Separating tubular series

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Let k be an algebraically closed field. We will consider finite dimensional k-algebras Λ and our aim is to describe some components or sometimes even all components of the Auslander-Reiten quiver of Λ. Components of the Auslander-Reiten quiver of Λ we just will call components of Λ. In case Λ is of tame representation, it seems that there is a large amount of components which are regular tubes. We recall from [3] that a tube is a translation quiver containing an oriented cycle, and with underlying topological space being of the form $S^1 \times \mathbb{R}_{\geq 0}$. The regular tubes are the translation quivers of the form $\mathbb{Z}\mathbb{A}_\infty/r$ with $r \geq 1$, and r is called the rank of the tube. Tubes usually occur in families indexed by some set I, and in this case, we will speak of a tubular I-series. In our investigation presented here, the index set I is always the projective line $\mathbb{P}_1 k$ over k. Given a tubular I-series $\mathcal{T}_i$ $(i \in I)$, with $\mathcal{T}_i$ regular of rank r_i, we associate with it a diagram, called its type, which is constructed as follows: We form the disjoint union of diagrams $\mathbb{A}_{r_i}$, with $i \in I$, choose in any $\mathbb{A}_{r_i}$ one particular endpoint, and identify all these endpoints in order to form a star. For example, given the path algebra Λ of an extended Dynkin diagram $\tilde{\Delta}$ with some orientation (where $\Delta = \mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ or $\mathbb{E}_8$), then the regular Λ-modules form a tubular $\mathbb{P}_1 k$-series, consisting of regular tubes, and the tables in [4] show that its type in just Δ (For example, in case $\Delta = \mathbb{E}_6$, the simple regular representations of an oriented $\tilde{\mathbb{E}}_6$-quiver form two τ-orbits of length 3, one τ-orbit of length 2, all other τ-orbits are of length 1, thus there are two regular tubes of rank 3, one of rank 2, and all others are of rank 1, so that the diagram of this tubular $\mathbb{P}_1 k$-series is the star $\mathbb{E}_6$). One objective of the present paper is to outline a direct proof of this fact.

Given a tubular I-series $\mathcal{T} = \dot{\bigcup}_{i\in I} \mathcal{T}_i$ formed by components of the algebra Λ, we will say that $\mathcal{T}$ is <u>separating</u> provided the remaining indecomposable Λ-modules fall into two disjoint classes $\mathcal{P}$, $\mathcal{Q}$ such that

(1) $\mathrm{Hom}(Q,P) = \mathrm{Hom}(Q,T) = \mathrm{Hom}(T,P) = 0$ for all $P \in \mathcal{P}$, $T \in \mathcal{T}$, $Q \in \mathcal{Q}$, and

(2) Given $i \in I$, any homomorphism $\varphi : P \longrightarrow Q$ can be factored through a direct sum of modules in $\mathcal{T}_i$. (We will say that $\mathcal{T}$ separates $\mathcal{P}$ from $\mathcal{Q}$.)

Of course, in case $\mathcal{T}$ is the tubular $\mathbb{P}_1 k$-series of all regular Λ-modules, where Λ is the path algebra of some oriented extended Dynkin diagram, then $\mathcal{T}$ is separating, with $\mathcal{P}$ the set of indecomposable preprojective modules, and $\mathcal{Q}$ the set of indecomposable preinjective modules.

Our main interest lies in a class $\mathcal{C}$ of algebras introduced in section 4, and we are going to give a complete description of the indecomposable C-modules, for any $C \in \mathcal{C}$. We will see that any algebra in $\mathcal{C}$ has countably many separating tubular $\mathbb{P}_1k$-series, all but two being of a fixed type, namely of type $\mathbb{T}_{2,2,2,2}$ (the case $(\mathbb{D}_{4\lambda})$), or $\mathbb{T}_{3,3,3}$ (the case $(\mathbb{E}_6)$), or $\mathbb{T}_{4,4,2}$ (the case $(\mathbb{E}_7)$), or $\mathbb{T}_{6,3,2}$ (the case $(\mathbb{E}_8)$), and only two additional components, a preprojective component and a preinjective component. Also, the dimension vectors of the indecomposable C-modules can be characterized as being the positive connected vectors x in the Grothendieck group $K_o(C)$ satisfying $q_C(x) = 0$ or 1, where q_C is a suitable quadratic form on $K_o(C)$, the socalled Euler characteristic.

In particular, it follows directly from our investigations of the algebras of type $(\mathbb{D}_{4\lambda})$ that the pattern of type $(\tilde{\mathbb{D}}_4,1)_\lambda$ (see [7]) is tame. This solves the one remaining case which had been left open in [7]. We only remark that in all cases $(\mathbb{D}_{4\lambda}),(\mathbb{E}_6),(\mathbb{E}_7),(\mathbb{E}_8)$, the determination of the indecomposable modules of any algebra of that type directly classifies the indecomposable representations of the patterns of that type.

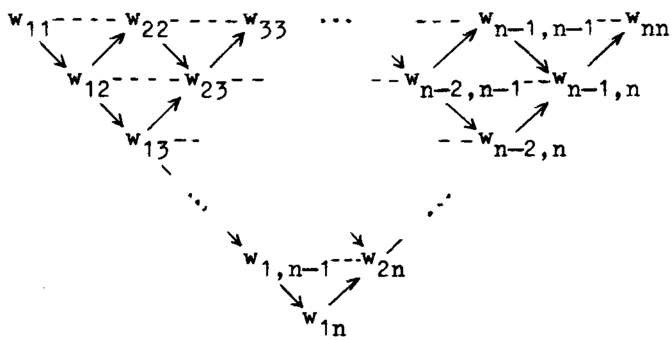
The notes give an outline of results with indications both of the method of proofs as well as of applications. A detailed account will appear in [8]. It should be noted that the author is strongly endebted to S. Brenner and M.C.R. Butler. Their ideas (both mathematical and philosophical) concerning the use of tilting functors for tame algebras like squids have influenced the present investigation [2]. The one-parameter series of indecomposable modules over algebras (or better, of representations of partially ordered sets) of type $(\mathbb{E}_6),(\mathbb{E}_7)$ and $(\mathbb{E}_8)$ first have been determined by Nazarova and Zavadskij [9], and Zavadskij has informed the author that he also obtained the classification of all indecomposable modules in these cases. The results were reported at Torun in December 1981, and, in spring 1982, at the Seminaire d'Algèbre Dubreil-Malliavin in Paris and at the mathematical institute of the Ukrainian Academy of Science in Kiev. The author is grateful to all these institutions for their hospitality and for the possibility to discuss the results. These discussions resulted in many improvements; in fact, the whole theory was transformed many times, and we hope that the form presented here is the most accessible one.

Notation: For a translation quiver Γ with translation τ, we denote by $\Gamma^{(o)}$ the set of its vertices, by $\Gamma^{(1)}$ the set of arrows, and by $\Gamma^{(2)}$ the graph of τ. Let Λ be a finite-dimensional algebra. The isomorphism class of a Λ-module M will be denoted by $[M]$, it is a vertex of the Auslander-Reiten-quiver $\Gamma(\Lambda)$ of Λ. Given an indecomposable projective module $P(i)$, we denote by $S(i)$ its top $P(i)/\mathrm{rad}P(i)$. $\mathbb{N}$ will denote the set of natural numbers $\{1,2,3,\dots\}$.

1 Construction of separating tubular series

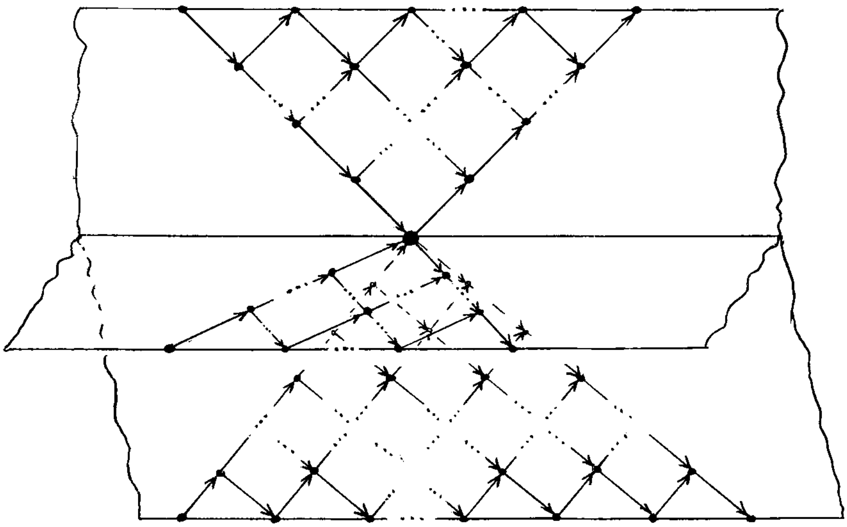
Given a translation quiver Γ, a full translation subquiver Δ of Γ is given by a set $\Delta^{(o)} \subseteq \Gamma^{(o)}$, and satisfying $\Delta^{(i)} = \Gamma^{(i)} \cap (\Delta^{(o)} \times \Delta^{(o)})$, for $i=1,2$. The full translation subquiver Δ of Γ will be called mesh complete, provided $\left[\frac{z}{x}\right] \in \Delta^{(2)}$ and $y \to z$ in $\Gamma^{(1)}$ implies $y \in \Delta^{(o)}$.

The Auslander-Reiten quiver of the linearly ordered quiver of type $\mathbb{A}_n$ will be denoted by $\Phi(n)$, its vertices are of the form w_{ij} with $1 \le i \le j \le n$, there is an arrow $w_{ij} \longrightarrow w_{i,j+1}$ for all $1 \le i \le j < n$, and an arrow $w_{ij} \longrightarrow w_{i+1,j}$ for all $1 \le i < j \le n$, and there are the extensions $\left[\frac{w_{ij}}{w_{i-1,j-1}}\right]$ for all $1 < i \le j \le n$. Note that the vertices w_{1j} are projective, the vertices w_{in} are injective, thus w_{1n} is the unique projective-injective vertex of $\Phi(n)$. The vertices w_{ij} with $1 < i \le j < n$ will be said to belong to the interior of $\Phi(n)$.



Given a vertex w of a translation quiver Γ, a mesh complete full subquiver Φ of Γ will be called a wing for w provided Φ is of the form $\Phi(n)$ for some $n \ge 2$ with w being the unique projective-injective vertex of Φ, and such that (*) for $x \to w$ in Γ, with $w \in \Phi$, and projective and not injective in Φ, the vertex x is not injective in Γ and $\tau^{-}x \in \Phi$, and dually (**) for $w \to y$ in Γ, with $w \in \Phi$ and injective and not projective in Φ, the vertex y is not projective in Γ and $\tau y \in \Phi$; the number n will be called the length of the wing.

Given a finite dimensional algebra Λ with Auslander-Reiten quiver $\Gamma(\Lambda)$, an indecomposable Λ-module W will be called a wing module provided given any arrow $[X] \longrightarrow [W]$ or any arrow $[W] \longrightarrow [X]$ in $\Gamma(\Lambda)$ there exists a wing for $[W]$ in $\Gamma(\Lambda)$ containing the arrow. Given a wing module M, with minimal right almost split map $\bigoplus_{i=1}^{s} X_i \longrightarrow M$, where all X_i are indecomposable, the vertices $[X_i]$ belong to pairwise different wings $\Phi^{(i)}$ for $[M]$, and let n_i be the length of the wing $\Phi^{(i)}$. The star $\mathbb{T}_{n_1 \dots n_s}$ will be called the type of the wing module. In case $s = 3$, the component containing a wing module of type $\mathbb{T}_{n_1, n_2, n_3}$ looks as follows:



A wing module W will be said to be separating provided the indecomposable Λ-modules X with $[X]$ not belonging to the interior of a wing for $[W]$ and different from $[W]$ itself, fall into two disjoint classes $\mathcal{U}$ and $\mathcal{V}$ such that

$$\mathrm{Hom}(V,U) = \mathrm{Hom}(V,W) = \mathrm{Hom}(W,U) = 0$$

for all $U \in \mathcal{U}$, $V \in \mathcal{V}$ and such that, moreover, any homomorphism $U \longrightarrow V$ with $U \in \mathcal{U}$, $V \in \mathcal{V}$ factors through a direct sum of copies of W.

Examples of separating wing modules of type $\mathbb{T}_{n_1 \dots n_s}$ occur in the preprojective component of a quiver with underlying graph of the form $\mathbb{T}_{n_1 \dots n_s}$. More general, most tilted algebras of type $\mathbb{T}_{n_1 \dots n_s}$ will also have separating wing modules of type $\mathbb{T}_{n_1 \dots n_s}$.

Given any (not necessarily indecomposable) Λ-module R, we denote by $\Lambda[R]$ the one-point extension of Λ by R, it is given by the following matrix algebra

$$\begin{bmatrix} \Lambda & R \\ 0 & k \end{bmatrix}.$$

Its modules are of the form $({}_\Lambda X, {}_k Y, \varphi : {}_\Lambda X \longleftarrow {}_\Lambda R \underset{k}{\otimes} Y)$, with ${}_\Lambda X$ being a Λ-module, ${}_k Y$ a k-vectorspace, and φ Λ-linear. In case $Y = 0$, we just deal with a Λ-module. All indecomposable projective $\Lambda[R]$-modules but one are, in fact, Λ-modules, the remaining one will be denoted by $P(\omega) = ({}_\Lambda R, k, 1_R)$, and we have $\operatorname{rad} P(\omega) = R$. Given a homomorphism $\rho : {}_\Lambda R \longrightarrow {}_\Lambda X$, we denote by $X(\rho) = ({}_\Lambda X, k, \rho)$, thus $X(\rho)$ is given by the following pushout diagram.

$$\begin{array}{ccc} R & \hookrightarrow & P(\omega) \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ X & \dashrightarrow & X(\rho). \end{array}$$

Note that if α is a non-zero element of k, then $X(\rho) \approx X(\alpha\rho)$. Thus, given an element $[\rho] \in \mathbb{P}\operatorname{Hom}(R,X)$, the module $X([\rho]) := X(\rho)$ is defined up to isomorphism.

Given an algebra Λ, we denote by $K_o(\Lambda)$ the Grothendieck group of all Λ-modules modulo exact sequences. It has a canonical basis given by the set of simple Λ-modules. In this way, $K_o(\Lambda)$ is a partially ordered abelian group. Given a Λ-module X, its residue class in $K_o(\Lambda)$ will be denoted by $\underline{\dim}\, X$, and called the dimension vector of X. In case Λ is of finite global dimension, there is a (usually non-symmetric) bilinear form on $K_o(\Lambda)$, given by

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle = \sum_{i \geq o} \dim_k \operatorname{Ext}^i(X,Y),$$

with $\operatorname{Ext}^o = \operatorname{Hom}$. The corresponding symmetrized bilinear form will be $(\ ,\ )$, thus

$$2(x,y) = \langle x,y\rangle + \langle y,x\rangle ,$$

and q_Λ denotes the quadratic form $q_\Lambda(x) = (x,x) = \langle x,x\rangle$.

Now we are able to state our main result concerning the construction of separating tubular series. Note that an algebra with a sincere, separating wing module always has global dimension ≤ 2.

Theorem. *Let Λ be a finite dimensional k-algebra with a sincere, separating wing module W of type $\mathbb{T}_{n_1 \ldots n_s}$. Let R be a (not necessarily indecomposable) module with*

$$\langle \underline{\dim}\, R, - \rangle = 2(\underline{\dim}\, W, -)$$

on $K_o(\Lambda)$, and assume that $\operatorname{proj.dim} W(\rho) \leq 1$ for all $0 \neq \rho \in \operatorname{Hom}(R,W)$. Consider the following linear form $\partial_W = \langle \underline{\dim}(W \oplus S(\omega)), - \rangle$ on $K_o(\Lambda[R])$, and let $\mathcal{P}_W, \mathcal{T}_W, \mathcal{Q}_W$ be the set of all indecomposable $\Lambda[R]$-modules Y satisfying $\partial_W(\underline{\dim}\, Y) < o$, $= 0$, or > 0, respectively. Then $\mathcal{T}_W$ is a tubular $\mathbb{P}_1k$-series of type $\mathbb{T}_{n_1 \ldots n_s}$ separating $\mathcal{P}_W$ from $\mathcal{Q}_W$.

In fact, $\mathrm{Hom}(R,W)$ is two-dimensional, thus we may identify $\mathbb{P}_1k = \mathbb{P}\,\mathrm{Hom}(R,W)$. Given $0 \neq \rho : R \longrightarrow W$, let $\mathcal{T}_W(\rho)$ be the component of $\Gamma(\Lambda[R])$ containing $W(\rho)$. Then $\mathcal{T}_W(\rho)$ is a regular tube. In case ρ factors through an irreducible map $X_i \longrightarrow W$, with X_i indecomposable, then $\mathcal{T}_W(\rho)$ is a tube of rank n_i, where n_i is the length of the wing for W containing $[X_i]$. In case ρ cannot be factored in this way, $\mathcal{T}_W(\rho)$ is a tube of rank 1.

The dimension vectors of the indecomposable modules in $\mathcal{T}_W$ can be numerically characterized as follows: If the indecomposable $\Lambda[R]$-module Y belongs to $\mathcal{T}_W$, then $q_{\Lambda[R]}(\underline{\dim}\,Y) = 0$ or 1, and, of course, $\partial_W(\underline{\dim}\,Y) = 0$. Conversely, given a positive element y in $K_0(\Lambda[R])$ with $\partial_W(y) = 0$ and $q_{\Lambda[R]}(y) = 1$, there is a unique indecomposable $\Lambda[R]$-module Y with $\underline{\dim}\,Y = y$, and given a positive element y in $K_0(\Lambda[R])$ with $\partial_W(y) = 0$ and $q_{\Lambda[R]}(y) = 1$, there is a $\mathbb{P}_1k$-family of type $\mathbb{T}_{n_1 \ldots n_s}$ consisting of indecomposable $\Lambda[R]$-modules Y with $\underline{\dim}\,Y = y$.

Remark 1. The assumptions of the theorem directly imply that R is projective or indecomposable. Namely, assume $R = R' \oplus R''$ with R' indecomposable and not projective, and with $R'' \neq 0$. We may suppose $\mathrm{Hom}(R'',W) \neq 0$, since either R'' has an indecomposable projective direct summand, and then $\mathrm{Hom}(R'',W) \neq 0$ due to the fact that W is sincere, or else we may exchange R' with an indecomposable summand of R''. Thus, let $0 \neq \rho : R'' \longrightarrow W$, and extend it to R by using the zero map on R'. Let $\pi : P \longrightarrow W$ be a projective cover of W. Then

$$\begin{array}{ccc} 0 \oplus R' \oplus R'' & \xrightarrow{\quad 1 \quad} & R' \oplus R'' \\ \Big\downarrow & & \Big\downarrow \left(\begin{smallmatrix} 0 \\ \rho \end{smallmatrix}\right) \\ P \oplus R' \oplus R'' & \xrightarrow{\left(\begin{smallmatrix} \pi \\ 0 \\ \rho \end{smallmatrix}\right)} & W \end{array}$$

is a projective cover of $W(\left(\begin{smallmatrix} 0 \\ \rho \end{smallmatrix}\right))$, and its kernel has $(R',0,0)$ as a direct summand, thus $\mathrm{proj.dim.}\ W(\left(\begin{smallmatrix} 0 \\ \rho \end{smallmatrix}\right)) \geq 2$.

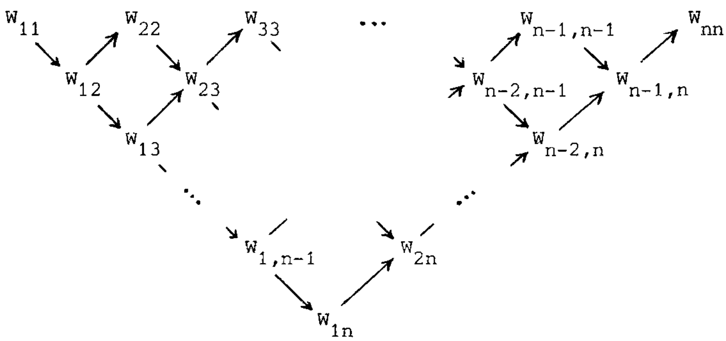
Remark 2. The condition $\mathrm{proj.dim.}\ W(\rho) \leq 1$ for $0 \neq \rho : R \longrightarrow W$ is not always easy to check. However, in some cases, it will be straightforward that this condition is satisfied. First of all, if R is projective then $\mathrm{proj.dim.}\ W \leq 1$ implies that $\mathrm{proj.dim.}\ W(\rho) \leq 1$ for any $\rho : R \longrightarrow W$. [Namely, $W(\rho)$ is an extension of W by the simple module $P(\omega)/R$.]. Also, given $\rho : R \longrightarrow W$ with $\mathrm{Ker}\,\rho$

projective and proj.dim. $\mathrm{Cok}(\rho) \leq 1$, then proj.dim. $W(\rho) \leq 1$. [Namely, ρ induces an exact sequence

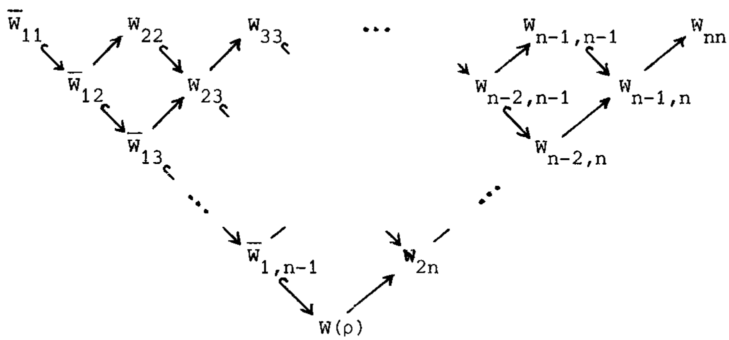
$$0 \longrightarrow \mathrm{Ker}(\rho) \hookrightarrow P(\omega) \rightarrow W(\rho) \longrightarrow \mathrm{Cok}(\rho) \longrightarrow 0$$

and thus $W(\rho)$ is an extension of $P(\omega)/\mathrm{Ker}(\rho)$ by $\mathrm{Cok}(\rho)$.]. In particular, if Λ is hereditary, and all proper submodules of R are projective, then proj.dim. $W(\rho) \leq 1$ for all $0 \neq \rho : R \longrightarrow W$.

Outline of proof. Let us show in which way a wing of length n of W gives rise to a regular tube of rank n. The wing is given by the following diagram of indecomposable modules W_{ij} with $1 \leq i \leq j \leq n$ and irreducible maps



where $W_{1n} = W$. Here, the maps $W_{ij} \longrightarrow W_{i,j+1}$, $1 \leq i \leq j < n$, are monomorphisms, and we may assume that they are inclusions. The maps $W_{ij} \longrightarrow W_{i+1,j}$, $1 \leq i < j \leq n$ are surjective with kernel W_{ii}, and we may assume that $W_{ij} = W_{1j}/W_{1,j-i}$ (with $W_{1,0} = 0$), and that the maps $W_{ij} \longrightarrow W_{i+1,j}$, $1 \leq i < j \leq n$ are the canonical projections. In this case, the given diagram is fully commutative. Let us determine $\dim \mathrm{Hom}(R,W_{ij})$ for all i,j. Note that R is projective or indecomposable. It follows that the indecomposable summands of R different from W belong to $\mathcal{U}$ due to the fact that W is sincere and $\langle \underline{\dim}\, R, \underline{\dim}\, W\rangle = 2(\underline{\dim}\, W, \underline{\dim}\, W) = 2$. Using the equality $\langle \underline{\dim}\, R, \underline{\dim}\, W_{ij}\rangle = \langle \underline{\dim}\, R, \underline{\dim}\, W_{ij}\rangle + \langle \underline{\dim}\, W_{ij}, \underline{\dim}\, R\rangle$, it is easy to see that $\dim \mathrm{Hom}(R,W_{ij}) = 0$ for all $1 < i \leq j < n$, and $= 1$ in the remaining cases except for $i=1$, $j=n$, thus for $W_{1n} = W$, where $\dim \mathrm{Hom}(R,W_{1n}) = 2$. Given an indecomposable Λ-module X, we denote by $\overline{X}$ the following $\Lambda[R]$-module $({}_\Lambda X, \mathrm{Hom}(R,X), e)$ with $e : R \otimes \mathrm{Hom}(R,X) \longrightarrow X$ being the evaluation map. We consider now the following fully commutative diagram of $\Lambda[R]$-modules



where $\rho : R \longrightarrow W$ is a non-zero map factoring through W_{11}; more precisely, let $W(\rho) = (W, \mathrm{Hom}(R,W_{11}), e)$, with $e : R \otimes \mathrm{Hom}(R,W_{11}) \longrightarrow W_{11} \hookrightarrow W$ the evaluation map. Let us note that $W_{ij} = \overline{W}_{ij}$ for $1 < i \leq j < n$.

We claim that the diagram exhibited above has the following property: For all $1 \leq i \leq j < n$, the minimal left almost split map starting in $\overline{W}_{ij}$ is built up from maps in the diagram, and similarly for $1 < i \leq j \leq n$, the minimal right almost split map ending in W_{ij} also is built up from maps in the diagram. This is a direct consequence of the following lemma.

<u>Lemma.</u> Let X and Z be indecomposable Λ-modules and let $f : X \longrightarrow Y$ be a minimal left almost split Λ-map. Then

$$\begin{array}{ccc} R \otimes \mathrm{Hom}(R,X) & \xrightarrow{\;1\;} & R \otimes \mathrm{Hom}(R,X) \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle ef} \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

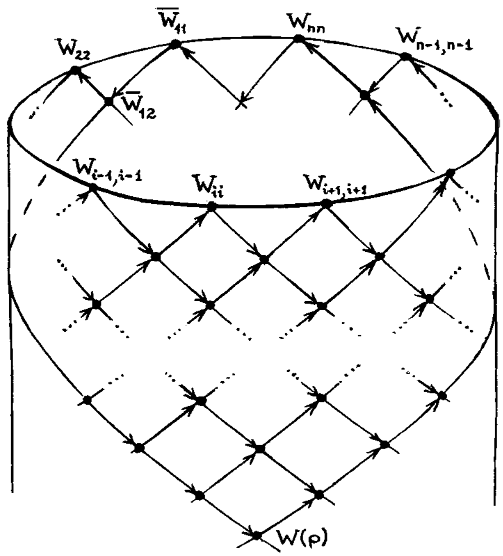
is a minimal left almost split $\Lambda[R]$-map. Also, let $g : Y \longrightarrow Z$ be a minimal right almost split Λ-map, and $e : R \otimes \mathrm{Hom}(R,Y) \longrightarrow Y$ the evaluation map. Then

$$\begin{array}{ccc} R \otimes \mathrm{Ker}(\mathrm{Hom}(R,g)) & \longrightarrow & 0 \\ {\scriptstyle e}\downarrow & & \downarrow \\ Y & \xrightarrow{\;g\;} & Z \end{array}$$

is a minimal right almost split $\Lambda[R]$-map.

The proof ist straightforward.

As a consequence, we conclude that in the category of all $\Lambda[R]$-modules, $\tau W_{ii} = W_{i-1,i-1}$, for $3 \leq i \leq n$, and $\tau W_{22} = \overline{W}_{11}$. It remains to calculate $\tau \overline{W}_{11}$. We note that proj.dim. $\overline{W}_{11} \leq 1$, and that $\mathrm{Hom}(\overline{W}_{11}, \Lambda[R]) = 0$, thus $\underline{\dim}\, \tau\overline{W}_{11} = c\, \underline{\dim}\, \overline{W}_{11}$, where c is the linear transformation of $K_o(\Lambda[R])$ given by $c\, \underline{\dim}\, P(i) = -\dim I(i)$, for any indecomposable projective module $P(i)$ and any indecomposable injective module $I(i)$ satisfying $P(i)/\mathrm{rad}\, P(i) \approx \mathrm{soc}\, I(i)$. Now $c\,(\underline{\dim}\, W_{ii}) = \underline{\dim}\, W_{i-1,i-1}$ for $3 \leq i \leq n$, and $c(\underline{\dim}\, W_{22}) = \underline{\dim}(W_{11} \oplus S(\omega))$, thus $c\, \underline{\dim}\, W(\rho) = \underline{\dim}\, W(\rho)$ implies $c(\underline{\dim}\, \overline{W}_{11}) = \underline{\dim}\, W_{nn}$. Since W_{nn} is the only indecomposable Λ-module with dimension vector $\underline{\dim}\, W_{nn}$, it follows that $\tau\overline{W}_{11} = W_{nn}$. This shows that the category of $\Lambda[R]$-modules contains the following τ-orbit: $W_{nn}, W_{n-1,n-1}, \ldots, W_{33}, W_{22}, \overline{W}_{11}, W_{nn}, \ldots$, and now it is easy to see that the corresponding component of $\Lambda[R]$ is a regular tube of rank n.



2. First Application: Maximal modules

We will say that the module category of Λ is directed, provided Λ is representation finite and

$$[X] \leq [Y] \iff \mathrm{Hom}(X,Y) \neq 0$$

defines a partial ordering on the set of isomorphism classes of indecomposable Λ-modules. An indecomposable Λ-module is said to be maximal provided its dimension vector is maximal in the set of all dimension vectors of indecomposable Λ-modules.

Lemma. Let W be a maximal indecomposable Λ-module, where Λ is a finite-dimensional algebra with directed module category, and assume W is sincere. Then there exists a projective Λ-module R with $\langle \underline{\dim}\, R, - \rangle = 2(\underline{\dim}\, W, -)$, and R is uniquely determined by W.

Proof. Since the module category of Λ is directed, and there exists an indecomposable sincere Λ-module, the indecomposable Λ-modules correspond bijectively to the positive roots of q_Λ, under $\underline{\dim}$, see [5]. Let $S(i)$, $1 \leq i \leq n$, be the simple Λ-modules, $e_i = \underline{\dim}\, S(i)$, $w = \underline{\dim}\, W$.
Now $w = \sum_{i=1}^{n} w_i e_i$ is a maximal root, thus $d_i = 2(w, e_i) \geq 0$ for all i. Let $P(i)$ be the indecomposable projective module with top $S(i)$, and $R = \bigoplus_{i=1}^{n} P(i)^{d_i}$. Then, for any Λ-module X with $\underline{\dim}\, X = x = \sum_{i=1}^{n} x_i e_i$,

$$\langle \underline{\dim}\, R, \underline{\dim}\, X \rangle = \sum_{i=1}^{n} d_i \,\mathrm{Hom}(P(i), X) = \sum_{i=1}^{n} d_i x_i$$

$$= 2(w, \Sigma x_i e_i) = 2(\underline{\dim}\, W, \underline{\dim}\, X).$$

Of course, $\underline{\dim}\, R$ is uniquely determined by $\langle \underline{\dim}\, R, - \rangle$, and it determines uniquely the projective module R.

Remark 1. (Ovsienko [6]) Note that R is the direct sum of at most two indecomposable projective modules. Namely, either $d_i = 1$, $w_i = 2$ for some i, and then $d_j = 0$ for all $j \neq i$, or else $d_i = d_j = 1$, $w_i = w_j = 1$ for some $i \neq j$, and then $d_t = 0$ otherwise.

Remark 2. Given an algebra with directed module category, any indecomposable and sincere module W satisfies $\mathrm{proj.dim}_\Lambda W \leq 1$. Thus, given in addition any projective module R and a homomorphism $\rho : R \longrightarrow W$, it follows that

$\text{proj.dim}_{\Lambda[R]} W(\rho) \leq 1$.

In order to be able to apply the construction theorem for separating tubular series, we have to restrict to the case that the τ-orbit of [W] is the only possible branching point in the orbit quiver of Λ. Thus, we deal with the following assumptions:

Λ is a finite dimensional k-algebra, with directed module category.

W is a sincere and maximal indecomposable Λ-module, and its τ-orbit is the only possible branching point of the orbit quiver of Λ.

In this case, let R be the uniquely determined projective Λ-module with $\langle \underline{\dim}\, R,-\rangle = 2(\underline{\dim}\, W,-)$. Then we obtain for $\Lambda[R]$ a tubular $\mathbb{P}_1$k-series $\mathcal{T}_W$ separating $\mathcal{P}_W$ from $\mathcal{Q}_W$, and being characterized by ∂_W. The type of this tubular series is given by the underlying graph of the orbit quiver of Λ.

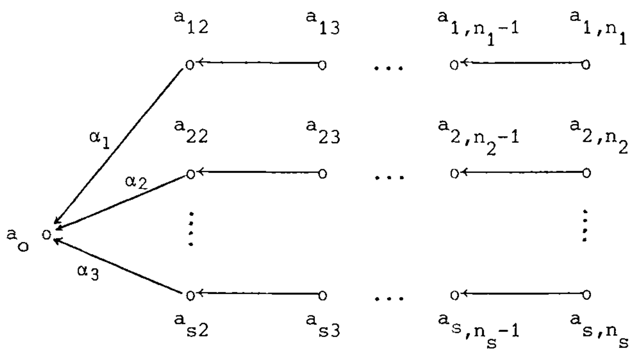
In particular, we can consider the case of Λ being the path algebra of a quiver of type $\mathbb{T} = \mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$, or $\mathbb{E}_8$, and W the unique maximal Λ-module. Then $\Lambda[R]$ is the path algebra of a quiver with underlying graph being the corresponding extended Dynkin diagram $\mathbb{T} = \tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$, respectively, and ∂_W is (a scalar multiple of) the usual defect function.

In this case, $\mathcal{P}_W$ is the set of all indecomposable preprojective, $\mathcal{Q}_W$ the set of all indecomposable preinjective $\Lambda[R]$-modules, whereas $\mathcal{T}_W$ is the set of all indecomposable regular modules. The type of the tubular series $\mathcal{T}_W$ is given by the underlying graph of the orbit quiver of Λ, and this graph is nothing but $\mathbb{T}$. Thus, we obtain a direct proof for the fact that the tubular type of the path algebra of a quiver of extended Dynkin type $\tilde{\mathbb{T}}$ has to be just $\mathbb{T}$.

3. Separating tubular series of arbitrary type

Our aim in this section is to construct an algebra with a separating tubular series of given type $\mathbb{T}_{n_1\ldots n_s}$.

We endow $\mathbb{T}_{n_1\ldots n_s}$ with the socalled subspace orientation, thus we deal with the following quiver



and Λ will denote its path algebra. Note that the indecomposable injective Λ-module $W = I(a_o)$ corresponding to the vertex a_o is a separating wing module of type $\mathbb{T}_{n_1\ldots n_s}$, its dimension vector ist

$$w = 1 \begin{array}{l} 11 \ldots 1 \\ 11 \ldots 1 \\ \vdots \\ 11 \ldots 1 \end{array} .$$

Now, a Λ-module R satisfies $\langle R,-\rangle = 2(w,-)$ if and only if the dimension vector of R is of the form

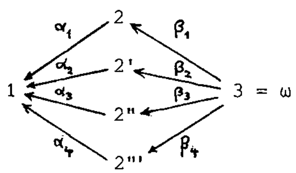
$$r = 2 \begin{array}{l} 11 \ldots 1 \\ 11 \ldots 1 \\ \vdots \\ 11 \ldots 1 \end{array} .$$

Also, given a module R with $\underline{\dim}\, R = r$, then any proper submodule of R is projective if and only if all linear transformations occuring in R are injective, and the linear transformations $\alpha_1,\ldots,\alpha_s$ have pairwise different images; a module satisfying these conditions will be said to be generic. Since Λ is hereditary, it follows that for any generic R with $\underline{\dim}\, R = r$, and any non-zero map $\rho : R \to W$, the projective dimension of $W(\rho)$ is ≤ 1. Thus, in this case, R and W satisfy the conditions of the construction theorem in section 1, and therefore $\mathcal{T}_W$ is a tubular $\mathbb{P}_1k$-series of type $\mathbb{T}_{n_1\ldots n_s}$ in the module category of $\Lambda[R]$.

4. Algebras of type $(\mathbb{D}_{4\lambda}), (\mathbb{E}_6), (\mathbb{E}_7), (\mathbb{E}_8)$.

We consider now certain special algebras and want to exhibit for any one of these a particular separating tubular series. The algebras will be given by quivers (vertices and solid arrows) with relations (usually marked by dotted lines, indicating the relation formed by the sum of all paths between the end points of the dotted line; in some cases, the relations will be written down explicitely). One particular vertex, always a source, is marked by ω. For the application of the construction theorem of section 1, the given algebras are those of the form $\Lambda[R]$, and Λ is obtained by deleting the vertex ω; note that $R = \text{rad}\, P(\omega)$. We also write down the dimension vector $\tilde{w} = \underline{\dim}(W \oplus S(\omega))$, where W as a Λ-module is a suitable sincere separating wing module of some type $\tilde{\Delta}$, such that W and R satisfy the conditions of the construction theorem. In this way, we obtain a separating tubular series of type $\tilde{\Delta}$, and $\Lambda[R]$ will be said to be of type (Δ).

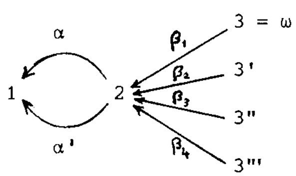
Case $(\mathbb{D}_{4\lambda})$ $(\lambda \in k\,,\ \lambda \neq 0,1)$

$C^{(1)} =$ 

$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0$

$\lambda\alpha_1\beta_1+\alpha_2\beta_2+\alpha_4\beta_4=0$

$\tilde{w}$	∂_0	∂_∞
1	1	-1
$1{}^{1}_{1}1$	$-2{}^{1}_{1}0$	$0{}^{-1}_{-1}2$
1	1	-1

$C^{(2)} =$ 

$\alpha\beta_1 = 0$

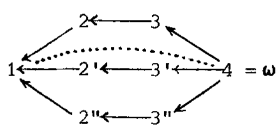
$\alpha'\beta_2 = 0$

$(\alpha-\alpha')\beta_3 = 0$

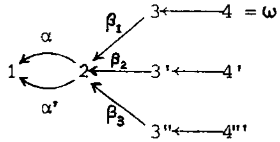
$(\lambda\alpha-\alpha')\beta_4 = 0$

$\tilde{w}$	∂_0	∂_∞
1	0	1
$24{}^{1}_{1}$	$-22{}^{0}_{0}$	$0-2{}^{1}_{1}$
1	0	1

Case $(\mathbb{E}_6)$

$C^{(1)} =$ 

$\tilde{w}$	∂_0	∂_∞
11	11	-1-1
1111	-3110	0-1-13
11	11	-1-1

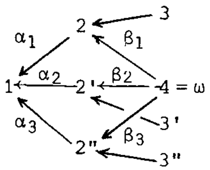
$C^{(2)} =$ 

$\alpha\beta_1 = 0$

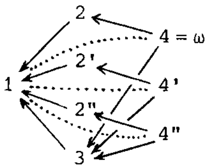
$\alpha'\beta_2 = 0$

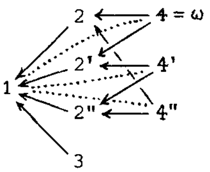
$(\alpha-\alpha')\beta_3 = 0$

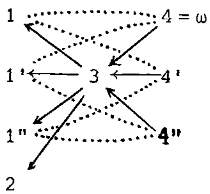
$\tilde{w}$	∂_0	∂_∞
21	00	11
3621	-3300	0-311
21	00	11

$C^{(3)}=$ 

$\alpha_1\beta_1=\alpha_2\beta_2=\alpha_3\beta_3$

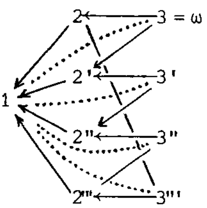
$C^{(4)}=$ 

$C^{(5)}=$ 

$C^{(6)}=$ 

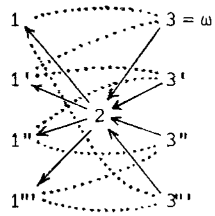
	$\tilde{w}$	∂_0	∂_∞
$C^{(3)}$	1 2 2 1 2 1 2 1 1	2 -6 2 0 2 2 2 2	-2 1 0 -2 3 -2 1 1
$C^{(4)}$	21 3 21 21 3	30 -6 30 30 3	-12 0 -12 -12 -3
$C^{(5)}$	21 2 21 21 1	10 -2 10 10 1	-11 0 -11 -11 0
$C^{(6)}$	2 1 2 6 1 2 1 3	-1 0 -1 20 -1 0 -1	0 2 0 -22 0 2 -2

Case E_7

$C^{(1)}=$ 

	$\tilde{w}$	∂_0	∂_∞
$C^{(1)}$	21 2 21 21 21	10 -2 10 10 10	-11 0 -11 -11 -11

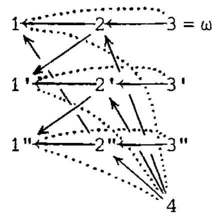
$C^{(2)} =$



$\tilde{w}$	∂_0	∂_∞
1 1	-1 0	0 1
1 1	-1 0	0 1
4	2	-2
1 1	-1 0	0 1
1 1	-1 0	0 1

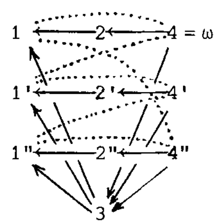
<u>Case</u> $\mathbb{E}_8$

$C^{(1)} =$



$\tilde{w}$	∂_0	∂_∞
241	-220	0-21
241	-220	0-21
241	-220	0-21
3	0	3

$C^{(2)} =$



$\tilde{w}$	∂_0	∂_∞
111	-210	0-12
111	-210	0-12
111	-210	0-12
3	3	-3

For any such algebra C, we also have noted the coefficients of two linear forms $\partial_0, \partial_\infty : K_0(\Lambda[R]) \longrightarrow \mathbb{Z}$ which will be of interest. Note that the restriction C_0 of C to the support of ∂_0, as well as the restriction C_∞ of C to the support of ∂_∞, are tame hereditary algebras. Note that always C is obtained from C_0 by a sequence of simple tubular extensions, and from C_∞ by a sequence of cosimple tubular extensions (see [3] for the definition). We remark that the two algebras C_0 and C_∞ are uniquely determined by C and that $\partial_0, \partial_\infty$ are, up to scalar multiples, the usual defect forms for these algebras. We have arranged $\partial_0, \partial_\infty$ in such a way that ∂_W is just a scalar multiple of $\partial_0 + \partial_\infty$.

Note that by definition <u>dim</u> W is obtained from $\tilde{w}$ by deleting the component of the vertex ω. In this way, we obtain the dimension vector of an indecomposable Λ-module belonging to the preinjective component of Λ, and therefore W is uniquely determined by <u>dim</u> W. Also, the explicit construction of the corresponding preinjective component of Λ (which is a routine procedure) shows that W is a separating wing module of type $\tilde{\Delta}$. Of course, it is also straightforward to check that the two linear forms $\langle \underline{\dim} R,-\rangle$ and $2(\underline{\dim} W,-)$ on $K_o(\Lambda)$ coincide. Thus, it only remains to be seen that proj.dim $W(\rho) \leq 1$ for all $0 \neq \rho : R \longrightarrow W$. In order to deal with this question, we recall that any given algebra C is obtained from a tame hereditary algebra C_∞ by a sequence of cosimple tubular extensions, and that $\partial_\infty(\tilde{w}) < 0$ in all cases. As a consequence, given any $0 \neq \rho : R \to W$ then $W(\rho)$ is an indecomposable $\Lambda[R]$-module not belonging to a component of $\Lambda[R]$ containing injective modules. Thus, we can invoke the dual of the following lemma:

Lemma. Let B be obtained from a tame concealed algebra by a sequence of simple tubular extensions, and let M be an indecomposable B-module belonging to a component of B containing no indecomposable projective B-module. Then inj. dim $M \leq 1$.

Proof, by induction on the number of simple tubular extensions. If B itself is a tame concealed algebra, then there are at most finitely many indecomposable modules with injective dimension ≥ 2 and all belong to the preprojective component. Now assume the result is true for B, take a ray module V in $\mathcal{T}(B,A)$, and let $C = B[V,n]$ for some n. Given a C-module ${}_C Y$, let Y' denote the maximal B-submodule of Y. It is obvious that inj.dim. $Y/Y' \leq 1$. Now let ${}_C Y$ be indecomposable, and assume that $\mathrm{Ext}^2_C(S(i),Y') \neq o$ for some simple C-module $S(i)$. If $S(i)$ is in fact a B-module, then also $\mathrm{Ext}^2_B(S(i),Y') \neq o$, since a minimal projective resolution of $S(i)$ considered as a C-module only contains B-modules, thus Y' contains at least one indecomposable direct summand in a component of B containing a projective module, and therefore Y' is indecomposable and the component of C containing Y also contains a projective module. Now assume $S(i)$ is not a B-module, and let $P(i)$ be its projective cover. There is only one possibility for $P(i)$, namely rad $P(i)$ has to be the direct sum of V and a projective module (since otherwise proj. dim $S(i) \leq 1$). Thus $\mathrm{Ext}^2_C(S(i),Y') = \mathrm{Ext}^1_C(\mathrm{rad}\, P(i),Y') = \mathrm{Ext}^1_C(V,Y') = \mathrm{Ext}^1_B(V,Y')$, and this is nonzero only in case $\mathrm{Hom}(Y',\tau V) \neq o$. However, this implies that Y' contains at least one indecomposable summand which is either in the preprojective component of B or in the component of B containing V, and therefore again Y' is indecomposable and the component of C containing Y is either the preprojective component of C or else the component of C containing V. This finishes the proof of the lemma.

Remark: In all cases considered, fixing both Λ and R, the module W is not the only wing module satisfying both conditions

$$\langle \underline{\dim}\ R,-\rangle = 2(\underline{\dim}\ W,-)$$

and $\text{proj.-dim.}W(\rho) \leq 1$ for all $o \neq \rho : R \longrightarrow W$. Actually, there are infinitely many wing modules having these properties, and all are of the same type. This shows that we obtain infinitely many separating tubular series in the category of all $\Lambda[R]$-modules, all being of the same type. However, for the moment, we are satisfied with the single wing module W exhibited above and with the tubular series produced by W, since we will encounter in the next section an algorithm which produces not only infinitely many, but actually <u>all</u> tubular series in the category of all $\Lambda[R]$-modules.

5. The elementary shrinking functor.

We consider the set $\mathcal{C}$ of algebras consisting of those exhibited in the last section as well as their duals and given an algebra $C \in \mathcal{C}$, we construct further separating tubular series by using appropriate functors with values in the category ${}_C\mathcal{M}$.

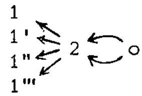
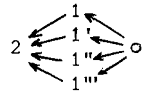
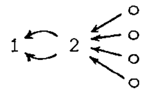
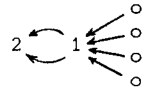
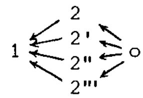
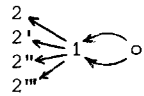
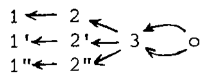
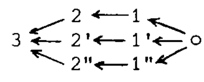
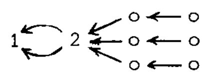
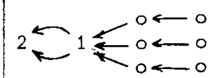
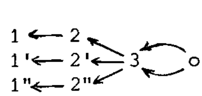
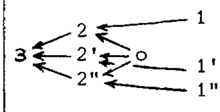
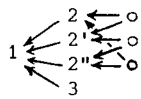
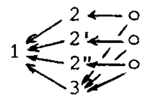
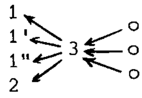
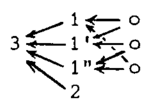
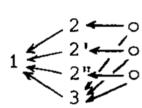
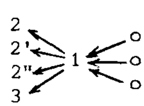
The functors used will be compositions of the following ones which we will call the elementary left shrinking functors, denoted by φ_ℓ, and which are defined on the module category ${}_{C_\ell}\mathcal{M}$, with C_ℓ again in $\mathcal{C}$. Always, C and C_ℓ will be tubular extensions of tame hereditary algebras C_o and $(C_\ell)_o$ of the same type, and the image of a C_ℓ-module M under the functor φ_ℓ restricted to C_o will be obtained from the restriction of M to $(C_\ell)_o$ by a sequence of Bernstein-Gelfand-Ponomarev reflection functors [1], whereas the restriction of $M\phi_\ell$ to the complement of C_o will be unchanged.

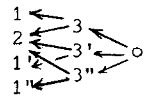
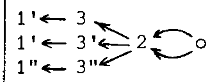
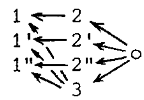
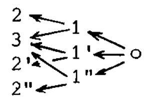
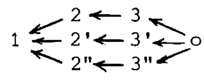
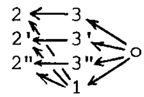
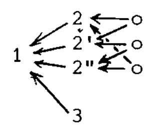
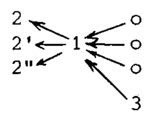
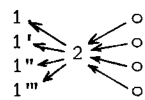
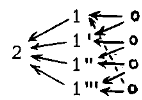
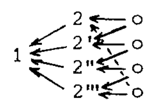
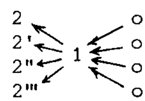
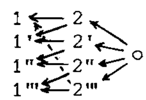
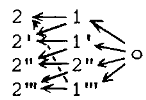
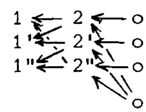
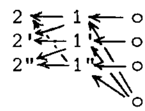
We recall the relevant definitions: Given the path algebra A of a quiver Σ, and a a sink of Σ, let Σ' be obtained from Σ by changing the orientation of all arrows ending in a, and A' the path algebra of Σ'. By $S_a^+ : {}_A\mathcal{M} \to {}_{A'}\mathcal{M}$ we denote the corresponding Bernstein-Gelfand-Ponomarev reflection functor; it annihilates the simple A-module $S_A(a)$, and gives an equivalence between the full subcategory of all A-modules without direct summands of the form $S_A(a)$, onto the full subcategory of all A'-modules without direct summands of the form $S_{A'}(a)$. Given an A-module V, we may extend S_a^+ to a functor

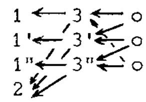
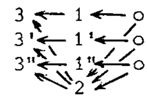
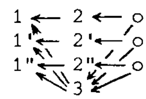
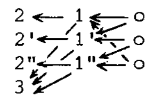
$$A[V] \longrightarrow A'[S_a^+V] ,$$

also denoted by S_a^+, as follows: Given an $A[V]$-module $({}_AX, {}_kY, \alpha : {}_AV \otimes_k Y \longrightarrow {}_AX)$, its image will be $(S_a^+X, Y, S_a^+\alpha)$. Given a sink-sequence $a_1, \ldots, a_n$ of Σ (called (+)-admissible sequence in [1]), we denote the composition of $S_{a_1}^+, \ldots, S_{a_n}^+$ just by $S_{a_1 \ldots a_n}^+$.

With these notations, the elementary left shrinking functors $\varphi_\ell : {}_{C_\ell}\mathcal{M} \longrightarrow {}_C\mathcal{M}$ are the following functors. The algebras C, C_ℓ are listed by the underlying quiver, for the relations we refer to the previous section, and the vertices of C_o and $(C_\ell)_o$ are indexed in order to identify φ_ℓ.

	φ_ℓ	C_ℓ	C
Case $\mathbb{D}_{4\lambda}$			
	$s^+_{11'1''1'''}$		
	s^+_1		
	s^+_1		
Case $\mathbb{E}_6$			
	$s^+_{11'1''22'2''11'1''}$		
	s^+_1		
	$s^+_{11'1''22'2''311'1''22'2''11'1''}$		
	$s^+_{122'2''3}$		
	$s^+_{11'1''2}$		
	s^+_1		

φ_ℓ	C_ℓ	C
s_2^+		
$s_{11'1''}^+$		
s_1^+		
$s_{122'2''313}^+$		
<u>Case</u> $\mathbb{E}_7$		
$s_{11'1''1'''}^+$		
s_1^+		
$s_{11'1''1'''}^+$		
<u>Case</u> $\mathbb{E}_8$		
$s_{11'1''}^+$		

φ_ℓ	C_ℓ	C
$S^+_{11'1''2}$		
$S^+_{11'1''}$		

Note that in the cases $\widetilde{\mathbb{D}}_{4\lambda}, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$, the given functor is given by the composition of the reflection functors of all possible sinks.

The main property of these elementary left shrinking functors relates to the tubular series constructed in the previous section: Recall that for any algebra $C \in \mathcal{C}$, we have constructed in the last section a tubular series which we will denote by $\mathcal{T}_1$, and the indecomposable C-modules not belonging to $\mathcal{T}_1$ fall into two disjoint classes $\mathcal{P}_1$, $\mathcal{Q}_1$ of modules such that $\mathcal{T}_1$ separates from $\mathcal{P}_1$ from $\mathcal{Q}_1$. In fact, given $\alpha, \beta \in \mathbb{N}$, let $\partial_{\alpha:\beta} = \alpha\partial_o + \beta\partial_\infty$, this is a linear form on $K_o(C)$. For example, for $\alpha = \beta = 1$, we just obtain $\partial_{1:1} = \partial_1$. For arbitrary $\alpha, \beta \in \mathbb{N}$, let $\mathcal{P}_{\frac{\alpha}{\beta}}$, $\mathcal{T}_{\frac{\alpha}{\beta}}$, $\mathcal{Q}_{\frac{\alpha}{\beta}}$ be the set of indecomposable C-modules M satisfying $\partial_{\alpha:\beta}(\underline{\dim}\ M) < o$, or $= o$, or $> o$, respectively. Also,we know that C is obtained from C_o by a sequence of simple tubular extensions, thus C has a preprojective component which we denote by $\mathcal{P}_o$, and a tubular series $\mathcal{T}_o$ being obtained from the tubular series of all regular C_o-modules by ray insertions [3], and we denote by $\mathcal{Q}_o$ the set of indecomposable C-modules not belonging to $\mathcal{P}_o$ or $\mathcal{T}_o$ (note that $\partial_o(\underline{\dim}\ M) < o$ characterizes those indecomposable modules M belonging to $\mathcal{P}_o$, that $\partial_o(\underline{\dim}\ M) = o$ for all M in $\mathcal{T}_o$ and that all but finitely many modules in $\mathcal{Q}_o$ satisfy $\partial_o(\underline{\dim}\ M) > o)$. Similarly, using that C is obtained from C_∞ by a sequence of cosimple tubular extensions, there is a preinjective component, denoted by $\mathcal{Q}_\infty$, a tubular series $\mathcal{T}_\infty$ obtained from the tubular series in ${}_{C_\infty}\mathcal{M}$ by coray insertions, and the remaining indecomposable module will form the set $\mathcal{P}_\infty$. Note that these subcategories $\mathcal{P}_\gamma$, $\mathcal{T}_\gamma$, $\mathcal{Q}_\gamma$, with $\gamma \in \mathbb{Q}_{\geq o} \,\dot{\cup}\, \{\infty\}$ are defined for any algebra $C \in \mathcal{C}$, however it does not seem to be necessary to make a reference to C, since it always should be clear in which category ${}_C\mathcal{M}$ we are working.

<u>Proposition.</u> Let $\varphi_\ell : {}_{C_\ell}\mathcal{M} \longrightarrow {}_C\mathcal{M}$ be an elementary left shrinking functor. The functor φ_ℓ annihilates a finite set $\mathcal{A}$ of modules in $\mathcal{P}_o$, and induces an equivalence between the full subcategory of all C_ℓ-modules without a direct summand in $\mathcal{A}$, and the image category of φ_ℓ. The image of $\mathcal{P}_\infty$ under φ_ℓ is just $\mathcal{P}_1$, the image of $\mathcal{T}_\infty$ under φ_ℓ is contained in $\mathcal{T}_1$, and any tube of $\mathcal{T}_1$ is obtained by ray-insertion from the image of a tube in $\mathcal{T}_\infty$ under φ_ℓ. Also, for any $\gamma \in \mathbb{Q}_{\geq 0}$, the functor φ_ℓ induces an equivalence of $\mathcal{T}_\gamma$ with $\mathcal{T}_{\gamma\varphi_\ell}$, where $\frac{\alpha}{\beta}\varphi_\ell = \frac{\alpha}{\alpha+\beta}$ for $\alpha,\beta \in \mathbb{N}$.

As a consequence, we obtain in ${}_C\mathcal{M}$ a new separating tubular $\mathbb{P}_1k$-series, namely $\mathcal{T}_{\frac{1}{2}}$ (being the image of the separating tubular series $\mathcal{T}_1$ in ${}_{C_\ell}\mathcal{M}$).

For any algebra $C \in \mathcal{C}$, there also is defined a corresponding elementary <u>right</u> shrinking functor $\varphi_r : {}_{C_r}\mathcal{M} \longrightarrow {}_C\mathcal{M}$. Note that with C also the dual algebra C^* is contained in $\mathcal{C}$, thus denoting by $\varphi_\ell(C^*) : {}_{(C^*)_\ell}\mathcal{M} \rightarrow {}_{C^*}\mathcal{M}$ the elementary left shrinking functor for the algebra C^*, let $C_r = ((C^*)_\ell)^*$, and $\varphi_r : {}_{C_r}\mathcal{M} \rightarrow {}_C\mathcal{M}$ the dual functor of $\varphi_\ell(C^*)$.

6. Self-reproduction of separating tubular series

Let I be the set of positive rational numbers, and define $\varphi_\ell, \varphi_r : I \longrightarrow I$ by $\frac{\alpha}{\beta}\varphi_\ell = \frac{\alpha}{\alpha+\beta}$, $\frac{\alpha}{\beta}\varphi_r = \frac{\alpha+\beta}{\alpha}$ for $\alpha,\beta \in \mathbb{N}$.

Lemma. Any element of I can be written in a unique way as $1\varphi_{i_1} \dots \varphi_{i_n}$, with $i_1,\dots,i_n \in \{\ell,r\}$.

The proof is be induction on $N(\frac{\alpha}{\beta}) := \alpha+\beta$, where $\alpha,\beta \in \mathbb{N}$ are without common disisor. Note that $N(\gamma\varphi_\ell) = N(\gamma\varphi_r) > N(\gamma)$ for any $\gamma \in I$. The smallest possible value of N is 2, in this case the assertion is obvious. Now assume the assertion is true for any γ with $N(\gamma) < m$, and let $N(\frac{\alpha}{\beta}) = \alpha+\beta = m$. If $\alpha < \beta$, then $\frac{\alpha}{\beta} = \frac{\alpha}{(\beta-\alpha)+\alpha} = \frac{\alpha}{\beta-\alpha}\varphi_\ell$ and $N(\frac{\alpha}{\beta-\alpha}) < m$. Using induction, we see that $\frac{\alpha}{\beta}$ can be written as $1\varphi_{i_1} \dots \varphi_{i_n}\varphi_\ell$, and also that this expression is unique. Similarly, we argue in case $\alpha > \beta$.

Consider again any algebra $C \in \mathcal{C}$, and let $\gamma \in I$. The lemma above shows that $\gamma = 1\varphi_{i_1} \dots \varphi_{i_n}$ for some $i_1,\dots,i_n \in \{\ell,r\}$ and it follows from the proposition in the previous section (and the dual assertion) that the restriction of the composite functor

$$\varphi_{i_1} \dots \varphi_{i_n} : C_{i_n\dots i_1}{}^M \longrightarrow C^M$$

defines an equivalence from $\mathcal{T}_1$ in $C_{i_n\dots i_1}{}^M$ onto $\mathcal{T}_\gamma$ in C^M. As a consequence, $\mathcal{T}_\gamma$ is a tubular $\mathbb{P}_1k$-series and also it follows that $\mathcal{T}_\gamma$ separates $\mathcal{P}_\gamma$ from $\mathcal{Q}_\gamma$. In this way, we obtain for any $\gamma \in I$ a separating tubular $\mathbb{P}_1k$-series.

On the other hand, we have noted above that the structure of $\mathcal{P}_o$ and $\mathcal{T}_o$, as well as of $\mathcal{T}_\infty$ and $\mathcal{Q}_\infty$, is completely known due to previous investigations. Thus, we only have to consider $\mathcal{Q}_o \cap \mathcal{P}_\infty$. We claim that $\mathcal{Q}_o \cap \mathcal{P}_\infty$ is the disjoint union of all $\mathcal{T}_\gamma$, with $\gamma \in I$. Namely, given an indecomposable module X in $\mathcal{Q}_o \cap \mathcal{P}_\infty$, then $\partial_o(\underline{\dim}\, X) > o$, $\partial_\infty(\underline{\dim}\, X) < o$, thus let $\gamma = -\frac{\partial_\infty(\underline{\dim}\, X)}{\partial_o(\underline{\dim}\, X)}$, then obviously X belongs to $\mathcal{T}_\gamma$. Altogether, we see:

Theorem 1. Any algebra $C \in \mathcal{C}$ of type (Δ) has the following components:

(1) a preprojective component $\mathcal{P}_o$ (containing precisely the preprojective C_o-modules).

(2) a separating tubular $\mathbb{P}_1 k$-series $\mathcal{T}_o$ (obtained from the tubular series of C_o by ray-insertions).

(3) for any $\gamma \in I$, a separating tubular $\mathbb{P}_1 k$-series $\mathcal{T}_\gamma$ of type (Δ) (consisting of all indecomposable modules X with $\partial_{\alpha:\beta}(\underline{\dim}\, X) = o$, where $\gamma = \frac{\alpha}{\beta}$ and $\alpha, \beta \in \mathbb{N}$).

(2)* a separating tubular $\mathbb{P}_1 k$-series $\mathcal{T}_\infty$ (obtained from the tubular series of C_∞ by coray insertions).

(1)* a preinjective component $\mathcal{Q}_\infty$ (containing precisely the preinjective C_∞-modules).

Considering in the same way positive roots of q_C instead of indecomposable C-modules, we see:

Theorem 2. Let C be an algebra in $\mathcal{C}$. For any indecomposable C-module M, the dimension vector $\underline{\dim}\, M$ is a connected positive vector in $K_o(C)$ satisfying $q_C(\underline{\dim}\, M) = 0$ or 1. Conversely, given a positive connected vector x in $K_o(C)$ with $q_C(x) = 1$, there exists precisely one isomorphism class of indecomposable C-modules M with $\underline{\dim}\, M = x$. Given a positive connected vector x in $K_o(C)$ with $q_C(x) = 0$, there is a one-parameter family of isomorphism classes of indecomposable C-modules M satisfying $\underline{\dim}\, M = x$.

Here, a vector in $K_o(C)$ is said to be connected provided its support is a connected subset of the quiver of C. (An example of a non-connected positive root is furnished by $\begin{smallmatrix} & & 0 & 1 \\ 1 & 1 & 0 & 0 \\ & & 0 & 0 \end{smallmatrix}$ for the algebra $C^{(2)}$ of case $(\tilde{\mathbb{E}}_6)$.)

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