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EIGENVALUES OF COXETER TRANSFORMATIONS AND THE GELFAND-KIRILLOV DIMENSION OF THE PREPROJECTIVE ALGEBRAS

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ABSTRACT. The spectral radius of a Coxeter transformation is shown to be an eigenvalue which can be expressed in terms of lengths of certain positive roots of the corresponding valued graph. This result is used to determine the Gelfand-Kirillov dimension of the preprojective algebras: This dimension is equal to 0, 1 or ∞ according to whether the underlying graph is Dynkin, Euclidean or otherwise.

Let $C = (c_{ij})$ be an $n \times n$ Cartan matrix, that is $c_{ii} = 2$ for $1 \leq i \leq n$, $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$, and $c_{ij} \leq 0$ is integral for all $i \neq j$. Throughout the paper, C is assumed to be indecomposable and its graph not to be Dynkin. Recall that the associated (in general, not symmetrizable) valued graph is given by n vertices and edges $i \bullet\!\!-\!\!\bullet j$ with valuation $(-c_{ij}, -c_{ji})$ for $i \neq j$ and $c_{ij} \neq 0$ (and with no loops); see [3].

DEFINITION. The reflections $r_s \in \mathrm{GL}(\mathbf{C}^n)$, $1 \leq s \leq n$, are defined by $r_s\mathbf{x} = \mathbf{y} = (y_i)$, where $y_i = x_i$ for $i \neq s$ and $y_s = -x_s - \sum_i c_{is}x_i$. The element $c = r_n \cdots r_2 r_1$ is called a *Coxeter transformation*.

Note that with a c, also c^{-1} is a Coxeter transformation (corresponding to a Cartan matrix obtained from C by a permutation of rows and columns). Recall that the *spectral radius* of a linear transformation c is the maximal $|\lambda|$ of all eigenvalues λ of c. The *degree* $d(\lambda)$ of an eigenvalue λ of c is the maximal size of the Jordan λ-blocks of c.

PROPOSITION 1. *The spectral radius ρ of c is an eigenvalue of c and $d(\rho) \geq d(\lambda)$ for all eigenvalues λ of c with $|\lambda| = \rho$.*

The proof of Proposition 1 follows from Theorem 3.1 of J. S. Vandergraft [8], as soon as we construct a (closed) cone with nonempty interior in $\mathbf{R}^n$ which is mapped by c into itself. In fact, we shall consider c^{-1} instead of c. The authors are indebted to L. Elsner for helpful discussions concerning eigenvalues of linear transformations.

REMARK. Applying Proposition 1 to c^{-1}, we conclude that there is an eigenvalue $1/\rho'$ of c with the least absolute value and $d(1/\rho') \geq d(\lambda)$ for all eigenvalues λ of c

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with $|\lambda| = 1/\rho'$. If the Cartan matrix is symmetrizable, then $\rho' = \rho$. For, then a quadratic form is defined which is c-invariant. For nonsymmetric Cartan matrices, in general $\rho' \neq \rho$. If C defines a Euclidean graph, then $\rho' = \rho = 1$ [3] and therefore all eigenvalues lie on the unit circle. This is, of course, also trivially true when C defines a Dynkin graph (for, c is of finite order). In the case that C defines a tree, sharper results on the eigenvalues of the corresponding Coxeter transformations were obtained by N. A' Campo [1] and V. F. Subbotin and R. B. Stekolščik [7]; however, they do not generalize to the general situation of nonsymmetrizable matrices.

Define the *preprojective vectors* (or "preprojective roots") $\mathbf{p}(s, t)$, $1 \leq s \leq n$, $t \geq 0$, of C by

$$\mathbf{p}(s, t) = c^{-t} r_1 r_2 \cdots r_{s-1} \mathbf{s},$$

where $\mathbf{s} = (\delta_{is})_i$. For $\mathbf{x} = (x_i) \in \mathbf{C}^n$, we write $|\mathbf{x}| = \sum_i |x_i|$.

The fact that the graph defined by C is not Dynkin is equivalent to the statement that all $\mathbf{p}(s, t)$ are distinct. This, in turn, is equivalent to the statement that all $\mathbf{p}(s, t)$ are positive; see [3].

Now, define the (preprojective) closed cone K as the closed convex hull of all $\mathbf{p}(s, t)$. Clearly, $c^{-1}K \subseteq K$. Moreover, since the projective vectors $\{\mathbf{p}(s, 0) \mid 1 \leq s \leq n\}$ form a basis of $\mathbf{C}^n$, the interior K^0 of K is nonempty.

PROPOSITION 2. *For any* $\mathbf{p} = \mathbf{p}(s, t)$, *we have*

$$\rho = \lim_{m \to \infty} \sqrt[m]{|c^{-m}\mathbf{p}|}\,.$$

REMARK. The limit

$$\lim_{m \to \infty} \sqrt[m]{|c^{-m}\mathbf{x}|}$$

clearly measures the growth of the vector $\mathbf{x}$ with respect to c and can be called the "*growth number*" of $\mathbf{x}$; or, if $\mathbf{x} = \mathbf{dim}\, X$ for some module X, the growth number of X. (In general, one may consider $\mathbf{dim}(\mathrm{Tr}\, D)^m X$ instead of $c^{-m}\, \mathbf{dim}\, X$ and call

$$\overline{\lim} \sqrt[m]{|\mathbf{dim}(\mathrm{Tr}\, D)^m X|}$$

the growth number of X [7].)

COROLLARY. *All preprojective vectors have equal growth number. If the graph defined by* C *is Euclidean, this number is equal to* 1. *If the graph is neither Dynkin nor Euclidean, it is greater than* 1.

PROOF OF PROPOSITION 2. Let $\mathbf{e}_{ij}$, $1 \leq i \leq q$, $1 \leq j \leq n_i$, be a Jordan basis of $\mathbf{C}^n$ with respect to which c^{-1} is the product of q Jordan $n_i \times n_i$ matrices of the form

$$\begin{bmatrix} \lambda_i & \lambda_i & & & \\ & \lambda_i & \lambda_i & & 0 \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & 0 & & & \lambda_i & \lambda_i \\ & & & & & \lambda_i \end{bmatrix};$$

moreover, in view of Proposition 1, we assume that $\rho = \lambda_1 > |\lambda_i|$ for all eigenvalues λ_i, $1 \leq i \leq q$. Thus

$$c^{-1}\mathbf{e}_{i,j} = \lambda_i(\mathbf{e}_{i,j} + \mathbf{e}_{i,j-1});$$

here $\mathbf{e}_{i,r} = \mathbf{0}$ for $r < 0$. Let $\mathbf{p} = \sum_{i,j} \alpha_{i,j}\mathbf{e}_{i,j}$; consequently,

$$c^{-m}\mathbf{p} = \sum_{i,j} \alpha_{i,j}\lambda_i^m\left[\mathbf{e}_{i,j} + \binom{m}{1}\mathbf{e}_{i,j-1} + \cdots + \binom{m}{s}\mathbf{e}_{i,j-s} + \cdots\right].$$

Now, take $d = \max d(\lambda_i)$ and $s_m = \binom{m}{d}$; thus, for large m, $1/s_m \cdot \binom{m}{s} \leq 1$ for all $\mathbf{e}_{i,j-s} \neq \mathbf{0}$. Consequently, since

$$\lim_{m\to\infty} \sqrt[m]{s_m} = 1,$$

$$\lim_{m\to\infty} \sqrt[m]{|c^{-m}\mathbf{p}|} = \lambda_1 \lim_{m\to\infty} \sqrt[m]{|c^{-m}\mathbf{p}|/s_m\lambda_1^m}$$

$$= \lambda_1 \lim_{m\to\infty} \sqrt[m]{\sum_{i,j} |\alpha_{i,j}||\lambda_i/\lambda_1|^m \quad d} = \lambda_1 = \rho.$$

As an application of the previous results, we are going to determine the Gelfand-Kirillov dimension of the preprojective algebra $\Pi(\mathfrak{M})$ of a modulated graph $\mathfrak{M}$ (see [4]).

THEOREM. *The Gelfand-Kirillov dimension* Dim $\Pi(\mathfrak{M})$ *of the preprojective algebra* $\Pi(\mathfrak{M})$ *of a modulated graph* $\mathfrak{M}$ *is equal to* 0 *if the graph defined by* $\mathfrak{M}$ *is Dynkin,* 1 *if it is Euclidean and* ∞ *otherwise.*

REMARK. Thus, choosing a particular admissible orientation Ω of $\mathfrak{M}$, Dim $\Pi(\mathfrak{M}) = 0$ if the respective tensor algebra $R(\mathfrak{M}, \Omega)$ is of finite representation type, Dim $\Pi(\mathfrak{M}) = 1$, if $R(\mathfrak{M}, \Omega)$ is of tame representation type and Dim $\Pi(\mathfrak{M}) = \infty$ otherwise (that is if $R(\mathfrak{M}, \Omega)$ is of wild representation type).

PROOF OF THEOREM. For the calculation of the Gelfand-Kirillov dimension of $\Pi = \Pi(\mathfrak{M})$, we need a finite-dimensional k-subspace V of Π which generates Π. Choose an admissible orientation Ω, and let $R = R(\mathfrak{M}, \Omega)$. Then R is a finite-dimensional hereditary k-algebra, and Π_R decomposes into the direct sum of all preprojective right R-modules, each occurring with multiplicity one:

$$\Pi_R = \bigoplus_{t>0} \bigoplus_{1\leq s\leq n} P(s, t);$$

see [4]. Let

$$V^m = \bigoplus_{0<t\leq m} \bigoplus_{1\leq s\leq n} P(s, t);$$

then V^0 can be identified with R_R and we put $V = V^1$. It follows from [4] that

$$V^{m+1} = V^m \cdot V^1 \quad \text{for all } m.$$

[Namely, we can identify $P(s, t)$ with

$$\bigoplus_{1\leq r\leq n} \operatorname{Hom}(P(r, 0), P(s, t)),$$

and in this way, we see that the product $P(s, t)\,P(s', t')$ is contained in $P(s', t + t')$ and that any element of $P(s', t + 1)$ can be written as a sum of elements from the various products $P(s, t)\,P(s', 1)$ with $1 \leq s \leq n$.]

Let $f = \max_{1 \le s \le n} \dim_k F_s$; here, F_s are the division rings given in the modulation $\mathfrak{M}$. Thus, f is the maximal k-dimension of a simple R-module, and therefore

$$|\mathbf{dim}\, M| \le \dim_k M \le f|\mathbf{dim}\, M|$$

for any R-module M. This shows that we may consider

$$\overline{\lim_{m\to\infty}} \frac{\log|\mathbf{dim}\, V^m|}{\log m}$$

instead of

$$\overline{\lim_{m\to\infty}} \frac{\log \dim_k V^m}{\log m}$$

in the definition of the Gelfand-Kirillov dimension. Now,

$$|\mathbf{dim}\, V^m| = \sum_{t=0}^{m} \sum_{s=0}^{n} |\mathbf{dim}\, P(s,t)| = \sum_{t=0}^{m} \sum_{s=0}^{n} |\mathbf{p}(s,t)|$$
$$\ge |\mathbf{p}(s,m)| = |c^{-m}\mathbf{p}(s,0)| \quad \text{for any } 1 \le s \le n.$$

Now,

$$\lim_{m\to\infty} \frac{\log|\mathbf{p}(s,m)|}{\log m} = \lim_{m\to\infty} \frac{m}{\log m} \cdot \frac{\log|\mathbf{p}(s,m)|}{m} = \infty,$$

since

$$\lim_{m\to\infty} \frac{\log|\mathbf{p}(s,m)|}{m} = \log \rho$$

by Proposition 2. Hence, if the graph defined by $\mathfrak{M}$ is neither Dynkin nor Euclidean, we see that Dim $\Pi = \infty$.

In the case that the graph is Euclidean let $\mathbf{h}$ be the standard homogeneous vector; thus $\mathbf{h}$ is the minimal positive integral vector stable under c. Furthermore, there is a natural number u such that

$$c^u\mathbf{x} = \mathbf{x} + \partial(\mathbf{x})\mathbf{h}$$

for some linear functional ∂ (called the defect); see [3]. In this case, we use the subspace $W = V^{u-1}$ in place of V; thus, $W^m = V^{m(u-1)}$. Let $d = \partial(\mathbf{dim}\, W)$. Then

$$\mathbf{dim}\, W^m = \mathbf{dim}\, W + (m-1)\, d\mathbf{h},$$

and therefore

$$\text{Dim } \Pi = \lim_{m\to\infty} \frac{\log|\mathbf{dim}\, W^m|}{\log m} = 1,$$

because

$$\lim_{m\to\infty} \frac{\log(m-1)\, d|\mathbf{h}|}{\log m} = 1.$$

If the graph of $\mathfrak{M}$ is Dynkin, trivially Dim $\Pi = 0$.

REMARK 1. In their paper, W. Borho and H. Kraft [2] define the superdimension DIM A of an algebra A as

$$\overline{\lim_{m\to\infty}} \frac{\log\log \dim_k V^m}{\log m}.$$

It is easy to see that DIM $\Pi(\mathfrak{R}) = 0$ if the graph defined by $\mathfrak{R}$ is Dynkin or Euclidean, and that DIM $\Pi(\mathfrak{R}) = 1$ otherwise.

REMARK 2. We also note the analogy to a theorem of V. Kac [9] which asserts that the Lie algebra associated to a Cartan matrix C is of finite growth if and only if C is Dynkin or Euclidean. However, whereas the theorem of Kac measures really the increase of the dimensions of the root spaces for the imaginary roots (all others have dimension 1), the theorem above measures the increase of the components of certain real roots with respect to a fixed basis of the root system.

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