding into M_R which is a representation equivalence with the full subcategory of images, thus R is wild. (The corresponding case of the local algebra $R = k\langle X,Y\rangle / (X^2+Y^2, YX, Y^3)$ has been treated in [34]; the proof that the functor defined above is a representation equivalence, can be found in that paper).

As a consequence, if $\Gamma = \tilde{A}_{pq}$, and M_Γ is indecomposable of period q and regular length q+1, then $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ is wild. For, we may suppose that M_Γ is of the form



but then the corresponding algebra is given by



with relations $\alpha_p\ldots\alpha_1\alpha' = \beta_q\ldots\beta_1\beta'$, $\beta_1\alpha' = 0$ specializes to the previous case by shrinking the arrows α_i,β_i with $i \geq 2$.

We consider now the general case. Let Γ be a connected quiver of tame type, and $M_\Gamma = (M_i,\varphi_\alpha)$ a regular representation of Γ, such that $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ is not wild.

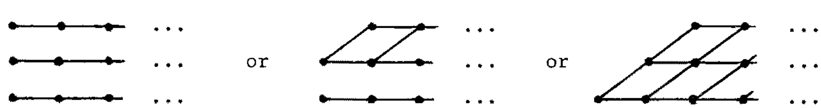
First, we claim that M cannot be the direct sum of two isomorphic simple regular modules. For, if $M = N \oplus N$ with N simple regular, let N(2) be indecomposable regular with regular socle N and regular length 2, then we can apply the previous lemma to $X = \mathrm{Hom}(M,N)$ and $Y = \mathrm{Hom}(M,N(2))$ in the vectorspace category $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$, and get a contradiction.

Similarly, M cannot be the direct sum of three non-zero regular modules. For, assume $M = N \oplus N' \oplus N''$, with N, N', N'' simple regular. By the result above, N, N', N'' have to be pairwise non-isomorphic. Again, denote by N(m) the indecomposable regular module of regular length m with regular socle N. Then we obtain in $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ as full subcategories partially ordered sets of the form (n, n, n), namely, take just all $\mathrm{Hom}(M_\Gamma,N(m))$, $\mathrm{Hom}(M_\Gamma,N'(m))$, $\mathrm{Hom}(M_\Gamma,N''(m))$, with $m \leq n$. But according to the theorem of Nazarova, this is impossible. In general, M will map onto $N \oplus N' \oplus N''$, and then $\mathrm{Hom}(N \oplus N' \oplus N'',\mathcal{M})$ can be considered as a subcategory of $\mathrm{Hom}(M,\mathcal{M})$.

Next, we note that $\dim M_i \leq 2$ for all vertices i. Namely, let I(i) be the indecomposable injective representation corresponding to the vertex i. Then $\dim \mathrm{Hom}(M,I(i)) = \dim M_i$, and the endomorphism ring of $\mathrm{Hom}(M,I(i))$ as an object in $\mathrm{Hom}(M,\mathcal{M}_\Gamma)$ is k. Thus we can apply the lemma in 2.4.

Also, we see that M cannot have two isomorphic regular composition factors. Namely, by previous considerations, we know that for a decomposition $M = M' \oplus M''$, the regular composition factors of M' are pairwise different from the regular composition factors of M''.
Thus, assume M is indecomposable and has two isomorphic regular composition factors, say of the form S with period t, such that M is of regular length $\geq t+1$. Actually, we may suppose that the regular length of M is $t+1$. In case Γ is of type $\tilde{A}_{pq}$, we have $t = 1$, p, or q. The last two cases are impossible, as we have seen above. Similarly, the first case is impossible, since also in this case $\mathcal{M}({}_k M_\Gamma)$ would specialize to some $\mathcal{M}({}_k N_\Delta)$, with N_Δ of type $(\tilde{A}_{11}, {}^{1}_{1})$. In cases $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, one easily observes that for an indecomposable regular module M of period t and regular length $t+1$, always at least one component of M is of dimension ≥ 3.

It follows that the regular length of M is ≤ 2. Namely, assume the regular length of M is equal to 3. Since the regular composition factors of M are pairwise non-isomorphic, the full subcategory of $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$ consisting of all objects $\mathrm{Hom}(M,X)$ with X regular, is of the form



depending whether M has 3, 2, or 1 indecomposable summands, respectively. But in all cases, it is easy to find subsets of the form (n,n,n) for all $n \in \mathbb{N}$.

If there exists an arrow $\underset{r}{\circ} \xrightarrow{\alpha} \underset{s}{\circ}$ in Γ with $\dim M_r = \dim M_s = 2$, and $\varphi_\alpha : M_r \longrightarrow M_s$ an isomorphism, then $\Gamma = \tilde{A}_{1q}$, the vertex r is a source, s is a sink, and $M = M' \oplus M''$, with M' simple regular of period 1, and M'' simple regular of period q, and $M' \not\approx M''$. Namely, only in case $\Gamma = \tilde{A}_{1q}$, with r a source, and s a sink, there is an additional path from r to s. Otherwise, there is just one path from r to s given by α, and thus $\mathrm{Hom}(I(s),I(r))$ is one-dimensional, and therefore also $\mathrm{Hom}(X,Y)$ can be at most one-dimensional, where $X = \mathrm{Hom}(M,I(s))$, $Y = \mathrm{Hom}(M,I(r))$. The assumption on φ_α to be bijective, implies that also the dual map φ_α^* is bijective, but φ_α^* can be identified with an element in $\mathrm{Hom}(X,Y)$. By the lemma at the beginning of this section we can conclude that $\mathrm{Hom}(M,\mathcal{M})$ is wild.

Finally we note that either M is of type $(\tilde{A}_{pq},p,q)$, say $M = M' \oplus M''$ with $\dim M' = 1 \begin{smallmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{smallmatrix} 1$, $\dim M'' = 1 \begin{smallmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{smallmatrix} 1$, or else at most one component of M may be two-dimensional. Namely, in all other cases one easily observes that for $\dim M_r = \dim M_s = 2$ with different vertices r,s, there is a chain of arrows $\circ \xrightarrow{\alpha_i} \circ$ joining r and s such that all φ_{α_i} are isomorphisms. This is clear for $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, and also for $\tilde{D}_n$, with M indecomposable of period 2. Note that the case $\tilde{D}_n$ with M of period $n-2$ cannot occur, since we recall that M is of regular length at most 2.

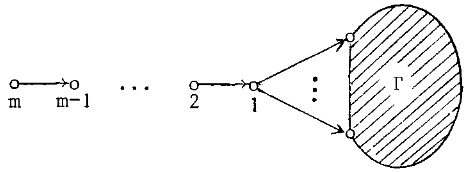
We have shown that M has regular length ≤ 2, not two isomorphic regular composition factors, that all Auslander-Reiten translates $A^m M$ have at most two-dimensional components, and that either M is of type $(\tilde{A}_{pq},p,q)$ or else that at most one component of M is two-dimensional. Now these conditions exclude all cases not listed in theorem 3, but the cases $(\tilde{A}_{pq},1)$ and $(\tilde{D}_n,2)$. For these two situations, we have calculated the pattern in section 3.3. It follows that for large values of p,q,n, the corresponding vectorspace categories are wild, using the theorem of Nazarova, and that they are tame precisely in the listed cases. This finishes the proof.

3.7. Some components of the Auslander-Reiten quiver

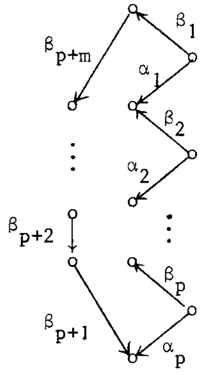
In the preceding sections, we have discussed the question under what conditions a regular enlargement of a tame quiver is tame, again. In case R is even domestic, we want to describe the category of R-modules in more detail. In fact, we will describe certain types of components of the Auslander-Reiten quiver of regular enlargements of tame quivers.

Thus, let Γ be a tame connected quiver, and ${}_T M_\Gamma$ a bimodule with M_Γ regular. If we consider $M({}_T M_\Gamma)$, then we know from 2.6 that nearly all components of the Auslander-Reiten quiver of M_Γ remain unchanged in the Auslander-Reiten quiver of $M({}_T M_\Gamma)$. Namely, the only ones which can, and will, be changed are the preinjective component of M_Γ, and those regular components of M_Γ which contain direct summands of M_Γ.

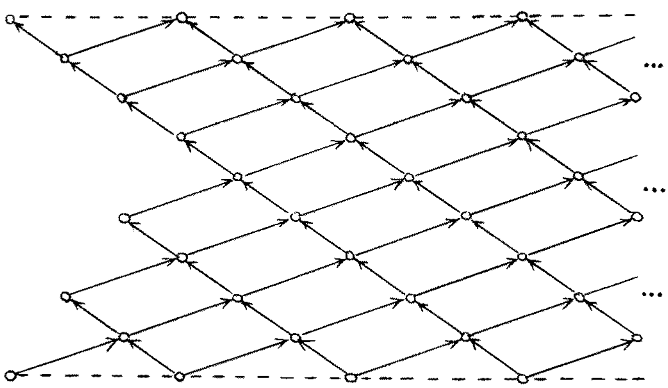
Let us consider first the case where M_Γ is simple regular, with Γ a tame connected quiver, and T being of type A_m, so that R is given by a quiver with relations of the following form



where M_R reduces to $M({}_T M_\Gamma)$. We describe now the component of the Auslander-Reiten quiver of R containing M. Let p be the period of M_Γ. Denote by Σ_{pm} the following quiver



As in 2.2, we form $\mathbb{N}\Sigma_{pm}$, but we reverse now the orientation of all the arrows of the form (β_i, z) and (α_i^*, z), with $z \in \mathbb{N}$. Denote the new quiver by $(\mathbb{N}\,\Sigma_{pm})'$. For example, for $p = 3$, $m = 3$, we obtain the quiver $(\mathbb{N}\,\Sigma_{33})'$

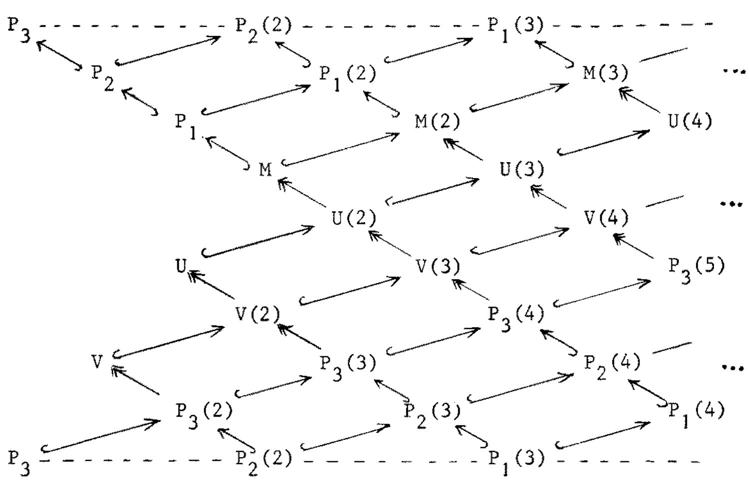


where the two dotted lines have to be identified in order to form a cylinder.

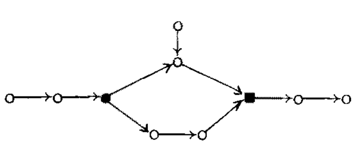
Lemma 1. The component of the Auslander-Reiten quiver for R which contains M, is of the form $(\mathbb{N}\,\Sigma_{pm})'$.

For the proof, we consider first the ring R' obtained from R by reversing the orientation of all the arrows in the left arm, so that $\mathcal{M}_{R'}$ is of the form $\mathcal{M}({}_k(M \oplus N)_{\Gamma\dot{\cup}\Delta})$, where Δ is the quiver $\underset{2}{\circ}\!\longrightarrow\!\underset{3}{\circ} \dots \underset{m-1}{\circ}\!\longrightarrow\!\underset{m}{\circ}$, and $\underline{\dim}\, N_\Delta = (1\ 1\ \dots\ 1\ 1)$. The vectorspace category $\mathrm{Hom}(M \oplus N, \mathcal{M}_{\Gamma\dot{\cup}\Delta})$ is the disjoint union of $\mathrm{Hom}(M,\mathcal{M}_\Gamma)$ and a chain with m-1 elements. Now denote by $\mathcal{R}$ the category of regular Γ-modules, by $\mathcal{J}$ the category of preinjective Γ-modules. Now, as we know, $\mathrm{Hom}(M,\mathcal{R})$ is the additive category of a chain, and any non-zero map $M \longrightarrow I$, with I preinjective, can be factored along this chain. This implies that any indecomposable object (U,φ,X) in $\mathcal{M}({}_k(M \oplus N)_{\Gamma\dot{\cup}\Delta})$ where $\varphi : U \otimes_k (M \oplus N) \longrightarrow X$, either $X|\Gamma$ is preinjective or regular. Also, in case $X|\Gamma$ is regular, then $X|\Gamma$ is indecomposable, and U is one-dimensional. From this it follows easily that the indecomposable objects (U,φ,X) with $X|\Gamma$ regular, form one component. If we translate this from R' to R (using reflection functors), we similarly see that the indecomposable R-modules Y with $Y|\Gamma$ in the same component $\mathcal{C}$ of $\mathcal{M}_\Gamma$ as M_Γ, form one component of the Auslander-Reiten quiver of $\mathcal{M}_R$ and that those not belonging to $\mathcal{C}$ are of the form $M^i(z) = (M(z) \oplus P(i))\,/\,M$ with $z \in \mathbb{N}$, $1 \le i \le m$, where $M(z)$ is the indecomposable regular module with regular length z and regular socle M, and P(i) is the indecomposable projective module corresponding to i. There are obvious maps between the modules in $\mathcal{C}$ and the $M^i(z)$, so that we obtain a configuration of modules and maps of the form $(\mathbb{N}\ \Sigma_{pm})'$, and it is easy to see that any other map between these modules is a composition of the given maps. This then implies that we really have constructed in this way the component completely, as we wanted to do.

For example, for $p = 3$, $m = 3$, denote AM by U and A^2M by V, and use the notation M(z), and $M^i(z) = (M(z) \oplus P(i))/M$ as above. Then the corresponding component is as follows:



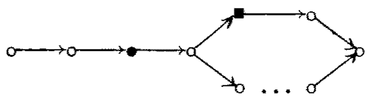
Such a component exists for


,

here, Γ is an $\tilde{E}_6$-quiver, and the dimension types of M, U, V are

$$\underline{\dim}\, M = \begin{array}{ccc} & 0 & \\ & 1 & \\ 000 & & 111 \\ & 11 & \end{array}, \quad \underline{\dim}\, U = \begin{array}{ccc} & 0 & \\ & 0 & \\ 000 & & 100 \\ & 01 & \end{array}, \quad \underline{\dim}\, V = \begin{array}{ccc} & 1 & \\ & 1 & \\ 000 & & 110 \\ & 00 & \end{array}.$$

Similarly, we have such a component for



with Γ being of type $\tilde{A}_{3q}$, and with

$$\underline{\dim}\, M = \begin{smallmatrix} & 0 & 0 & \\ 001 & & & 1 \\ & 1\ldots 1 & & \end{smallmatrix}, \quad \underline{\dim}\, U = \begin{smallmatrix} & 1 & 0 & \\ 000 & & & 0 \\ & 0\ldots 0 & & \end{smallmatrix}, \quad \underline{\dim}\, V = \begin{smallmatrix} & 0 & 1 & \\ 000 & & & 0 \\ & 0\ldots 0 & & \end{smallmatrix}.$$

Note that the components of the form $(\mathbb{N}\,\Sigma_{pm})'$ are rather curious: they contain m indecomposable projective modules, and every other indecomposable module belonging to the component is obtained from them by applying some A^{-i}. In this respect, they are similar to the components of preprojective modules of quivers without relations. On the other hand, whereas for an indecomposable preprojective module P we always have $\operatorname{End}(P) = k$, nearly all indecomposable objects in such a component have non-trivial nilpotent endomorphisms. However, we note that at least the following is true: any nilpotent endomorphism, or, more general, any non-invertible homomorphism between two indecomposable objects in the component, is a composition of irreducible maps.

In the situation above, we have seen that for the indecomposable objects (U,φ,X) in $\mathcal{M}({}_k(M \oplus N)_{\Gamma\dot{\cup}\Delta})$, always $X|\Gamma$ is either regular or preinjective. This is a rather strong assertion, as the following lemma shows.

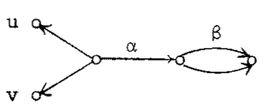
Lemma 2. Let Γ be a tame connected quiver, Δ some quiver, let M_Γ be non-zero regular, and N_Δ arbitrary. Then for any indecomposable object (U,φ,X) in $\mathcal{M}({}_k(M \oplus N)_{\Gamma\dot{\cup}\Delta})$, the restriction $X|\Gamma$ is either regular or preinjective, if and only if M_Γ is simple regular and $\operatorname{Hom}(N_\Delta,\mathcal{M}_\Delta)$ is the additive category of a chain.

Proof. Assume M is not simple regular, say with a simple regular submodule M_1. Let $\varphi_1 : M_1 \to I$ be a non-zero homomorphism, with I indecomposable injective, and extend it to a homomorphism $\varphi : M \to I$. Let $\pi : M \to M/M_1$ be the projection, and consider the object $(k, \binom{\varphi}{\pi}, I \oplus M/N_1)$ in $\mathcal{M}({}_kM_\Gamma)$. It is easy to see that it is indecomposable, but $I \oplus M/N_1$ is neither preinjective nor regular.

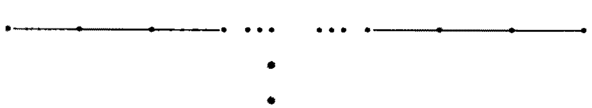
Also, if $\operatorname{Hom}(N_\Delta,\mathcal{M}_\Delta)$ is not the additive category of a chain, then it either contains one two-dimensional object Y with endomorphism ring of dimension ≤ 2, or two one-dimensional incomparable objects Y_1, Y_2. In $\mathrm{Hom}(M_\Gamma, \mathcal{M}_\Gamma)$, we always have as non-zero objects $\mathrm{Hom}(M_\Gamma, M_\Gamma) =: X$, and $\mathrm{Hom}(M_\Gamma, I_\Gamma) =: X'$, where I_Γ is some indecomposable injective module. It is well-known (and easy to see) that there exists a subspace U of the object $X \oplus X' \oplus Y$, or $X \oplus X' \oplus Y_1 \oplus Y_2$, respectively, of the vectorspace category $\mathrm{Hom}(M \oplus N, \mathcal{M}_{\Gamma \dot{\cup} \Delta})$, with inclusion map ι, such that the tripel $(U, \iota, X \oplus X' \oplus Y)$ or $(U, \iota, X \oplus X' \oplus Y_1 \oplus Y_2)$ is indecomposable.

Let us consider now some cases of regular enlargements $\mathcal{M}({}_k(M \oplus N)_{\Gamma \dot{\cup} \Delta})$ where there exist indecomposable objects (U, φ, X) with X being the direct sum of a non-zero regular and a non-zero preinjective module. We only will consider domestic regular enlargements of a quiver of the form $\tilde{A}_{pq}$, but we hope that these examples will shed some light on the general situation. The examples will cover at least all cases of domestic regular enlargements which occur in Theorem 1, thus, in this way, we finish our program of giving a complete description of the module categories of the domestic one-relation algebras occurring in Theorem 1.

We start with the following one-relation algebra

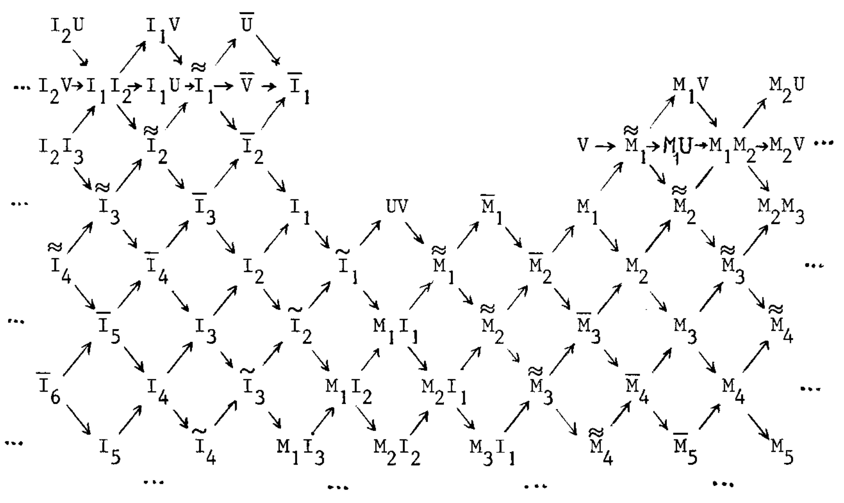
R :  $\beta\alpha = 0.$

Here, Γ is the $\tilde{A}_{11}$-quiver with M_Γ being of dimension type $(1\ 1)$, and Δ the disjoint union of the two A_1-quivers consisting of the vertices u, v, with U, V being the corresponding simple modules, and $M_R \approx \mathcal{M}({}_k(M \oplus U \oplus V)_{\Gamma \dot{\cup} \Delta})$. The vectorspace category $\mathrm{Hom}(M \oplus U \oplus V, \mathcal{M}_{\Gamma \dot{\cup} \Delta})$ is of the form



where the long chain is formed by the objects $\mathrm{Hom}(M, M_i)$ and $\mathrm{Hom}(M, I_i)$, with M_i being the indecomposable regular representation of Γ with regular socle M and regular length i, and I_i the indecomposable representation of Γ of dimension type $(i\ i-1)$. This shows that we can construct indecomposable R-modules as follows: Let X_1, X_2 be non-isomorphic indecomposable $\Gamma \dot{\cup} \Delta$-modules with

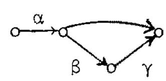
$\mathrm{Hom}(M \oplus U \oplus V, X_i)$ one-dimensional, say generated by φ_i. Then we denote by $\overline{X}_i$ the R-module (k, φ_i, X_i). If $\mathrm{Hom}(M \oplus U \oplus V, X_1)$ and $\mathrm{Hom}(M \oplus U \oplus V, X_2)$ are incomparable, let X_1X_2 denote the R-module $\left(k, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, X_1 \oplus X_2\right)$, otherwise let X_1X_2 be the R-module $\left(k^2, \begin{pmatrix}\varphi_1 & 0 & \pi_U & 0\\ 0 & \varphi_2 & \pi_U & \pi_V\end{pmatrix}, X_1 \oplus X_2 \oplus U \oplus V\right)$, with π_U, π_V denoting the canonical projections from $M \oplus U \oplus V$ onto U or V, respectively. Note that in the last case, X_1, X_2 both have to be of the form M_i or I_j. Finally, if X is again of one of the forms M_i or I_j, we denote by $\tilde{X}$ the R-module $(k, (\varphi\ \pi_U\ \pi_V), X \oplus U \oplus V)$, and by $\approx\!\!X$ the R-module $\left(k^2, \begin{pmatrix}\varphi & \pi_U & 0\\ \varphi & 0 & \pi_V\end{pmatrix}, X \oplus U \oplus V\right)$. With this notation, the component of the Auslander-Reiten quiver of R containing M is as follows:



Note that in M_R, there is a chain of irreducible maps from any I_i to any M_j, for example a chain of length 6 from I_1 to $M_1 = M$; whereas in M_Γ, the modules I_i and M_j belong to different components. Thus, two Auslander-Reiten components of M_Γ are joined to form, together with additional modules, a single component. Also note that this component has the following property: there are non-invertible maps between modules of the component which cannot be expressed as sums of compositions of irreducible maps, namely all maps from a module of the form M_i to a module I_j.

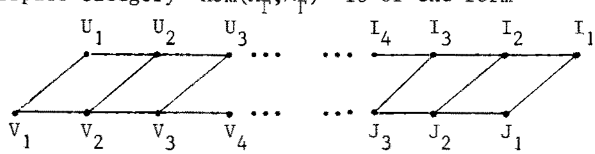
For the remaining two examples, Γ will be the $\tilde{A}_{12}$-quiver , and we will consider the two simple regular modules U, V of period 2, with $\underline{\dim}\, U = (1\ 0\ 1)$, $\underline{\dim}\, V = (0\ 1\ 0)$. Let U_i be the indecomposable regular module with regular socle U and regular length i, and similarly V_i indecomposable regular with regular socle V and regular length i. We denote by I_i the indecomposable Γ-module with dimension type $(i, i-1, i-1)$, and by J_i that of dimension type $(i, i, i-1)$.

First, let $M = V_2$, and consider the regular enlargement of Γ by M. We obtain the one-relation algebra R given by

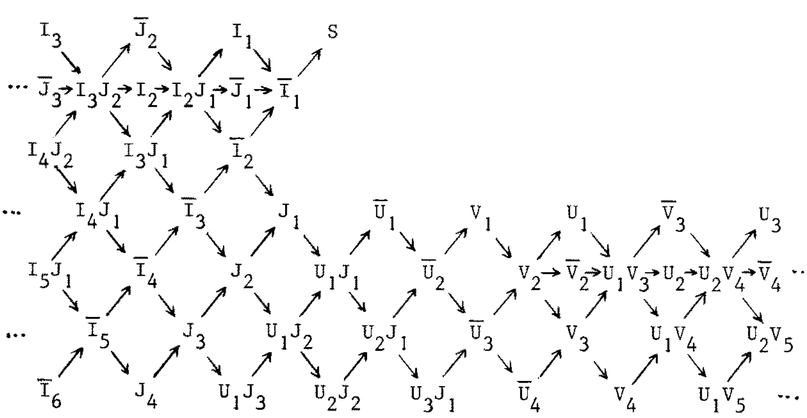


, $\gamma\beta\alpha = 0$.

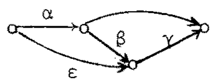
The vectorspace category $\mathrm{Hom}(M_\Gamma, M_\Gamma)$ is of the form



where we have added to a point of the form $\mathrm{Hom}(M,X)$ the symbol X. The Auslander-Reiten component of R containing M is as follows (where we use a similar notation for indecomposable R-modules as in the previous example, and where S denote the simple R-module of dimension type $(1\ 0\ 0\ 0)$):

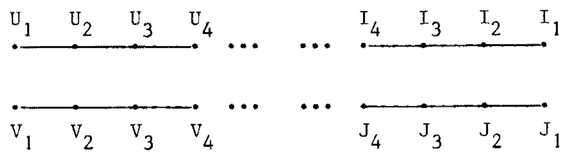


Next, let $M' = U_1 \oplus V_1$, thus the enlargement of Γ by M' is the algebra R' given by

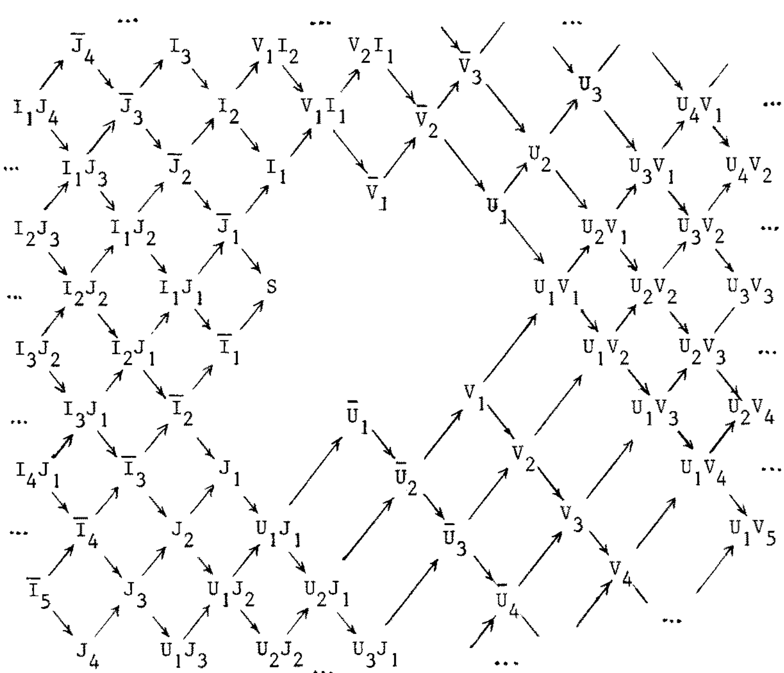


$\beta\alpha = 0, \quad \gamma\varepsilon = 0.$

The vectorspace category $\mathrm{Hom}(M'_\Gamma, M_\Gamma)$ is



where again we have denoted the object $\mathrm{Hom}(M,X)$ just by the symbol X. The Auslander-Reiten component of R' containing U and V is as follows:



3.8. Further examples of non-domestic tame algebras

With the help of Theorem 3, we can construct a large amount of non-domestic tame algebras, and we want to mention at least some of the algebras which arise in this way.

First, consider an arbitrary enlargement R of a quiver Γ, say by M_Γ, thus $\mathcal{M}_R$ reduces to $\mathcal{M}({}_kM_\Gamma)$. If Γ is a quiver with n vertices and m arrows, then R is defined by a quiver with n+1 vertices, m+g arrows, and h relations, where g is the number of homogeneous generators and h the number of homogeneous relations of M. Recall that for a module M with minimal projective resolution

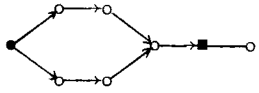
$$\bigoplus_{j=1}^{h} Q_j \longrightarrow \bigoplus_{i=1}^{g} P_i \longrightarrow M \longrightarrow 0$$

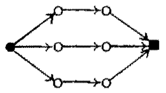
and P_i, Q_j indecomposable projective, one calls g the number of homogeneous generators, and h the number of homogeneous relations of M.

As a consequence, we see that such enlargements very seldom will be one-relation algebras. For example, we know that there are no regular enlargements of a tame quiver of type $(\tilde{E}_6,2)$ which are one-relation algebras. In writing down quivers with more relations, we will use the following convention: For any pair consisting of a black circle and a black square with an oriented path from the circle to the square, we have to take the relation given by the sum of all paths from the circle to the square. Any additional relation will be given separately (in our examples, the starting point usually will be one of the black dots, the end point a white square). Again, we do not write down the orientation of the arrows which do not appear in relations, since we can use the obvious reflection functors in order to carry one possible orientation into any other. As a consequence, any of the diagrams below stands for a certain number of isomorphism classes of algebras (for a fixed base field, this number depends on the number of edges without orientation, and the corresponding symmetry group). Let us write down all regular enlargements of tame quivers with pattern of similarity type $\textcircled{E_6}$.

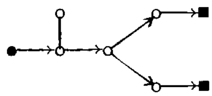
<u>The case</u> $(\tilde{E}_6,2)$. This is the list of all regular enlargements of a quiver of type $\tilde{E}_6$ by a simple regular module of period 2.

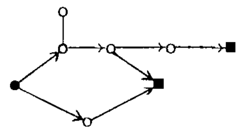
Quivers with one relation:

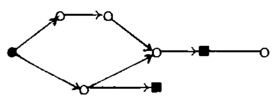


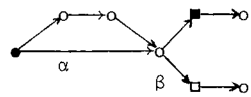


Quivers with two relations:

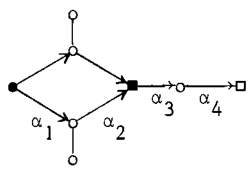




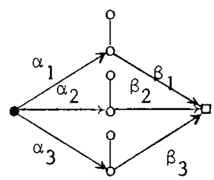




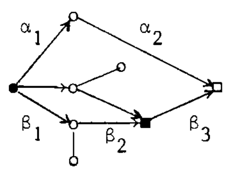
$\beta\alpha = 0$



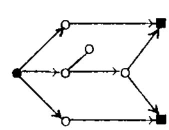
$\alpha_4\alpha_3\alpha_2\alpha_1 = 0$

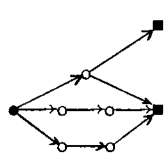


$\beta_1\alpha_1 = \beta_2\alpha_2 = \beta_3\alpha_3$

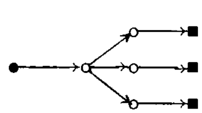


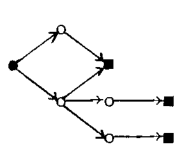
$\alpha_2\alpha_1 = \beta_3\beta_2\beta_1$

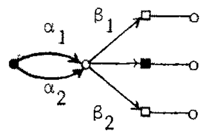




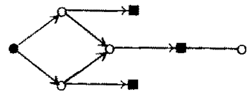
Quivers with three relations:

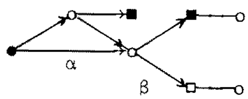




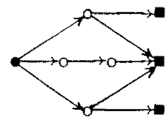


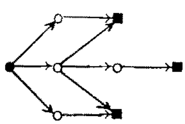
$\beta_1\alpha_1 = 0 = \beta_2\alpha_2$

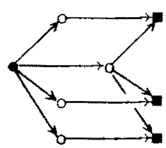




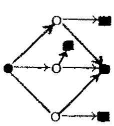
$\alpha\beta = 0$



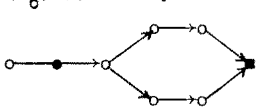




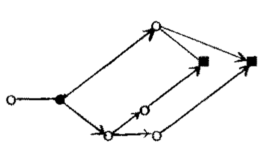
One quiver with four relations:

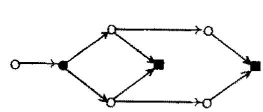


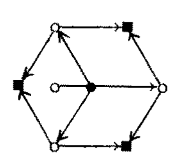
<u>The case</u> $(\widetilde{A}_{33},1)$. Besides the duals of the first algebras of type $(\widetilde{E}_6,2)$, namely

 ,

there are just three additional cases





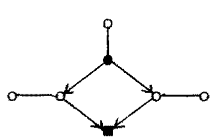


We have seen above that there are 19 different possibilities for regular enlargements of tame quivers of type $(\widetilde{E}_6,2)$. Note that this counts only the essentially different possibilities, not taking into account the orientation of the arms. For some other pattern, let us give the corresponding numbers of essentially different possibilities of regular enlargements of tame quivers:

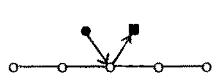
$(\widetilde{E}_6,2)$	$\overline{(\widetilde{E}_6,3)}$	$(\widetilde{E}_7,3)$	$\overline{(\widetilde{E}_7,4)}$	$(\widetilde{E}_8,5)$
20	9	61	28	202

Next, assume that S is the concealment of a tame connected quiver Γ, and M_S a regular S-module. Then the pattern of $\mathrm{Hom}(M_S,\mathcal{M}_S)$ is the same as that of $\mathrm{Hom}(N_\Gamma,\mathcal{M}_\Gamma)$, where N_Γ a corresponding Γ-module. Namely, we know that there is an equivalence η between a cofinite subcategory $\mathcal{U}$ of $\mathcal{M}_\Gamma$ and a cofinite subcategory of $\mathcal{M}_R$. We can assume that all regular Γ-modules lie in $\mathcal{U}$, and then η gives an equivalence between the regular Γ-modules and the regular R-modules. Thus, let $N = \eta^{-1}(M)$. We use now a remark in 2.3 in order to see that the vectorspace categories $\mathrm{Hom}(M_S,\mathcal{M}_S)$ and $\mathrm{Hom}(N_\Gamma,\mathcal{M}_\Gamma)$ belong to the same pattern. Thus, a classification of the tame regular enlargements of tame connected quiver (Theorem 3), immediately also applies to regular enlargements of tame concealed quivers.

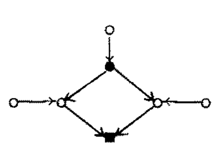
We consider just two examples of concealed quivers of type $\widetilde{E}_6$, and there the regular modules of type $(\widetilde{E}_6,2)$, namely



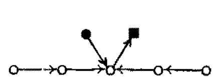
and



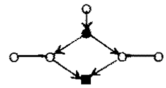
For one orientation, we write down the dimension types of the two simple regular modules of period 2.

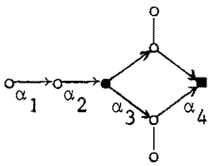


: $\begin{matrix} & & 0 & & \\ & & 1 & & \\ 1 & 1 & & 1 & 1 \\ & & 1 & & \end{matrix}$ and $\begin{matrix} & & 1 & & \\ & & 1 & & \\ 0 & 1 & & 1 & 0 \\ & & 0 & & \end{matrix}$,

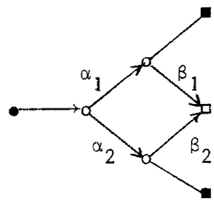


: $\begin{matrix} & & 0\ 0 & & \\ 0 & 1 & 1 & 1 & 0 \end{matrix}$ and $\begin{matrix} & & 1\ 1 & & \\ 1 & 1 & 2 & 1 & 1 \end{matrix}$.

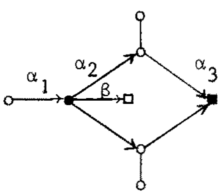
The possible extensions of  by a simple regular module of period 2 are as follows:



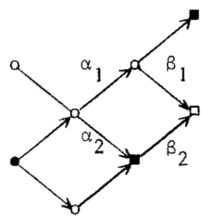
$\alpha_4\alpha_3\alpha_2\alpha_1 = 0$



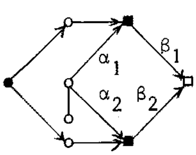
$\beta_1\alpha_1 = \beta_2\alpha_2$



$\alpha_3\alpha_2\alpha_1 = 0, \quad \beta\alpha_1 = 0$

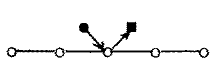


$\beta_1\alpha_1 = \beta_2\alpha_2$

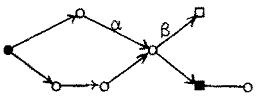


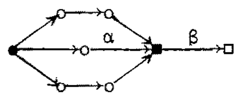
$\beta_1\alpha_1 = \beta_2\alpha_2$.

The first of the cases is the dual one of an algebra considered above (namely an $(\widetilde{E}_6,2)$-extension of a quiver), the others are new.

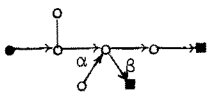
Similarly, for  we obtain the following extensions by a simple regular module of period 2. Always, we have $\beta\alpha = 0$.

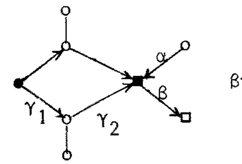
Quivers with two relations:



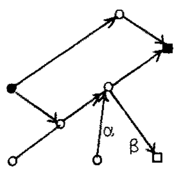


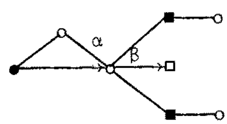
Quivers with three relations:

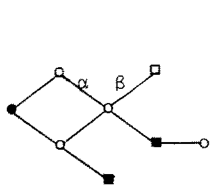


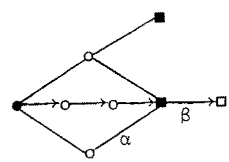


$\beta\gamma_2\gamma_1 = 0$

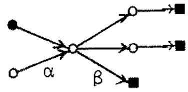


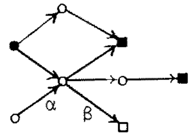


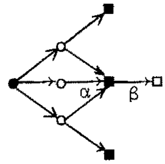


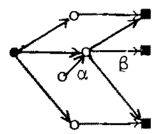


Quivers with four relations:

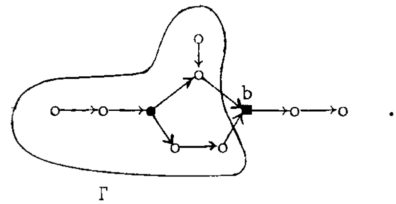








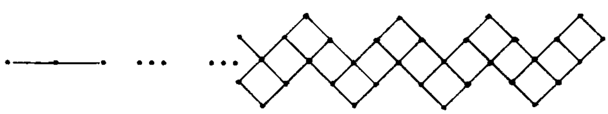
Finally, we show that we can combine regular enlargements and regular "co"-enlargements without much difficulty. Consider the following one-relation algebra

R : 

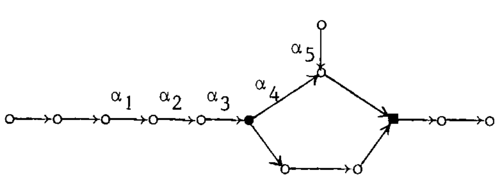
We claim the following: Given an indecomposable R-module X with $X_b \neq 0$, then either its restriction $X|\Gamma$ to Γ is preprojective, or else X belongs to a Auslander-Reiten component $\mathcal{C}$ which is the dual of a component of the form $(\mathbb{N}\ \Sigma_3)'$ discussed in 3.7. This follows immediately from 3.7, where we have described the module category of the opposite algebra R^{op} in great detail: As a consequence, if we choose for M_R one of the following indecomposable modules I, U^*, V^* with

$$\underline{\dim} I = \begin{matrix} & & & 0 & & & \\ & & & 1 & & & \\ 1 & 1 & 1 & & 1 & 1 & 1 \\ & & 1 & 1 & & & \end{matrix}, \quad \underline{\dim} U^* = \begin{matrix} & & & 0 & & & \\ & & & 0 & & & \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ & & 1 & 0 & & & \end{matrix}, \quad \underline{\dim} V^* = \begin{matrix} & & & 1 & & & \\ & & & 1 & & & \\ 0 & 1 & 1 & & 0 & 0 & 0 \\ & & 0 & 0 & & & \end{matrix},$$

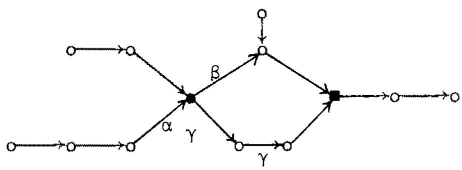
then the vectorspace category $\mathrm{Hom}(M_R, M_R)$ belongs to the pattern



Namely, if $\mathrm{Hom}(M_R, X_R) \neq 0$ for some indecomposable R-module X_R, then either X_R belongs to $\mathcal{C}$, or else X is in fact a Γ-module and X_Γ is preinjective. The right part of the pattern comes from the preinjective Γ-modules, and it is easy to see that we obtain just a chain of objects $\mathrm{Hom}(M_R, X_R)$ with X_R indecomposable in $\mathcal{C}$. In order to use a non-domestic pattern, we consider $M \times N$, where N_Δ is the minimal faithful representation of a A_2-quiver Δ. The corresponding enlargements of R by $M \times N$ lead to the following quivers with relations:

M = I  $\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1 = 0$

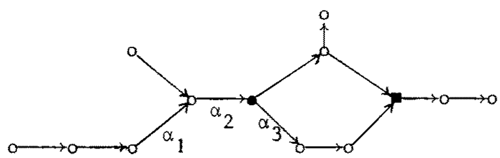
$M = U^*$



$\beta\alpha = 0$

$\gamma_1\gamma_2\alpha = 0$

$M = V^*$



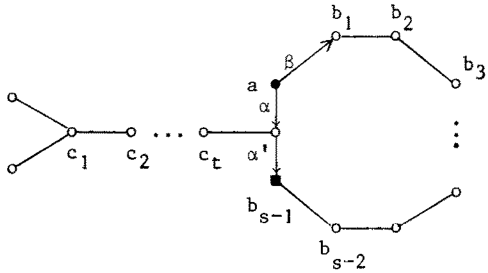
$\alpha_3\alpha_2\alpha_1 = 0$

It follows that these quivers with relations are tame, and non-domestic. However note that in any case there is just a countable number of indecomposable representations X with $X_a \neq 0$ for all vertices a.

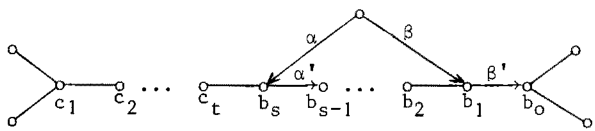
3.9 Two special one-relation algebras

In our investigation of the tame one-relation algebras, two algebras were not yet touched, and they have to be considered separately. They are not concealed quivers, since they will turn out to be non-domestic (and there are no non-domestic tame quivers). Also, it can be seen easily that they are not enlargements of a tame quiver by some module. However, we will see that they are specializations of algebras which are enlargements of a tame quiver by some regular module, so that we can use Theorem 3.

The first one-relation algebra R which we have to consider is given by



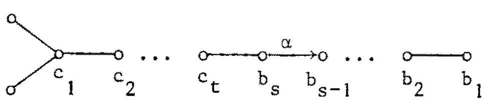
We can assume that β is directed as $\underset{a}{\circ}\xrightarrow{\beta}\underset{b_1}{\circ}$, so that a is a source. Namely, otherwise there has to exist a source b_m and a path $\underset{b_m}{\circ}\longrightarrow\underset{b_{m-1}}{\circ}\cdots\underset{b_2}{\circ}\to\underset{b_1}{\circ}\to\underset{a}{\circ}$, since we have excluded oriented cycles, and then we can apply a product $\sigma_{b_1}\sigma_{b_2}\ldots\sigma_{b_m}$ of reflections in order to obtain a new orientation with a being a source. Now we see that R is a specialization of the following quiver with two relations



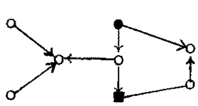
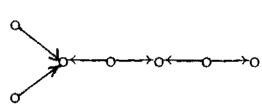
with $\alpha'\alpha = 0$, $\beta'\beta = 0$

and this is a regular enlargement of a quiver Γ' of type $\tilde{D}_n$, with $n = s+t+4$, namely by the regular module $P(b_s)/P(b_{s-1}) \oplus P(b_1)/P(b_o)$, thus of type $(\tilde{D}_n,(n-2) \oplus (n-2))$. It follows that R is tame, but we cannot decide, in this way, whether R is domestic or not.

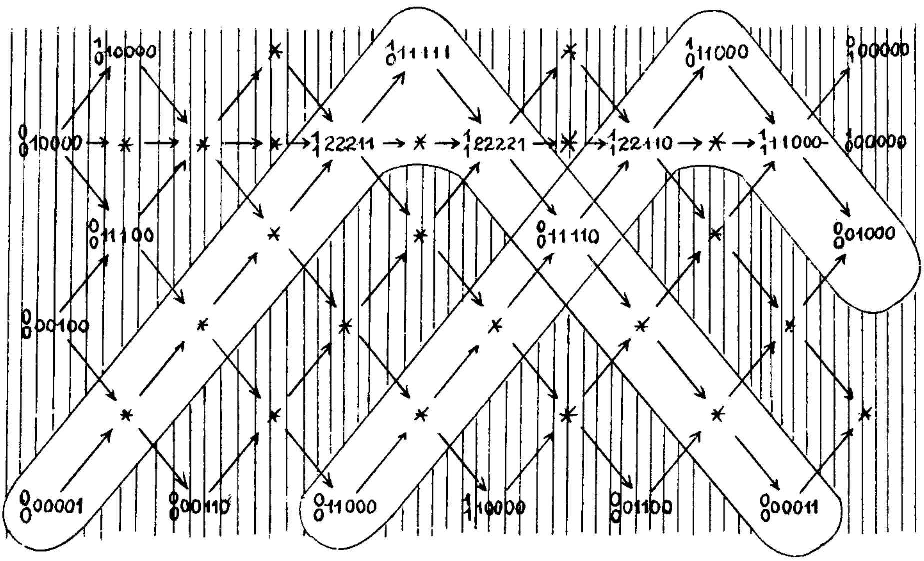
In order to see that R is non-domestic, we write R as an enlargement of a quiver of <u>finite</u> type by some module. Let Γ be the quiver



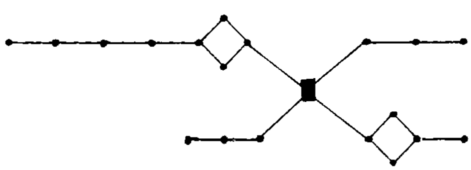
and $M = P(b_s)/P(b_{s-1}) \oplus P(b_1)$. Then $\mathcal{M}_R = \mathcal{M}({}_kM_\Gamma)$, and we have to calculate the vectorspace category $\mathrm{Hom}(M_\Gamma,\mathcal{M}_\Gamma)$. It turns out that this is a subcategory of a vectorspace category with pattern (D_n'), (this follows from the fact that Γ is a subquiver of Γ' and M, as Γ'-module, is precisely the regular Γ'-module considered above), and this subcategory is itself non-domestic. Let us give the calculation in one example: Let

R :  , thus Γ =  ,

and M is the direct sum of the indecomposable representations of dimension type $\begin{smallmatrix}0\\0\end{smallmatrix}\,1\;1\;0\;0\;0$ and $\begin{smallmatrix}0\\0\end{smallmatrix}\,0\;0\;0\;0\;1$. The Auslander-Reiten quiver of Γ is as follows

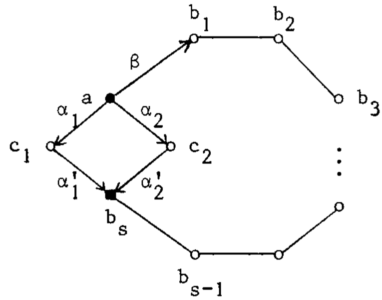


and we have indicated both $\mathrm{Hom}({}^{0}_{0}11000, M_\Gamma)$ and $\mathrm{Hom}({}^{0}_{0}00001, M_\Gamma)$, thus $\mathrm{Hom}(M, M_\Gamma)$ is



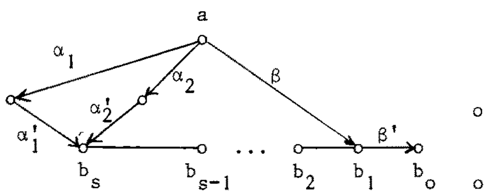
We have to invoke now the classification of the subspace of such a vectorspace category which follows from [29], in order to see that there are infinitely many series, thus it is non-domestic.

The second algebra S which we did not consider yet is



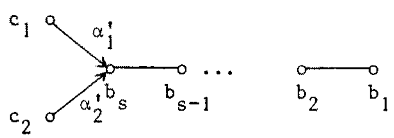
where again we can assume that an orientation is choosen with $\underset{a}{\circ} \overset{\beta}{\longleftarrow} \underset{b_1}{\circ}$.
Now we see that S is a specialization of the quiver with two

relations



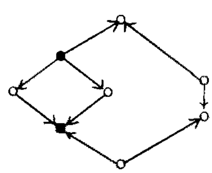
with $\alpha_1'\alpha_1 = \alpha_2'\alpha_2$, $\beta'\beta = 0$

This is a regular enlargement of a $\tilde{D}_n$-quiver, where $n = s+4$, of type $(\tilde{D}_n, (n-2) \oplus (n-2))$. Thus S is tame. As in the previous case, we can write S as an enlargement of a quiver of finite type, namely,

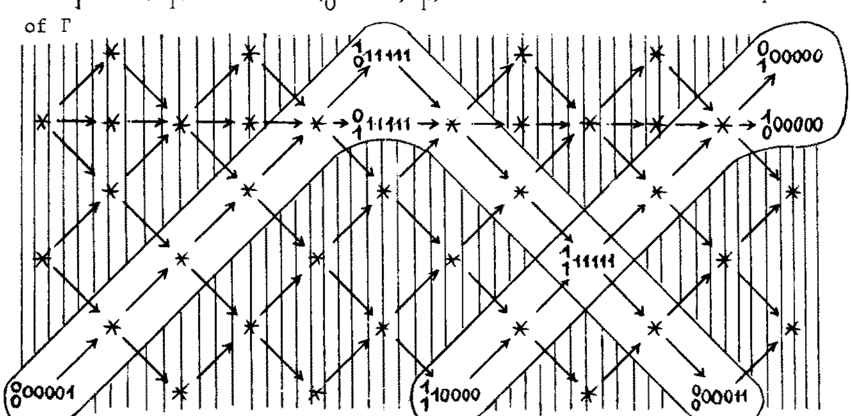


by the module $N = (P(c_1) \oplus P(c_2))/P(b_s) \oplus P(b_1)$. For example, in the case of

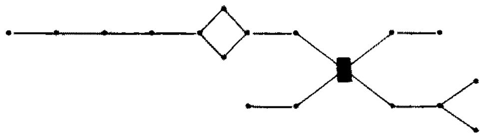
S :



we obtain the same quiver Γ as in the special case of R considered above, this time N is the direct sum of ${}^{1}_{1}\,1\;0\;0\;0\;0$ and ${}^{0}_{0}\,0\;0\;0\;0\;1$. Let us indicate again the vectorspace categories $\mathrm{Hom}({}^{1}_{1}10000, M_\Gamma)$ and $\mathrm{Hom}({}^{0}_{0}00001, M_\Gamma)$ in the Auslander-Reiten quiver of Γ



As a consequence, the vectorspace category $\mathrm{Hom}(N_\Gamma, M_\Gamma)$ is



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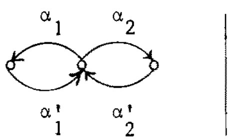
and consequently, non-domestic.

In order to show that S is tame, we have referred to Nazarova and Rojter [29]. The main example and the starting point for the theory developped in that paper was the quiver with relation



$$\alpha_1'\alpha_1 = \alpha_2'\alpha_2 ,$$

the determination of its indecomposable representations having been posed before as a problem by Gelfand. Note that this quiver with relation is a specialization of the situation above: we have to shrink all arrows outside the commutative square.

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