Kawada's theorem

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Kawadas's theorem solved the Köthe problem for basic finite-dimensional algebras: It characterizes completely those finite-dimensional algebras for which any indecomposable module has squarefree socle and squarefree top, and describes the possible indecomposable modules. This seems to be the most elaborate result of the classical representation theory (prior to the introduction of the new combinatorical and homological tools: quivers, partially ordered sets, vectorspace categories, Auslander-Reiten sequences). However, apparently his work was not appreciated at that time.

These are the revised notes of parts of a series of lectures given at the meeting on abelian groups and modules in Trento (Italy), 1980. They are centered around the second part of Kawada's theorem: the shapes of the indecomposable modules over a Kawada algebra.

1. Köthe algebras and algebras of finite representation type

Recall the following important property of abelian groups, thus of $\mathbb{Z}$-modules: every finitely generated module is a direct sum of cyclic modules. Köthe showed that the only commutative finite-dimensional algebras which have this property are the uniserial ones, and he posed the question to classify also the non-commutative finite-dimensional algebras with this property [11]. An algebra for which any finitely generated left or right module is a direct sum of cyclic modules, is now called a Köthe-algebra, and a classification of these algebras seems to be rather difficult. In fact, for a solution one would need a classification of all algebras of finite representation type, as well as some further insight into the structure of the modules over a given algebra of finite representation type.

(1.1) Notation. Let k be a (commutative) field, and A a finite-dimensional k-algebra (associative, with 1). We want to investigate the representations of A, thus we consider A-modules (usually, we will deal with finite-dimensional left A-modules and call them just modules). Always, homomorphisms will be written on the opposite side as scalars, thus the composition of $f : {}_AX \to {}_AY$ and $g : {}_AY \to {}_AZ$ will be denoted by fg . Given any module M , we denote by $\mathrm{rad}M$ the radical of M, it is the intersection of all maximal submodules, and call $M/\mathrm{rad}M =: \mathrm{top}M$ the top of M . If $\mathrm{rad}M = 0$, then M is called semisimple. Also, let $\mathrm{soc}M$ be the socle of M , it is the sum of all simple submodules of M . Any finite-dimensional A-module M has a composition series

$$0 = M_0 \subset M_1 \subset \dots \subset M_\ell = M \quad ,$$

with M_i/M_{i-1} simple, for all $1 \le i \le \ell$. The M_i/M_{i-1} are called composition factors, and the number ℓ is called the length of M, denoted by $|M|$. (The module $grM := \bigoplus_{i=1}^{\ell} M_i/M_{i-1}$ will be called the graded module corresponding to M; we will need this construction later.) We choose a fixed ordering $S(1),\dots,S(n)$ of the simple A-modules, and denote by $(\underline{\dim}M)_i$ the number of composition factors of M isomorphic to $S(i)$, this number is independent of the given composition series (theorem of Jordan-Hölder). In this way, we obtain an n-tupel $\underline{\dim}M$, called the dimension type of M. If M is semisimple and $(\underline{\dim}M)_i \leq 1$ for all i, then M is called squarefree. A module is semisimple and squarefree if and only if it is the direct sum of pairwise non-isomorphic simple modules. Again, assuming M to be finite-dimensional, then we can write M as a direct sum $M = \bigoplus_{i=1}^{m} M_i$ of indecomposable modules, and such a decomposition is unique up to isomorphism (theorem of Krull-Schmidt). In order to know all finite-dimensional modules, we therefore may restrict to the indecomposable ones. Note that a finite-dimensional module M is indecomposable if and only if its endomorphism ring $End(M)$ is local. In particular, we always have the indecomposable direct summands of the left module ${}_AA$, we denote representatives of their isomorphism classes by $P(1),\dots,P(n)$, where $topP(i) = S(i)$, for $1 \leq i \leq n$. Thus, ${}_AA = \bigoplus_{i=1}^{n} P(i)^{p(i)}$ for some $p(i) \in \mathbb{N}$ (here, M^m denotes the direct sum of m copies of M), Note that we can calculate $\underline{\dim}M$ for any module M as follows:

$$(\underline{\dim}M)_i = |{}_{EndP(i)}Hom_A(P(i),M)| \quad .$$

The projective modules are the direct sums of various $P(i)$, they are the modules with the usual lifting property. For any module M, there exists (uniquely up to isomorphism) an epimorphism $\varphi : P \to M$ with P projective and with kernel contained in $radP$, it is called the projective cover. If $\varphi: P \to M$ is a projective cover, then $topP \approx topM$. The (left) A-module ${}_AM$ is called cyclic provided it is an epimorphic image of ${}_AA$. Note that for a cyclic module M, we have $|M| \leq |{}_AA|$.

(1.2) The module M is cyclic if and only if $(\underline{\dim}\ topM)_i \leq p(i)$ for all i. Namely, let $P \to M$ be a projective cover of M. Then $P = \bigoplus_{i=1}^{n} P(i)^{m(i)}$, where $m(i) = (\underline{\dim}\ topM)_i$, since $topM = topP$. Now if $m(i) \leq p(i)$ for all i, then P is a direct summand of ${}_AA$, thus M is an epimorphic image of ${}_AA$. Conversely, if there exists an epimorphism ${}_AA \to M$, then P is isomorphic to a direct summand of ${}_AA$, thus $m(i) \leq p(i)$ for all i.

(1.3) The algebra A is said to be of finite representation type provided there are only finitely many indecomposable A-modules. (In this case, even the infinite-dimensional modules are direct sums of those finite-dimensional modules [17]). For example, the algebra $A = k[T]/(T^n)$, with $k[T]$ being the polynomial ring in one variable T and (T^n) the ideal generated by T^n, for some n, is of finite representation type: the only indecomposable modules being the modules $k[T]/(T^i)$, where $1 \leq i \leq n$. On the other hand, the three-dimensional algebra $k[T_1,T_2]/(T_1^2,T_1T_2,T_2^2)$ is not of finite representation type. There is a general theorem due to Rojter [18] which asserts that a finite-dimensional algebra with a bound on the length of the indecomposable modules, is necessarily of finite representation type. In particular, any Köthe algebra A has to be of finite representation type (here, $|_A A|$ is a bound for the length of the indecomposable A-modules).

(1.5) Conversely, one may ask when an algebra A of finite representation type actually is a Köthe algebra. Let $M_1,\dots,M_m$ be the indecomposable left A-modules. As we have seen above, M_j is cyclic if and only if $(\underline{\dim}\ \mathrm{top} M_j)_i \le p(i)$, for all i. For any (left) module M, let $M^* = \mathrm{Hom}_k(M,k)$ be its dual module, it is a right A-module. Note that $M_1^*,\dots,M_m^*$ are the indecomposable right modules, and it follows that M_j^* is cyclic if and only if $(\underline{\dim}\ \mathrm{soc} M_j)_i \le p(i)$, for all i. Let $q_A(i)$ be the maximum of all $(\underline{\dim}\ \mathrm{top} M_j)_i$ and all $(\underline{\dim}\ \mathrm{soc} M_j)_i$, where $1 \le j \le m$. Then, A is a Köthe algebra if and only if $q_A(i) \le p(i)$, for all i. If we replace A by a Morita equivalent algebra A', then there is a canonical bijection between the A-modules and the A'-modules. In particular, we may index the simple A-modules and the simple A'-modules in the same way. With A, also A' is of finite representation type, and $q_A(i) = q_{A'}(i)$. However, the numbers $p(i) = p_A(i)$ can be changed arbitrarily, by choosen an appropriate Morita equivalent algebra. For example, for the ring M(d,A) of all $d \times d$-matrices over A, we have $p_{M(d,A)}(i) = dp_A(i)$, for all i. As a consequence, we see: Any algebra of finite representation type is Morita equivalent to a Köthe algebra.

(1.6) If $p_A(i) = 1$ for all i, then A is called a basic algebra. For any algebra A, there exists (uniquely up to isomorphism) a basic algebra A_o which is Morita equivalent to A. The following conditions now obviously are equivalent for an algebra A:

(i) A_o is a Köthe algebra.

(ii) Any algebra Morita equivalent to A is a Köthe algebra.

(iii) Any indecomposable A-module has squarefree top and squarefree socle.

An algebra A satisfying these conditions will be called a Kawada algebra.

2. The work of Kawada

These algebras which we now call Kawada algebras, were thoroughly inyestigated by Y. Kawada around 1960. He both gave a characterization of these algebras in terms of their indecomposable projective modules, as well as a full classification of the possible indecomposable modules.

(2.1) In 1960, Kawada reported his results at a meeting of the Mathematical Society of Japan, and a survey appeared in 1961 in two parts [1] : "The purpose of this paper is to announce that Köthe's problem mentioned above is completely solved for the case of self-basic algebras." This survey contains a set of 19 conditions which characterize Kawada algebras, as well as the list of the possible indecomposable modules.

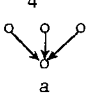
One may formulate these two results separately, as Kawada did it in his survey. His proof however derives both results at the same time. This proof is published in a series of three papers [1] amounting altogether to 255 pages, and devoted just to this one theorem.

(2.2) The 19 conditions. These conditions are formulated in terms of the indecomposable projective A-modules and their submodules and factor modules. Let us give some examples: Condition VI has the shortest formulation (we use the notation introduced in 1): For any primitive idempotent e , the A-module Ae(radA)e is serial. Some of the conditions are, however, rather clumsy. We quote condition X:

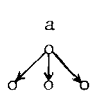
> **X.** Assume that $Ae_{\lambda_1}g_1$ is a module such that $Ne_{\lambda_1}g_1 = Ae_\varepsilon te_{\lambda_1}g_1 + Ae_\rho we_{\lambda_1}g_1$ where $Ae_\varepsilon te_{\lambda_1}g_1$ is uni-serial, $Ae_\varepsilon te_{\lambda_1}g_1 \frown Ae_\rho we_{\lambda_1}g_1 = N^m e_\varepsilon te_{\lambda_1}g_1 = Ae_\sigma uwe_{\lambda_1}g_1 \neq 0 (m \geqq 1)$, $Ne_\rho we_{\lambda_1}g_1 = Ae_\sigma uwe_{\lambda_1}g_1 \oplus Ae_\tau vwe_{\lambda_1}g_1$ where $Ae_\tau vwe_{\lambda_1}g_1$ is uni-serial, and $S(Ae_{\lambda_1}g_1) = Ae_\sigma uwe_{\lambda_1}g_1 \oplus N^k e_\tau vwe_{\lambda_1}g_1 (k \geqq 0)$. Assume that $Ae_{\lambda_2}g_2$ is a non-simple module whose socle is isomorphic to $N^k e_\tau vwe_{\lambda_1}g_1$. Let φ be an isomorphism which maps $S(Ae_{\lambda_2}g_2)$ onto $N^k e_\tau vwe_{\lambda_1}g_1 + Ae_\varepsilon te_{\lambda_1}g_1 / Ae_\varepsilon te_{\lambda_1}g_1$ considered as a submodule of $Ae_{\lambda_1}g_1 / Ae_\varepsilon te_{\lambda_1}g_1$. Then φ is extendable; more precisely, either φ is extendable to a monomorphism $\Phi_1 : Ae_{\lambda_2}g_2 \to Ae_{\lambda_1}g_1 / Ae_\varepsilon te_{\lambda_1}g_1$, or φ^{-1} is extendable to a monomorphism $\Phi_2 : Ae_{\lambda_1}g_1 / Ae_\varepsilon te_{\lambda_1}g_1 \to Ae_{\lambda_2}g_2$.

(Here, the elements e_* are primitive idempotents, N = radA, and S(M) denotes the socle of M.)

Of course, one may reformulate these conditions in terms of the quiver with relations which defines A , at least in case the base field is algebraically closed. Then the conditions are more easy to visualize. For example, it is clear that any vertex a can have at most 4 neighbors, with at most two arrows having a as endpoint, and at most two arrows having a as starting point. Namely, otherwise, we obtain a subquiver of type D_4 with one of the orientations



or



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in the first case, we obtain an indecomposable module with socle $S(a)^2$, namely with dimension type $\begin{smallmatrix}1 & 1 & 1\\ & 2 & \end{smallmatrix}$, in the second case, we obtain an indecomposable module with top $S(a)^2$. Also, we see that we have to expect a rather long list of conditions. For example, we have to exclude subquivers of the form E_6 (with no relation) with all possible orientations. This is easy to formulate if one can use a diagrammatic language, however, it amounts to a large number of awkward conditions in terms of idempotents and serial modules.

(2.3) The possible indecomposable modules. The second part of Kawada's theorem describes completely the shape of the indecomposable modules over a Kawada algebra. Kawada first devides the indecomposable projective modules into 5 different types and then lists 38 possibilities of forming indecomposable modules as amalgamations of indecomposable projective modules. We want to present this list in a slightly different form. In order to do this, we first introduce the notion of the shape of a module.

3. The shape of a module

In order to define the shape of a module, we have to develop some of the machinery presently available in representation theory. Given a finite-dimensional algebra A, we will make use of its Auslander-Reiten-species $\Gamma(A)$, and the universal covering $\tilde{\Gamma}(A)$ of $\Gamma(A)$, as defined by Gabriel and Riedtmann. Since for an algebra A of finite representation type, $\tilde{\Gamma}(A)$ is the Auslander-Reiten-species of some "locally finite-dimensional algebra", we always have to take into account certain infinite-dimensional algebras (such an algebra will not contain a unit element).

(3.1) Locally finite-dimensional algebras. The k-algebra C is said to be locally finite-dimensional provided there exists a set $\{e_i \mid i \in I\}$ of orthogonal idempotents e_i of C such that $C = \bigoplus_{i,j\in I} e_i C e_j$, with Ce_i and e_iC finite-dimensional for every $i \in I$. For a C-module ${}_CM$, we require $CM = M$, or, equivalently, $M = \bigoplus_{i\in I} e_iM$. All modules considered will be assumed to be finite-dimensional over k. Note that we may and will assume that the idempotents e_i all are primitive, so that the left modules Ce_i are indecomposable. In case Ce_i and Ce_j are isomorphic as left C-modules only for $i = j$, we call C basic. As for finite-dimensional algebras, two locally finite-dimensional algebras will be called Morita equivalent in case their module categories are equivalent. And, given C, locally finite-dimensional, there exists a basic locally finite-dimensional algebra C_o which is Morita equivalent to C.

Assume now that C is locally finite-dimensional. For any module M, we define its support algebra $C(M)$ as the factor algebra of C modulo the ideal $\langle e \mid eM = 0,\ e^2 = e \rangle$ generated by all idempotents e with $eM = 0$. With M also $C(M)$ is finite-dimensional over k. Note that M is a sincere $C(M)$-module (i. e. no idempotent $\neq 0$ annihilates M). The $C(M)$-modules will be considered as C-modules, and the set of $C(M)$-modules is closed under submodules, factor modules and extensions.

Clearly, the modules $P(i) = Ce_i$ are indecomposable and projective, and the modules $I(i) = (e_iC)^*$ indecomposable and injective. There is a categorial equivalence ν between the category of (finite-dimensional) projective modules and the category of (finite-dimensional) injective modules, with $\nu P(i) = I(i)$, the Nakayama functor (see [5]). Namely, maps $P(i) = Ce_i \to Ce_j = P(j)$ are given by right multiplication with elements from C of the form e_ice_j. But left multiplication by e_ice_j also gives a map $e_jC \to e_iC$, thus let $\nu(\cdot e_ice_j) = (e_ice_j\cdot)^*$. For any C-module M, there exists a projective cover $P(M) \twoheadrightarrow M$, and an injective envelop $M \hookrightarrow I(M)$.

We always have $\nu P = I(\mathrm{top}P)$ for P projective.

Let us now describe the Auslander-Reiten translation: Let M be a C-module, and

$$P_1 \xrightarrow{p} P_o \longrightarrow M \longrightarrow 0$$

the first terms of a minimal projective resolution of M . Then, by definition, $\tau M = \mathrm{Ker}\,\nu(p)$, thus we have the exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu(p)} \nu P_o \quad .$$

Let $X = P_o \oplus P_1 \oplus \nu P_1 \oplus \nu P_o$. Then all modules involved in the construction of $\tau M = \tau_C M$ are in fact C(X)-modules, and we see that we have $\tau_{C(X)}M = \tau M$. Now assume that M is indecomposable and not projective. For the finite-dimensional algebra $C(X)$, we have an Auslander-Reiten sequence

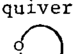
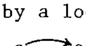
$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$$

and we claim that this sequence has the usual lifting properties with respect to all C-modules, not only the C(X)-modules. Namely , given a C-module Y , then we also may consider $C(X \oplus Y)$. Since $\tau_{C(X \oplus Y)}M = \tau M$, the Auslander-Reiten sequence ending with M , in the category of $C(X \oplus Y)$-modules, must be the given sequence (since it is characterized as being a socle element of ${}_{\mathrm{End}(M)}\mathrm{Ext}^1(M,\tau M)$. Thus, we may call the given sequence an <u>Auslander-Reiten sequence</u> in the category of C-modules.

Finally, we also will need the <u>dimension type</u> of a C-module. Let $\{P(i) \mid i \in I_o\}$ be a complete set of pairwise non-isomorphic indecomposable projective modules. The dimension type $\underline{\dim} M$ of the C-module M will be an I_o-tuple, with

$$(\underline{\dim} M)_i = |{}_{\mathrm{End}(P(i))}\mathrm{Hom}_C(P(i),M)| \quad ,$$

or, equivalently, the number of composition factors of M of the form $\mathrm{top}\,P(i)$.

(3.2) <u>Translation species</u>. We first need the notion of a translation quiver. A quiver (Γ_o,Γ_1) is given by a set Γ_o of "vertices" and a set Γ_1 of "arrows", to any arrow being assigned its starting point and its end point in Γ_o . A translation quiver (Γ_o,Γ_1,τ) is given by a locally finite quiver (Γ_o,Γ_1) without loops  or multiple arrows  and with an injective function $\tau : \Gamma'_o \longrightarrow \Gamma_o$ where Γ'_o is a subset of Γ_o , such that for any $y \in \Gamma'_o$, we have $y^- = (\tau y)^+$. Here, y^- denotes the set of starting points of arrows with end point y , and y^+ the set of end points of arrows with starting point y . We denote by Γ'_1 the set of arrows $\alpha : x \to y$ with $y \in \Gamma'_o$. For every arrow $\alpha : x \to y$ in Γ'_1 , there is a unique arrow $\tau y \to x$, and it will be denoted by $\sigma\alpha$. A translation species $(\Gamma_o,\Gamma_1,F,N,\tau,\chi)$ is given by the following data: (Γ_o,Γ_1,τ) is a translation quiver. For every $y \in \Gamma_o$, there is given a division ring $F(y)$, and for every arrow $x \xrightarrow{\alpha} y$ in Γ_1 , there is given an $F(x)$-$F(y)$-bimodule $N(\alpha) = N(x,y)$, finite-dimensional on either side. Next, for every $y \in \Gamma'_o$, there also is given an isomorphism $\tau_y : F(y) \to F(\tau y)$. Finally, let $\alpha : x \to y$ be in Γ_1' , we may consider $N(\sigma\alpha)$ as an $F(y)$-$F(x)$-bimodule using the isomorphism τ_y . There is given a non-degenerate bilinear form

$$\chi_\alpha = \chi_{xy} : {}_{F(x)}N(\alpha)_{F(y)} \otimes {}_{F(y)}N(\sigma\alpha)_{F(x)} \to {}_{F(x)}F(x)_{F(x)} \quad .$$

Note that $N(\sigma\alpha)$ is isomorphic as an $F(y)$-$F(x)$-bimodule to the left-dual $\mathrm{Hom}_{F(x)}(N(\sigma\alpha),F(x))$ of $N(\alpha)$, using the bilinear form χ_α . [However, in general it may not be possible to identify the division rings in an α-orbit in such a way that all maps τ_y are identity maps, since the τ-orbit may be closed. Similarly, it may not be possible to identify $N(\sigma\alpha)$ with $\mathrm{Hom}_{F(x)}(N(\alpha),F(x))$ for all α , so that the bilinear forms χ_α are the evaluation maps.] Also, the bilinear form χ_{xy} determines a unique element c_{xy} in $N(\tau y,x) \underset{F(x)}{\otimes} N(x,y)$ as follows: let $n_1,\ldots,n_t$ be a basis of ${}_{F(x)}N(x,y)$ and $\varphi_1,\ldots,\varphi_t$ the dual basis with respect to χ_{xy} , then $c_{xy} = \Sigma\, \varphi_i \otimes n_i$. This element c_{xy} is called the canonical element [2] .

Given a translation species $\Gamma = (\Gamma_o,\Gamma_1,F,N,\tau,\chi)$, we can construct the tensor category $\otimes\Gamma$ over Γ . It has Γ_o as set of objects. Given a pair $x,y \in \Gamma_o$, let $W(x,y)$ be the set of (oriented) paths from x to y in (Γ_o,Γ_1) , and if $w = (\underset{x}{o} \xrightarrow{\alpha_1} o \xrightarrow{\alpha_2} o \ldots o \xrightarrow{\alpha_r} \underset{y}{o})$ is a path in $W(x,y)$, let $N(w) = N(\alpha_1) \otimes N(\alpha_2) \otimes \ldots \otimes N(\alpha_r)$, the tensor products being taken with respect to the various $F(*)$; note that for the constant path w_x at the point x we have $N(w_x) = F(x)$. Now, for $x,y \in \Gamma_o$ define as set of homomorphisms from x to y the set $\bigoplus_{w\in W(x,y)} N(w)$, the composition being given by the tensor product. The mesh category $\diamond\Gamma$ is defined as the factor category of $\otimes\Gamma$ modulo the ideal generated by the elements $\sum_{x\in y^-} c_{xy}$ with $y \in \Gamma_o'$.

(3.3) The universal covering of translation species. Let (Γ_o,Γ_1,τ) be a translation quiver. A covering of (Γ_o,Γ_1,τ) is given by a translation quiver (Δ_o,Δ_1,τ) and a quiver map $\pi : (\Delta_o,\Delta_1) \to (\Gamma_o,\Gamma_1)$ which is compatible with τ , with $\pi^{-1}(\Gamma_o') = \Delta_o'$ and such that the induced maps $x^+ \to (\pi x)^+$ and $x^- \to (\pi x)^-$ are bijective, for any $x \in \Delta_o$, see [6] . Now assume a translation species $(\Gamma_o,\Gamma_1,F,N,\tau,\chi)$ is given, and (Δ_o,Δ_1,τ) is a covering of (Γ_o,Γ_1,τ) with covering map π . Then we can construct a translation species $(\Delta_o,\Delta_1,F,N,\tau,\chi)$ with underlying translation quiver (Γ_o,Γ_1,τ) as follows: For $y \in \Delta_o$, let $F(y) = F(\pi y)$; for $\alpha \in \Delta_1$, let $N(\alpha) = N(\pi\alpha)$; for $y \in \Delta_o'$, let $\tau_y = \tau_{\pi y} : F(y) = F(\pi y) \to F(\tau\pi y) = F(\pi\tau y) = F(\tau y)$, and for α in Δ_1' , let $\chi_\alpha = \chi_{\pi\alpha}$. Thus, all the data of a translation species are lifted back via π . We call $(\Delta_o,\Delta_1,F,N,\tau,\chi)$ a covering of $(\Gamma_o,\Gamma_1,F,N,\tau,\chi)$.

Any translation quiver (Γ_o,Γ_1,τ) has a universal covering, as Gabriel and Riedtmann (see [6]) have shown; it will be denoted by $(\widetilde{\Gamma}_o,\widetilde{\Gamma}_1,\tau)$. Given a translation species

$\Gamma = (\Gamma_o,\Gamma_1,F,N,\tau,\chi)$, we therefore can consider $\tilde{\Gamma} = (\tilde{\Gamma}_o,\tilde{\Gamma}_1,F,N,\tau,\chi)$ and call it the universal covering of Γ . Note that there exists a group G of automorphisms of $(\tilde{\Gamma}_o,\tilde{\Gamma}_1,\tau)$ such that $(\Gamma_o,\Gamma_1,\tau) = (\tilde{\Gamma}_o,\tilde{\Gamma}_1,\tau)/G$. By construction, G is also a group of automorphisms of the translation species $\tilde{\Gamma}$, and clearly $\Gamma = \tilde{\Gamma}/G$.

Note that for the universal covering $(\tilde{\Gamma}_o,\tilde{\Gamma}_1,\tau)$ of a translation quiver, neither any τ-orbit of $\tilde{\Gamma}_o$, nor any σ-orbit of $\tilde{\Gamma}_1$ is closed. Therefore, for the universal covering $(\tilde{\Gamma}_o,\tilde{\Gamma}_1,F,N,\tau,\chi)$, we can assume that all maps τ_y for $y \in \tilde{\Gamma}'_o$ are identity maps (we choose one representative y in any τ-orbit of $\tilde{\Gamma}_o$, fix $F(y)$, and identify the division rings corresponding to the remaining vertices in this τ-orbit with $F(y)$, using the maps τ_*). Similarly, in any σ-orbit of $\tilde{\Gamma}_1$, we select one arrow α , fix the corresponding bimodule $N(\alpha)$, and replace the bimodules corresponding to the remaining arrows in this σ-orbit by suitable dualized forms of $N(\alpha)$.

(3.4) Auslander-Reiten-species. Let C be a locally finite-dimensional algebra. We denote by $\Gamma(C) = (\Gamma_o,\Gamma_1,F,N,\tau,\chi)$ its Auslander-Reiten-species, which is defined as follows: Γ_o is a fixed set of representatives of the isomorphism classes of indecomposable C-modules. For any $X \in \Gamma_o$, let $F(X) = \mathrm{End}(X)/\mathrm{rad}\,\mathrm{End}(X)$, the residue division ring of $\mathrm{End}(X)$, and τX its Auslander-Reiten translate. For $X,Y \in \Gamma_o$, let $N(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ the bimodule of irreducible maps [15], it is an $F(X)$-$F(Y)$-bimodule; given $X,Y \in \Gamma_o$, there is an arrow $X \to Y$ in Γ_1 if and only if $N(X,Y) \neq 0$; also, Γ'_o is the set of non-projective modules in Γ_o , and τ is the Auslander-Reiten translation.

In order to be able to define the isomorphisms τ_Y and the bilinear forms χ_{XY} , we have to consider Auslander-Reiten sequences. Let $Y \in \Gamma'_o$. Since Y is indecomposable and non-projective, there exists an Auslander-Reiten sequence

$$0 \longrightarrow \tau Y \xrightarrow{(f_{X,i})} \bigoplus_{X\in Y^-} X^{d(X)} \xrightarrow{(g_{X,i})} Y \longrightarrow 0 \quad ,$$

here, both $(f_{X,i})$ and $(g_{X,i})$ are indexed by $X \in Y^-$ and $1 \leq i \leq d(X)$. We fix some $X \in Y^-$ and write $f_i = f_{X,i}$, $g_i = g_{X,i}$. Then $(f_i)_i$ gives modulo $\mathrm{rad}^2(\tau Y,X)$ a basis of $N(\tau Y,X)_{F(X)}$, and $(g_i)_i$ gives modulo $\mathrm{rad}^2(X,Y)$ a basis of ${}_{F(X)}N(X,Y)$, see [15]. Any automorphism of Y lifts to an automorphism of the middle term $\bigoplus_{X\in Y^-} X^{d(X)}$, and therefore induces an automorphism of τY . In this way, we define an isomorphism $\tau_Y : F(Y) \to F(\tau Y)$. More precisely, given $h \in F(Y)$ and g_i , there are elements $h_{ij} \in F(X)$ satisfying $\sum_j h_{ij} g_j = g_i h$, and then $\tau(h)\cdot f_j = \sum_i f_i h_{ij}$.

Now define for $u_i,v_i \in F(X)$

$$\chi_{XY}\Big(\sum_i u_i g_i \ , \ \sum f_i v_i\Big) = \sum_i u_i v_i \quad ,$$

and note that this factors over the tensor product $N(X,Y) \otimes_{F(Y)} N(\tau Y,X)$. Namely, if

$h \in F(Y)$, then

$$\chi_{XY}(g_i h, f_j) = \chi_{XY}(\sum_r h_{ir} g_r, f_j) = h_{ij}$$

and

$$\chi_{XY}(g_i, \tau(h) f_j) = \chi_{XY}(g_i, \sum_r f_r h_{rj}) = h_{ij} \quad .$$

Thus, χ_{XY} is a bilinear form on ${}_{F(X)}N(X,Y) \otimes_{F(Y)} N(\tau Y,X)_{F(X)}$ with values in $F(X)$.

(3.5) Coverings of an Auslander-Reiten species. Given a finite-dimensional algebra A of finite representation type, we may consider coverings Δ of the Auslander-Reiten species $\Gamma(A)$ of A. Gabriel and Riedtmann [6] have shown (at least in the case when k is algebraically closed) that Δ is again the Auslander-Reiten species of a suitable locally finite-dimensional algebra. In fact, one can construct such an algebra $R(\Delta)$ as follows: First, consider the mesh category $C = \diamond\Delta$ over Δ. Given any (not necessarily finite) family J of objects in C, let $\mathrm{End}_o(J)$ be the ring of all row-and-column finite matrices indexed by J, with entries in the x-y-position (where $x,y \in J$) from $\mathrm{Hom}_C(x,y)$, and with matrix multiplication, using the composition in C. Now, let $R(\Delta) = \mathrm{End}_o(\Delta_o \setminus \Delta_o')$. (Note that $\Gamma_o \setminus \Gamma_o'$ is a complete set of indecomposable projective A-modules, and we want that $\Delta_o \setminus \Delta_o'$ becomes a complete set of indecomposable projective $R(\Delta)$-modules.) The result of Gabriel and Riedtmann can be formulated as follows:

Proposition: Let A be a finite-dimensional algebra of finite representation type. Let Δ be a covering of the Auslander-Reiten species $\Gamma(A)$ of A. Then $R(\Delta)$ is locally finite-dimensional, and $\Gamma(R(\Delta)) = \Delta$.

Two special cases of this proposition are of particular interest. First of all, let $\Delta = \Gamma(A)$. Note that the algebra A and $R(\Gamma(A))$ are not necessarily isomorphic, not even in case k is algebraically closed. Algebras of the form $A = R(\Gamma(A))$ have been called standard, and for any algebra A of finite representation type, there is associated $R(\Gamma(A))$, its standard form. (Note that for algebraically closed k, the algebra $R(\Gamma(A))$ is nothing else than Kupisch's "Stamm-Algebra" [14] of the Auslander algebra of A.)

The other special case we are interested in is the case of $\Delta = \tilde{\Gamma}(A)$, the universal covering of $\Gamma(A)$. We will denote $R(\tilde{\Gamma}(A))$ by $\tilde{A}$. Note that $\tilde{A}$ usually will not be finite-dimensional. This was the reason for considering the more general class of locally finite-dimensional algebras from the beginning. Also note that $\tilde{A}$ can be shown to be the only basic locally finite-dimensional algebra satisfying $\Gamma(\tilde{A}) = \tilde{\Gamma}(A)$.

(3.6) The type of an $\tilde{A}$-module. Let A be a finite-dimensional algebra and $\tilde{A}$ the basic locally finite-dimensional algebra with $\Gamma(\tilde{A}) = \tilde{\Gamma}(A)$. Recall that for any $\tilde{A}$-module X, the support algebra of X is denoted by $\tilde{A}(X)$. Now $\tilde{A}(X)$ is a finite-dimensional algebra, of finite representation type, since with $\tilde{A}$ also $\tilde{A}(X)$

is of bounded representation type. Also the Auslander-Reiten quiver of $\widetilde{A}(X)$ has no oriented cycles, since otherwise we would obtain an oriented cycle in $\widetilde{\Gamma}(A)$. As a consequence, we can apply the results of [8]. In particular, we see that if X,Y are indecomposable $\widetilde{A}$-modules with $\underline{\dim} X = \underline{\dim}\, Y$, then X,Y are isomorphic. Also, we have the following: By definition, the $\widetilde{A}(X)$-module X is always sincere. Thus, if X is indecomposable, then X is a faithful $\widetilde{A}(X)$-module. Both results rest on the fact that for X indecomposable, the algebra $\widetilde{A}(X)$ is a tilted algebra in the sense of [8]. In fact, we have the following proposition [8]:

Proposition. Let X be an indecomposable $\widetilde{A}$-module. Then there exists a basic, hereditary, finite-dimensional algebra H, a tilting module T_H, and a primitive idempotent e of H such that $\widetilde{A}(X) = \operatorname{End}(T_H)$, ${}_{\widetilde{A}(X)}X = {}_{\widetilde{A}(X)}Te$ and moreover He is the only simple projective H-module. Also, H, T_H, and e are uniquely determined by X.

The algebra H is called the type of X.

Let us indicate in which way H,T_H,e are constructed: One constructs a "complete slice" $\mathcal{T}$ in the Auslander-Reiten quiver of $\widetilde{A}(X)$ with X being the unique sink of $\mathcal{T}$, and defines $T = \bigoplus_{T_i \in \mathcal{T}} T_i$, $H = \operatorname{End}({}_{\widetilde{A}(X)}T)$, with e being the projection onto the direct summand X of T.

(3.7) The shape of an A-module. Again, let A be a finite-dimensional algebra, and $\widetilde{A}$ the basic locally finite-dimensional algebra with $\widetilde{\Gamma}(A) = \Gamma(\widetilde{A})$. Also, denote by π the covering map $\widetilde{\Gamma}(A) \to \Gamma(A)$, note that π is surjective, and let G be the group of automorphisms of $\widetilde{\Gamma}(A)$ such that $\widetilde{\Gamma}(A)/G = \Gamma(A)$.

Now let M be an indecomposable A-module. Since π is surjective, there exists some $\widetilde{M}$ in $\widetilde{\Gamma}_o$ such that $\pi(\widetilde{M}) = M$. Let $\widetilde{A}(\widetilde{M})$ be the support algebra of $\widetilde{M}$, and consider $\widetilde{M}$ as an $\widetilde{A}(\widetilde{M})$-module. The pair $(\widetilde{A}(\widetilde{M}),\widetilde{M})$, or also the pair $(\widetilde{A}(\widetilde{M}),\underline{\dim}\widetilde{M})$ will be called the shape of M. Note that the shape of M is independent of the choice of $\widetilde{M}$. For, any other inverse image of M under π is of the form $\widetilde{M}^g$, with $g \in G$, and clearly g defines an isomorphism between the corresponding support algebras and also between the $\widetilde{A}(\widetilde{M})$-module $\widetilde{M}$ and the $\widetilde{A}(\widetilde{M}^g)$-module $\widetilde{M}^g$.

(3.8) Example. Let A be the matrix algebra

$$A = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & c \end{pmatrix} \;\middle|\; a \in \mathbb{R}\,;\, b,c,d \in \mathbb{C} \right\} .$$

There are two simple A-modules $S(1)$ and $S(2)$, with $\operatorname{End} S(1) = \mathbb{R}$, $\operatorname{End} S(2) = \mathbb{C}$. The A-modules which do not split off a copy of $S(1)$ are given by a $\mathbb{C}$-vectorspace V endowed with an endomorphism φ satisfying $\varphi^2 = 0$, and an $\mathbb{R}$-subspace U contained in the kernel of φ, thus we will use the notation (V,U,φ). There are the indecomposable projective modules

$$P(1) = (\mathbb{C}, \mathbb{R}, 0) \ , \ P(2) = (\mathbb{C}\mathbb{C}, 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \quad ,$$

the indecomposable injective modules

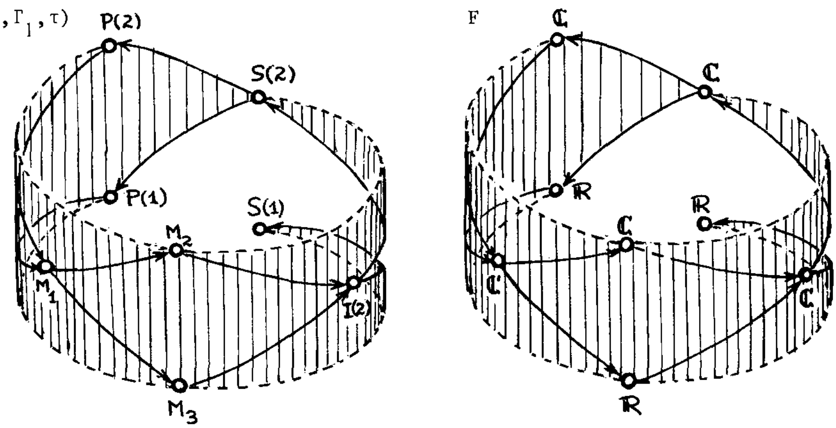
$$I(1) = S(1) \ , \ I(2) = (\mathbb{C}\mathbb{C}, \ \mathbb{C}0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \quad ,$$

and three additional indecomposable modules of length > 1

$$M_1 = (\mathbb{C}\mathbb{C}\mathbb{C}, \ \mathbb{R}\,\mathbb{R}\,0 \ , \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}) \ , \ M_2 = (\mathbb{C}, \mathbb{C}, 0) \ , \ M_3 = (\mathbb{C}\mathbb{C}, \ \mathbb{R}0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \quad .$$

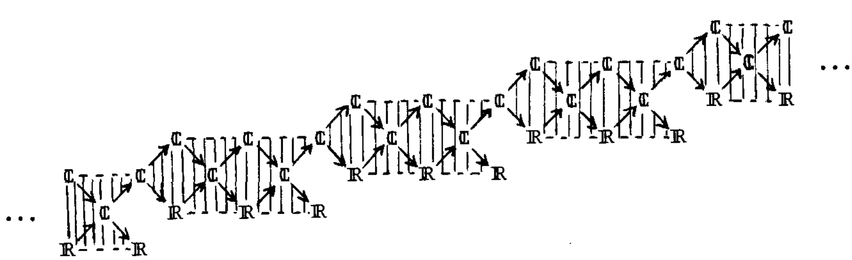
The Auslander-Reiten species of A is of the form

$(\Gamma_0, \Gamma_1, \tau)$ F

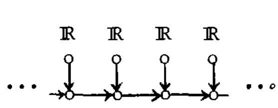


The arrows $\alpha : x \to y$ are always endowed with the bimodule $N(\alpha) = {}_{F(x)}\mathbb{C}_{F(y)}$, the bilinearforms ${}_{\mathbb{C}}\mathbb{C}_{F(y)} \otimes {}_{F(y)}\mathbb{C}_{\mathbb{C}} \to {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ are the multiplication map, those of the form ${}_{\mathbb{R}}\mathbb{C}_{\mathbb{C}} \otimes {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}} \to {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}$ are given by a projection ${}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}} \to {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}$.

The universal covering $\widetilde{\Gamma}(A)$ is of the form

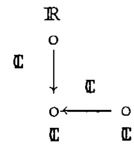


with the same description of the bimodules and bilinear forms. Finally, $\widetilde{A}$ is given by the species



again with bimodules ${}_{\mathbb{R}}\mathbb{C}_{\mathbb{C}}$ and ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$, and with all compositions being zero.

Let us determine the shape of the A-module M_1 . An inverse image of M_1 under the covering map is given by $\widetilde{M}_1$ with dimension type $\cdots {}^{0\ 2\ 0\ 0}_{0\ 2\ 1\ 0} \cdots$, thus $\widetilde{A}(\widetilde{M})$ is the hereditary algebra

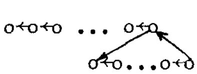
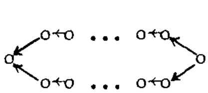
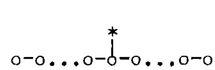
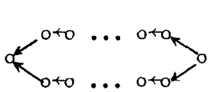
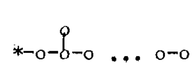


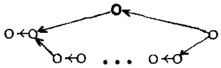
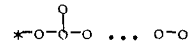
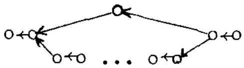
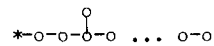
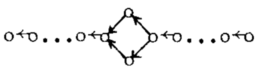
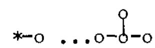
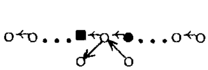
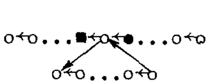
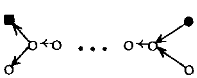
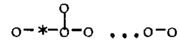
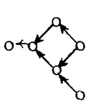
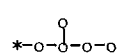
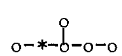
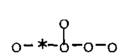
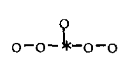
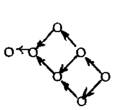
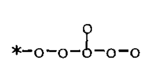
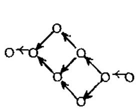
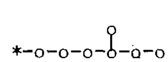
and ${}_{\widetilde{A}(\widetilde{M}_1)}\widetilde{M}_1$ is the indecomposable representation of dimension type ${}^{2}_{2\ 1}$.

4. The shapes of the indecomposable modules of a Kawada algebra

(4.1) Let us now give the list of the indecomposable modules for a Kawada algebra. We will write down all possible shapes $(\widetilde{A}(\widetilde{M}),\underline{\dim}\widetilde{M})$ of such modules. In fact, it turns out that always $\widetilde{A}(\widetilde{M})$ is the path algebra of a fully commutative quiver with at most one zero-relation over a division ring F ; thus, we only list this quiver and mark the starting point of a zero-relation by ● , the endpoint by ■ . Note that this result is only the second part of Kawada's theorem.

Theorem (Kawada). *Let A be a Kawada algebra, and M indecomposable. Then the shape of M is one of the following or its dual:*

$\widetilde{A}(\widetilde{M})$	$\underline{\dim}\widetilde{M}$	type	Kawada's notation
o–o–o ... o–o	111 ... 11	A_n	I-1, I-2, II-3·1, II-3·2, II-3·3, II-3·4
	111 ... 111 1	D_n	I-4·1, II-2·1, II-1·1, I-3·1
	111 ... 121 1 ⋮ 121 ... 221 1		I-4·2 II-2·2, II-2·3 II-1·2 I-3·2, I-3·3
	111 ... 11 11 ... 11	D_n	IV-2·1, III-4·2
	111 ... 12 11 ... 11 ⋮ 122 ... 22 11 ... 11		IV-2·2, IV-2·3 III-4·3, III-4·4
	11 ... 11 1 1 11 ... 11		III-4·1
	11 ... 11 1 1 11 ... 11		V-1

	1 11 1 11 ... 11		III-3·1, III-1
	1 11 11 11 ... 11		III-3·1
	1 11...11 11...11 1		III-1, III-2
	11 ... 121 ... 11 11		II-2·4
	11 ... 121 ... 11 11...11		IV-2·4
	1 1 22 ... 22 1 1		II-1·3
	1 11 1 1 1		III-3·2, IV-1·1
	1 11 1 2 1		III-3·3, IV-1·2
	1 12 1 2 1		III-3·4, IV-1·3
	2 12 1 2 1		III-3·5, IV-1·4
	1 11 1 1 1 1		V
	1 11 1 1 11 1		V

An edge in the quiver of $\tilde{A}(\tilde{M})$ means that there is an arow with arbitary orientation. In all cases, the type of $\tilde{M}$ is the path algebra H of a quiver without cycles. We have listed these quivers, the unique sink being marked by * (except in the first cases). Note that modules with shape of type A_n are also called strings.

(4.2) Note however that there are algebras with all indecomposable modules having shapes as in the list, without being a Kawada algebra. For example, the path algebra of $\circ \longrightarrow \circ \circlearrowleft \alpha$ with $\alpha^2 = 0$ is not a Kawada algebra, whereas all its indecomposable modules are strings. However, under the assumption that all indecomposable modules have shapes as in the list, it is not difficult to check for any of these modules both top and socle, and thus to verify directly whether it is a Kawada algebra or not.

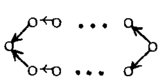
(4.3) Let us outline a direct proof of the theorem. First, one notes that *with an indecomposable A-module M also the $\tilde{A}$-module $\tilde{M}$ has squarefree top and squarefree socle*. As a consequence, we see that for a Kawada-algebra A , also the algebras $\tilde{A}(\tilde{M})$ are Kawada algebras. Thus, we may assume that A is a tilted algebra with an indecomposable sincere representation, and at the same time a Kawada algebra, and have to show that A is one of the algebras in the list. (Note that it is easy to check that all these algebras are Kawada algebras and that all their indecomposable modules are listed, using the inductive construction of the corresponding Auslander-Reiten quiver, as outlined in [5].) Now one uses induction on the number of simple A-modules: Given A , we can write it as a one-point-extension of a Kawada algebra B by a B-module $_BX$, see [16], and, by induction, we know all indecomposable B-modules. Since A is a Kawada algebra, the vectorspace category $\mathrm{Hom}(_BX, _BM)$ actually is of the form addS for some partially ordered set S , and in addition, the width of S must be ≤ 2 . Now it is a rather elementary, however tedious, exercise to check all possibilities.

5. Appendix: The reception of the work of Kawada

Kawada's theorem was the last result in a sequence of investigations of special classes of algebras of finite representation type. These investigations started with Köthe and Nakayama who studied the serial algebras, and they were continued for example by Yoshii and Tachikawa. All these investigations aimed at an internal characterization of algebras whose modules decompose in a predictable way. However, after the work of Kawada, this type of problem must have appeared as a dead end: First of all, the length of his proof was rather surprising. And what was the result? 19 really horrible conditions which are difficult to check and which did not seem to give much insight into the problem. As a consequence, for a long time, there were no further attempts to deal with algebras of finite representation type, the work of Kawada was forgotten.

Some of Kawada's results were rediscovered later, and usually not in a simpler form. His methods involve a large number of different ways of amalgamation of modules in order to form large indecomposable modules, and also different ways of splitting off certain types of modules in order to decompose a given module. Several of these techniques were needed later by different authors and had to be introduced again. In particular, the decomposition of modules which are direct sums of strings has been investigated thoroughly (strings also have been called V-modules [13]), they play a rather dominant role in representation theory. We note however that not all algebras of finite representation type with only strings as indecomposable modules are Kawada algebras (see 4.2).

The most important Kawada algebras are perhaps the blocks of group algebras with cyclic defect group (in particular, the group algebras of groups with cyclic p-Sylow group over a field of characteristic p). These algebras were investigated by Dade, Janusz and Kupisch. Using deep character theoretical results of Dade, both Janusz [9] and Kupisch [12,13] determined the structure first of the indecomposable projective modules, they are of shape

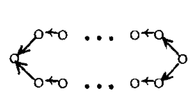
 or o←o←o ... o←o ,

and then of the remaining modules: they are strings. After having derived the structure of the indecomposable projective modules, one could have applied Kawada's theorem.

A special class of Kawada algebras (which includes the blocks of group algebras with cyclic defect group) have been considered recently [4]: algebras of distributive module type. Recall that a module is said to be distributive in case its lattice of submodules is distributive. Note that a module M over a finite-dimensional algebra is distributive if and only if for every pair of submodules $0 \subseteq U \subseteq V \subseteq M$ with V/U semisimple, this module V/U is squarefree. The finite dimensional algebra A is said to be of distributive module type provided any indecomposable module is distributive. Clearly, algebras of distributive module type are Kawada algebras. Thus, we can apply Kawada's theorem. Note that the shape of a distributive module is again a distributive module, and the only quivers with relations occuring in Kawada's list for which all indecomposable representations are distributive, are

o–o–o ... o–o

and the commutative quiver

 .

There also is a recent survey on the Köthe problem (which there is called the σ-cyclic problem, and correspondingly Köthe rings there are called σ-cyclic rings), with a "look to the future". It was presented at the 1978 annual AMS-meeting and then also published. This survey does have a reference to the papers [1] of Kawada, but it refers to them as follows: "Kawada gave a determination of a very special case of the σ-cyclic problem (e. g. radical square zero, and every indecomposable cyclic embeds in R), but even then some 19 conditions were deemed necessary and sufficient." The number of conditions is the right one, but everything else is pure fantasy (actually, under the mentioned assumptions, the problem would be very easy [10]). On the other hand, the author poses the problem to do what Kawada actually did: "Call a ring property P Morita stable if every ring Morita equivalent to a ring with P also has P . ... It would be a reasonable conjecture that any semi-perfect Morita stable σ-cyclic ring is uniserial." At least Nakayama gave a counter example to such a conjecture, and we have seen above the large variety of possible shapes of modules found by Kawada. A look to the past is sometimes valuable.

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