The derived category of a tubular algebra

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Tubular algebras are rather special algebras of global dimension 2 with 6,8,9 or 10 simple modules, but their module categories seem to be of wider interest. For a definition, we refer to [8]; we note that typical examples are the canonical tubular algebras, these are the canonical algebras of type (2,2,2,2), (3,3,3), (4,4,2) and (6,3,2); a description of these canonical algebras by quivers and relations will be recalled below. The aim of this note is to outline that previous results of d'Este and the authors can be combined in order to obtain a rather complete description of the derived category $D^b(A\text{-mod})$ of a tubular algebra A. For a definition of the derived category $D^b(A)$ (of bounded complexes) over an abelian category A we refer to the original article by Verdier [9]. We will freely use the notation and terminology of [8].

In a first step we note that it is sufficient to consider the case of a canonical tubular algebra.

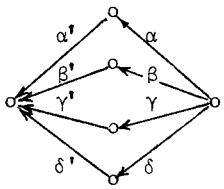
1. Reduction to canonical tubular algebras C

Given a tubular algebra A, there exists a canonical tubular algebra C of the same type with an equivalence $D^b(A\text{-mod}) \approx D^b(C\text{-mod})$ of triangulated categories.

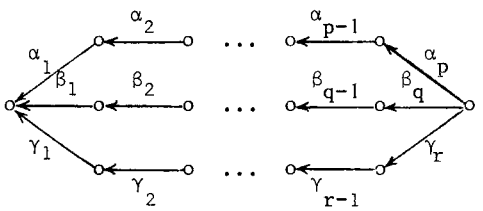
Proof. According to [5,6], the derived category does not change under tilting. Let A be a tubular algebra of type $\mathbf{T}$. According to [8], 5.7.3, there is a tubular extension B of a tame concealed canonical algebra, of extension type $\mathbb{T}$, and a left shrinking functor, thus a tilting functor A-mod $\longrightarrow$ B-mod. According to [8], 4.8.1, we know that B^{op} is the one point extension of a tame concealed bush algebra of branching type $\mathbf{T}$ by a coordinate module. According to [8], 5.7.2, there is a canonical algebra C' and a left shrinking functor B^{op}-mod $\longrightarrow$ C'-mod. Let $C = (C')^{op}$, then C again is a canonical algebra, and of type $\mathbb{T}$, and there is a tilting functor C-mod $\longrightarrow$ B-mod. Altogether, we have $D^b(A\text{-mod}) \approx D^b(B\text{-mod}) \approx D^b(C\text{-mod})$.

2. Description of $\hat{C}$-mod

We consider now the case of C a canonical tubular algebra, say of type $\mathbb{T}$. Actually, instead of dealing with $D^b(C\text{-mod})$, we consider the category $\hat{C}$-mod, since the categories $D^b(C\text{-mod})$ and $\hat{C}$-mod are equivalent (even as triangulated categories), according to [5,6]. Let us recall the structure of a canonical tubular algebra and the construction of $\hat{C}$. Let k be an algebraically closed field. The canonical algebras of type $(2,2,2,2)$ are defined by the quiver



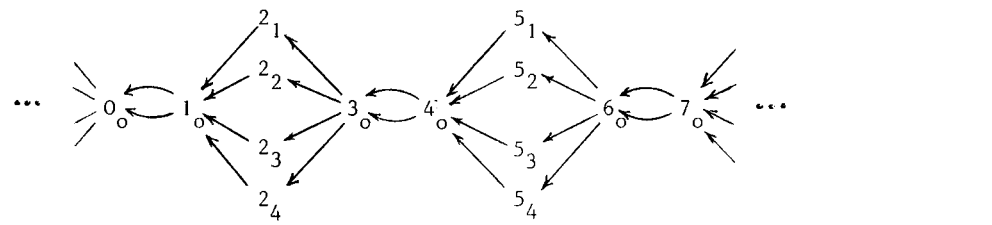
with $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$, $\alpha\alpha' + \lambda\beta\beta' + \delta\delta' = 0$, where λ is some fixed element in $k \setminus \{0,1\}$ (for different λ,λ', we usually obtain non-isomorphic algebras, the only isomorphisms are for $\lambda' = 1-\lambda$, $\frac{1}{\lambda}$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{\lambda-1}$ and $\frac{\lambda-1}{\lambda}$). The canonical algebras of type (p,q,r) are given by the quiver

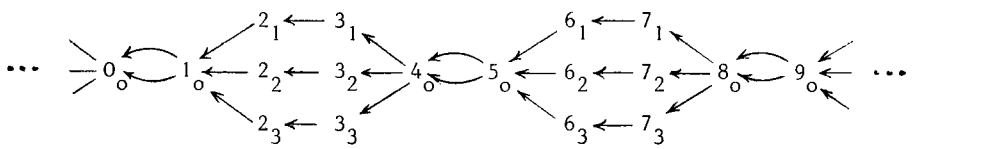


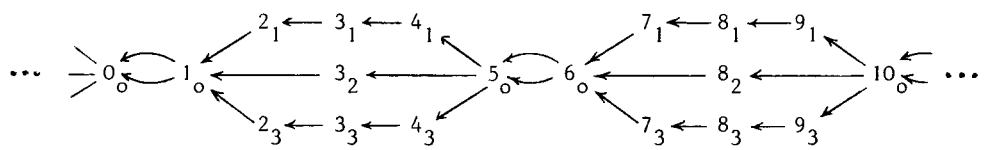
with $\alpha_p\alpha_{p-1} \cdots \alpha_1 + \beta_q\beta_{q-1} \cdots \beta_1 + \gamma_r\gamma_{r-1} \cdots \gamma_1 = 0$; the only tubular ones are those of type $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$. Given a finite dimensional algebra A, with $Q = \mathrm{Hom}(A,k)$, considered as an A-A-bimodule, Hughes - Waschbüsch [7] have introduced the (infinite dimensional) algebra

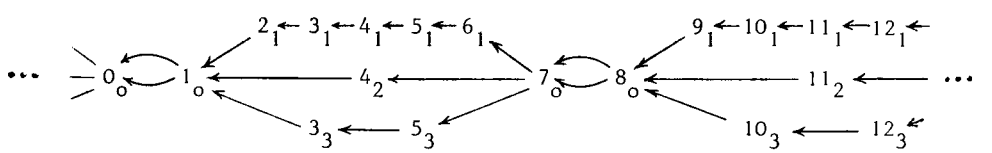
$$\hat{A} = \begin{bmatrix} \ddots & \ddots & & 0 \\ & A(i-1) & Q(i) & \\ & & A(i) & Q(i+1) \\ & & & A(i+1) & \ddots \\ 0 & & & & \ddots \end{bmatrix}$$

of all double-infinite matrices having only finitely many non-zero entries, on the main diagonal being from copies $A(i)$ of A, those on the upper next diagonal being from copies $Q(i)$ of Q, with $i \in \mathbb{Z}$, and with multiplication given by the A-A-bimodule structure on Q, and zero composition $Q \otimes Q \longrightarrow 0$. The quiver Δ of $\hat{C}$, where C is a canonical tubular algebra, is given as follows:









In case C is of type $(2,2,2,2)$, $(3,3,3)$, $(4,4,2)$, or $(6,3,2)$, let us denote by d the number $2,3,4$, or 6, respectively, and let $d' = d+1$. We denote by Δ_{3n} the full subquiver of Δ given by the vertices $(nd')_o$ and $(nd'+1)_o$, it is a copy of the Kronecker quiver. We denote by Δ_{3n+1} the full subquiver given by the vertices a_i, with $nd'+1 \leq a \leq nd'+d$, and all possible i; it is a subspace quiver of type $\mathbb{T}$. Finally, we denote by Δ_{3n+2} the full subquiver given by the vertices a_i, with $nd'+2 \leq a \leq (n+1)d'$, and all possible i; it is a factorspace quiver of type $\mathbb{T}$. If $m \leq m'$ are integers, let $\Delta_{mm'}$ be the full subquiver of Δ given by the vertices in the union of all $\Delta_{m''}$, with $m \leq m'' \leq m'$.

Similarly, let $\Delta_{m\infty}$ be the full subquiver of Δ given by the vertices in $\Delta_{m'}$, with $m \leq m'$, and $\Delta_{-\infty,m}$ the full subquiver of Δ given by the vertices in $\Delta_{m'}$, with $m' \leq m$. We denote by C_m the restriction of $\hat{C}$ to Δ_m, and by $C_{mm'}$ the restriction of $\hat{C}$ to $\Delta_{mm'}$. Note that the algebras $C_{3n+1,3n+2}$ are isomorphic canonical algebras, the algebras $C_{3n,3n+1}$ are isomorphic "left squids", and the algebras $C_{3n+2,3n+3}$ are isomorphic "right squids" [1]; all these algebras $C_{m,m+1}$ are tubular of type $\mathbb{T}$. With these notations, we can collect the information available for the category $\hat{C}$-mod.

We fix some $m \in \mathbf{Z}$. Since C_m is a tame hereditary algebra, we may speak of preprojective, regular, and preinjective C_m-modules. The minimal positive radical vector of $K_o(C_m)$, considered as an element of $K_o(\hat{C})$, will be denoted by h_m. Let $\mathcal{T}_m$ be the module class given by all $\hat{C}$-modules with restriction to C_m being non-zero and regular. Let $\mathcal{P}_m$ be the $C_{-\infty,m}$-modules with restriction to C_m being preprojective, and $\mathcal{Q}_m$ the $C_{m,\infty}$-modules with restriction to C_m being preinjective. Then

$$\hat{C}\text{-mod} = \mathcal{P}_m \vee \mathcal{T}_m \vee \mathcal{Q}_m ,$$

$\mathcal{T}_m$ is a tubular family separating $\mathcal{P}_m$ from $\mathcal{Q}_m$, being obtained from the tubular family of C_m by ray insertions and coray insertions, and all modules in $\mathcal{T}_m$ are actually $C_{m-1,m+1}$-modules.

Proof. We use [3]. Since $C_{m-1,m+1}$ is a tubular extension and a tubular coextension of the tame hereditary algebra C_m, the category $C_{m-1,m+1}$-mod can be written in the form

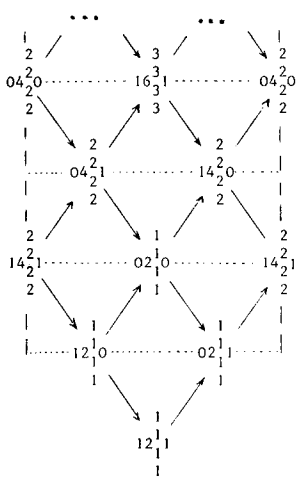
$$C_{m-1,m+1}\text{-mod} = \mathcal{P}'_m \vee \mathcal{T}_m \vee \mathcal{Q}'_m$$

where $\mathcal{T}_m$ is the tubular family obtained from the tubular family of C_m by ray insertions and coray insertions, where $\mathcal{P}'_m$ contains only $C_{m-1,m}$-modules with restriction to C_m being preprojective, and $\mathcal{Q}'_m$ contains only $C_{m,m+1}$-modules with restriction to C_m being preinjective, and $\mathcal{T}_m$ separates $\mathcal{P}'_m$ from $\mathcal{Q}'_m$. Note that the restriction of any indecomposable module in $\mathcal{T}_m$ to C_m is non-zero, since the simple projective $C_{m-1,m-1}$-modules belong to $\mathcal{P}'_m$, the simple injective $C_{m-1,m-1}$-modules belong to $\mathcal{Q}'_m$. Thus $\mathcal{P}'_m$ contains all $C_{m-1,m}$-modules with restriction to C_m preprojective, and $\mathcal{Q}'_m$ contains all $C_{m,m+1}$-modules with restriction to C_m preinjective. We obtain $C_{m-1,\infty}$

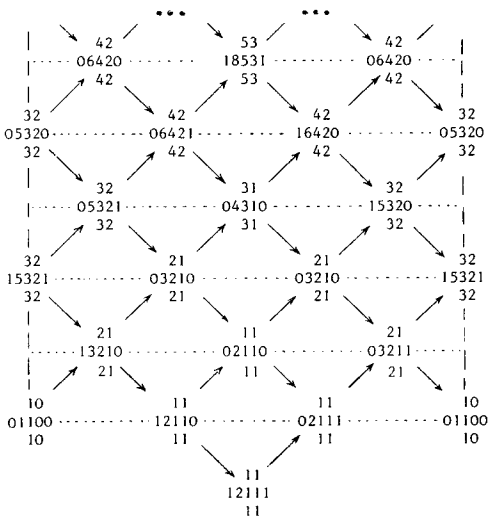
from $C_{m-1,m+1}$ by the successive one-point extensions using modules with restriction to $C_{m-1,m+1}$ belonging to $\mathcal{Q}'_m$; in this way both $\mathcal{P}'_m$ and $\mathcal{T}_m$ remain untouched as unions of components of the Auslander-Reiten quiver, whereas the additional modules together with those in $\mathcal{Q}'_m$ give $\mathcal{Q}_m$, and $\mathcal{T}_m$ separates $\mathcal{P}'_m$ from $\mathcal{Q}_m$. We obtain $\hat{C}$ from $C_{m-1,\infty}$ by successive one-point co-extensions using modules with restriction to $C_{m-1,\infty}$ belonging to $\mathcal{P}'_m$, thus now $\mathcal{T}_m$ and $\mathcal{Q}_m$ remain untouched, and the additional modules together with those in $\mathcal{P}'_m$ give $\mathcal{P}_m$.

Before we proceed, let us desribe in more detail the structure of the tubular families $\mathcal{T}_m$. They are indexed over the projective line $\mathbb{P}_1 k$, thus $\mathcal{T}_m = \bigvee_{\rho \in \mathbb{P}_1 k} \mathcal{T}_m(\rho)$. Note that any indecomposable module in $\mathcal{T}_m$ is either τ-periodic or else projective-injective. First, consider the case $m = 3n+1$. In this case, all but one of the tubes are stable, thus contain only C_{3n+1}-modules. The remaining one, say for the index $\rho = \infty$ is obtained from a stable tube in C_{3n+1}-mod of rank $d-1$ by inserting one ray and one coray. The tube $\mathcal{T}_m(\infty)$ has a unique projective-injective vertex, and all other vertices are stable. We indicate the shape of $\mathcal{T}_m(\infty)$ for the various cases, replacing any vertex by the corresponding dimension vector in $K_o(C_{m-1,m+1})$; of course, the vertical boundary lines have to be identified in order to obtain a tube.

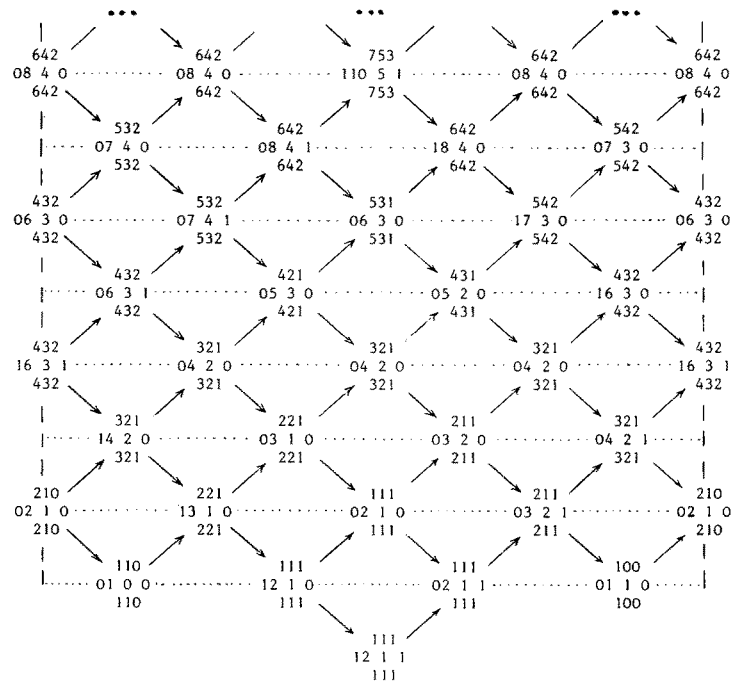
$\mathbb{T} = (2,2,2,2)$



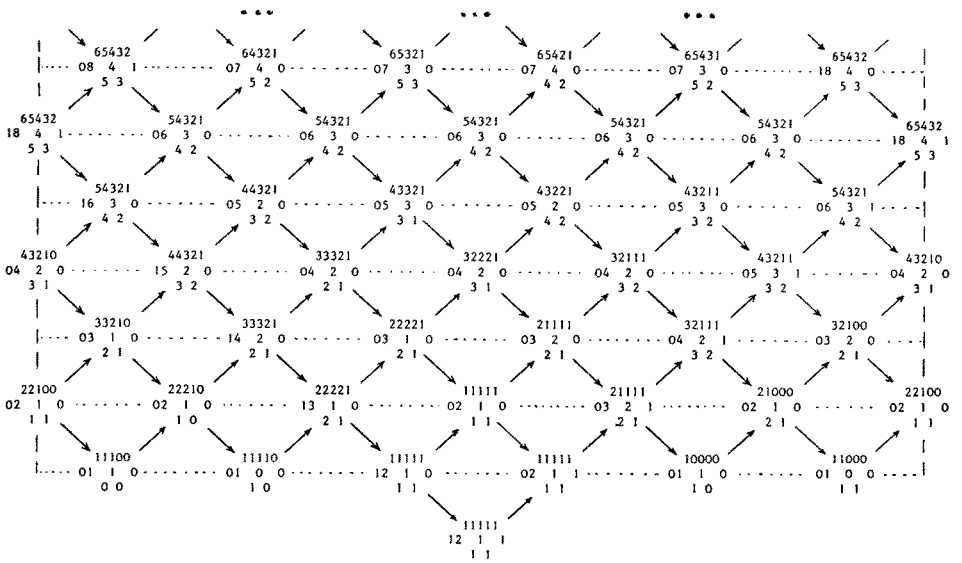
$\mathbb{T} = (3,3,3)$



$\mathbb{T} = (4,4,2)$



$\mathbb{T} = (6,3,2)$

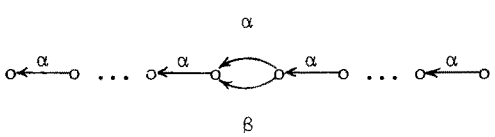


In the stable category $\hat{C}$-mod, we obtain from $\mathcal{T}_m(\infty)$ a standard stable tube of rank d. Consequently, in $\hat{C}$-mod, we obtain from $\mathcal{T}_m$ a standard stable tubular family of type $\mathbb{T}$. In particular, all non-projective indecomposable modules in $\mathcal{T}_m$ are τ-periodic of period d.

The case $m = 3n+2$ is similar to the case $m = 3n+1$; actually, the algebras $C_{3n-1,3n+3}$ are opposite to those of the form $C_{3n,3n+2}$.

So let us consider the case $m = 3n$. In this case all stable tubes of $\mathcal{T}_m$ are homogeneous (i.e. of the form $\mathbb{ZA}_\infty/1$). For $\mathbb{T} = (2,2,2,2)$, there are four non-stable tubes, everyone containing just one projective-injective vertex. For $\mathbb{T} = (p,q,r)$, there are three non-stable tubes, containing $p-1$, $q-1$, and $r-1$ projective-injective vertices, respectively. In the stable category $\hat{C}$-mod, we obtain from the non-stable tubes in $\mathcal{T}_m$ four standard stable tubes of rank 2, in case $\mathbb{T} = (2,2,2,2)$, and three standard stable tubes of rank p,q,r, in case $\mathbb{T} = (p,q,r)$, thus again $\mathcal{T}_m$ gives rise, in $\hat{C}$-mod, to a standard stable tubular family of type $\mathbb{T}$. Again we want to indicate the shape of the exceptional tubes $\mathcal{T}_m(\rho)$. In case the stable rank of $\mathcal{T}_m(\rho)$ is p, the modules in $\mathcal{T}_m(\rho)$ are defined over a subalgebra given by the restriction of $\hat{C}$ to a full (convex) subquiver of the form

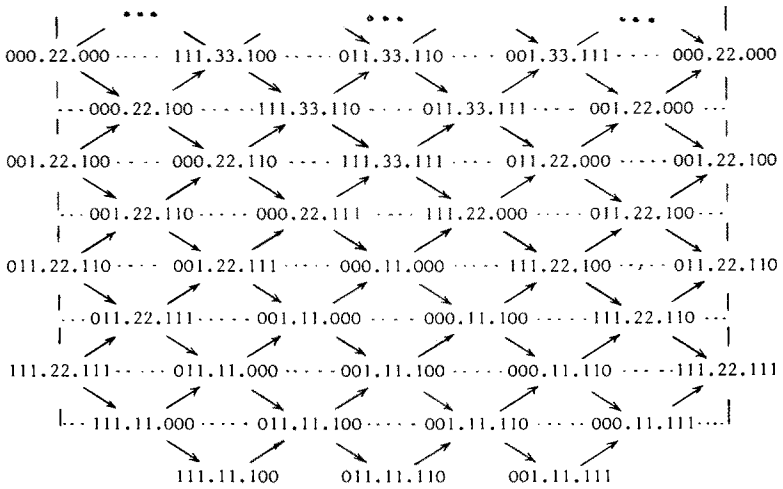
(*)



with 2p vertices (p to the left and p to the right of the double arrow), and there are the following relations:

(**) $$\alpha\beta = 0, \quad \beta\alpha = 0, \quad \alpha^{p+2} = 0.$$

Of course, any algebra with quiver (*) and relations (**), different from the Kronecker algebra, has precisely one non-stable tube. For example, for the quiver (*) with both four vertices to the left, and to the right of the double arrow, and the relations (**), the non-stable tube is of the following form (again, the vertical boundary lines have to be identified):



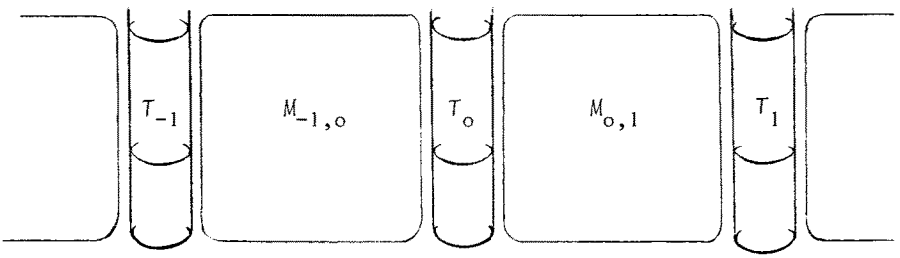
We consider now simulteneously the various $m \in \mathbb{Z}$. Let $M_{m,m+1}$ be the set of $C_{m,m+1}$-modules with restriction to C_m being preinjective, and with restriction to C_{m+1} being preprojective, thus

$$M_{m,m+1} = P_{m+1} \cap Q_m,$$

and

$$\hat{C}\text{-mod} = \bigvee_{m \in \mathbb{Z}} T_m \vee \bigvee_{m \in \mathbb{Z}} M_{m,m+1}.$$

The categorical structure may be visualized as follows:



with maps only from left to right (and inside the individual module classes). Note that the indecomposable modules in T_m have support in $\Delta_{m-1,m+1}$, those in $M_{m,m+1}$ have support in $\Delta_{m,m+1}$. In particular, all indecomposable $\hat{C}$-modules have bounded support.

We remark that our account on the decomposition of $\hat{C}$-mod into the module classes $\mathcal{T}_m$ and $\mathcal{M}_{m,m+1}$ follows closely the treatment given by Gabriella d'Este in her Oberwolfach talk 1981 [2]. The module classes $\mathcal{M}_{m,m+1}$ have been described in section 5.2 of [8]. As a first invariant of an indecomposable module X in $\mathcal{M}_{m,m+1}$, its index has been defined in [8], it is an element of $\mathbb{Q}^+$. In our case, it seems more advisable to consider instead the $\wedge$-index, obtained from the index by changing the normalisation; the $\wedge$-index of a module in $\mathcal{M}_{m,m+1}$ is an element of the rational interval $\mathbb{Q}_m^{m+1} = \{\gamma \in \mathbb{Q} \mid m < \gamma < m+1\}$. The definition of the $\wedge$-index will be given below.

Recall that we have denoted by h_m the minimal positive radical vector of $K_o(C_m)$. Let $d_m = d$, for $m \equiv 0 (\mathrm{mod}\ 3)$, and $= 1$ otherwise. Given $\gamma \in \mathbb{Q}_m^{m+1}$, say $\gamma = m + \frac{\alpha}{\beta}$ with integers $0 < \alpha < \beta$, let h_γ be the minimal positive vector in $K_o(\hat{C})$ which is a rational multiple of $(\beta-\alpha)d_m h_m + \alpha d_{m+1} h_{m+1}$.

[Note that this definition of h_γ differs in two ways from that in [8]. First of all, γ is renormalized, as mentioned above, it is the $\wedge$-index of h_γ. Second,our minimality condition implies that the coefficients of h_γ are relative prime; in contrast, in [8] α,β were supposed to be relative prime, and then $\beta h_m + \alpha h_{m+1}$ was considered.]

On $K_o(\hat{C})$, there is defined the usual bilinear form $\langle -,- \rangle$ by

$$\langle e(a), e(b) \rangle = \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i(E(a),E(b)),$$

where $E(a)$ is the simple $\hat{C}$-module corresponding to the vertex a of Δ, and $e(a) = \underline{\dim}\, E(a)$. Note that the sum is indeed finite, since the restriction of $\hat{C}$ to any finite subquiver of Δ has finite global dimension, and we can evaluate $\mathrm{Ext}^i(E(a),E(b))$, by restricting to any full convex subquiver containing both a and b. The corresponding quadratic form is denoted by $\hat{\chi}$, thus $\hat{\chi}(x) = \langle x,x \rangle$. For any $\gamma \in \mathbb{Q}$, we denote by ι_γ the linear form $\iota_\gamma = \langle h_\gamma, - \rangle : K_o(\hat{C}) \longrightarrow \mathbb{Z}$. Also, let $\bar{C}_\gamma$ be defined as follows: For $\gamma = m \in \mathbb{Z}$, let $\bar{C}_m = C_{m-1,m+1}$, and for $\gamma \in \mathbb{Q}_m^{m+1}$ with $m \in \mathbb{Z}$, let $\bar{C}_\gamma = C_{m,m+1}$. With these notations,let

$$K_o(\hat{C})_\gamma = K_o(\bar{C}_\gamma) \cap \mathrm{Ker}\ \iota_\gamma.$$

For $\gamma \notin \mathbb{Z}$, define $\mathcal{T}_\gamma$ as the module class given by the indecomposable $\hat{C}$-modules X with $\underline{\dim}\, X \in K_o(\hat{C})_\gamma$. Thus, if $\gamma \in \mathbb{Q}_m^{m+1}$ (with $m \in \mathbb{Z}$),

then T_γ is given by the indecomposable $C_{m,m+1}$-modules X with $\langle h_\gamma, \underline{\dim}\, X\rangle = 0$; and we call these the $C_{m,m+1}$-modules with $\wedge$-index γ. Then, [8] asserts that

$$M_{m,m+1} = \bigvee_{\gamma \in \mathbb{Q}_m^{m+1}} T_\gamma,$$

and that T_γ is a standard stable tubular family of type $\mathbb{T}$, and is controlled by the restriction of $\hat{\chi}$ to $K_o(\hat{C})_\gamma$. Also, T_γ is separating, and it separates P_γ from Q_γ, where

$$(+) \qquad P_\gamma = \bigvee_{\beta<\gamma} T_\beta \quad , \quad Q_\gamma = \bigvee_{\gamma<\delta} T_\delta \ .$$

[According to [8], T_γ separates $P_\gamma \cap C_{m,m+1}$-mod from $Q_\gamma \cap C_{m,m+1}$-mod. Using the separation property of T_m and T_{m+1}, it easily follows that T_γ separates P_γ from Q_γ.]

Altogether, we see that

$$\hat{C}\text{-mod} = \bigvee_{\gamma \in \mathbb{Q}} T_\gamma,$$

where all T_γ are separating tubular families, separating P_γ from Q_γ, with P_γ and Q_γ given by (+), and that the stable tubular type of any T_γ is $\mathbb{T}$.

We should add the following remark. By definition, for $\gamma \notin \mathbb{Z}$, the module class T_γ is given by all indecomposable $\hat{C}$-modules X with $\underline{\dim}\, X \in K_o(\hat{C})_\gamma$. If $\gamma = m \in \mathbb{Z}$, and X is an indecomposable module in T_m, then clearly $\underline{\dim}\, X \in K_o(\hat{C})_m$; however, not all indecomposable $\hat{C}$-modules X with $\underline{\dim}\, X \in K_o(\hat{C})_m$ will belong to T_m, one needs in addition the condition that the restriction of X to Δ_m is non-zero. (For example, $E(2_1)$ is a $\bar{C}_o$-module and satisfies $\langle h_o, \underline{\dim}\, E(2_1)\rangle = 0$, however, $E(2_1)$ belongs to $T_{1/2}$).

We end this section by determing the position of the simple $\hat{C}$-modules. This will be needed in section 4, and it also gives a reason for the chosen normalization of the $\wedge$-index. Note that for any simple $\hat{C}$-module E, there is precisely one integer m such that E has support in $\Delta_m \cap \Delta_{m+1}$. (For the vertices $(nd')_o$, take $m = 3n-1$, for $(nd'+1)_o$, take $m = 3n$, and for the vertices a_i with $nd'+1 \leq a \leq nd'+d$, take $m = 3n+1$).

Lemma: Let E be a simple $\hat{C}$-module, with support in $\Delta_m \cap \Delta_{m+1}$. Then E considered as a $\hat{C}$-module, belongs to $\mathcal{T}_\gamma$ with $\gamma = m + \frac{1}{2}$.

Proof. One only has to verify that $\langle h_m + h_{m+1}, \underline{\dim}\, E\rangle = 0$.

We have obtained in this way an explicit description of $\hat{C}$-mod, and therefore also of $\hat{C}$-$\underline{\text{mod}}$. All components of $\hat{C}$-$\underline{\text{mod}}$ are stable tubes of rank a divisor of d, they form separating standard $\mathbb{P}_1 k$-families of type $\mathbb{T}$, and the set of these families may be indexed over $\mathbb{Q}$, in a rather natural way. Since $D^b(C\text{-mod}) \approx \hat{C}$-$\underline{\text{mod}}$, this could finish our investigation. However, the description of $D^b(C\text{-mod})$ outlined above is given in terms of $K_o(\hat{C})$, and it seems advisable to use more intrinsic invariants.

3. The additive function $\dim^A$ on $\hat{A}$-mod.

Consider an arbitrary finite dimensional algebra A. The algebra $\hat{A}$ has countably many subalgebras $A(i)$ isomorphic to A, and given a vertex a of A, we denote by $a(i)$ the corresponding vertex of $A(i)$. In this way we obtain all vertices of $\hat{A}$. In particular, $K_o(\hat{A}) = \bigoplus_{i \in \mathbb{Z}} K_o(A(i))$. We identify A with $A(o)$. We denote by $\hat{\nu}$ the canonical shift isomorphism, sending $A(i)$ to $A(i+1)$, thus $a(i)$ to $a(i+1)$. It induces a self-equivalence on $\hat{A}$-mod, again denoted by $\hat{\nu}$. Given a vertex b of $\hat{A}$, denote by $\hat{P}(b)$ the indecomposable projective $\hat{A}$-module with top the simple $\hat{A}$-module corresponding to b, and let $\hat{p}(b) = \underline{\dim}\, \hat{P}(b)$. Note that $\hat{P}(b)$ is also an indecomposable injective $\hat{A}$-module, and its socle is the simple A-module corresponding to $\nu^{-1}b$. Since $\hat{\nu}\hat{P}(b) = \hat{P}(\hat{\nu}b)$ for all vertices b of $\hat{A}$, it easily follows that $\hat{\nu}$ is the Nakayama functor for $\hat{A}$. We denote by $P(\hat{A})$ the subgroup of $K_o(A)$ generated by the dimension vectors $\hat{p}(b)$, with b a vertex of $\hat{A}$. If we denote by $p_A(a)$ the dimension vector of the indecomposable projective A-module with top corresponding to a, and by $q_A(a)$ that of the indecomposable injective A-module with socle corresponding to a, we have

3.1. $$\hat{p}(a(i)) = q_A(a)\hat{\nu}^{i-1} + p_A(a)\hat{\nu}^i \ .$$

(Applying $\hat{\nu}$ to an element x of $K_o(\hat{A})$, we write $\hat{\nu}$ to the right of x; since we think of x as a row vector.)

Assume now the Cartan matrix C_A of A is invertible over $\mathbb{Z}$ (for example, this is satisfied in case $\mathrm{gl.dim.}A < \infty$). Recall that the columns of C_A are given by $p_A(a)^T$, the rows by $q_A(a)$, and that for an invertible Cartan matrix C_A, the Coxeter matrix is defined by $\Phi_A = -C_A^{-T}C_A$. First, we note that under our assumption of C_A being invertible,

3.2. $$K_o(\hat{A}) = K_o(A) \oplus P(\hat{A}) \ .$$

(For, using 3.1 for $i \geq 1$, we see that all these $\hat{\nu}^i p_A(a)$ belong to $K_o(A) + P(\hat{A})$, therefore $K_o(A(i)) \subseteq K_o(A) + P(\hat{A})$. Using 3.1 for $i \leq o$, it follows that all $\hat{\nu}^{i-1} q_A(a)$ belong to $K_o(A) + P(\hat{A})$, therefore $K_o(A(i-1)) \subseteq K_o(A) + P(\hat{A})$.) Next, we observe the following:

3.3. For any $x \in K_o(A)$ and all $i \in \mathbb{Z}$, we have $x\hat{\nu}^i \equiv x\Phi_A^i (\mathrm{mod}\ P(\hat{A}))$.

Proof: Since $P(\hat{A})$ is stable under $\hat{\nu}$, we obtain from $\hat{\nu}$ a linear automorphism of $K_o(\hat{A})/P(\hat{A}) = K_o(A)$, which we denote by $\bar{\nu}$. Since $\bar{\nu}$ and Φ_A are linear automorphisms, it is sufficient to consider the case $i = 1$, and that $x = p_A(a)$ for some vertex a of A. But 3.1 gives $p_A(a)\bar{\nu} = -q_A(a) = p_A(a)\Phi_A$.)

We denote the projection of $K_o(\hat{A})$ onto $K_o(A)$ with kernel $P(\hat{A})$ by π_A, and given an $\hat{A}$-module X, let $\underline{\dim}^A X = (\underline{\dim}\, X)\pi_A$. Note that $\underline{\dim}^A$ vanishes on all projective $\hat{A}$-modules and takes values in $K_o(A)$. Also note that $\underline{\dim}^A$ is an additive function on the stable Auslander-Reiten quiver of $\hat{A}$. Let Σ denote Heller's suspension functor on $\hat{A}$-mod, thus $\Sigma X \approx I/X$, where I is an injective envelope of X.

3.4. For any $\hat{A}$-module X, we have $\underline{\dim}^A \Sigma X = -\underline{\dim}^A X$.

(For, let $\Sigma X = I/X$, with I an injective (= projective) $\hat{A}$-module. Use the additivity of $\underline{\dim}^A$ on exact sequences and that $\underline{\dim}^A I = o$). Combining 3.3 and 3.4, we obtain:

3.5. For any $\hat{A}$-module X, we have $\underline{\dim}^A \hat{\tau} X = \underline{\dim}^A \hat{\nu} X = (\underline{\dim}^A X)\Phi_A$.

(Proof: It is well-known that $\hat{\tau} = \Sigma^{-2}\hat{\nu}$, see [4], thus $\underline{\dim}^A \hat{\tau} X = \underline{\dim}^A \Sigma^{-2}\hat{\nu} X = \underline{\dim}^A \hat{\nu} X$, according to 3.4. Since $\underline{\dim}^A \hat{\nu} X = (\underline{\dim}^A X)\hat{\nu}$, we can apply 3.3 and obtain the second equality).

The last assertion seems to be remarkable since it shows that up to $P(\hat{A})$, we can determine the dimension vector of $\hat{\tau} X$ by using the Coxeter transformation Φ_A of A, without any further restriction. For A of finite global dimension, we will give a different interpretation of this result, at the end of the paper.

4. Description of $\hat{C}$-mod in terms of $\underline{\dim}^C$.

We return to the case of the canonical tubular algebras C. As above, we identify C with $C(o)$, and denote by χ_C the usual quadratic form on $K_o(C)$; of course, this is just the restriction of $\hat{\chi}$ to $K_o(C)$. We consider the projection $\pi_C : K_o(\hat{C}) \longrightarrow K_o(C)$ with kernel $P(\hat{C})$.

4.1. If $x,y \in K_o(C_{m,m+1})$ for some $m \in \mathbb{Z}$, then

$$\langle x\pi_C, y\pi_C \rangle = \langle x,y \rangle .$$

Proof. The case $m = 1$ is trivial. Let us consider the case $m = o$. For any $x \in K_o(C_{o1})$, the difference $x - x\pi_C$ is an integral multiple of $\hat{p}((d+1)_o)$, say $x - x\pi_C = x_o\hat{p}((d+1)_o)$, and $y - y\pi_C = y_o\hat{p}((d+1)_o)$, with $x_o, y_o \in \mathbb{Z}$. For any $\hat{C}$-module M, we have

$$\langle \hat{p}((d+1)_o), \underline{\dim}\, M \rangle = \dim \operatorname{Hom}(\hat{P}((d+1)_o), M),$$

since $\hat{P}((d+1)_o)$ is projective. Thus $\langle \hat{p}((d+1)_o), y \rangle = 0$ for all $y \in K_o(C_{o1})$. Similarly,

$$\langle \underline{\dim}\, M, \hat{p}((d+1)_o) \rangle = \dim \operatorname{Hom}(M, \hat{P}((d+1)_o)),$$

for any $\hat{C}$-module M, since $\hat{P}((d+1)_o)$ is injective. Thus $\langle z, \hat{p}((d+1)_o) \rangle = 0$ for all $z \in K_o(C_{12})$. Thus

$$\begin{aligned} \langle x,y \rangle &= \langle x\pi_C + x_o\hat{p}((d+1)_o), y\pi_C + y_o\hat{p}((d+1)_o) \rangle \\ &= \langle x\pi_C, y\pi_C \rangle + y_o \langle x\pi_C, \hat{p}((d+1)_o) \rangle + x_o \langle \hat{p}((d+1)_o), y \rangle \\ &= \langle x\pi_C, y\pi_C \rangle . \end{aligned}$$

This finishes the case $m = 0$. Dually, the case $m = 2$ also holds. The general case now follows using 3.3: Let $m = 3n+t$ with $0 \leq t \leq 2$, $n \in \mathbb{Z}$, and $x,y \in K_o(C_{m,m+1})$. Then $x\nu^{-n}$, $y\nu^{-n} \in K_o(C_{t,t+1})$. Since $\langle -,- \rangle$ is ν-invariant, and the restriction of $\langle -,- \rangle$ to $K_o(C)$ is Φ_C-invariant, we have

$$\begin{aligned} \langle x,y \rangle &= \langle x\nu^{-n}, y\nu^{-n} \rangle = \langle x\nu^{-n}\pi_C, y\nu^{-n}\pi_C \rangle \\ &= \langle x\pi_C\Phi_C^{-n}, y\pi_C\Phi_C^{-n} \rangle = \langle x\pi_C, y\pi_C \rangle , \end{aligned}$$

this finishes the proof. As a direct consequence, we obtain

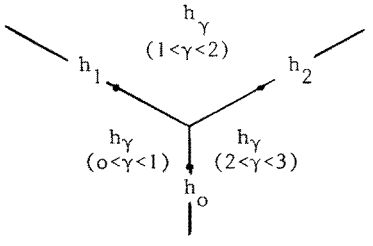
4.1'. The restriction of π_C to $K_o(C_{m,m+1})$ is an isometry from $(K_o(C_{m,m+1}), \hat{\chi} \mid K_o(C_{m,m+1}))$ onto $(K_o(C), \chi_C)$.

Proof. The surjectivity follows directly from the form of the elements $\hat{p}(a)$ which generate the kernel $P(\hat{C})$ of π_C. Since $K_o(C_{m,m+1})$ and $K_o(C)$ have the same rank, the restriction of π_C to $K_o(C_{m,m+1})$ is an isomorphism, thus an isometry according to 4.1.

Let $h^C_\gamma = h_\gamma \pi_C, \iota^C_\gamma = \langle h^C_\gamma, - \rangle : K_o(C) \longrightarrow \mathbf{Z}$, and $K_o(C)_\gamma = \operatorname{Ker} \iota^C_\gamma$. we easily verify:

$$h^C_{3n+1} = h_1, \quad h^C_{3n+2} = h_2, \quad h^C_{3n} = -\frac{1}{d}(h_1+h_2)$$

(for $n = o$, the first two equalities are trivial, the third is an easy calculation. The general case follows from 3.3, using the fact that both h_1 and h_2 are Φ_C-invariant). Since π_C maps $K_o(C_{m,m+1})$ isomorphically onto $K_o(C)$, it follows that for $\gamma = m + \frac{\alpha}{\beta}$, with integers m, α, β, $0 \leq \alpha < \beta$, the vector h^C_γ is the minimal element of $K_o(C)$ which is a positive rational multiple of $(\beta-\alpha)d_m h^C_m + \alpha d_{m+1} h^C_{m+1}$. In particular, all h^C_γ belong to the radical $\operatorname{rad}\chi_C$ of χ_C. Note that $h^C_\gamma = h^C_{\gamma+3}$ for all $\gamma \in \mathbb{Q}$, and that any non-zero element of $\operatorname{rad}\chi_C$ can be written in a unique way as a positive integral multiple of some h^C_γ, with $0 \leq \gamma < 3$. It seems convenient to visualize the plane $\operatorname{rad}\chi_C$ as follows:



We recall that Φ_C has order d, and given $x \in K_o(C)$, we denote $\partial(x) = \sum_{i=o}^{d-1} x\Phi_C^i$. Since $\partial(x)$ is Φ_C-invariant, it belongs to $\operatorname{rad}\chi_C$. Let $K_o(C)_o$ the set of elements $x \in K_o(C)$ with $\partial(x) = 0$. Note that $K_o(C)_o$ is a linear subspace of dimension 4,6,7, or 8, respectively (it is the kernel of the surjective linear map $\partial = \sum_{i=o}^{d-1} \Phi_C^i : K_o(C) \longrightarrow \operatorname{rad}\chi_C$). Recall that an element x of $K_o(C)_o$ is called a root of χ_C provided $\chi_C(x) = 1$.

4.2. All roots of χ_C lie outside of $K_o(C)_o$.

Proof. Let x be a root. We can write x in the form $y + uh_1 + vh_{3/2}$ with $u,v \in \mathbb{Z}$, and y vanishing on the vertices d_1 and $(d+1)_o$. With x also y is a root, and we claim that $\mathcal{O}(y) \notin d\cdot\mathrm{rad}\chi_C = \mathbb{Z}dh_1 + \mathbb{Z}dh_{3/2}$. Since $\mathcal{O}(h_1) = dh_1$, $\mathcal{O}/h_{3/2} = dh_{3/2}$, it then follows that $\mathcal{O}(x) \notin d\cdot\mathrm{rad}\chi_C$, in particular, $\mathcal{O}(x) \neq 0$. Let Δ' be the full subquiver of Δ_1 obtained by deleting the vertex d_1, thus Δ' is of the form $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, respectively, and y is a root for Δ'. In particular, the absolute value of the coefficient $y(1_o)$ of y at 1_o is bounded by d. Given a vertex x of Δ', denote the corresponding base vector by $e(x)$. Note that $\mathcal{O}(e(1_o)) = -h_1$, and $\mathcal{O}(e(x)) = h_{3/2}$ for the remaining vertices x of Δ'. It follows that $\mathcal{O}(y) = -y(1_o)h_1 + wh_{3/2}$, where $w = \sum_{a\neq 1_o} y(a)$. Thus, if $0 \neq |y(1_o)| < d$, then $\mathcal{O}(y) \notin d\cdot\mathrm{rad}\chi_C$. If $|y(1_o)| = d$, then we consider the root $z = y - y(1_o)h_1$, and it is sufficient to show that $\mathcal{O}(z) \notin d\cdot\mathrm{rad}\chi_C$. Thus, consider a root z with support in $\Delta_1 \cap \Delta_2$. Since $\Delta_1 \cap \Delta_2$ is the disjoint union of quivers of the form $\mathbb{A}_s$, with $s \leq 5$, and since $\mathcal{O}(e(x)) = h_{3/2}$ for all vertices $x \in \Delta_1 \cap \Delta_2$, it follows that $\mathcal{O}(z) = wh_{3/2}$ with $w = \sum_a z(a)$, and $1 \leq |w| \leq 5$, thus also in this case $\mathcal{O}(z) \notin d\cdot\mathrm{rad}\chi_C$.

4.3. $K_o(C)_o = \{x \in K_o(C) \mid \langle y,x\rangle = o \text{ for all } y \in \mathrm{rad}\chi_C\}$.

Proof. Let $x \in K_o(C)$, with $\langle y,x\rangle = o$ for all $y \in \mathrm{rad}\chi_C$. Then $o = \langle y,x\rangle = \langle y\Phi_C^i, x\Phi_C^i\rangle = \langle y, x\Phi_C^i\rangle$, since $\langle -,-\rangle$ is Φ_C-invariant and $y\Phi_C = y$ for $y \in \mathrm{rad}\chi_C$. Thus $\langle y,\mathcal{O}(x)\rangle = o$ for all $y \in \mathrm{rad}\chi_C$. However, $\mathcal{O}(x) \in \mathrm{rad}\chi_C$ implies that $\mathcal{O}(x) = o$, since $\langle -,-\rangle$ is non-degenerate on $\mathrm{rad}\chi_C$ (for, $\langle h_\gamma, h_\gamma\rangle = o$, and $\langle h_1,h_2\rangle = d^2$), therefore $x \in K_o(C)_o$. Since both spaces $K_o(C)_o$ and $\{x \mid \langle y,x\rangle = o \text{ for all } y \in \mathrm{rad}\chi_C\}$ have codimension 2, it follows that we have equality.

Let $\iota_\gamma^C = \langle h_\gamma^C, -\rangle : K_o(C) \longrightarrow \mathbb{Z}$, and $K_o(C)_\gamma = \mathrm{Ker}\,\iota_\gamma^C$. This a family of hyperplanes of $K_o(C)$, with $K_o(C)_\gamma = K_o(C)_{\gamma+3/2}$. If h_γ^C and h_δ^C are linearly independent (thus, if $\gamma-\delta$ is not an integral multiple of $\frac{3}{2}$), then $K_o(C)_\gamma \cap K_o(C)_\delta = K_o(C)_o$, according to 4.3. Also, let $K_o(C)_\gamma^+$ be the set of elements x of $K_o(C)$ with $\mathcal{O}(x)$ a positive multiple of h_γ^C. We have $K_o(C)_\gamma^+ = K_o(C)_\delta^+$ provided $\gamma-\delta$ is an integral multiple of 3. We obtain in this way a decomposition of $K_o(C)$ into pairwise disjoint subsets

$$K_o(C) = K_o(C)_o \cup \bigcup_{o\leq\gamma<3} K_o(C)_\gamma^+ .$$

4.4. For any $\gamma \in \mathbb{Q}$, we have $K_o(C)_\gamma = K_o(C)^+_{-\gamma} \cup K_o(C)_o \cup K_o(C)^+_\gamma$.

Proof. Given $x \in K_o(C)_\gamma$, then $\langle h^C_\gamma, x\Phi^i_C\rangle = \langle h^C_\gamma \Phi^i_C, x\Phi^i_C\rangle = \langle h^C_\gamma, x\rangle = 0$, thus with x also $\mathcal{O}(x)$ belongs to $K_o(C)_\gamma$. However, $K_o(C)_\gamma \cap \operatorname{rad}\chi_C$ is the subgroup generated by h^C_γ, therefore $\mathcal{O}(x)$ is a multiple of h^C_γ. Conversely, we know already that $K_o(C)_o \subseteq K_o(C)_\gamma$. Thus, let $\mathcal{O}(x)$ be a non-zero multiple h^C_γ. If $\langle h^C_{\gamma+1}, x\rangle$ would be zero, then by the previous consideration, $\mathcal{O}(x)$ is a multiple of $h^C_{\gamma+1}$, impossible. Thus, there are integers u,v with $u \neq o$ and $\langle uh^C_\gamma + vh^C_{\gamma+1}, x\rangle = o$. Now, $uh^C_\gamma + vh^C_{\gamma+1}$ is a nonzero element of $\operatorname{rad}\chi_C$, thus a multiple of some h^C_δ. The consideration above shows that $\mathcal{O}(x)$ is a multiple of h^C_δ, thus $h^C_\delta = \pm h^C_\gamma$, and therefore $\langle h^C_\gamma, x\rangle = 0$.

In order to determine the structure of $(K_o(C)_\gamma, \chi_C \mid K_o(C)_\gamma)$, we use the following consequence of 4.1 and 4.1':

4.1" For any $\gamma \in \mathbb{Q}$, the map π_C maps $K_o(\hat{C})_\gamma$ onto $K(C)_\gamma$, and this is an isomorphism and an isometry (with respect to the restrictions of $\hat{\chi}$ and χ_C) in case $\gamma \notin \mathbb{Z}$.

4.5. For any $\gamma \in \mathbb{Q}$, the restriction of χ_C to $K_o(C)_\gamma$ is the radical product $\frac{|r|}{s}\chi_s$ of t quadratic forms of type $\tilde{\mathbb{A}}_{n_s-1}$, $1 \leq s \leq t$, where $\mathbb{T} = (n_1,\ldots,n_t)$.

Proof. We may assume $\gamma \notin \mathbb{Z}$, since $K_o(C)_\gamma = K_o(C)_{\gamma+3/2}$. Let $\gamma \in Q^{m+1}_m$. According to [8], we know that the restriction of $\hat{\chi}$ to $K_o(\hat{C})_\gamma = K_o(C_{m,m+1}) \cap \operatorname{Ker} \iota_\gamma$ is of the stated form, thus the same holds for the restriction of χ_C to $K_o(C)_\gamma$, according to 4.1".

4.6. Let $\gamma \in \mathbb{Q}$, and X a non-projective module in $\mathcal{T}_\gamma$. Then $\underline{\dim}^C X$ belongs to $K_o(C)^+_\gamma$.

Proof. Since $\underline{\dim}^C$ is additive, and vanishes on projective modules, we can assume that X is indecomposable. We use the equality $\underline{\dim}^C \hat{\tau} X = (\underline{\dim}^C X)\Phi_C$ established in 3.5. If the component containing X does not contain an indecomposable projective module, then $\sum_{i=o}^{d-1} \underline{\dim}\, \hat{\tau}^i X$ is a positive integral multiple of h_γ. Thus, assume $\gamma \in \mathbb{Z}$, and that the component containing X contains the indecomposable projective modules $\hat{P}_1,\ldots,\hat{P}_s$, say with dimension vectors $\hat{p}_i = \underline{\dim}\, \hat{P}_i$,

and let $\hat{p} = \sum_{i=1}^{s} \hat{p}_i$. A glance at the various possible cases immediately yields that always $\sum_{i=o}^{d-1} \underline{\dim}\, \hat{\tau}^i X - \hat{p}$ is an integral multiple of h_γ. (Actually, it is sufficient to check this for $X = \operatorname{rad} \hat{P}_1$, and then to use induction on the distance from the mouth of the tube). Application of π_C gives the desired result.

For $\gamma \notin \mathbb{Z}$, we now can formulate precisely in which way $\underline{\dim}^C$ and χ_C control $\mathcal{T}_\gamma$.

4.7. Let $\gamma \notin \mathbb{Z}$. The map $\underline{\dim}^C$ maps the set of indecomposable modules in $\mathcal{T}_\gamma$ onto the set of roots and radical vectors in $K_o(C)^+_\gamma$. For any root x in $K_o(C)^+_\gamma$, there is precisely one isomorphism class of indecomposable modules X in $\mathcal{T}_\gamma$ with $\underline{\dim}^C X = x$. For any radical vector x in $K_o(C)^+_\gamma$, there is a one-parameter family of indecomposable modules X in $\mathcal{T}_\gamma$ with $\underline{\dim}^C X = x$.

Let us define an increasing map $\sigma : \mathbb{Q} \longrightarrow \mathbb{Q}$ by

$$\sigma(m+\tfrac{\alpha}{\beta}) = \begin{cases} m+1 + \dfrac{\beta-\alpha}{2\beta-3\alpha} & 0 \le 2\alpha \le \beta \\ \text{for} & \\ m+2 + \dfrac{2\alpha-\beta}{3\alpha-\beta} & 0 \le \alpha \le \beta \le 2\alpha \end{cases}$$

where $m,\alpha,\beta \in \mathbb{Z}$, $\beta \neq o$. Note that $\sigma(m) = m + \frac{3}{2}$ and $\sigma(m+\frac{1}{2}) = \sigma(m+2)$. The reason for introducing this map is the following property:

4.8. $h^C_{\sigma\gamma} = - h^C_\gamma$, for any $\gamma \in \mathbb{Q}$.

Proof. Denote $d_m h^C_m$ by h'_m, and note that $h'_m + h'_{m+1} + h'_{m+2} = 0$, and $h'_m = h'_{m+3}$ for all $m \in \mathbb{Z}$. Let $h(m,\alpha,\beta) = (\beta-\alpha)h'_m + \alpha h'_{m+1}$. First, let $0 \le 2\alpha \le \beta$. Then

$$h(m,\alpha,\beta) + h(m+1,\beta-\alpha,2\beta-3\alpha) = (\beta-\alpha)h'_m + \alpha h'_{m+1} + (\beta-2\alpha)h'_{m+1} + (\beta-\alpha)h'_{m+1}$$

$$= (\beta-\alpha)h'_m + \alpha h'_{m+1} + (\beta-2\alpha)h'_{m+1} + (\beta-\alpha)(-h'_m-h'_{m+1}) = 0.$$

Similarly, for $0 \leq \alpha \leq \beta \leq 2\alpha$,

$$h(m,\alpha,\beta) + h(m+2,2\alpha-\beta,3\alpha-\beta) = (\beta-\alpha)h'_m + \alpha h'_{m+1} + \alpha h'_{m+2} + (2\alpha-\beta)h'_{m+3}$$

$$= (\beta-\alpha)h'_m + \alpha h'_{m+1} + \alpha(-h'_m-h'_{m+1}) + (2\alpha-\beta)h'_m = 0.$$

Note that for $\gamma = m + \frac{\alpha}{\beta}$ with $m,\alpha,\beta \in \mathbb{Z}$, $0 \leq \alpha \leq \beta$, and $\beta \neq 0$, the vector h_γ is the minimal vector in $K_o(C_{m,m+1})$ which is a positive rational multiple of $(\beta-\alpha)d_m h_m + \alpha d_{m+1} h_{m+1}$. Since π_C maps $K_o(C_{m,m+1})$ isomorphically onto $K_o(C)$, the vector h^C_γ is the minimal vector in $K_o(C)$ which is a positive rational multiple of $h(m,\alpha,\beta)$. The assertion now follows from the calculations above.

The mapping σ can be used in order to express the shift given by Heller's suspension functor. Given a module class X in $\hat{C}$-mod, we denote by $\underline{X}$ the corresponding object class in $\hat{C}$-$\underline{\text{mod}}$.

4.9. For any $\gamma \in \mathbb{Z}$, we have $\Sigma(\underline{T}_\gamma) = \underline{T}_{\sigma\gamma}$.

Proof. It is sufficient to show $\Sigma(\underline{T}_\gamma) \subseteq \underline{T}_{\sigma\gamma}$, since σ is invertible and Σ a self-equivalence on $\hat{C}$-$\underline{\text{mod}}$. Consider first the case $\gamma = m \in \mathbb{Z}$. Let X be an indecomposable non-projective module in T_m. If E is a simple submodule of X, then its support must lie in Δ_{m-1}, according to the lemma at the end of section (1) (and using the fact that $\mathrm{Hom}(T_\delta, T_m) = 0$ for $\delta > m$). Thus, the support of the injective envelope of X, and also the support of ΣX, have to be contained in $\Delta_{m-1,m+2}$. Assume ΣX belongs to T_β. Since the simple $C_{m-1,m+2}$-modules belong to the T_δ with $\delta = m - \frac{1}{2}, m + \frac{1}{2}, m + \frac{3}{2}, m + \frac{5}{2}$, we must have $\beta \leq m + \frac{5}{2}$. Of course, also $m \leq \beta$. On the other hand, according to 3.4, $\underline{\dim}^C \Sigma X = -\underline{\dim}^C X$, thus

$$\sum_{i=o}^{d-1} (\underline{\dim}^C \Sigma X)\Phi_C^i = -\sum_{i=o}^{d-1} (\underline{\dim}^C X)\Phi_C^i$$

is a positive multiple of $-h^C_m = h^C_{\sigma m}$. It follows that the difference of β and $\sigma m = m + \frac{3}{2}$ is an integral multiple of 3. Since in addition we know that $m \leq \beta \leq m + \frac{5}{2}$, it follows that $\beta = \sigma m$.

A similar argument works in case $\gamma \notin \mathbb{Z}$. Let $\gamma \in \mathbb{Q}_m^{m+1}$, and X indecomposable in T_γ. The socle of X has support in Δ_m, thus the support

of ΣX is in $\Delta_{m,m+3}$. Let ΣX belong to $\mathcal{T}_\beta$, thus $\beta \leq m + \frac{7}{2}$. Since any indecomposable injective module I with $\mathrm{Hom}(X,I) \neq 0$ belongs to some $\mathcal{T}_\delta$ with $\delta \geq m+1$, we see that $m+1 \leq \beta \leq m + \frac{7}{2}$. As above, the difference of β and σm has to be an integral multiple of 3, and now $m + \frac{3}{2} \leq \sigma m \leq m + \frac{5}{2}$. Again, it follows that the only possibility is $\beta = \sigma m$.

Of particular interest is the case $\gamma = m \in \mathbf{Z}$, since it provides us with the description of $\mathcal{T}_{-m}$ in terms of roots and null-vectors, similar to that in 4.7. In this way, we can extend 4.7 to all $\gamma \in \mathbb{Q}$ (of course, we also may use a case-by-case investigation, using the structure of the non-stable tubes as exhibited in (1).

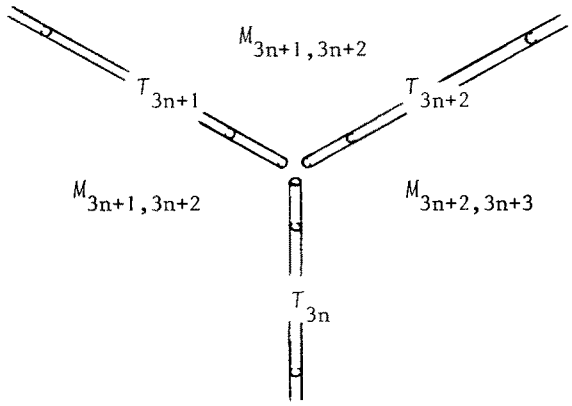
Theorem. Let $\gamma \in \mathbb{Q}$. The map $\underline{\dim}^C$ maps the set of indecomposable non-projective modules in $\mathcal{T}_\gamma$ onto the set of roots and radical vectors in $K_o(C)^+_\gamma$. For any root x in $K_o(C)^+_\gamma$, there is precisely one isomorphism class of indecomposable (and non-projective) modules X in $\mathcal{T}_\gamma$ with $\underline{\dim}^C X = x$. For any radical vector x in $K_o(C)^+_\gamma$, there is a one-parameter family of indecomposable (and non-projective) modules X in $\mathcal{T}_\gamma$ with $\underline{\dim}^C X = x$.

Proof. We only have to consider the case $\gamma = m \in \mathbb{Z}$. We use the Heller suspension functor Σ which gives an equivalence of $\mathcal{T}_{-m}$ and $\mathcal{T}_{-\sigma m} = \mathcal{T}_{\sigma m}$. According to 3.4, there is the following commutative diagram

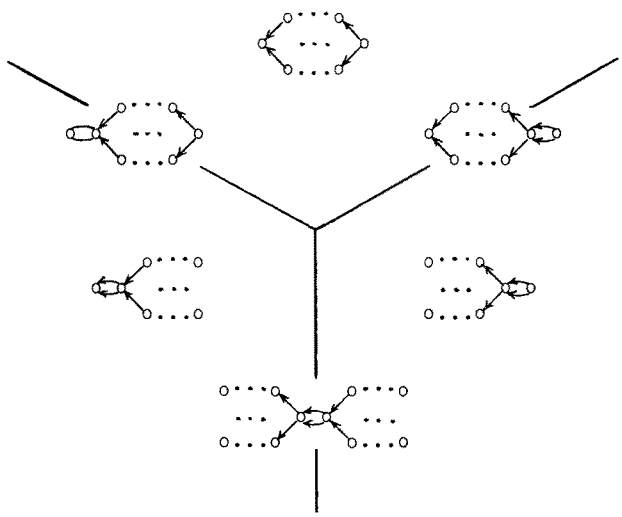
$$\begin{array}{ccc} & \overset{\Sigma}{\longrightarrow} & \\ \mathcal{T}_{-m} & \longrightarrow & \mathcal{T}_{\sigma m} \\ \underline{\dim}^C \downarrow & & \downarrow \underline{\dim}^C \\ K_o(C)^+_m & \overset{-1}{\longrightarrow} & K_o(C)^+_{\sigma m} \end{array}$$

Of course, the map -1 is an isometry from $(K_o(C), \chi_C)$ to itself, and it maps $K_o(C)^+_m$ onto $K_o(C)^+_{\sigma m}$. Since $\sigma m \notin \mathbb{Z}$, we can apply 4.7 to $\mathcal{T}_{\sigma m}$, and the assertion for $\mathcal{T}_{\sigma m}$ carries over to the corresponding assertion for $\mathcal{T}_{-m}$.

We can visualize the category $\hat{C}$-mod in terms of $\underline{\dim}^C$ as follows, using the plane $\mathrm{rad}\chi_C$ as index set:



Also, we indicate the shape of the support of T_m and $M_{m,m+1}$:



5. The derived category $D^b(C\text{-mod})$

Consider again an arbitrary finite dimensional algebra A. It has been shown in [6] (and is easy to see) that we can identify $K_o(A)$ and the Grothendieck group $K_o(D^b(A\text{-mod}))$ of $D^b(A\text{-mod})$ as a triangulated category. (The Grothendieck group of a triangulated category $\mathcal{A}$ is given by F/R, with F the free abelian group with basis the set of isomorphism classes $[X]$ of objects X of $\mathcal{A}$, and R the subgroup of F generated by the elements $[X] - [Y] + [Z]$, where $X \to Y \to Z \to TX$ is a triangle in $\mathcal{A}$). Given an object $X^\cdot$ in $D^b(A\text{-mod})$, we denote by $\underline{\dim}\, X^\cdot$ the corresponding element in $K_o(D^b(A\text{-mod})) = K_o(A)$. Note that there is a canonical embedding of A-mod into $D^b(A\text{-mod})$ (as the full subcategory of complexes concentrated in degree zero), and the restriction of $\underline{\dim}$ to this full subcategory coincides with the usual dimension vector function. Also, for an arbitrary complex $X^\cdot$, we have $\underline{\dim}\, X^\cdot = \sum_i (-1)^i \underline{\dim}\, X^i$.

Assume now that A has finite global dimension. In this case, it has been shown in [6] that $D^b(A\text{-mod})$ and $\hat{A}\text{-}\underline{\text{mod}}$ are equivalent as triangulated categories, and that there exists such an equivalence η which is the identity on A-mod (embedded into $D^b(A\text{-mod})$ as the complexes concentrated in degree zero, and embedded into $\hat{A}\text{-}\underline{\text{mod}}$ as $A(o)$-mod). There is the following commutative diagram

$$\begin{array}{ccc} D^b(A\text{-mod}) & \xrightarrow{\eta} & \hat{A}\text{-}\underline{\text{mod}} \\ & \searrow^{\underline{\dim}} \quad \swarrow_{\underline{\dim}^A} & \\ & K_o(A) & \end{array}$$

Proof. Both $\underline{\dim}$ and $\underline{\dim}^A$ coincide on A-mod with the usual dimension-vector function. Any object in $D^b(A\text{-mod})$ can be obtained by forming successive mapping cones, starting from objects in A-mod. Therefore, it is sufficient to show that $\underline{\dim}^A$ is additive on triangles. However, the triangles in $\hat{A}\text{-}\underline{\text{mod}}$ are obtained by starting with a map $X \longrightarrow Y$, and an injective envelope I of X in $\hat{A}$-mod, and forming an induced exact sequence in $\hat{A}$-mod

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I & \longrightarrow & \Sigma X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X & \longrightarrow & 0 \end{array}$$

Then, $X \to Y \to Z \to \Sigma X$ is a triangle in $\hat{A}$-$\underline{\text{mod}}$, and the lower exact sequence gives

$$\begin{aligned} \underline{\dim}^A Z &= \underline{\dim}^A Y + \underline{\dim}^A \Sigma X \\ &= \underline{\dim}^A Y - \dim^A X. \end{aligned}$$

As a consequence, we see that the description of $\hat{C}$-$\underline{\text{mod}}$ in terms of $\underline{\dim}^C$ as given in (3) is just the description of $D^b(C\text{-mod})$ in terms of $\underline{\dim}$, and this is the description which we were aiming at.

We add a remark concerning Auslander-Reiten triangles in $D^b(A\text{-mod})$, where A is a finite dimensional algebra of finite global dimension. A triangle $X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v} Z^\cdot \xrightarrow{w} TX^\cdot$ in $D^b(\text{mod } A)$ is called an Auslander-Reiten triangle provided $X^\cdot, Z^\cdot$ are both indecomposable, $w \neq o$, and the following equivalent conditions are satisfied: (i) for all $f : X^\cdot \to V^\cdot$, f not split mono, there exists $f' : Y^\cdot \to V^\cdot$ with $uf' = f$; (ii) for all $g : W^\cdot \to Z^\cdot$, g not split epi, there exists $g' : W^\cdot \to Y^\cdot$ with $g'v = g$; (iii) for all $h_1 : U_1^\cdot \to Z^\cdot$, h_1 not split epi $\Rightarrow h_1 w = 0$, and (iv) for all $h_2 : TX^\cdot \to U_2^\cdot$, h_2 not split mono $\Rightarrow wh_2 = 0$. The Auslander-Reiten sequences in $\hat{A}$-mod give rise to Auslander-Reiten triangles in $\hat{A}$-$\underline{\text{mod}}$, and therefore in $D^b(A\text{-mod})$. In this way, the existence of Auslander-Reiten triangles in $D^b(A\text{-mod})$ has been established in [6]. However, we also may copy the existence proof for Auslander-Reiten sequences, as outlined in [4], in order to show directly the existence of Auslander-Reiten triangles in $D^b(A\text{-mod})$, and, at the same time, obtain the numerical criterion of 2.5.

There is a natural transformation $\alpha_Y : D\,\text{Hom}(Y,-) \longrightarrow \text{Hom}(-,\nu Y)$, where ν is the Nakayama functor, and D the duality with respect to the base field k, such that α_Y is invertible, in case Y is projective. An object in $D^b(A\text{-mod})$ can be written in the form $P^\cdot$, where $P^\cdot$ is a bounded complex of projective A-modules. Now assume $P^\cdot$ is indecomposable in $D^b(A\text{-mod})$, and let $\varphi \in D\,\text{Hom}(P^\cdot,P^\cdot)$ be a non-zero linear form on $\text{Hom}(P^\cdot,P^\cdot) = \text{End}(P^\cdot)$ which vanishes on the radical $\text{rad End}(P^\cdot,P^\cdot)$. We consider the image $\alpha_{P^\cdot}(\varphi)$ of φ under $\alpha_{P^\cdot}$, it is a non-zero map $P^\cdot \longrightarrow \nu P^\cdot$ which has the following properties: Given an indecomposable object $X^\cdot$ in $D^b(A\text{-mod})$, and a non-invertible map $\xi : X^\cdot \longrightarrow P^\cdot$, or a non-invertible map $\eta : \nu P^\cdot \longrightarrow X^\cdot$, then $\xi\alpha_{P^\cdot}(\varphi) = o$, or $\alpha_{P^\cdot}(\varphi)\eta = o$, respectively. Let $C(T^{-1}\alpha_{P^\cdot}(\varphi))$ be the mapping cone of $T^{-1}\alpha_{P^\cdot}(\varphi)$. It follows that $T^{-1}\nu P^\cdot \longrightarrow C(T^{-1}\alpha_{P^\cdot}(\varphi)) \longrightarrow P^\cdot \xrightarrow{\alpha_{P^\cdot}(\varphi)} \nu P^\cdot$ is an Auslander-Reiten triangle.

We denote $T^{-1}\nu P^\cdot$ by $\tau P^\cdot$. We have $\underline{\dim}^A \nu P^\cdot = (\underline{\dim}^A P^\cdot)C_A^{-T}C_A$ therefore $\underline{\dim}^A \tau P^\cdot = \underline{\dim}^A T^{-1}\nu P^\cdot = -\underline{\dim}^A \nu P^\cdot = -(\underline{\dim}^A P^\cdot)C_A^{-T}C_A = (\underline{\dim}^A P^\cdot)\Phi_A$.

Appendix: The category $T(C)$-mod.

Our investigation of $\hat{C}$-mod also establishes the structure of the category $T(C)$-mod, where $T(C)$ is the socalled trivial extension of C. We recall that $T(C) = C \ltimes Q$ has the additive structure $C \oplus Q$, and the multiplication is defined by the formula $(c_1,q_1)(c_2,q_2) = (c_1c_2, c_1q_2+q_1c_2)$, for $c_1,c_2 \in C$, $q_1,q_2 \in Q$. Alternatively, $T(C)$ may be considered as $\hat{C}/\nu$. Here, we consider $\hat{C}$ not as an algebra, but rather as a locally finite-dimensional k-category, and $\hat{C}/\nu$ is the quotient in the category of all locally finite-dimensional k-categories,see [10]. Since the indecomposable $\hat{C}$-modules have bounded support, and ν acts freely on the set of isomorphism classes of indecomposable $\hat{C}$-modules, it follows that $T(C)$-mod can be identified with $\hat{C}$-mod/ν. (This was pointed out by G. d'Este in [2]; for a recent general account, see [11]). As a consequence, the indecomposable $T(C)$-modules are in one-to-one correspondance with the indecomposable $\hat{C}$-modules X in T_γ, with $0 \leq \gamma < 3$.

Note that the algebras $T(C)$ have the following property: given any Auslander-Reiten sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $T(C)$-mod, then the middle term Y is the direct sum of at most two indecomposable direct summands.

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