REPRESENTATION THEORY OF FINITE-DIMENSIONAL ALGEBRAS

DURHAM LECTURES 1985

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Let k be an algebraically closed field and A a finite-dimensional k-algebra (associative, with 1). We consider finite-dimensional left A-modules, and call them just modules; the category of all A-modules will be denoted by A-mod. Any module can be written as a (finite) direct sum of indecomposable modules, and the theorem of Krull-Schmidt asserts that such a decomposition is essentially unique: it is unique up to isomorphism. For many purposes it therefore is sufficient to deal only with indecomposable modules. The main problems of the representation theory of finite-dimensional algebras are the following:

- to develop methods for constructing indecomposable modules,
- to look for suitable invariants in order to be able to identify indecomposable modules,
- to show that a given list of indecomposable modules is complete: that it contains a representative of any isomorphism class.

Typical invariants of a module M are the socalled Jordan-Hölder multiplicities: the algebra A has only finitely many simple modules, say $E_1,\dots,E_n$, and we may denote by $(\underline{\dim}\, M)_i$ the multiplicity of E_i occurring in a composition series of M (this is well-known to be an invariant of the isomorphism class of M). The vector $\underline{\dim}\, M$ obtained in this way is called the <u>dimension vector</u> of M. So one may ask for a description of the possible dimension vectors of indecomposable modules for a given algebra, and, having fixed a particular dimension vector, for a description of all indecomposable modules having this dimension vector.

One of the first questions usually will be that about the number of isomorphism classes of indecomposable modules. There may be only finitely many isomorphism classes of indecomposable A-modules, and then A is said to be representation-finite. Examples of representation-finite algebras are first of all the semi-simple ones, but also the algebras of all upper triangular matrices of given size, and there is a vast literature on representation-finite algebras. In case there are infinitely many isomorphism classes of indecomposable A-modules, there are actually always one-parameter families of isomorphism classes of indecomposable A-modules, as was conjectured by Brauer and Thrall. If there exists a two-parameter family of isomorphism classes of indecomposable A-modules, A is said to be wild, otherwise tame. The study of representation-infinite algebras is still in the beginning, only some types of examples seem to be well understood. We will present below several results which are independent of the representation type and exhibit some examples of tame algebras. In addition, we will pose a number of open problems which seem to be worthwhile to study.

A general reference for the terminology used here are our lecture notes [Ri2]. Those notes should also be consulted for the precise attribution of most of the results presented here. Only in case we deal with results which fell out of the scope of [Ri2] or which where not yet available at that time, we will indicate the source. Our aim in these lecturesis to give an introduction to the representation theory of finite-dimensional algebras. In particular, we are going to direct the interest towards the main results presented in [Ri2]. In addition, we will report on some recent investigations which are contained in the papers [RV], [Ri3], [Ha] and [HR].

LECTURE 1

THE AUSLANDER-REITEN QUIVER

It will be necessary to consider besides categories of the form A-mod also some related categories, for example full subcategories of A-mod (which are closed under direct sums and direct summands), or the categories of representations of partially ordered sets, or derived categories. Always, the categories which we will deal with will be k-additive categories (thus, additive categories with k operating centrally on the Hom-sets and such that all $\mathrm{Hom}(X,Y)$ are finite-dimensional k-vector-spaces) with split idempotents; and we call such a category a <u>Krull-Schmidt category</u> (note that in a Krull-Schmidt category, any object is a finite direct sum of indecomposable objects, and such a decomposition is unique up to isomorphism).

We start with the basic notions. Given an indecomposable object in a Krull-Schmidt category we call a map $f : X \longrightarrow Y$ a <u>source map</u> for X (the usual name would be "minimal left almost split map") provided the following three conditions are satisfied: first, f is not split mono; second, given any map $f' : X \longrightarrow Y'$ which is not split mono, there is $\eta : Y \longrightarrow Y'$ with $f' = f\eta$; and third, any $\zeta : Y \longrightarrow Y$ with $f = f\zeta$ is an automorphism. There is the following dual notion: Given an indecomposable module Z, we call a map $g : Y \longrightarrow Z$ a <u>sink map</u> for Z (or a "minimal right almost split map") provided, first, g is not split epi; second, given any map $g' : Y' \longrightarrow Z$ which is not split epi, there is η with $g' = \eta g$; and third, any $\zeta : Y \longrightarrow Y$ with $g = \zeta g$ is an automorphism. In case we deal with K = A-mod where A is a representation-finite algebra, it is not surprising to see that source maps and sink maps exist. The following remarkable theorem asserts that they always do exist in module categories, independent of the representation type:

<u>THEOREM</u> (M. Auslander, I. Reiten). Let A be a finite-dimensional k-algebra. For any indecomposable module M, there exists a source map and a sink map in A-mod, and both are unique up to isomorphism.

Let Z be indecomposable with sink map $g : Y \longrightarrow Z$. Either Z is projective, then we may take for Y the radical $\mathrm{rad}\, Z$ of Z and for g the inclusion map. Or, if Z is not projective, then g is epi, its kernel $\mathrm{Ker}\, g$ is indecomposable and will be denoted by τZ, and the inclusion map $\tau Z \longrightarrow Y$ is a source map.

Dually, let X' be indecomposable with source map $f' : X' \longrightarrow Y'$. Either X' is injective, then we may take $Y' = X'/\mathrm{soc}\,X'$, and f' the canonical epimorphism. Or, if X' is not injective, then f' is mono, its cokernel $\mathrm{Cok}\, f'$ is indecomposable and will be denoted by $\tau^{-}X'$, and the canonical epimorphism $Y' \longrightarrow \tau^{-}X'$ is a sink map.

Starting with a non-projective indecomposable module Z, or with a non-injective indecomposable module X, we obtain a non-split exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

with f a source map for X, and g a sink map for Z, with $X = \tau Z$ and $Z = \tau^{-}X$. Sequences of this kind are called Auslander-Reiten sequences. Now, in such an Auslander-Reiten sequence, both modules X, and Z are indecomposable, whereas Y usually is not. We decompose $Y = \bigoplus_i Y_i$, with all Y_i indecomposable, and rewrite the sequence above in the form

$$(*) \qquad 0 \longrightarrow X \xrightarrow{(f_i)_i} \bigoplus_i Y_i \xrightarrow{(g_i)_i} Z \longrightarrow 0 .$$

The maps $f_i : X \longrightarrow Y_i$ are irreducible (we recall the definition below), and we obtain in this way sufficiently many irreducible maps starting in X. The maps $g_i : Y_i \longrightarrow Z$ also are irreducible, and we obtain in this way sufficiently many irreducible maps ending in Z.

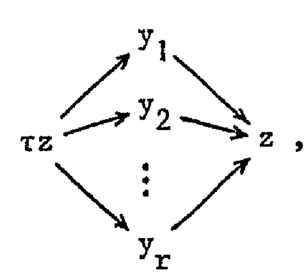
Consider a general Krull-Schmidt category K. If M,M' are indecomposable objects in K, denote by $\mathrm{rad}(M,M')$ the set of non-invertible maps $M \longrightarrow M'$. If M,M' are arbitrary, say with decompositions $M = \bigoplus M_i$, $M' = \bigoplus M'_j$ into indecomposables, let $\mathrm{rad}(M,M') = \bigoplus_{i,j} \mathrm{rad}(M_i,M'_j)$. We obtain in this way an ideal rad in the category K. We define $\mathrm{rad}^d(M,M')$ as the set of maps $M \longrightarrow M'$ which can be written as compositions of d maps all belonging to rad, and let $\mathrm{rad}^\infty = \bigcap_{d\in\mathbb{N}} \mathrm{rad}^d$. If M,M' are indecomposable objects, the maps in $\mathrm{rad}(M,M') \setminus \mathrm{rad}^2(M,M')$ are just the irreducible maps, and the factorspace $\mathrm{Irr}(M,M') = \mathrm{rad}(M,M')/\mathrm{rad}^2(M,M')$ is called the bimodule of irreducible maps.

We have noted above that in the module category A-mod, an Auslander-Reiten sequence (*) displays sufficiently many irreducible maps starting in X or ending in Z. In fact, assume that M is indecomposable, and that $\mathrm{Irr}(X,M)$ is of dimension d. Then, precisely d of the summands Y_i of Y are isomorphic to M, say $Y_1 = \ldots = Y_d = M$, and the residue classes of the maps $f_1,\ldots,f_d$ form a basis of $\mathrm{Irr}(X,M)$, whereas the residue classes of the maps $g_1,\ldots,g_d$ form a basis of $\mathrm{Irr}(M,Z)$. In particular, we always have

$$(**) \qquad \dim_k \mathrm{Irr}(\tau Z,M) = \dim_k \mathrm{Irr}(M,Z)$$

for M,Z indecomposable, and Z not projective.

With these preparations, we are going to define the Auslander-Reiten quiver Γ_A of A. We have noted in the introduction that the main object of the representation theory is the study of the set of isomorphism classes of indecomposable modules, and we denote this set by $(\Gamma_A)_o$. We want to endow this set with more structure in order to gain insight into its properties, and the theory expounded above shows that we may consider it as the set of vertices of a socalled translation quiver. Now, a <u>quiver</u> is nothing else than an oriented graph with possible multiple arrows and loops, thus of the form $Q = (Q_o,Q_1,s,e)$, where Q_o,Q_1 are two sets, and $s,e : Q_1 \longrightarrow Q_o$ set maps; the elements of Q_o are called vertices or points, those of Q_1 arrows, and given $\alpha \in Q_1$, then $s(\alpha)$ is called its starting point and $e(\alpha)$ its end point, pictured as follows: $s(\alpha) \xrightarrow{\alpha} e(\alpha)$. A <u>translation quiver</u> Γ is a locally finite quiver with an additional bijection $\tau = \tau_\Gamma : \Gamma_o' \longrightarrow \Gamma_o''$ of two subsets of Γ_o such that for $y \in \Gamma_o$, $z \in \Gamma_o'$, the number of arrows from y to z coincides with the number of arrows from τz to y. (In case we actually fix bijections σ from the set of arrows $y \longrightarrow z$ onto the set of arrows $\tau z \longrightarrow y$, we speak of a <u>polarized</u> translation quiver). A translation quiver may be thought of as being built from small units, the socalled "meshes"



they are defined for any $z \in \Gamma_o'$ (some of the y_i may coincide). The vertices in $\Gamma_o \setminus \Gamma_o'$ are called <u>projective</u> vertices, those in $\Gamma_o \setminus \Gamma_o''$ are called <u>injective</u> vertices. We return to the case of a finite-dimensional algebra A. The isomorphism class of a module M will be denoted by $[M]$. We have already noted that the vertices of Γ_A are of the form

$[X]$, with X an indecomposable module. For X,Y indecomposable, the number of arrows $[X] \longrightarrow [Y]$ in Γ_A is given by $\dim_k \operatorname{Irr}(X,Y)$, thus there is at least one arrow $[X] \to [Y]$ iff there exists an irreducible map $X \longrightarrow Y$. As translation we use the function τ, with $\tau[Z] = [\tau Z]$ which is defined for Z indecomposable and non projective. The formula (**) asserts that indeed we obtain in this way a translation quiver, the <u>Auslander-Reiten quiver</u> Γ_A of A. Of course, the projective vertices of Γ_A are just the vertices of the form $[P]$, with P indecomposable projective, the injective vertices are those of the form $[Q]$ with Q indecomposable injective. We observe that Γ_A has only finitely many projective and finitely many injective vertices, and these numbers coincide.

The structure of the Auslander-Reiten quiver Γ_A of finite-dimensional algebras should be studied rather carefully. Usually, Γ_A will decompose into several components and one may ask for the possible translation quivers occurring as components of Auslander-Reiten quivers. If X is an indecomposable module, and Γ a component of Γ_A, containing $[X]$, we just will say that X belongs to Γ. The components which do not contain projective or injective vertices will be said to be <u>regular</u>; of course, all but finitely many components are regular. Since Γ_A is always locally finite, any component is either finite or countable. In fact, for representation-infinite (and connected) algebras, there are no finite components:

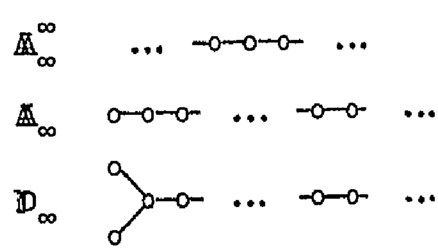
<u>THEOREM</u> (Auslander). Assume A is connected, and that Γ is a component of Γ_A containing only modules of bounded length. Then A is representation finite and $\Gamma_A = \Gamma$.

This strengthens Rojter's theorem which had established the first Brauer-Thrall conjecture ("bounded representation type implies finite representation type"). It also gives an effective method for showing that a given finite list $M_1,\ldots,M_n$ of indecomposable modules is complete: it is sufficient to show that given an irreducible map $M_i \longrightarrow X$ or $X \longrightarrow M_i$ with X indecomposable, then X already occurs in the list. The completeness may be shown in the following way: first, one checks that all indecomposable projective modules occur in the list; second, that the list is closed under τ^- and contains all indecomposable summands of the corresponding Auslander-Reiten sequences (i.e. if M_j is not injective, and (**) is an Auslander-Reiten sequence with $X = M_j$, then Z and all

the Y_i should belong to the list). We observe that this procedure refrains from the necessity of decomposing large modules, an operation which would produce a lot of difficulties. The only modules to be decomposed are those occurring as middle terms of Auslander-Reiten sequences (and, according to a theorem of Bautista-Brenner for representation-finite algebras, these are direct sums of at most four indecomposable modules).
We pose the following problem:

Problem 1. Let Γ be a component of some Auslander-Reiten quiver, and d a natural number. Is the number of isomorphism classes of indecomposable modules X in Γ of length d always finite?

Given any quiver Δ, one may construct a translation quiver $\mathbb{Z}\Delta$ as follows: the set of vertices of $\mathbb{Z}\Delta$ is given by $\mathbb{Z}\times\Delta_0$; for any arrow $\alpha : a \longrightarrow b$ in Δ, there are the arrows $(z,\alpha) : (z,a) \longrightarrow (z,b)$ and $(z,\alpha)' : (z,b) \longrightarrow (z+1,a)$ for all $z \in \mathbb{Z}$, and the translation is defined by $\tau(z,a) = (z-1,a)$. This is a regular translation quiver, and any regular translation quiver can be obtained from some $\mathbb{Z}\Delta$, even with Δ a tree, as factor quiver $\mathbb{Z}\Delta/G$ with respect to some automorphism group G of $\mathbb{Z}\Delta$. Note that for Δ an oriented tree, we obtain isomorphic translation quivers when changing the orientation, thus, in this case, we do not have to specify the orientation. Of particular interest for representation theory are the translation quivers $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{Z}\mathbb{A}_\infty^\infty$ and $\mathbb{Z}\mathbb{D}_\infty$, where $\mathbb{A}_\infty^\infty$ has as vertices the integers, and edges i—$i+1$, for $i \in \mathbb{Z}$, the graph $\mathbb{A}_\infty$ is the full subgraph of $\mathbb{A}_\infty^\infty$ given by the non-negative integers, and $\mathbb{D}_\infty$ is obtained from $\mathbb{A}_\infty$ by adding a vertex $0'$ and an edge $0'$—1. Thus,



The automorphisms of $\mathbb{Z}\mathbb{A}_\infty$ are of the form τ^n, with $n \in \mathbb{Z}$, and for $n \geq 1$ we denote $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$ just by $\mathbb{Z}\mathbb{A}_\infty/n$. Components of the form $\mathbb{Z}\mathbb{A}_\infty/n$ with $n \geq 1$ will be called regular tubes, those with $n = 1$ are said to be homogeneous tubes.

The regular components of a hereditary algebra A are all of the form $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/n$, with $n \geq 1$, they are tubes in case A is tame, and then almost all are homogeneous, whereas all regular components are of the form $\mathbb{Z}\mathbb{A}_\infty$, in case A is wild [Ri1]. For the group algebra kG of a dihedral 2-group G, with k of characteristic 2, there are countably many components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$, all other regular components are homogeneous tubes, the number of homogeneous tubes is equal to the cardinality of k [BSh]. For a semidihedral group G, and k again of characteristic 2, there are countably many components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$, of the form $\mathbb{Z}\mathbb{D}_\infty$, and of the form $\mathbb{Z}\mathbb{A}_\infty/2$; the remaining ones are homogeneous tubes, the number of such components again is equal to the cardinality of k. We will see in the next lecture that for nearly all finite quivers Δ, there are algebras having a component of the form $\mathbb{Z}\Delta$.

A vertex x of a translation quiver which satisfies $\tau^n x = x$ for some $n \geq 1$ is said to be <u>periodic</u>. Of course, all vertices of $\mathbb{Z}\mathbb{A}_\infty/n$ are periodic. And there is the following converse

<u>PROPOSITION</u>. If a regular component of an Auslander-Reiten quiver contains a periodic vertex, then it is a regular tube.

We pose the following problem:

<u>Problem 2</u>. Let A be any finite-dimensional algebra. Is it true that all but finitely many components of Γ_A are of the form $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{Z}\mathbb{A}_\infty/n$, $\mathbb{Z}\mathbb{A}_\infty^\infty$, and $\mathbb{Z}\mathbb{D}_\infty$? Is it true that all but at most countably many components of Γ_A are of the form $\mathbb{Z}\mathbb{A}_\infty$ and $\mathbb{Z}\mathbb{A}_\infty/1$?

We note that the first question has a positive answer for group algebras, according to the investigations of Webb [W]. Note that a positive answer to the first question implies that at most finitely many components of Γ_A can have multiple arrows, and consequently that for M,M' indecomposable, $\dim \operatorname{Irr}(M,M') \leq 1$, except for at most countably many modules M,M'. On the other hand, it is easy to construct examples of components with multiple arrows: We later will discuss preprojective components, and there is no difficulty to exhibit such components with an arbitrarily large number of arrows between two vertices. In the next lecture, we will see that in the same way we also may construct regular components with an arbitrarily large number of arrows between two vertices.

Given a polarized translation quiver Γ, there is defined its <u>mesh category</u> $k(\Gamma)$, as follows: first, we construct the path category $k\Gamma$

of Γ, it is an additive category whose indecomposable objects are just the vertices of Γ, the space $\mathrm{Hom}_{k\Gamma}(a,b)$ of morphisms from a to b is the k-vectorspace with basis the set of all paths from a to b in Γ, and the composition of morphisms is induced from the usual composition of paths. The mesh category $k(\Gamma)$ is the factor category of $k\Gamma$ modulo the ideal generated by the socalled mesh relations: these are the elements of the form $\Sigma\, \sigma(\alpha)\alpha$, the summation being done over all arrows α with fixed non-projective endpoint.

A component Γ of the Auslander-Reiten quiver Γ_A of an algebra A will be said to be standard provided the full additive subcategory of A-mod whose indecomposable objects are the modules in Γ is equivalent to $k(\Gamma)$. In this and the next lecture, we will provide methods for constructing standard components, but we should stress already here that standard components seem to occur only scarcely.

Problem 3. Let A be any finite-dimensional algebra. Is it true that any standard regular component is either a regular tube or of the form $\mathbb{Z}\Delta$, with Δ a finite quiver without oriented cycles?

Given a translation quiver Γ, it will be of interest to consider integer valued functions on the set Γ_o of vertices of Γ. A function $f : \Gamma_o \longrightarrow \mathbb{Z}$ is called an additive function on Γ provided

$$f(z) + f(\tau z) = \sum_{e(\alpha)=z} f(s(\alpha))$$

for any non-projective vertex z. A typical example of an additive function on an Auslander-Reiten quiver Γ_A is the length function which attaches to each vertex $[X]$ the length of the module X. This function has additional properties, giving rise to the following definition: an additive function f on the translation quiver Γ is called a length function provided

$$f(x) \geq 1 \quad \text{for all } x \in \Gamma_o$$
$$f(p) = 1 + \sum_{e(\alpha)=p} f(s(\alpha)) \text{ for any projective vertex } p,$$
$$f(q) = 1 + \sum_{s(\alpha)=q} f(e(\alpha)) \text{ for any injective vertex } q.$$

We note however that on an Auslander-Reiten quiver Γ_A, there usually will exist length functions different from the function $[X] \mapsto \mathrm{length}(X)$.

A translation quiver Γ is said to be preprojective provided the following three conditions are satisfied:

(1) There is no cyclic path in Γ.

(2) Any τ-orbit contains a projective vertex.

(3) There are only finitely many τ-orbits.

There is the following criterion for preprojective translation quivers occurring as components of Auslander-Reiten quivers of algebras:

PROPOSITION. A prepojective translation quiver admits at most one length function. A connected preprojective translation quiver occurs as a component of some Auslander-Reiten quiver Γ_A if and only if it admits a length function.

Also, there is the following result:

PROPOSITION. A preprojective component of an Auslander-Reiten quiver Γ_A always is standard.

Given a Krull-Schmidt category K, we construct full subcategories ${}_dK$, for d an integer ≥ -1, or $d = \infty$, as follows: Let ${}_{-1}K$ be the subcategory $\langle o \rangle$. If ${}_{d-1}K$ is already defined, let ${}_dK$ be the full subcategory of all objects Z of K which have the property that any indecomposable object Y with $\mathrm{rad}(Y,Z) \neq 0$ belongs to ${}_{d-1}K$. Finally, let ${}_\infty K$ be the union of all ${}_dK$, $d \in \mathbb{N}$. In case $K = A\text{-mod}$, the indecomposable modules in ${}_oK$ are just the simple projective modules; and an indecomposable module belongs to ${}_1K$ if and only if it is projective and its radical is semisimple and projective. Clearly, if $[X] \longrightarrow [Y]$ is an arrow in Γ_A, and Y belongs to ${}_\infty(A\text{-mod})$, then also X belongs to ${}_\infty(A\text{-mod})$. Actually, given a translation quiver Γ, we may define in a similar way full translation subquivers ${}_d\Gamma$, where d is an integer ≥ -1, or $d = \infty$ (by definition, ${}_{-1}\Gamma$ is the empty quiver, a vertex z of Γ belongs to ${}_d\Gamma$ if and only if every vertex y with an arrow $y \longrightarrow z$ belongs to ${}_{d-1}\Gamma$, and ${}_\infty\Gamma$ is the union of all ${}_d\Gamma$, $d \in \mathbb{N}$), and it is easy to see that a vertex $[X]$ of Γ_A belongs to ${}_d\Gamma_A$ if and only if X belongs to ${}_d(A\text{-mod})$, for any d. We are interested in the question under what conditions ${}_\infty\Gamma_A$ is a component, or at least a union of components of Γ_A. Note that ${}_\infty\Gamma_A$ is non-empty if and only if there exists a simple projective A-module. Always ${}_\infty\Gamma_A$ is a preprojective translation quiver, and any component of Γ_A which is preprojective, is contained in ${}_\infty\Gamma_A$. The following criterion is easily verified: ${}_\infty\Gamma_A$ is a union of (preprojective) components if and only if the following condition is satisfied:

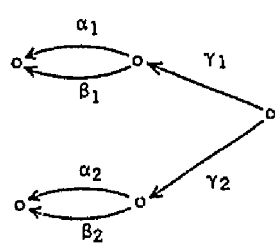
(P) If P is an indecomposable projective A-module, and some indecomposable direct summand of rad P belongs to ${}_\infty$(A-mod), then rad P belongs to ${}_\infty$(A-mod).

An immediate consequence is the following: assume the radical of any indecomposable projective A-module is indecomposable or zero, and that there exists at least one simple projective A-module, then ${}_\infty\Gamma_A$ is non-empty and a union of preprojective components of Γ_A.

Examples of classes of algebras which have preprojective components are the following:

- the hereditary algebras
- the concealed algebras (see lecture 2)
- the canonical algebras (see lecture 3).

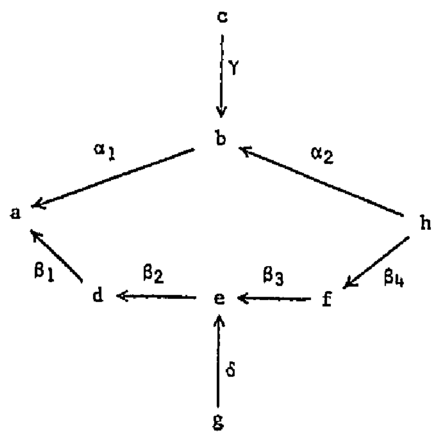
One should observe that the Auslander-Reiten quiver of an algebra A usually will not have any preprojective component at all, but even for A being connected, there may be more than one preprojective component, as the following example shows:



$\gamma_1\alpha_1 = 0, \qquad \gamma_2\alpha_2 = 0$

(Here, and in the following, it often will be convenient to exhibit an algebra A by a quiver with relations, this means that the opposite algebra A^{op} is obtained from the path algebra of the quiver by factoring out the ideal generated by the given relations. In this way, the category A-mod is just the category of all representations of the quiver which satisfy the given relations! We should recall that a representation V of a quiver Q is given by a set of (finite-dimensional) vectorspaces V_x, indexed by the vertices x of Q, and linear maps $V_\alpha : V_{s(\alpha)} \longrightarrow V_{e(\alpha)}$, indexed by the arrows α of Q.)

Let us outline the actual construction of a preprojective component by considering the following example:



$$\alpha_2\alpha_1 = \beta_4\beta_3\beta_2\beta_1$$

First, one has to determine the indecomposable projective modules and their radicals. Any vertex x of the quiver Q of A determines a simple module $E(x)$ (if we consider $E(x)$ as a representation of Q, the vectorspace indexed by x is k, all others are 0), and we denote by $P(x)$ the projective cover of x (considering $P(x)$ as a representation of Q, the vectorspace $P(x)_y$ is obtained from the free vectorspace with basis all paths from x to y as a factorspace by taking into account the given relations). In this way, we obtain all simple and all indecomposable projective modules. Note that for any representation V of Q, the dimension of V_x is just the Jordan-Hölder multiplicity of $E(x)$ in V, and it seems to be rather suggestive to arrange the entries of the dimension vector in the form of the quiver. Actually, it will be convenient to denote V just by $\underline{\dim}\, V$, provided V is the only indecomposable module with this dimension vector. (The indecomposable modules in $_\infty$(A-mod) always have this property, an account of this result will be given in the second lecture). In our case, we obtain the following list of indecomposable projective modules and their radicals:

<table>
<tr><td>x</td><td>a</td><td>b</td><td>c</td><td>d</td><td>e</td><td>f</td><td>g</td><td>h</td></tr>
<tr><td>P(x)</td><td>o
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</table>

The module $P(a)$ is simple projective, and all rad $P(x)$, $x \neq a$, are indesomposable, thus the criterion mentioned above asserts that A-mod has a preprojective component.

We are going to construct inductively ${}_d(A\text{-mod})$ and ${}_d\Gamma_A$. The indecomposable modules in ${}_o(A\text{-mod})$ are the simple projective ones. (In our case, ${}_o\Gamma_A$ consists of the single vertex $[P(a)]$. In particular, ${}_\infty\Gamma_A$ will be connected.) Suppose we have already constructed ${}_d(A\text{-mod})$, and ${}_d\Gamma_A$, for some $d \geq o$. We single out the indecomposable modules X in ${}_d(A\text{-mod})$ which satisfy the following properties:

i) $x = [X]$ is an injective vertex of ${}_d\Gamma_A$, and there is an arrow $x \longrightarrow y$ in ${}_d\Gamma_A$, with y not belonging to ${}_{d-1}\Gamma_A$.

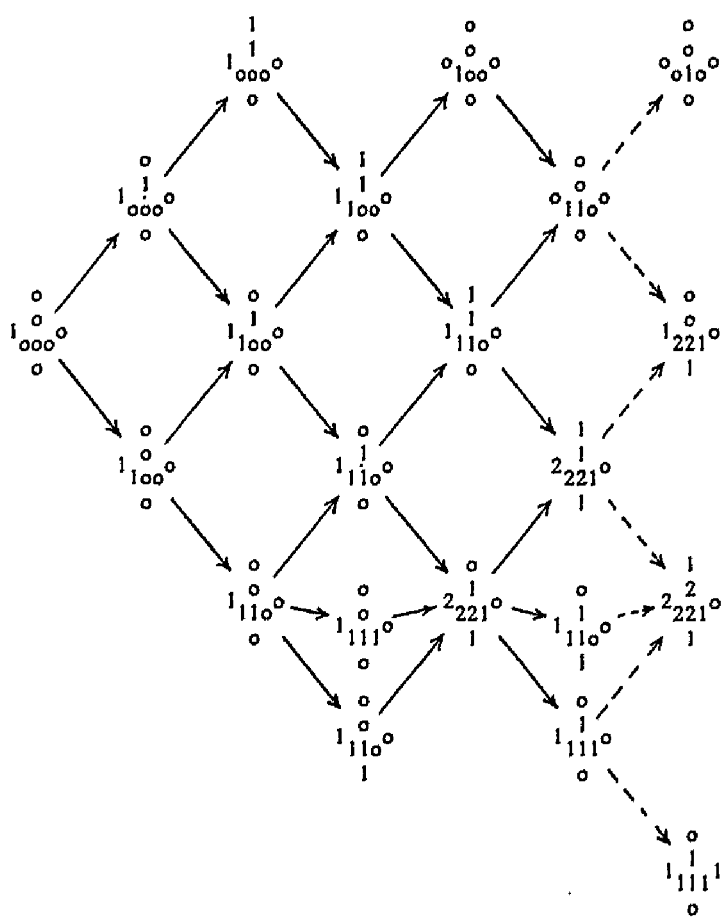
ii) If M is an indecomposable module in ${}_d(A\text{-mod})$ with an arrow $[M] \longrightarrow x$ in ${}_d\Gamma_A$, then either M is an injective A-module, or else $[M]$ is not injective in ${}_d\Gamma_A$.

iii) If X is a direct summand of rad P, with P indecomposable projective, then P belongs to ${}_d(A\text{-mod})$.

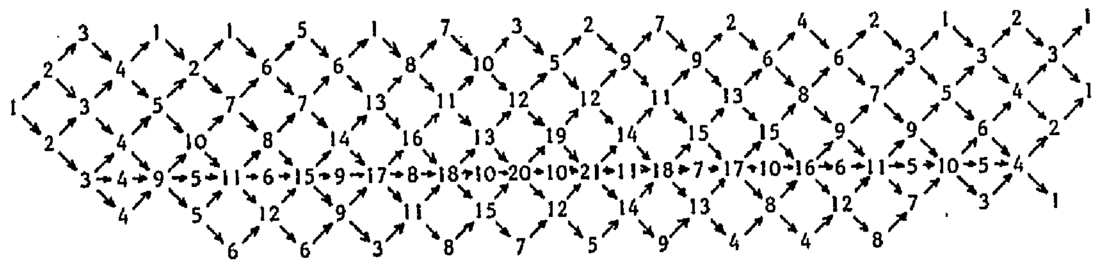
Suppose X satisfies i), ii), iii). Consider the vector

$$- \underline{\dim}\, X + \sum_\alpha \underline{\dim}\, Y_\alpha ,$$

where the summation extends over all arrows $\alpha : [M] \longrightarrow [Y_\alpha]$ starting in $[X]$. In case this vector is not positive, X is injective, thus the τ-orbit of $[X]$ in Γ_A ends in $[X]$. Or else, this vector is the dimension vector $\underline{\dim}\, \tau^- X$, and $\tau^- X$ belongs to ${}_{d+1}(A\text{-mod})$; in this case, we obtain $[\tau^- X]$ as a vertex of ${}_{d+1}\Gamma_A$ outside ${}_d\Gamma_A$, with $\tau[\tau^- X] = [X]$ (the arrows ending in $[\tau^- X]$ are uniquely determined by the condition on ${}_{d+1}\Gamma_A$ to be a translation quiver). In this way, we will obtain several new vertices in ${}_{d+1}\Gamma_A$ all of which are non-projective. In case there exist indecomposable projective modules P not belonging to ${}_d(A\text{-mod})$, we have to check whether there exists such a P with rad P in ${}_d(A\text{-mod})$. Thus assume P is indecomposable projective, does not belong to ${}_d(A\text{-mod})$, rad $P = \oplus Y_i^{n_i}$ with all Y_i indecomposable, and pairwise non-isomorphic, and all Y_i in ${}_d(A\text{-mod})$. In this case, $[P]$ is a vertex of ${}_{d+1}\Gamma_A$ outside ${}_d\Gamma_A$, and there are n_i arrows $[Y_i] \longrightarrow [P]$. This finishes the construction of ${}_{d+1}\Gamma_A$. Let us consider our example: suppose we have already constructed ${}_5\Gamma_A$, namely the part depicted below by solid arrows, and we are going to construct ${}_6\Gamma_A$. There are three indecomposable modules X which satisfy the conditions i), ii), iii), namely those with dimension vectors $\begin{smallmatrix} & 0 & \\ & 0 & \\ 0 & 100 & 0 \\ & 0 & \end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & 110 & 0 \\ & 0 & \end{smallmatrix}$, and $\begin{smallmatrix} & 0 & \\ & 1 & \\ 2 & 221 & 0 \\ & 1 & \end{smallmatrix}$, and the calculation of dimension vectors shows that none of these is injective (of course, this could be checked also directly!). In this way, we obtain three non-projective vertices in ${}_6\Gamma_A$ outside ${}_5\Gamma_A$. Also, the indecomposable module Y with $\underline{\dim}\, Y = \begin{smallmatrix} & 0 & \\ & 1 & \\ 1 & 111 & 0 \\ & 0 & \end{smallmatrix}$ is just rad $P(h)$, thus ${}_6\Gamma_A$ contains in addition $[P(h)]$, and there is just one arrow ending in $[P(h)]$, and this arrow starts in $[Y]$:



Continuing in this way, we obtain the complete component. In our case, the component turns out to be finite, thus it is the whole Auslander-Reiten quiver Γ_A. Actually, as soon as we know the position of the projective vertices, we may as well work solely with the length function: it only remains to determine whether any vertex is an injective one, and this can be read off from the values of the length function. In our case, we obtain the following values of the length function:



Looking at the Auslander-Reiten quiver Γ_A of an algebra A, the indecomposable A-modules seem to be converted into mere vertices of a graph. So, an obvious question is under what conditions one may recover the modules from the combinatorial data. In case we deal with a preprojective component Γ, everything works out very well: Let X be an indecomposable module. We decompose ${}_AA = \oplus P_i$, with P_i indecomposable (and projective), and A may be identified with the endomorphism ring $\mathrm{End}({}_AA) = \mathrm{End}(\oplus P_i)$, thus A is given by the various $\mathrm{Hom}(P_i,P_j)$ and their compositions. As a k-space, there are the canonical isomorphisms

$$X \approx \mathrm{Hom}({}_AA,X) \approx \oplus \mathrm{Hom}(P_i,X),$$

and the A-module structure on X corresponds to the operation of $\oplus \mathrm{Hom}(P_i,P_j)$ on $\oplus \mathrm{Hom}(P_i,X)$, using the composition of maps. Assume now that X belongs to Γ. Then, any P_i with $\mathrm{Hom}(P_i,X) \neq o$ also belongs to Γ, since Γ is a preprojective component. Also, Γ is standard, thus we may calculate in $k(\Gamma)$ all the non-zero $\mathrm{Hom}(P_i,X)$, all $\mathrm{Hom}(P_s,P_t)$ where both P_s,P_t belong to Γ, and all the corresponding compositions. We may reformulate this as follows: Let e be an idempotent of A such that all the indecomposable direct summands of Ae belong to Γ, whereas none of $A(1-e)$ belongs to Γ. Then, first of all, $(1-e)Ae = o$, therefore $A(1-e)$ is a twosided ideal, and the algebra $A/A(1-e) \approx eAe$ can be calculated in $k(\Gamma)$. Also, given an indecomposable module X in Γ, we have $(1-e)X = o$, thus X as an A-module is in fact an $A/A(1-e)$-module, and X, as an $A/A(1-e)$-module, can be calculated in $k(\Gamma)$. Let us return to the example considered above. We have calculated the values of the length function, and we see that there is a unique indecomposable module of maximal length, namely $\tau^{-7}P(e)$. The explicit calculation of dimension vectors shows that $\underline{\dim}\ \tau^{-7}P(e) = \begin{smallmatrix} & & 2 & & \\ & & 3 & & \\ 2 & 3 & 4 & 3 & 2 \\ & & 2 & & \end{smallmatrix}$. As a quiver representation, this module is given by four 2-dimensional, three 3-dimensional, and one 4-dimensional vectorspace. Later in this lecture, we will draw attention to these vectorspaces and suitably defined subspaces.

Translation quivers do not only arise in the representation theory of finite-dimensional algebras, but also for many related structures. We will consider in this lecture a second case: the Auslander-Reiten quiver of a finite partially ordered set. First, we recall the main notions of the representation theory of partially ordered sets. Let S be a finite partially ordered set. An S-<u>space</u> is of the form $V = (V_\omega; V_s)_{s \in S}$, where V_ω is a vectorspace, all V_s, $s \in S$, are subspaces of V_ω, and $V_s \subseteq V_t$ for $s \le t$. The space V_ω is called the total space of V, its dimension is denoted by $\underline{\dim}_\omega V$. A map $f : V \longrightarrow W$ of S-spaces is given by a k-linear map $f_\omega : V_\omega \longrightarrow W_\omega$ satisfying $V_s f_\omega \subseteq W_s$, for all $s \in S$; the restriction of f_ω to V_s will be denoted by $f_s : V_s \longrightarrow W_s$ and we may write $f = (f_\omega; f_s)$. It will be convenient to consider besides S also the partially ordered set S^+ obtained from S by adding a new element ω with $s < \omega$ for all $s \in S$. We denote by $\ell(S)$ the category of all S-spaces with finite-dimensional total space. It is a Krull-Schmidt category with short exact sequences, a short exact sequence is of the form (f,g), where $f : V' \longrightarrow V$, $g : V \longrightarrow V''$ are maps of S-spaces such that all the sequences

$$0 \longrightarrow V_t' \xrightarrow{f_t} V_t \xrightarrow{g_t} V_t'' \longrightarrow 0$$

are exact, for $t \in S^+$. If (f,g) is a short exact sequence, then f may be called a proper mono, and g a proper epi. Note that $f : V' \longrightarrow V$ is proper mono if and only if f_ω is mono and $V_s' f_s = V_\omega' f \cap V_s$. And, $g : V \longrightarrow V''$ is proper epi if and only if all g_s, $s \in S^+$, are epi. For $s \in S^+$, we define an S-space $P_S(s)$ by $(P_S(s))_t = k$ for all $t \ge s$, and $= 0$ otherwise. These S-spaces behave similar to the indecomposable projective modules: they are the only indecomposable S-spaces which have the usual lifting property with respect to all proper epis. Similarly, we may consider the indecomposable S-spaces which have the extension property with respect to the proper monos: they are denoted by $Q_S(s)$ with $s \in S \cup \{\omega'\}$. Here, $(Q_S(\omega'))_t = k$ for all $t \in S^+$, and, for $s \in S$, we have $(Q_S(s))_t = k$ provided $t \nleq s$ and $= 0$ otherwise. Since $\ell(S)$ is a Krull-Schmidt category, we may speak of source maps and sink maps in $\ell(S)$, and there is the corresponding result which seems to be due to Bautista:

<u>THEOREM</u>. For any indecomposable S-space V, there exists a source map and a sink map, and both are unique up to isomorphism.

Let Z be an indecomposable S-space with sink map $g : Y \longrightarrow Z$. Either $Z = P_S(\omega)$, then $Y = 0$. Or, $Z = P_S(s)$ for some $s \in S$, then Y has one-dimensional total space; in particular, Y is indecomposable. Or, if Z is not of the form $P_S(s)$ for any $s \in S^+$, then there is an exact sequence (f,g) of S-spaces, say with $f : X \longrightarrow Y$, the S-space X is indecomposable and f is a source map.

Let X' be an indecomposable S-space with source map $f' : X' \longrightarrow Y'$. Either $X' = Q_S(\omega')$, then $Y' = 0$. Or, $X' = Q_S(s)$ for some $s \in S$, then Y' has one-dimensional total space; in particular, Y' is indecomposable. Or, if X' is not of the form $Q_S(s)$ for any $s \in S \cup \{\omega'\}$, then there is an exact sequence (f',g') of S-spaces, say with $g' : Y' \longrightarrow Z'$, the S-space Z' is indecomposable and g' is a sink map.

Of course, as in the case of a module category, the short exact sequences (f,g) with $f : X \longrightarrow Y$ a source map, $g : Y \longrightarrow Z$ a sink map, are called Auslander-Reiten sequences, we write $X = \tau Z$, $Z = \tau^- X$, and there is the same relation between such an Auslander-Reiten sequence and irreducible maps starting in X or ending in Z. In particular, the Auslander-Reiten sequences in $\ell(S)$ show that we may define a translation quiver Γ_S in the same way, as we have defined the Auslander-Reiten quiver of an algebra, and Γ_S is called the Auslander-Reiten quiver of the partially ordered set S. Note that the function $\underline{\dim}_\omega$ is an additive function on Γ_S.

In contrast to the case of an algebra, the Auslander-Reiten quiver of a finite partially ordered set S always has a preprojective component and this component is just ${}_\infty\Gamma_S$. The reason is very simple: the projective vertices of Γ_S are of the form $[P_S(s)]$, $s \in S^+$, there is a unique source in Γ_S, namely $[P_S(\omega)]$, and any other projective vertex has a unique direct predecessor in Γ_S, since, as we have seen, the S-space Y occurring in the sink map $Y \longrightarrow P_S(s)$, for $s \in S$, is indecomposable. The translation quivers of the form ${}_{\infty}\Gamma_S$ can be characterized as follows.

Given a preprojective translation quiver Γ with a unique source ω, there exists a unique additive function h_Γ such that $h_\Gamma(\omega) = 1$, and $h_\Gamma(p) = \sum_{e(\alpha)=p} h_\Gamma(s(\alpha))$, for any projective vertex $p \neq \omega$ of Γ (the existence and unicity is shown by induction on $d \in \mathbb{N}$, where $\Gamma = {}_d\Gamma$, and then follows for $\Gamma = {}_{\infty}\Gamma$). A preprojective translation quiver Γ with a unique source is called a <u>left hammock</u> provided h_Γ takes values in $\mathbb{N}_1$ and satisfies $h_\Gamma(q) \geq \sum_{s(\alpha)=q} h_\Gamma(e(\alpha))$, for all injective vertices q of Γ, and, in this case, h_Γ is called the hammock function on Γ. A left hammock Γ is said to be <u>thin</u> provided $h_\Gamma(p) = 1$ for any projective vertex p of Γ.

<u>PROPOSITION</u>. ([RV]) If S is a finite partially ordered set, then ${}_{\infty}\Gamma_S$ is a thin left hammock, and the hammock function on ${}_{\infty}\Gamma_S$ is just the function $\underline{\dim}_\omega$. Conversely, any thin left hammock occurs in this way.

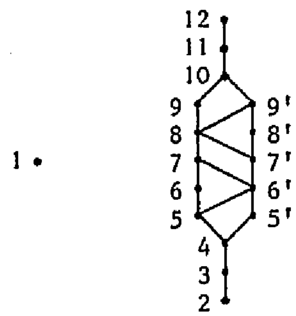
In particular, we see that the preprojective component ${}_{\infty}\Gamma_S$ of Γ_S has no multiple arrows. We may pose the following problem:

<u>Problem 4</u>. Let X,Y be indecomposable S-spaces. Is always $\dim \operatorname{Irr}(X,Y) \leq 1$?

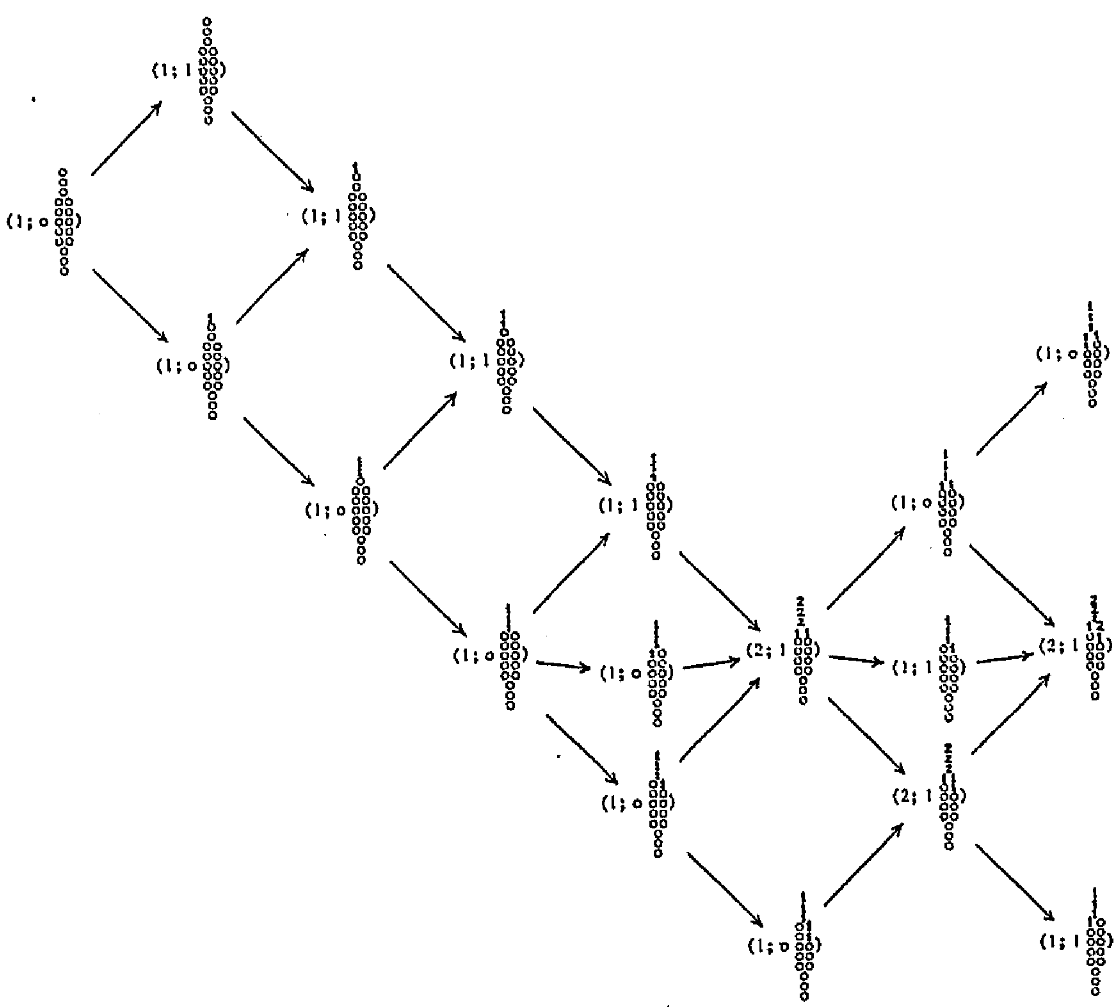
As in the case of an algebra, a component Γ of Γ_S may be said to be <u>standard</u> provided the full additive subcategory of $\ell(S)$ whose indecomposable S-spaces are the S-spaces belonging to Γ is equivalent to $k(\Gamma)$.

<u>PROPOSITION</u>. Let S be a finite partially ordered set. The component ${}_{\infty}\Gamma_S$ always is standard.

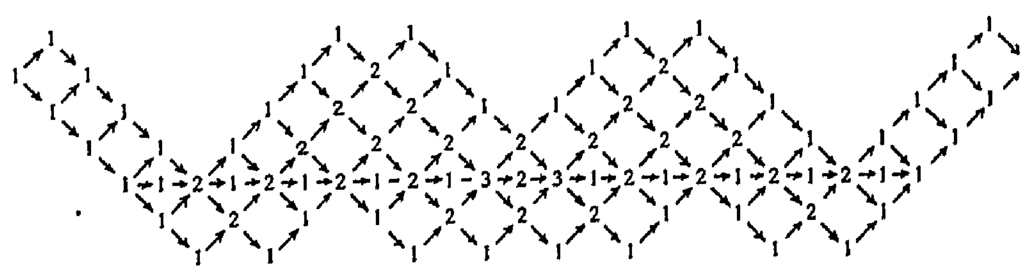
The actual construction of ${}_{\infty}\ell(S)$ and ${}_{\infty}\Gamma_S$ is done in the same way as in the case of an algebra. Again, it is convenient to work with integral vectors: given an S-space V, let $\underline{\dim}_S V = (\dim V_\omega; \dim V_s)_{s \in S}$, and we will arrange the entries $\dim V_s$ $(s \in S)$ in the form of S. (Recall that the first component $\dim V_\omega$ of $\underline{\dim}_S V$ also has been denoted by $\underline{\dim}_\omega V$.). For example, consider the partially ordered set S



we obtain for $_7\Gamma_S$ the following translation quiver:



For the example we have chosen here, it turns out that ${}_\infty\Gamma_S$ actually is finite, and (as in the case of an algebra) this implies ${}_\infty\Gamma_S = \Gamma_S$, thus there are only finitely many isomorphism classes of indecomposable S-spaces (of course, in this case, S is said to be representation finite). The complete Auslander-Reiten quiver Γ_S together with its hammock function $\underline{\dim}_\omega$ looks as follows:



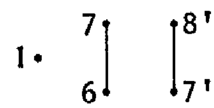
Also here, we may ask whether it is possible to recover an S-space from the component it belongs to, and its position in this component. Again, we will see that this is possible in case we deal with preprojective components! Given any S-space V, we may consider the vectorspace $\mathrm{Hom}(P_S(\omega),V)$ and its subspaces $\mathrm{Hom}(P_S(\omega),P_S(s))\mathrm{Hom}(P_S(s),V)$, defined for any $s \in S$. In this way, we actually obtain an S-space $(\mathrm{Hom}(P_S(\omega),V);\ \mathrm{Hom}(P_S(\omega),P_S(s))\mathrm{Hom}(P_S(s),V))_{s\in S}$. We choose a non-zero element $\xi \in (P_S(\omega))_\omega = k$, the evaluation at ξ gives an isomorphism $\mathrm{Hom}(P_S(\omega),V) \longrightarrow V_\omega$, which maps the subspace $\mathrm{Hom}(P_S(\omega),P_S(s))\mathrm{Hom}(P_S(s),V)$ onto V_s, for any $s \in S$, thus

$$V = (V_\omega;V_s)_{s\in S} \approx (\mathrm{Hom}(P_S(\omega),V);\mathrm{Hom}(P_S(\omega),P_S(s))\mathrm{Hom}(P_S(s),V))_{s\in S}.$$

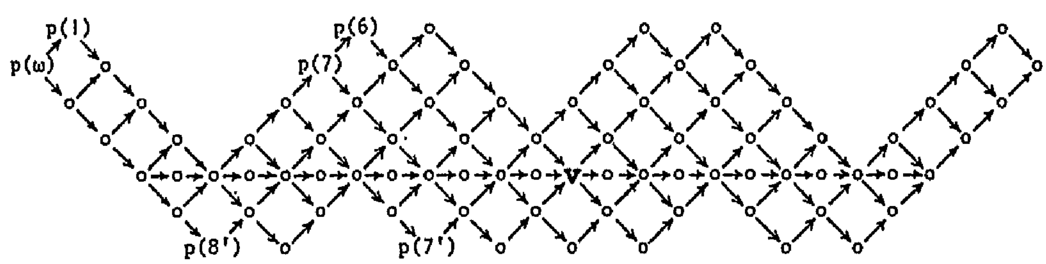
If V belongs to ${}_\infty\ell(S)$, then $\mathrm{Hom}(P_S(s),V)$ can be non-zero only in case $P_S(s)$ belongs to ${}_\infty\ell(S)$, and, since ${}_\infty\Gamma_S$ is standard, we can calculate for any $P_S(s)$ in ${}_\infty\ell(S)$ the spaces $\mathrm{Hom}(P_S(\omega),P_S(s)),\mathrm{Hom}(P_S(s),V)$, and the composition, inside $k({}_\infty\Gamma_S)$, whereas for $P_S(s)$ outside ${}_\infty\ell(S)$, we have both $\mathrm{Hom}(P_S(s),V) = 0$ and $\mathrm{Hom}_{k(\Gamma_S)}([P_S(s)],[V]) = 0$. Altogether, we conclude that for V in ${}_\infty\Gamma(S)$, we have

$$V = (V_\omega;V_s)_{s\in S} \approx (\mathrm{Hom}_{k(\Gamma_S)}(p(\omega),v);\mathrm{Hom}_{k(\Gamma_S)}(p(\omega),p(s))\mathrm{Hom}_{k(\Gamma_S)}(p(s),v))_{s\in S}$$

where $p(s) = [P_S(s)]$ for any $s \in S^+$, and $v = [V]$. Of course, we should keep in mind that the partially ordered set S itself can be recovered from $k(\Gamma)$ only partially, what we can recover is the full subset $\{s \in S \mid P_S(s) \in {}_\infty\ell(S)\}$, it is a filter in S. Given any S-space V, let us define its <u>support</u> S_V as the set of all $s \in S$ such that $\sum_{t<s} V_t \subsetneqq V_s$, with the induced ordering. Clearly, we may recover V if we know just V_ω and the subspaces V_s, $s \in S_V$. For example, in the special case considered above, the S-space $V = \tau^{-6}P_S(10)$ has as support the subset



of S, so it is completely determined by its total space which we may assume to be $\mathrm{Hom}_{k(\Gamma_S)}(p(\omega),v)$, and the subspaces $\mathrm{Hom}_{k(\Gamma_S)}(p(\omega),p(s))\mathrm{Hom}_{k(\Gamma_S)}(p(s),v)$, for $s = 1, 6, 7, 7'$, and $8'$. Let us display the vertices $p(s)$ for $s = \omega, 1, 6, 7, 6', 8'$; and, in addition, the vertex v inside Γ_S:



Note that the dimension vector of V, considered as an S_V-space, is $(3;2^{22}_{11})$.

A finite left hammock is just called a <u>hammock</u>. For left hammocks, there is the following result on the growth of the hammock function along a τ-orbit, and we will outline below that this implies that any hammock is thin.

<u>PROPOSITION</u>. ([RV]) Let H be a left hammock, and x a vertex of H. If x is injective, then $h_H(x) = 1$.* If x is not injective, then $h_H(x) \leq h_H(\tau^- x) + 1$.

This result has many consequences. First of all, given an injective vertex q of a left hammock H, then q can be starting point of at most one arrow, and if there is an arrow $q \longrightarrow y$, then also $h_H(y) = 1$. Consequently, a hammock has a unique sink. It follows easily that the opposite translation quiver H^* of a hammock H, is a left hammock again, and in fact, a thin one. Thus, by symmetry, a hammock is always thin. We conclude that the hammocks are just the Auslander-Reiten quivers of the representation-finite partially ordered sets.

As an application, let us outline a relation between the Auslander-Reiten quivers of representation-finite algebras and representation-finite partially ordered sets. In fact, first consider only representation-directed algebras; by definition, these are those representation-finite algebras A whose Auslander-Reiten quiver Γ_A has no oriented cycles, or equivalently, those algebras A for which Γ_A is a preprojective translation quiver. Let $P(x)$ be an indecomposable projective A-module, say $P(x) = Ae(x)$ for some primitive idempotent $e(x)$ in A, and let $A_x = A/\langle e(x)\rangle$, where $\langle e(x)\rangle$ is the twosided ideal of A generated by $e(x)$. Of course, the A_x-modules are just those A-modules M satisfying $\mathrm{Hom}(P(x),M) = 0$, thus those A-modules M which do not contain $E(x) = P(x)/\mathrm{rad}\, P(x)$ as a composition factor. Given A-modules M,N and two maps $f,g : M \longrightarrow N$, define $f \underset{x}{\sim} g$ iff $f-g$ factors through an A_x-module, thus iff the restriction of $f-g$ vanishes on $e(x)M$, and this is equivalent to $\mathrm{Hom}(P(x),f-g) = 0$. The factor category of A-mod obtained in this way is denoted by $\mathcal{H}(x)$, its objects are the same as those of A-mod, and

$$\mathrm{Hom}_{\mathcal{H}(x)}(M,N) = \mathrm{Hom}(M,N)/\underset{x}{\sim}\ .$$

Note that the indecomposable objects in $\mathcal{H}(x)$ are given by those indecomposable modules M with $\mathrm{Hom}(P(x),M) \neq 0$. If we denote by $H(x)$ the full translation subquiver of Γ_A with vertices of the form $[M]$, where M is an indecomposable module satisfying $\mathrm{Hom}(P(x),M) \neq 0$, then one sees rather easily that $\mathcal{H}(x)$ is equivalent to $k(H(x))$, since A-mod is equivalent to $k(\Gamma_A)$. Let us investigate the translation quiver $H(x)$: With Γ_A also $H(x)$ is a preprojective translation quiver, and obviously $H(x)$ has a unique source, namely $[P(x)]$. Consider the function $[M] \longrightarrow \dim \mathrm{Hom}(P(x),M)$ on $H(x)$. Since $P(x)$ is projective, this is an additive function even on all of Γ_A. Also $\dim \mathrm{Hom}(P(x),P,(x)) = 1$, whereas for $P(y)$ indecomposable projective, and $[P(y)] \neq [P(x)]$, we have $\dim \mathrm{Hom}(P(x),P(y)) = \dim \mathrm{Hom}(P(x), \mathrm{rad}\, P(y))$. It follows that this function is just $h_{H(x)}$. Consequently, $h_{H(x)}$ takes values in $\mathbb{N}_1$. Also, for Q indecomposable injective, we have $\dim \mathrm{Hom}(P(x),Q) \geq \dim \mathrm{Hom}(P(x),Q/\mathrm{soc}\, Q)$, thus $h_{H(x)}$ satisfies all the properties of a hammock function, thus $H(x)$ is a finite left hammock, thus a hammock. We can now use the previous result which asserts that $H(x)$ is the Auslander-Reiten quiver $\Gamma_{S(x)}$ of some representation finite partially ordered set. Thus, there is the following theorem:

<u>THEOREM</u>. ([RV]) For any indecomposable projective module $P(x)$ of a representation-directed algebra, there exists a partially ordered set $S(x)$ such that $\mathcal{H}(x)$ and $\ell(S(x))$ are equivalent categories, and $H(x)$ and $\Gamma_{S(x)}$ isomorphic translation quivers.

Note that the hammock function on $H(x)$ is $[M] \longmapsto \dim \mathrm{Hom}(P(x),M)$, whereas the hammock function on $\Gamma_{S(x)}$ is $[V] \longmapsto \underline{\dim}_{\omega} V$, and the unicity of hammock functions asserts that these functions are equal when we identify $H(x)$ and $\Gamma_{S(x)}$. There is the following corollary:

<u>COROLLARY</u>. If M is an indecomposable module over a representation-directed algebra, then all entries of the dimension vector $\underline{\dim}\, M$ are bounded by 6.

<u>Proof</u>. The entries of $\underline{\dim}\, M$ are $(\underline{\dim}\, M)_x = \dim \mathrm{Hom}(P(x),M)$; thus, if $(\underline{\dim}\, M)_x \neq 0$, then $[M]$ is in $H(x)$, and $(\underline{\dim}\, M)_x = h_{H(x)}([M])$. On the other hand, a well-known theorem of Klejner asserts that the total space of an indecomposable S-space, where S is representation finite, is bounded by 6. Since $S(x)$ is representation finite, we can use this theorem: the hammock function on $\Gamma_{S(x)}$ is bounded by 6.

The rather strange bound 6 first appeared in the theorem of Klejner. There is a general result due to Ovsienko from which one may deduce the assertion of the corollary without difficulty. There are also other proofs of the corollary known which however seem to be more awkward. Note that the proof above shows that the assertion concerning modules is a direct consequence of that concerning S-spaces.

Assume now that A is basic, let ${}_AA = \bigoplus_{x=1}^{n} P(x)$, thus $P(1),\ldots,P(n)$ is a complete list of the indecomposable projective modules. For any x, let $S(x)$ be the partially ordered set given by the theorem, with $H(x) \approx \ell(S(x))$, and $H(x) \approx \Gamma_{S(x)}$. If M is an indecomposable module, let $M_x = \mathrm{Hom}(P(x),M)$, and we identify M with $\oplus M_x$. If M is indecomposable and $M_x \neq 0$, then M is an indecomposable object in $H(x) \approx \ell(S(x))$, and the dimension of the $S(x)$-space $\bar{M}_x$ corresponding to M is $\dim M_x$, thus we may consider M_x itself as total space of this $S(x)$-space, thus $\bar{M}_x = (M_x;(\bar{M}_x)_s)_{s \in S(a)}$. Note the following:

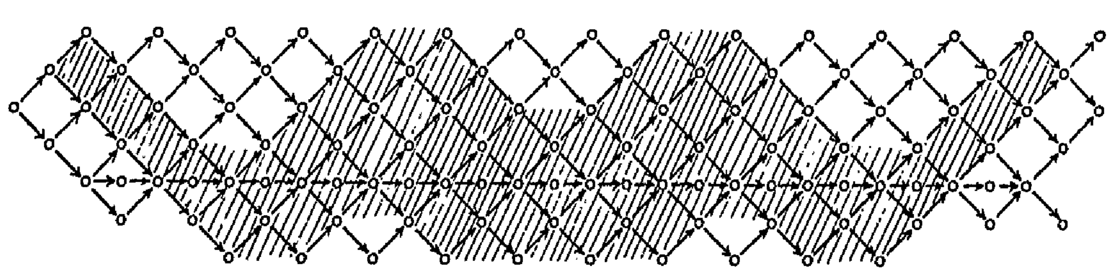
1) We obtain all indecomposable $S(x)$-spaces in the form $\bar{M}_x$, with M indecomposable (and $M_x \neq 0$).

2) If M is indecomposable, $M_x \neq 0$, then the isomorphism class of the $S(x)$-space $\bar{M}_x$ determines the isomorphism class of M uniquely.

3) Let $n(x)$ be the sum of the dimensions of the total spaces of all $S(x)$-spaces occurring in a complete list of indecomposable $S(x)$-spaces (thus, the sum of all values of the hammock function $h_{H(x)}$). Then $\sum_{x=1}^{n} n(x)$ is the sum of the dimensions of the modules occurring in a complete list of indecomposable modules.

Indeed the direct sum $\oplus M$ of all modules M occurring in a complete list of indecomposable modules can be written in the form $\bigoplus_M \bigoplus_x M_x$, and the dimension of $\bigoplus_M M_x$ ist just $n(x)$.

As an illustration of the theorem, we consider again the algebra A which has served before as example; as we have seen, this algebra is representation-directed. If we focus on the hammock $H(b)$ defined by the vertex b, we obtain the following shaded part of Γ_A:



but this is just the hammock which was exhibited above as the Auslander-Reiten quiver Γ_S of some specific partially ordered set, thus $S = S(b)$.

In which way does one obtain $S(x)$ in general? The elements of $S(x)$ are the projective vertices of $H(x)$, thus they are of the form $[U]$, where U is an indecomposable module, with $\mathrm{Hom}(P(x),U) \neq 0$, and $\mathrm{Hom}(P(x),\tau U) = 0$. The fact that the hammock $H(x)$ is thin means just that for such a module U, the space $\mathrm{Hom}(P(x),U)$ is 1-dimensional. Given two indecomposable modules U,U' with $[U],[U']$ in $S(x)$, we have $[U] \geq [U']$ in $S(x)$ if and only if $\mathrm{Hom}_{H(x)}(U,U') \neq 0$, thus if and only if $\mathrm{Hom}(P(x),U)\mathrm{Hom}(U,U') \neq 0$.

Similarly, given an indecomposable module M with $M_x = \mathrm{Hom}(P(x),M) \neq 0$, and $[U] \in S(x)$, we can write down $(\bar{M}_x)_{[U]}$ as follows:

$$(\bar{M}_x)_{[U]} = \mathrm{Hom}(P(x),U)\mathrm{Hom}(U,M) \subseteq \mathrm{Hom}(P(x),M) = M_x.$$

Take a projective presentation of U,

$$\bigoplus_{i=1}^{r} P(y_i) \xrightarrow{[\gamma_{ij}]_{ij}} \bigoplus_{j=o}^{t} P(x_j) \xrightarrow{[\varepsilon_j]_j} U \longrightarrow 0$$

with $x_o = x$, and $\varepsilon_o \neq o$ (note that, in general, a minimal projective presentation will not satisfy these conditions!). Then

$$(\bar{M})_{[U]} = \{m_o \in M_x \mid \exists\ m_j \in M_{x_j},\ 1 \leq j \leq t \text{ with } \sum_{j=o}^{t} \gamma_{ij}\, m_j = 0, \text{ for all } i\}.$$

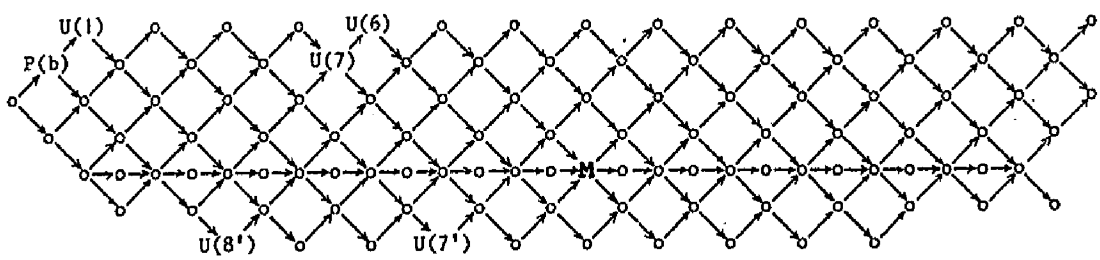
Let us outline the proof: since $\mathrm{Hom}(P(x),U)$ is 1-dimensional, $\mathrm{Hom}(P(x),U) = k\varepsilon_o$, and therefore $\mathrm{Hom}(P(x),U)\mathrm{Hom}(U,M) = \varepsilon_o\,\mathrm{Hom}(U,M)$. Our presentation induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}(U,M) \longrightarrow \bigoplus_{j=o}^{t} \mathrm{Hom}(P(x_j),M) \longrightarrow \bigoplus_{i=1}^{r} \mathrm{Hom}(P(y_i),M).$$

Here, $\eta \in \mathrm{Hom}(U,M)$ is mapped to the tuple $(\varepsilon_o\eta,\dots,\varepsilon_t\eta)$ which satisfies $\sum_j \gamma_{ij}\, \varepsilon_j\eta = 0$ for all i. Now, any element in $\mathrm{Hom}(P(x),U)\mathrm{Hom}(U,M)$ can be written in the form $\varepsilon_o\eta$ with $\eta \in \mathrm{Hom}(U,M)$, thus define $m_j = \varepsilon_j\eta$. Conversely, assume there are given $m_j' \in M_{x_j}$ with $\sum_{j=o}^{t} \gamma_{ij}\, m_j' = 0$, for all i. The exactness of the sequence above gives $\eta' \in \mathrm{Hom}(U,M)$ with

$m_j' = \varepsilon_j \eta'$ in particular, m_o' belongs to $\varepsilon_o \mathrm{Hom}(U,M) = \mathrm{Hom}(P(x),U)\mathrm{Hom}(U,M)$.

We return to our specific example, and consider the A-module $M = \tau^{-7}P(e)$, with dimension vector $\underline{\dim}\, M = \begin{smallmatrix} & & 2 & & \\ & & 3 & & \\ 2 & 3 & 4 & 3 & 2 \\ & & 2 & & \end{smallmatrix}$. We want to determine the $S(b)$-space $\bar{M}_b$. A comparison of the position of $[M]$ in the hammock $H(b)$ reveals that $[M]$, considered as an element of $H(b) = \Gamma_{S(b)}$ is just $[\tau^{-6}P_{S(b)}(10)]$. For any $s \in S(b)$, let $U(s)$ be the indecomposable module with $[U(s)]$ in $H(b)$, and $[U(s)] = [P_S(s)]$ as an element of $H(b) = \Gamma_{S(b)}$. We recall that the support of $\tau^{-6}P_{S(b)}(10)$ is the full subset of $S(b)$ given by the elements $\{1,6,7,7',8'\}$, and that the dimension vector of $\tau^{-6}P_{S(b)}(10)$, restricted to its support, is $(3;2^{22}_{11})$. We are dealing with the following displayed modules:



The recipe for determining $(\bar{M}_b)_{[U(s)]}$ easily yields the following:

s	$\underline{\dim}\, U(s)$	$(\bar{M}_b)_{[U(s)]}$	s	$\underline{\dim}\, U(s)$	$(\bar{M}_b)_{[U(s)]}$
1	$\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 1 & 0 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix}$	γM_c	8'	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 1 & 1 & 1 & 1 & 1 \\ & & 0 & & \end{smallmatrix}$	$\alpha_2 M_h$
7	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 1 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{smallmatrix}$	$\alpha^{-1}\beta_1\beta_2(\beta_3 M_f \cap \delta M_g)$	7'	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{smallmatrix}$	$\alpha_2\beta_4^{-1}\beta_3^{-1}(\beta_2^{-1}O_d + M_g)$
6	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix}$	$\alpha_1^{-1}O_a$			

For example, in dealing with $U(7)$, we use the presentation

$$P(a) \oplus P(e) \xrightarrow{\begin{bmatrix}\alpha_1 & -\beta_1\beta_2\beta_3 & 0\\ 0 & \beta_3 & -\delta\end{bmatrix}} P(b) \oplus P(f) \oplus P(g) \longrightarrow U(7) \longrightarrow 0,$$

thus $(\bar{M}_b)_{[U(7)]}$ is given by the elements $x_o \in M_b$ such that there exist $x_1 \in M_f$, $x_2 \in M_g$ satisfying

$$\alpha_1 x_o - \beta_1\beta_2\beta_3 x_1 = 0, \quad \beta_3 x_1 - \delta x_2 = 0.$$

It follows that $x = x_o \in M_b$ belongs to $(\bar{M}_b)_{[U(7)]}$ if and only if

$$x \in \alpha_1^{-1}\beta_1\beta_2\beta_3\beta_3^{-1}\delta(M_g) = \alpha_1^{-1}\beta_1\beta_2(\beta_3 M_f \cap \delta M_g).$$

Altogether, we see that the hammock approach to representation directed algebras leads to an intrinsic interpretation of the use of subspace methods.

Let us add that the hammock approach is not restricted to representation-directed algebras. In case we deal with a representation-finite algebra A, we may use the well-established covering techniques. Actually, we will consider an arbitrary algebra A and an indecomposable projective A-module $P(a)$ provided there are only finitely many isomorphism classes of indecomposable modules M with $\mathrm{Hom}(P(a),M) \neq 0$. Of course, in this case, all indecomposable modules M with $\mathrm{Hom}(P(a),M) \neq 0$ belong to a single component. We use the filtration of $\mathrm{Hom}(P(a),M)$ given by the subspaces $M(d) = \mathrm{rad}^d(P(a),M)$. The dimension of $M(d)/M(d+1)$ counts the multiplicity of Jordan-Hölder factors of M of the form $P(a)/\mathrm{rad}\, P(a)$ which can be reached from $P(a)$ by means of maps in rad^d, but not in rad^{d+1}. We define a translation quiver $H(a)$ as follows: its vertices are the pairs $([M],d)$, where M is an indecomposable module with $M(d) \neq M(d+1)$. There are only arrows $([M],d) \longrightarrow ([N],e)$ for $e = d+1$, and the number of arrows is equal to $\dim_k \mathrm{Irr}(M,N)$. Finally, $([M],d)$ is projective if either $d \leq 1$, or both $d \geq 2$ and $(\tau M)(d-2) \neq (\tau M)(d-1)$; and $\tau([M],d) = ([\tau M],d-2)$, otherwise. For A representation-directed, the translation quiver constructed in this way is canonically isomorphic to the previously constructed hammock. Always, the translation quiver $H(a)$ is a hammock, thus, there exists a representation-finite partially ordered set $S(a)$ with $H(a) = \Gamma_{S(a)}$.

We end this lecture with two problems.

Problem 5. Which representation-finite partially ordered sets do occur in the form S(a)?

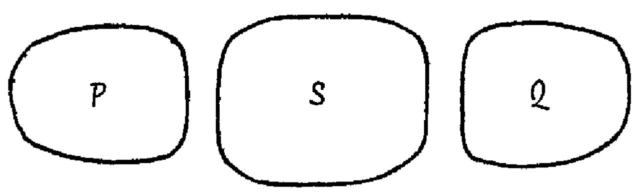
Note that there are examples of representation-finite partially ordered sets which cannot be realized in the form S(a).

Problem 6. Let A be a representation-finite algebra. Is it possible to construct the partially ordered sets S(a) without prior knowledge of Γ_A? And, what kind of relations do exist between the various S(a)?

LECTURE 2

SEPARATING SUBCATEGORIES: CONNECTING COMPONENTS AND SEPARATING TUBULAR FAMILIES

A full subcategory of A-mod which is closed under direct sums and direct summands will be called a module class. We will focus our attention to an interesting phenomenon, the existence of separating subcategories: for some kinds of algebras A, it turns out that there exist module classes $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}$ in A-mod such that any indecomposable A-module belongs to $\mathcal{P}$ or $\mathcal{S}$ or $\mathcal{Q}$ (thus, $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}$ exhaust the category A-mod), that $\operatorname{Hom}(X,Y) = 0$ provided one of the following conditions is satisfied: $X \in \mathcal{Q}$, $Y \in \mathcal{P}$; or $X \in \mathcal{Q}$, $Y \in \mathcal{S}$; or $X \in \mathcal{S}$, $Y \in \mathcal{P}$, and finally, given $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, then any map $P \longrightarrow Q$ can be factored through an object in $\mathcal{S}$. In this case, we call $\mathcal{S}$ a separating subcategory, separating $\mathcal{P}$ from $\mathcal{Q}$. We may visualize these subcategories $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}$ in the following way



with possible maps only going from left to right.

We are going to list some properties of separating subcategories. In order to do so, we will need some further definitions. A sequence $(X_o,X_1,\ldots,X_m)$ of indecomposable modules with $\operatorname{rad}(X_{i-1},X_i) \neq 0$, for all $1 \leq i \leq m$, will be called a path of length m in A-mod. In case there exists a path $(X_o,X_1,\ldots,X_m)$ in A-mod, we will write $X_o \preceq X_m$. Note that for any path $[X_o] \longrightarrow [X_1] \longrightarrow \ldots \longrightarrow [X_m]$ in Γ_A, the sequence $(X_o,X_1,\ldots,X_m)$ is a path in A-mod, whereas, obviously, the converse usually will not be true. Note that for a given path $(X_o,X_1,\ldots,X_m)$, the modules X_i may not even belong to the same component of Γ_A. A subcategory $\mathcal{M}$ of A-mod is said to be path-closed provided for any path $(X_o,X_1,\ldots,X_m)$ in A-mod, with X_o,X_m in $\mathcal{M}$, all X_i belong to $\mathcal{M}$. A module M is said to be sincere provided any simple module occurs as a composition factor of M. Note that any faithful module is sincere, but it is easy to construct sincere modules which are not faithful (unless A is semisimple): just take the direct sum of all simple modules. In general, there even will exist indecomposable sincere modules which are not faithful.

Assume now that $\mathcal{S}$ separates $\mathcal{P}$ from $\mathcal{Q}$. Then $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}$ all are path closed, $\mathcal{P}$ is closed under τ, whereas $\mathcal{Q}$ is closed under τ^-. Also, $\mathcal{S}$ is closed under τ if and only if $\mathcal{P}$ is closed under τ^-, and $\mathcal{S}$ is closed under τ^- if and only if $\mathcal{Q}$ is closed under τ. If A is connected, and $\mathcal{S} \neq \{0\}$, then $\mathcal{P}$ and $\mathcal{Q}$ are uniquely determined by $\mathcal{S}$. Assume now in addition that all projective modules belong to $\mathcal{P}$ and all injective ones to $\mathcal{Q}$. Then $\mathcal{S}$ contains sincere modules, and rather strong homological properties are satisfied by $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}$. Namely, if X belongs to $\mathcal{P}$ or to $\mathcal{S}$, then proj.dim.$X \leq 1$. (The proof rests on the well-known criterion that proj.dim $M \leq 1$ if and only if $\mathrm{Hom}(I,\tau M) = 0$ for any indecomposable injective module I). Similarly, if Y belongs to $\mathcal{S}$ or to $\mathcal{Q}$, then inj.dim. $Y \leq 1$. Since any submodule of a projective module belongs to $\mathcal{P}$, and the modules in $\mathcal{P}$ have projective dimension at most 1, we see that proj.dim $M \leq 2$ for any module M, thus the global dimension of A is at most 2.

The algebras which we will exhibit in this lecture often will have finite global dimension, usually the global dimension will be rather small. One of the important features of algebras of finite global dimension is the possibility of using quadratic forms. We denote by $K_0(A)$ the Grothendieck group of all A-modules modulo all short exact sequences. It is a free abelian group of finite rank, with the set of simple modules as a basis. With respect to this basis, we may identify $K_0(A)$ with $\mathbb{Z}^n$, and the element of $K_0(A)$ corresponding to the module M is just $\underline{\dim}\, M$. Assume now that A has finite global dimension, say gl.dim.$A = d$. Then it is easy to see that

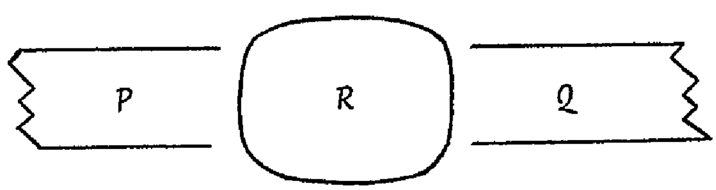
$$\langle \underline{\dim} M, \underline{\dim} N\rangle = \sum_{i=0}^{d} (-1)^i \dim \mathrm{Ext}^i(M,N)$$

(with $\mathrm{Ext}^0 = \mathrm{Hom}$) is well defined and extends to a bilinear form on $K_0(A)$. The corresponding quadratic form will be denoted by $\chi = \chi_A$, thus $\chi(x) = \langle x,x\rangle$ for $x \in K_0(A)$, but one should observe that $\langle -,-\rangle$ usually is non-symmetric. There is an easy way to calculate $\langle -,-\rangle$. Denote by P_i the projective cover of the simple module S_i, $1 \leq i \leq n$, and let C_A be the $n \times n$-matrix with i-j-entry given by $\dim \mathrm{Hom}(P(i),P(j))$, thus the j-th column is just $(\underline{\dim}\, P(j))^T$. This matrix C_A is called the Cartan matrix for A. For A of finite global dimension, C_A is invertible (even over $\mathbb{Z}$), and

$$\langle x,y\rangle = xC^{-T}y^T$$

for $x,y \in K_0(A)$.

The first examples of non-trivial separating subcategories to be known were found for hereditary algebras. Let us dwell for a moment on these algebras, so suppose A is a (finite-dimensional) connected hereditary algebra, thus the path algebra $A = k\Delta$ of some finite connected quiver without oriented cycles. As we have mentioned in the first lecture, A always has a preprojective component and this component is unique and embeds into $\mathbb{Z}\Delta$. We denote by $\mathcal{P}$ the module class whose indecomposable modules belong to the preprojective component; the modules in $\mathcal{P}$ will be said to be preprojective. Now, $k\Delta$ is representation-finite if and only if Δ is of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, and in this case, $\mathcal{P} = A\text{-mod}$. So assume $k\Delta$ is not representation-finite. Then, the preprojective component is of the form $\mathbb{N}\Delta$ (= the full translation subquiver of $\mathbb{Z}\Delta$ given by all vertices (z,x) with $z \in \mathbb{N}$). By duality, there is a similarly defined module class $\mathcal{Q}$, modules in $\mathcal{Q}$ will be said to be preinjective (with $k\Delta$, also $k\Delta^*$ is a hereditary algebra, and a $k\Delta$-module belongs to $\mathcal{Q}$ if and only if its k-dual is a preprojective $k\Delta^*$-module), and the component of Γ_A containing the indecomposable preinjective modules is of the form $(-\mathbb{N})\Delta$. There always are additional modules which have no indecomposable summand in either $\mathcal{P}$ or $\mathcal{Q}$, they will be said to be regular, and we denote by $\mathcal{R}$ the full subcategory of all regular modules. Then $\mathcal{R}$ is a separating subcategory, separating $\mathcal{P}$ from $\mathcal{Q}$. Taking into account the particular shape of the preprojective and the preinjective component, we may think of A-mod as being of the following form.



Of course, the regular components contain only regular modules, and the indecomposable regular modules all belong to regular components. We have mentioned in the first lecture that for these regular components, there are only very restrictive possibilities: they are regular tubes, in case Δ is an Euclidean quiver ($\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$), or else they are of the form $\mathbb{Z}\mathbb{A}_\infty$. Note that $k\Delta$ is tame if and only if Δ is Euclidean. The actual construction of the regular tubes in this case will be outlined later in this lecture.

Consider now again an arbitrary finite-dimensional algebra A. An indecomposable module M will be said to be directing provided there does not exist a path $(X_o, X_1, \ldots, X_n)$ of length $n \geq 1$ with $X_o = M = X_n$. Directing modules have very pleasant properties: Let M be directing, then:

(1) $\text{End}(M) = k$, $\text{Ext}^i(M,M) = 0$ for all $i \geq 1$.

A direct consequence of (1) is the following:

(1') If A has finite global dimension (so that χ_A is defined), $\chi_A(\underline{\dim}\, M) = 1$.

(2) $\underline{\dim}\, M$ determines M uniquely (more precisely: if also N is indecomposable, and $\underline{\dim}\, M = \underline{\dim}\, N$, then M and N are isomorphic; note that the module N is not assumed to be directing).

(3) The annihilator of M in A is generated, as an ideal, by idempotents. As a consequence, if M is sincere, then M is actually faithful.

(4) If M is sincere, then proj.dim.M ≤ 1, inj.dim.M ≤ 1, and gl.dim.A ≤ 2.

(5) If M is sincere, there are only finitely many indecomposable modules X with both $X \not\preceq M$ and $M \not\preceq X$.

Of course, one should keep in mind that for any indecomposable module N, we may look at the classes $\mathcal{P}(N) = \{X \mid X \preceq N\}$ of all predecessors and $\mathcal{Q}(N) = \{X \mid N \preceq X\}$ of all successors of N with respect to $\preceq$, and N will be directing if and only if N is the only module which belongs both to $\mathcal{P}(N)$ and to $\mathcal{Q}(N)$. The property (5) now asserts that for M sincere and directing, all but at most finitely many indecomposable modules will belong to $\mathcal{P}(N) \cup \mathcal{Q}(N)$.

Whereas the properties (1) to (4) of directing modules can be verified directly, we do not know any direct proof for (5). We will investigate below the algebras which have sincere directing modules rather carefully, and (5) will be an immediate consequence of these investigations; actually we will obtain an explicit bound. The main technical tool will be the tilting theory.

What are examples of directing modules? First of all, any indecomposable module in ${}_\infty(A\text{-mod})$ is directing: in particular, any indecomposable module belonging to a preprojective component is directing. Of course, there is the dual assertion: any indecomposable module belonging to a "preinjective" component is directing. We later will see that it is possible to construct also regular components which only contain directing modules. First, let us explain the main principles of tilting theory.

By definition, a <u>tilting module</u> is a module T satisfying the following three properties: its projective dimension is at most 1, the T-codimension of ${}_AA$ is at most one, and $\mathrm{Ext}^1(T,T) = 0$ (we say that T-codim $M \leq m$ provided there exists an exact sequence

$$0 \longrightarrow M \longrightarrow M_o \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_m \longrightarrow 0$$

with all M_i being direct sums of direct summands of T); in the definition of a tilting module the second condition may be replaced by the following: T has at least n pairwise non-isomorphic indecomposable direct summands where n is the number of simple A-modules (actually, the other conditions force that such a module can have at most n pairwise non-isomorphic indecomposable direct summands!) We should indicate that these conditions refer to T as an A-module, and write ${}_AT$ instead of T, since we will have to deal at the same time with T considered as a <u>right</u> B-module T_B, where $B = \mathrm{End}({}_AT)$. There is also the dual notion of a <u>cotilting module</u> S (its injective dimension is at most one, the S-dimension of any injective cogenerator is at most one, and $\mathrm{Ext}^1(S,S) = 0$).

Given any module M, let $\mathcal{G}(M)$ be the full subcategory of all modules generated by M, and $\mathcal{C}(M)$ that of all modules cogenerated by M. Also recall that a pair $(\mathcal{F},\mathcal{T})$ of full subcategories of A-mod is said to be a <u>torsion pair</u> provided $\mathcal{F}$ consists of all modules X with $\mathrm{Hom}(Y,X)=0$ for all $Y \in \mathcal{T}$, and $\mathcal{T}$ consists of all modules Y with $\mathrm{Hom}(Y,X) = 0$ for all $X \in \mathcal{F}$; in this case, the modules in $\mathcal{F}$ are said to be torsion free, those in $\mathcal{T}$ are said to be torsion (observe that our notation $(\mathcal{F},\mathcal{T})$ of a torsion pair presents first the class of torsion free, then the class of torsion modules, in contrast to a fairly standard usage in ring theory, however our notation seems to reflect better the underlying theme of "non-zero maps going from left to right"). The torsion pair $(\mathcal{F},\mathcal{T})$ is said to be <u>split</u> provided any indecomposable module belongs to $\mathcal{F}$ or to $\mathcal{T}$. Now we may formulate the main result of tilting theory:

<u>THEOREM</u> (Brenner, Butler). Let ${}_AT$ be a tilting module, and $B = \mathrm{End}({}_AT)$. Then the k-dual ${}_BS = D(T_B)$ of T_B is a cotilting module, $\mathrm{End}(T_B) = A$, the pairs $(\mathcal{C}(\tau_A T), \mathcal{G}({}_AT))$ and $(\mathcal{C}({}_BS), \mathcal{G}(\tau_B^- S))$ are torsion pairs, and $\mathcal{G}({}_AT)$ is equivalent to $\mathcal{C}({}_BS)$, and $\mathcal{C}(\tau_A T)$ is equivalent to $\mathcal{G}(\tau_B^- S)$.

Let us write down explicitly functors which provide the equivalences asserted in the theorem, note that they are restrictions of functors which are defined on the complete module categories. First of all, there is the functor $\mathrm{Hom}_A({}_AT_B,-)$: A-mod $\longrightarrow$ B-mod, it vanishes on $\mathcal{C}(\tau_A T)$, has image $\mathcal{C}({}_BS)$, and its restriction to $\mathcal{G}({}_AT)$ gives the equivalence $\mathcal{G}({}_AT) \longrightarrow \mathcal{C}({}_BS)$. Similarly, there is the functor $\mathrm{Ext}^1_A({}_AT_B,-)$: A-mod $\longrightarrow$ B-mod, it vanishes on $\mathcal{G}({}_AT)$, has image $\mathcal{G}(\tau^-_B S)$, and its restriction to $\mathcal{C}(\tau_A T)$ gives the second equivalence $\mathcal{C}(\tau_A T) \longrightarrow \mathcal{G}(\tau^-_B S)$. As reverse functors, use the restriction of ${}_AT_B\otimes-$ to $\mathcal{C}({}_BS)$, and of $\mathrm{Tor}^B_1({}_AT_B,-)$ to $\mathcal{G}(\tau^-_B S)$.

PROPOSITION: Let A be a hereditary algebra, ${}_AT$ a tilting module, $B = \mathrm{End}({}_AT)$, and ${}_BS = D(T_B)$. Then the torsion pair $(\mathcal{C}({}_BS), \mathcal{G}(\tau^-_B S))$ is split.

An algebra of the form $B = \mathrm{End}({}_AT)$, with ${}_AT$ a tilting module, and A hereditary, will be called a tilted algebra. The proposition above asserts that in this situation, we can recover all indecomposable B-modules N from indecomposable A-modules. For, if N belongs to $\mathcal{C}({}_BS)$, then $N = \mathrm{Hom}_A({}_AT_B, X)$ for some indecomposable A-module X in $\mathcal{G}({}_AT)$, whereas, if N does not belong to $\mathcal{C}({}_BS)$, then N belongs to $\mathcal{G}(\tau^-_B S)$, and therefore $N = \mathrm{Ext}^1_A({}_AT_B, X)$ for some indecomposable A-module X in $\mathcal{C}(\tau_A T)$. In this way, we obtain a bijection between the indecomposable B-modules and certain indecomposable A-modules, namely those which are either torsion, or torsion free with respect to our torsion pair.

There is the following criterion for an algebra B in order to be a tilted algebra. A module class $\mathcal{S}$ in B-mod will be called a slice provided $\mathcal{S}$ is path closed, contains a sincere module, and if, in addition, the following property is satisfied: given any Auslander-Reiten sequence $0 \to X \to Y \to Z \to 0$ in B-mod, then at most one of X,Z belongs to $\mathcal{S}$, and one of X,Z belongs to $\mathcal{S}$ in case an indecomposable direct summand of Y is in $\mathcal{S}$.

The module category of a tilted algebra always contains a slice: Let A be hereditary, ${}_AT$ a tilting module, and $B = \mathrm{End}({}_AT)$. Denote by $\mathcal{S}$ the set of B-modules of the form $\mathrm{Hom}_A({}_AT_B, I)$, where I is an injective A-module. Then $\mathcal{S}$ is a slice in B-mod. We may describe $\mathcal{S}$ also alternatively: Let ${}_BS = D(T_B)$. Then $\mathcal{S}$ is the set $\langle {}_BS \rangle$ of all modules which are direct sums of direct summands of ${}_BS$, therefore ${}_BS$ is called a slice module. (Thus, slice modules are the cotilting modules with hereditary endomorphism rings, and actually a module with hereditary endomorphism ring is a tilting module if and only if it is a cotilting module.)

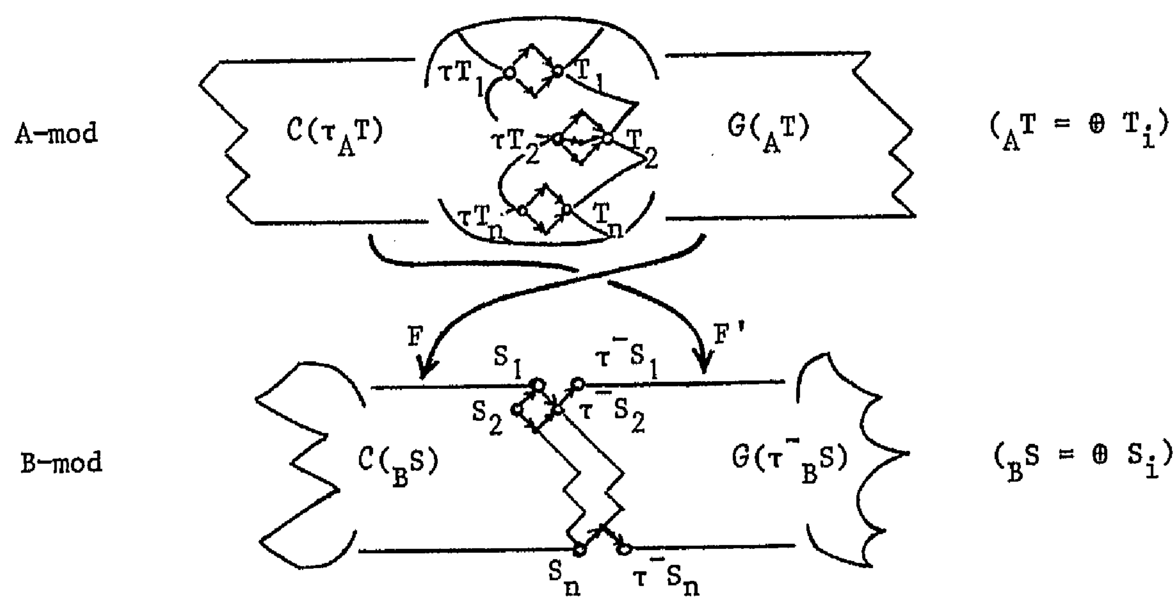
PROPOSITION: Let $\mathcal{S}$ be a slice in B-mod. Then $\mathcal{S} = \langle {}_B S \rangle$ for a slice module ${}_B S$; in particular, B is a tilted algebra.

This shows that B is a tilted algebra if and only if B-mod contains a slice.

Slices are separating subcategories! More precisely, let $\mathcal{S}$ be a slice in B-mod, say $\mathcal{S} = \langle {}_B S \rangle$, where ${}_B S$ is a slice module. Then $\mathcal{S}$ separates $\mathcal{C}(\tau_B S)$ from $\mathcal{G}(\tau_B^- S)$. Let us outline the proof. We have noted above that $(\mathcal{C}({}_B S), \mathcal{G}(\tau_B^- S))$ is a split torsion pair. Since the notion of a slice is self-dual, we see that similarly $(\mathcal{C}(\tau_B S), \mathcal{G}({}_B S))$ is a split torsion pair, and the definition of a slice implies that $\mathcal{G}({}_B S) \cap \mathcal{C}({}_B S) = \mathcal{S}$, and also that $\mathcal{G}(\tau_B^- S) \subseteq \mathcal{G}({}_B S)$. It follows that $\mathcal{C}(\tau_B S)$, $\mathcal{S}$, and $\mathcal{G}(\tau_B^- S)$ exhaust A-mod, and that the various zero-map conditions are satisfied. It remains to be seen that any map $f: P \longrightarrow Q$ with P in $\mathcal{C}(\tau_B S)$, Q in $\mathcal{G}(\tau_B^- S)$, factors through a module in $\mathcal{S}$. Note that for any P in $\mathcal{C}(\tau_B S)$, there exists an exact sequence $0 \to P \to S' \to S'' \to 0$, with both S', S'' in $\mathcal{S}$ (thus, S-codim $P \leq 1$). [Let ${}_B S = D({}_A T)$, where A is hereditary, ${}_A T$ a tilting module, and let $P = \mathrm{Hom}_A({}_A T_B, X)$, with X in $\mathcal{G}({}_A T)$. Choose an injective resolution $0 \to X \to I' \to I'' \to 0$ of X, and apply $F = \mathrm{Hom}_A({}_A T_B, -)$. Since $\mathrm{Ext}_A^1({}_A T, X) = 0$, the sequence $0 \to FX \to FI' \to$ $\to FI'' \to 0$ still is exact. It remains to observe that both FI', FI'' belong to $\mathcal{S}$.] With Q also $\tau^- Q$ belongs to $\mathcal{G}(\tau_B^- S)$, thus $\mathrm{Hom}(\tau^- Q, S'') = 0$, therefore $\mathrm{Ext}^1(S'', Q) = 0$. But this implies that f can be factored through the given inclusion $P \longrightarrow S'$.

Assume now that B is a connected tilted algebra, and let $\mathcal{S}$ be a slice in B-mod. Since B is connected, the indecomposable modules in $\mathcal{S}$ all belong to a fixed component Γ, and we want to look at such components for a while. Let ${}_B S$ be a slice module, with $\mathcal{S} = \langle {}_B S \rangle$, and we may assume that $A = \mathrm{End}({}_B S)$ is basic. Since A is hereditary, basic and connected, $A = k\Delta$ for some finite connected quiver Δ without oriented cycles. Now, $\mathcal{S}$ is equivalent to the category A-inj of all injective A-modules, and A-inj can be identified with the path category of Δ. On the other hand, constructing inductively the τ-translates, and the τ^--translates of the indecomposable direct summands of S, we see that they exhaust the component Γ. It follows that Γ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta$. Also, if we denote by $\overline{\mathcal{S}}$ the module class whose indecomposable modules are those belonging to Γ, then it is easy to see that $\overline{\mathcal{S}}$ is again a separating subcategory.

A component Γ containing the indecomposable modules of a slice S will be called a <u>connecting</u> component. The reason is the following: Let $S = \langle {}_BS \rangle$, $\mathrm{End}({}_BS) = A = k\Delta$, ${}_AT = D(S_A)$, and consider the functors $F = \mathrm{Hom}_A({}_AT_B, -)$ and $F' = \mathrm{Ext}^1_A({}_AT_B, -)$. The preinjective A-modules belonging to $G({}_AT)$ go under F to modules in $\overline{S} \cap C({}_BS)$, the preprojective A-modules which belong to $C(\tau_A T)$ go under F' to modules in $\overline{S} \cap G(\tau_B^- S)$, thus $\overline{S}$, in some sense, connects the preinjective and the preprojective component of A-mod:



Actually, in case (A is representation-infinite and) no non-zero direct summand of ${}_AT$ is preinjective, then all preinjective A-modules belong to $G({}_AT)$, and F gives an equivalence between the full subcategory of all preinjective A-modules, and $\overline{S} \cap C({}_BS)$. Similarly, if (A is representation-infinite and) no non-zero direct summand of ${}_AT$ is preprojective, then all preprojective A-modules belong to $C(\tau_A T)$, and F' gives an equivalence between the full subcategory of all preprojective A-modules and $\overline{S} \cap G(\tau_B^- S)$. In particular, for ${}_AT$ regular, Γ is isomorphic to $\mathbb{Z}\Delta$, thus regular. On the other hand, if ${}_AT$ is not regular, then Γ cannot be a regular component. (For, if ${}_AT$ has an indecomposable preinjective direct summand, say T_i, then $[F T_i]$ will be a projective vertex of Γ. Similarly, assume that T_j is an indecomposable preprojective direct summand of T; if T_j is not projective, then $[F'(\tau_A T_j)]$ is an injective vertex of Γ, if T_j is projective, then $[F(\nu T_j)]$ is an injective vertex of Γ, where νT_j is the injective envelope of $T_j / \mathrm{rad}\, T_j$).

In order to show the existence of components of the form $\mathbb{Z}\Delta$, it is sufficient to exhibit a regular tilting A-module, where $A = k\Delta$.

PROPOSITION: Let A be a finite-dimensional connected hereditary algebra, say $A = k\Delta$ for a quiver Δ. There exists a regular tilting A-module if and only if Δ is neither of Dynkin nor of Euclidean type, and has more than two vertices.

One direction of the proof is rather easy: In case Δ is of Dynkin or Euclidean type, one may prove without difficulties that $\mathbb{Z}\Delta$ does not allow an unbounded additive function, thus it is impossible to have a component of an Auslander-Reiten quiver of the form $\mathbb{Z}\Delta$. For $A = k\Delta$, where Δ has precisely two vertices, (and at least two arrows), any non-zero regular module X can be shown to satisfy $\mathrm{Ext}^1(X,X) \neq 0$, thus, also in this case, there cannot exist a regular tilting module. The converse implication is shown by constructing effectively regular tilting modules, see [Ri 3]. The argument shows that there cannot exist any algebra at all with a component of the form $\mathbb{Z}\Delta$, with Δ of Dynkin of Euclidean type. Since it is known that the Auslander-Reiten quiver of an algebra never contains a cyclic sectional path [BS], we obtain the following Corollary:

COROLLARY: Let Δ be a finite connected quiver, and $|\Delta_0| \neq 2$. Then, $\mathbb{Z}\Delta$ can be realised as a component of the Auslander-Reiten quiver Γ_A of some algebra A if and only if Δ has no oriented cycle and is neither of Dynkin nor of Euclidean type.

Problem 7. Let Δ be a quiver with two vertices, and at least three arrows, but no oriented cycles. Is it possible to realise $\mathbb{Z}\Delta$ as a component of some Γ_A?

We return to the investigation of sincere directing modules, say let M be a sincere directing B-module. Denote by $\mathcal{S}(M\to)$ the module class in B-mod such that an indecomposable B-module X belongs to $\mathcal{S}(M\to)$ if and only if first $M \preceq X$, and second, there does not exist an indecomposable non-projective B-module Z with both $M \preceq \tau Z$ and $Z \preceq X$. Similarly, denote by $\mathcal{S}(\to M)$ the module class in B-mod such that an indecomposable B-module Y belongs to $\mathcal{S}(\to M)$ if and only if $Y \preceq M$, and no indecomposable non-projective B-module Z satisfies both $Y \preceq \tau Z$ and $Z \preceq M$. Then one may show quite easily that both $\mathcal{S}(M\to)$ and $\mathcal{S}(\to M)$ are slices. In particular, B is a tilted algebra. Let $\mathcal{P}$, $\overline{\mathcal{S}}$, $\mathcal{Q}$ be the module classes in B-mod with $\overline{S}$ separating P from Q, and such that the indecomposable modules in $\overline{S}$ are those which belong to the same component as $S(M\rightarrow)$ and $S(\rightarrow M)$. If we realize $S(\rightarrow M)$ as the set of images of the injective A-modules under a functor $F: \mathrm{Hom}_A({}_AT_B,-)$, where A is hereditary, ${}_AT$ a tilting module, and $B = \mathrm{End}({}_AT)$, we see that $N \preceq M$ for all indecomposable B-modules N in the image of F. This shows that $N \preceq M$ for all indecomposable modules in P. Similarly, $M \preceq N$ for all indecomposable modules in Q. It follows that the indecomposable modules N with both $N \not\preceq M$ and $M \not\preceq N$ belong to $\overline{S}$. Let $A = k\Delta$, where Δ is a quiver with n vertices. The component of Γ_B given by the modules in $\overline{S}$ embeds into $\mathbb{Z}\Delta$, and the number of vertices in $\mathbb{Z}\Delta$ which are incomparable with a fixed vertex is bounded by $\frac{n(n-1)}{2}$. This shows that there are at most $\frac{n(n-1)}{2}$ indecomposable modules N satisfying both $N \preceq M$ and $M \preceq N$.

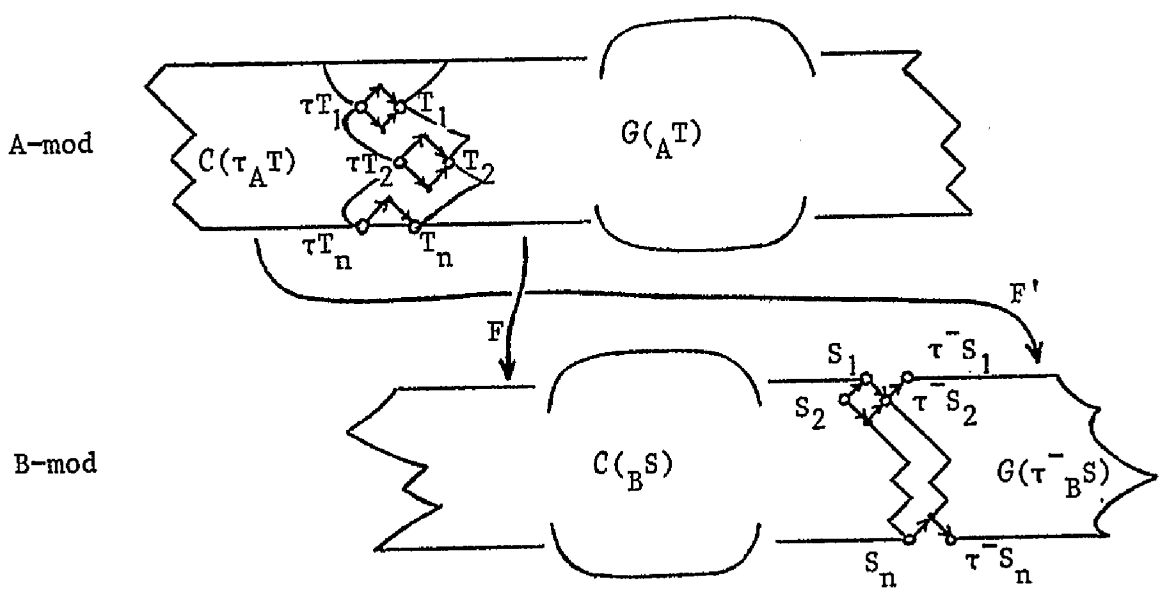
Let us stress that any indecomposable module in a connecting component is directing; in particular, the regular connecting components are regular components which only contain directing modules!

As we have seen, the structure of a connecting component is known, at least if it is a regular component. Of course, we are also interested in the remaining components of a tilted algebra.

<u>PROPOSITION.</u> A regular component of a tilted algebra which is not a connecting component is either a regular tube or of the form $\mathbb{Z}A_\infty$.

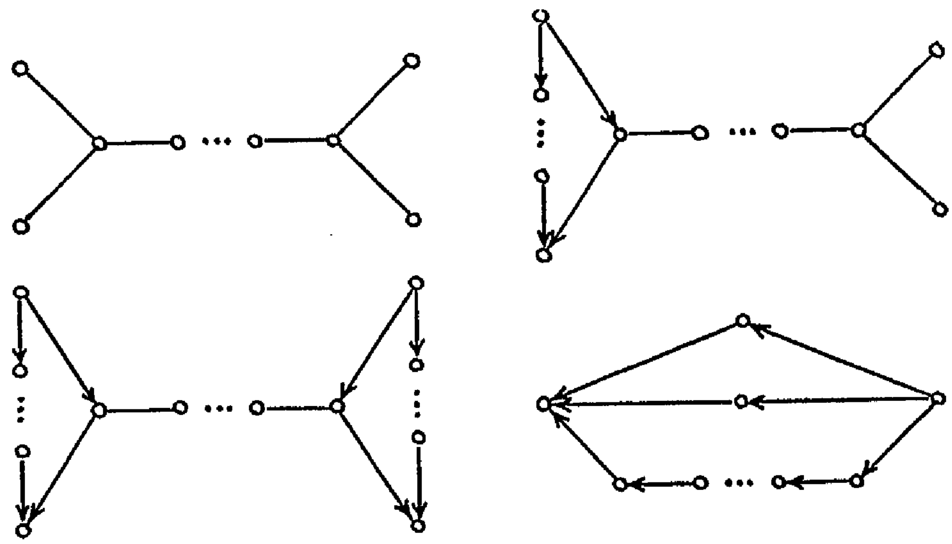
<u>Problem 8.</u> What are the possible structures of non-regular components of tilted algebras?

Of special interest are the endomorphism rings $B = \mathrm{End}({}_AT)$, where A again is hereditary, connected, and representation-infinite, and ${}_AT$ is a tilting module which is, in addition, preprojective. In this case, B will be called a <u>concealed</u> algebra. Note that for a preprojective tilting module ${}_AT$ (with A hereditary), all but only finitely many indecomposable A-modules belong to $G({}_AT)$, and therefore all but finitely many indecomposable B-modules belong to $C({}_BS)$, (where $B = \mathrm{End}({}_AT)$ and ${}_BS = D(T_B)$).



We see that a concealed algebra B has two non-regular components, one is a preprojective component (the images of the indecomposable preprojective modules in $G({}_A T)$ under F), the other a preinjective component (the modules which are either images under F of the indecomposable preinjective A-modules or images under F' of the indecomposable preprojective modules in $C(\tau_A T)$), and the remaining components of Γ_B correspond, under F, to the regular components of Γ_A. Thus, the module categories of concealed algebras look like "concealments" of the module categories of hereditary algebras.

The tame concealed algebras play a particular role in representation theory, they are the minimal representation-infinite algebras A which have a preprojective component. Of course, an algebra A is said to be minimal representation-infinite provided A itself is representation-infinite, whereas all proper factor algebras of A are representation-finite. Also, it is not difficult to see that any connected, representation-infinite algebra C with a preprojective component has a factor algebra which is a tame concealed algebra. The tame concealed algebras have been classified by Happel and Vossieck, and the list of these algebras has been reprinted nearly everywhere, so we content ourselves by mentioning only some of its features: these algebras may be divided into different types, according to the type of the corresponding hereditary algebra. The only tame concealed algebras of type $\tilde{\mathbb{A}}_n$ are the hereditary algebras of type $\tilde{\mathbb{A}}_n$; there are four kinds of tame concealed algebras of type $\tilde{\mathbb{D}}_n$:



Here, the unoriented edges may be oriented arbitrarily, and, as relations, one has to take the sum of all paths from some vertex to another, provided there are at least two such paths. In addition, there are finitely many algebras of types $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$, given by quivers with 7, 8, or 9 vertices, respectively. The number of isomorphism classes of algebras of types $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$ is 56, 437, and 3809, respectively. One of the reasons for the general interest in the list of Happel-Vossieck is its usefulness for checking finite representation type; in particular, the socalled Bongartz criterion makes use of this list (so the list is sometimes also referred to as the bazar of Bongartz-Happel-Vossieck).

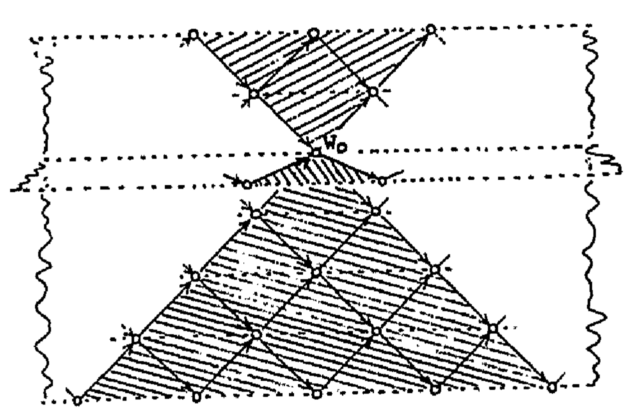
From the presentation above, one may have obtained a feeling of unsymmetry between the preprojective and the preinjective component of a concealed algebra B, but this is misleading: Also the preprojective component is a connecting component, since B can be written also as the endomorphism ring of a preinjective tilting module. In particular, we see that an algebra may have slices in two different components.

<u>PROPOSITION:</u> A (tilted) algebra B has at most two components containing slices, and if two, then B is a concealed algebra.

For the proof of this proposition, one needs additional techniques which we will report on in the third lecture. These techniques will shed more light on the connecting components and clarify the use of tilting functors.

Our next topic are the separating tubular families. Given an algebra A_o and an A_o-module R_o, the <u>one-point extension</u> $A_o[R_o]$ of A_o by R_o is the matrix algebra $A_o[R_o] = \begin{bmatrix} A_o & R_o \\ 0 & k \end{bmatrix}$, its elements are the matrices $\begin{bmatrix} a & r \\ 0 & b \end{bmatrix}$ with $a \in A_o$, $b \in k$, $r \in R_o$, subject to the usual addition and multiplication of matrices. The $A_o[R_o]$-modules can be written as triples $X = (X_o, X_\omega, \gamma_X)$, where X_o is an A_o-module, X_ω a k-space, and $\gamma_X : R_o \otimes_k X_\omega \longrightarrow X_o$ is A_o-linear. The $A_o[R_o]$-module $E(\omega) = (0,k,0)$ is simple and injective, and conversely, any algebra with a simple injective module can be written as a one-point extension.

We consider an algebra A_o with a sincere directing A_o-module W_o. In particular, A_o has global dimension at most 2, and we denote by C_o its Cartan matrix. As we have seen above, W_o belongs to a slice, say $\langle S_o \rangle$, where S_o is a slice A_o-module, and we can assume that its endomorphism ring is basic, thus of the form $k\Delta$. Note that the vertices of Δ correspond to the isomorphism classes of indecomposable summands of S_o; in particular, one of the vertices of Δ corresponds to $[W_o]$. We say that W_o is a <u>wing</u> module of type $(n_1,\ldots,n_t)$ provided the underlying graph of Δ is the star $\mathbb{T}_{n_1,\ldots,n_t}$ and $[W_o]$ corresponds to the center of the star. (The star $\mathbb{T}_{n_1,\ldots,n_t}$ is obtained from the disjoint union of copies of $\mathbb{A}_{n_i}$, $1 \leq i \leq t$, by choosing one endpoint in each $\mathbb{A}_{n_i}$, and identifying these endpoints to a single vertex, the center of the star). So assume now that W_o is a wing A_o-module. The connecting component of A_o-mod containing W_o has, in the vicinity of W_o, the following shape:



We have shaded the "wings" of W_o, these are given by those indecomposable modules which are successors of indecomposable modules in $\mathcal{S}(\to W_o)$, and predecessors of indecomposable modules in $\mathcal{S}(W_o \to)$. We will say that W_o is <u>dominated</u> by the A_o-module R_o provided, first of all,

$$\underline{\dim}\, R_o = (\underline{\dim}\, W_o)(I + C_o^{-1} C_o^T),$$

and second, for any $0 \neq \lambda : R_o \longrightarrow W_o$, the $A_o[R_o]$-module $W(\lambda) = (W_o, k, \lambda)$ satisfies proj.dim.$W(\lambda) \leq 1$. It is easy to construct algebras with wing modules, but not every wing module is dominated by some module. In case W_o is a wing A_o-module where A_o is representation-finite, and $\underline{\dim}\, W_o$ is a maximal root, W_o is always dominated by a projective module. Also other examples of dominated wing modules are known. In particular, the structure theory for the module category of a canonical algebra, as explained in the next lecture, rests on the use of certain dominated wing modules.

The separating subcategories which we are going to construct will be tubular families. By definition, a module class $\mathcal{T}$ is called a <u>regular tubular</u> $\mathbb{P}_1k$-<u>family</u> of type $(n_1,\ldots,n_t)$, provided $\mathcal{T}$ is equivalent to the mesh category $k(\Gamma)$, where Γ is the disjoint union of translation quivers $\Gamma(\rho)$, $\rho \in \mathbb{P}_1k$, each $\Gamma(\rho)$ being a regular tube, and such that t of these tubes are of the form $\mathbb{Z}\mathbb{A}_\infty / n_i$, $1 \leq i \leq t$, whereas the remaining ones all are homogeneous. Note that such a tubular family consists of various components $\mathcal{T}(\rho)$, one for each $\rho \in \mathbb{P}_1k$, with $\mathcal{T}(\rho)$ equivalent to $k(\Gamma(\rho))$. We will say that $\mathcal{T}$ is a <u>separating family</u>, say separating $\mathcal{P}$ from $\mathcal{Q}$, provided $\mathcal{T}$ is a separating subcategory, and, in addition, any map $f : P \longrightarrow Q$, with P in $\mathcal{P}$, Q in $\mathcal{Q}$ may be factored through an object of any of the module classes $\mathcal{T}(\rho)$, $\rho \in \mathbb{P}_1k$. Now we may state the main theorem:

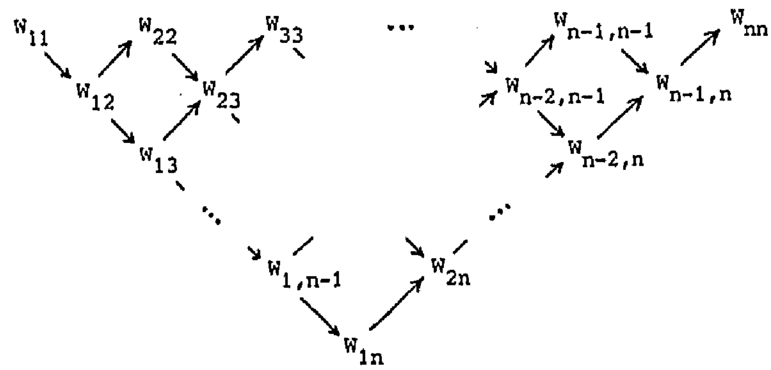
<u>THEOREM.</u> Let W_o be a sincere directing A_o-module, which is a wing module of type $(n_1,\ldots,n_t)$ and which is dominated by the A_o-module R_o. Let $A = A_o[R_o]$, and $w = \underline{\dim}\, W_o + \underline{\dim}(0,k,0) \in K_o(A)$. Denote by c_w the linear form $\iota_w = \langle w, - \rangle$ on $K_o(A)$. Let $\mathcal{P}_w$, $\mathcal{T}_w$, $\mathcal{Q}_w$ be the module classes in A-mod whose indecomposable modules are those indecomposable X which satisfy $\iota_w(\underline{\dim}\, X) < 0$, $= 0$, or > 0, respectively. Then $\mathcal{T}_w$ is a separating regular tubular $\mathbb{P}_1k$-family of type $(n_1,\ldots,n_t)$, separating $\mathcal{P}_w$ from $\mathcal{Q}_w$.

Let us show in which way a wing of W_o in A_o-mod gives rise to a tube in A-mod. We write $\langle -, - \rangle_o$ for the bilinear form and χ_o for the quadratic form on $K_o(A)$. The condition $\underline{\dim} R_o = (\underline{\dim} W_o)(I + C_o^{-1} C_o^T)$

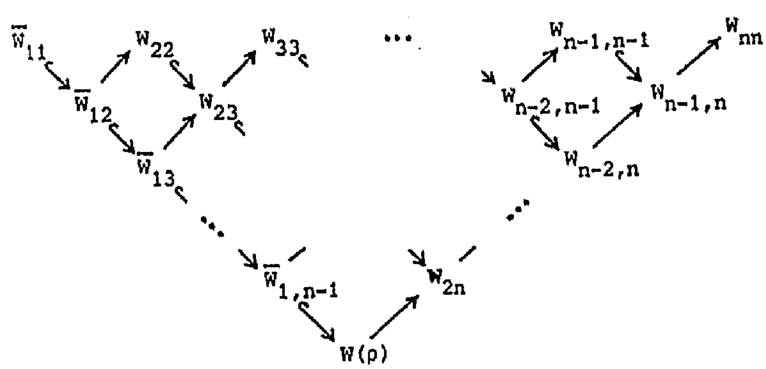
implies that

$$\langle \underline{\dim} R_o, \underline{\dim} W_o \rangle_o = (\underline{\dim} R_o)\, C_o^{-T} (\underline{\dim} W_o)^T$$
$$= (\underline{\dim} W_o)(\,C_o^{-T} + C_o^{-1})(\underline{\dim} W_o)^T$$
$$= 2\ \chi_o(\underline{\dim} W_o) = 2.$$

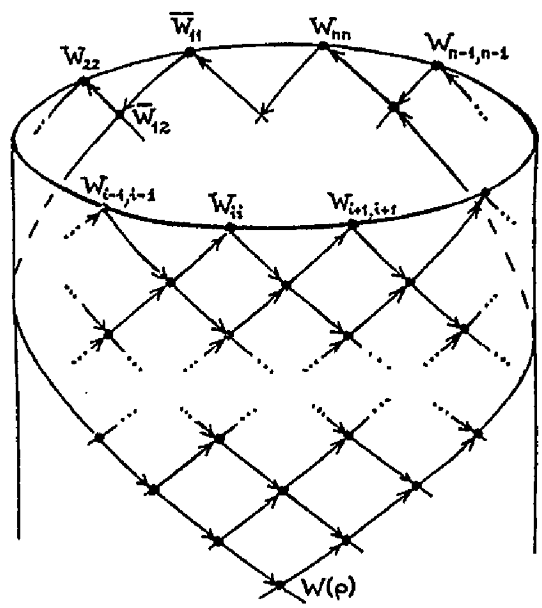
Taking into account also the second condition on R_o, it follows quite easily that $\dim \mathrm{Hom}(R_o, W_o) = 2$. A wing of W_o in A_o-mod is of the following form:



where all W_{ij} are indecomposable A_o-modules, $W_{1n} = W_o$, and the exhibited maps are irreducible. Again, using the domination conditions, one can show that $\dim \mathrm{Hom}(R_o, W_{ij}) = 1$ for $i = 0$, $1 \le j \le n-1$, and for $2 \le i \le n$, $j = n$, whereas $\mathrm{Hom}(R_o, W_{ij}) = 0$ for $2 \le i \le n-1$, $2 \le j \le n-1$. Up to scalar multiples, there is a unique non-zero map $\rho : R_o \longrightarrow W_o$ which factors through W_{11}. We may consider A_o-mod as a full subcategory of A-mod, identifying the A_o-module X_o with the triple $(X_o, 0, 0)$. Also, given an A_o-module Y_o, we write $\overline{Y}_o$ for the A-module $(Y_o, \mathrm{Hom}(R_o, Y_o), e)$, where $e : R_o \times \mathrm{Hom}(R, Y_o) \to Y_o$ is the evaluation map. In A-mod, we obtain from the diagram above the following one:



It is not difficult to see that the maps exhibited in the diagram again are irreducible, thus we look at part of a component $\mathcal{T}(\rho)$ of A-mod. A calculation shows that the dimension vector of the Auslander-Reiten translate $\tau_A \overline{W}_{11}$ is just $\underline{\dim}\, W_{nn}$. Since W_{nn} is uniquely determined by its dimension vector, this actually shows $\tau_A \overline{W}_{11} = W_{nn}$, thus W_{nn} is a τ-periodic module of period n. In this way, the component $\mathcal{T}(\rho)$ turns out to be a regular tube of the form $\mathbb{Z}\mathbb{A}_\infty / n$.



For any $0 \neq \rho \in \mathrm{Hom}(R_o, W_o)$, let $\mathcal{T}(\rho)$ be the component which contains $W(\rho)$. Then $\mathcal{T}(\rho)$ is a regular tube and $\mathcal{T}(\rho)$ is homogeneous if and only if ρ does not factor through an irreducible map $Y_o \longrightarrow W_o$, with Y_o indecomposable. Also, $\mathcal{T}(\rho) = \mathcal{T}(\rho')$ if and only if $\rho \sim \rho'$ in $\mathbb{P}\mathrm{Hom}(R_o, W_o)$. Thus, the index set for the tubular family constructed in this way is $\mathbb{P}\mathrm{Hom}(R_o, W_o) = \mathbb{P}_1 k$.

As a typical application, consider the case of $A_o = k\Delta^o$, where Δ^o is a quiver of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, and W_o the unique maximal indecomposable A_o-module. Note that always W_o is a wing module, and, as we have mentioned above, dominated by some projective module R_o. It turns out that $A = A_o[R_o]$ is the path algebra of a quiver of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, or $\widetilde{\mathbb{E}}_8$, respectively (the quiver of A is obtained from Δ^o by adding one vertex, and we obtain in this way the corresponding extended Dynkin diagram). Now, the type of the wing module is (2,2,n-2), in case Δ^o is of type $\mathbb{D}_n$, and (3,3,2), (4,3,2), (5,3,2) in case Δ is of type $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, respectively. For Δ^o of type $\mathbb{A}_n$, the type of W_o is of the form (p,q), with $p+q = n+1$, and depends on the orientation of Δ^o. In all cases, it follows that A has a regular tubular family $\mathcal{T} = \mathcal{T}_w$ of the same type, and $\mathcal{T}$ comprises just all regular modules. The modules in $\mathcal{P} = \mathcal{P}_w$ are the preprojective ones, those in $\mathcal{Q} = \mathcal{Q}_w$ the preinjective ones. In this way, we obtain the complete structure of the module category for any tame hereditary algebra. Any regular tubular family is, as a category in its own right, an abelian category which is serial (this means that any indecomposable object has a unique chain of subobjects). In particular, the category of all regular modules over a tame hereditary algebra is a serial abelian category.

There are also other cases of one-point extensions $A = A_o[R_o]$, where a separating subcategory $\mathcal{S}_o$ of A_o-mod gives rise to a separating subcategory $\mathcal{S}$ of A-mod. Assume that there is given a subcategory $\mathcal{S}_o$ of A_o-mod which separates $\mathcal{P}_o$ from $\mathcal{Q}_o$. The first case to be mentioned is that of R_o belonging to $\mathcal{Q}_o$. In this case, $\mathcal{S}_o$, considered as a subcategory of A-mod, is still separating: denote by $\mathcal{Q}$ the full subcategory of A-mod of all triples $(X_o, X_\omega, \gamma_X)$ with X_o in $\mathcal{Q}$. Then $\mathcal{S}_o$ separates $\mathcal{P}_o$ from $\mathcal{Q}$. The reason is that $\mathrm{Hom}(R_o, M_o) = 0$ for all M_o in $\mathcal{P}_o$ or $\mathcal{S}_o$. Here, two of the module classes $\mathcal{P}_o$, $\mathcal{S}_o$, $\mathcal{Q}_o$ are not changed at all when going from A_o-mod to A-mod, however one should observe that the change of $\mathcal{Q}_o$ to $\mathcal{Q}$ may be drastically. There is a second, less trivial case, the one-point extension $A = A_o[R_o]$, where R_o is a ray module in a component which is contained in $\mathcal{S}_o$.

A path $x_1 \longrightarrow x_2 \longrightarrow \dots \longrightarrow x_{n-1} \longrightarrow x_n$ in a translation quiver Γ is said to be sectional provided $\tau x_{i+1} \neq x_{i-1}$ for all possible i. A vertex v in Γ is said to be a ray vertex provided for any $i \in \mathbb{N}$, there is precisely one sectional path of length i and starting in v and such that the endpoints of the sectional paths of different length are different. Thus, given a ray vertex v, we obtain a unique infinite sectional path

$$v = v[1] \longrightarrow v[2] \longrightarrow \dots \longrightarrow v[i] \longrightarrow \dots$$

with pairwise different vertices $v[i]$, this path is called the ray starting at v. An indecomposable module V will be called a ray module provided the component Γ to which V belongs, is standard, and $[V]$ is a ray vertex in Γ. If V is a ray module, the indecomposable module M with $[M] = [V][i]$ will be denoted by $V[i]$.

Let V_o be a ray A_o-module and let Γ^o be the component to which V_o belongs. Let $A = A_o[V_o]$, and denote by Γ the component of Γ_A to which V_o, considered as an A-module, belongs. Then Γ contains all of Γ^o and, in addition, the modules $\overline{V_o[i]}$. The module $\overline{V}_o$ is again a ray module, and the ray starting in $[\overline{V}_o]$ is given by

$$[\overline{V}_o] \longrightarrow [\overline{V_o[2]}] \longrightarrow \dots \longrightarrow [\overline{V_o[i]}] \longrightarrow \dots .$$

Since this ray comprises just those indecomposable modules which belong to Γ and not to Γ^o, we say that Γ is obtained from Γ^o by <u>inserting a ray</u>. Assume now that V_o belongs to a separating subcategory $\mathcal{S}_o$, say separating $\mathcal{P}_o$ from $\mathcal{Q}_o$, and that, in addition, $\mathrm{Hom}(V_o,S_o) = 0$ for all indecomposable A_o-modules S_o in $\mathcal{S}_o$ which do not belong to Γ^o. This last condition is always satisfied in case $\mathcal{S}_o$ is standard. Denote by $\mathcal{S}$ the module class in A-mod which is obtained from $\mathcal{S}_o$ by adding the modules $\overline{V_o[i]}$ (and closing under direct sums), let $\mathcal{Q}$ be the set of all A-modules (X_o,X_ω,γ_X), with X_o having no direct summand in $\mathcal{P}_o$ or of the form $V_o[i]$. Then $\mathcal{S}$ is a separating subcategory, it separates $\mathcal{P}_o$ from $\mathcal{Q}$.

These last considerations may be applied to the case of A_o being a tame hereditary algebra, and V_o a regular ray A_o-module. We recall that the regular modules of a tame hereditary algebra form a regular tubular family. Now, the ray vertices of a regular tube $\mathbb{Z}\mathbb{A}_\infty / n$ are those vertices which belong to the "mouth"; these are those vertices of $\mathbb{Z}\mathbb{A}_\infty / n$ which are end points of precisely one arrow (and starting points of precisely one arrow), thus $\mathbb{Z}\mathbb{A}_\infty / n$ has precisely n ray vertices. Recall that the category of all regular A_o-modules is abelian, and the simple objects in this category are just the ray modules.

LECTURE 3
REPETITIVE ALGEBRAS

The third lecture is devoted to recent investigations of Happel. As we have seen in the previous lectures, the regular components of the Auslander-Reiten quiver of an algebra are usually rather well-behaved, whereas there are difficulties in dealing with non-regular components. Of course, given any translation quiver Γ, one may delete the τ-orbits which contain projective or injective vertices, in order to obtain a better behaved translation quiver Γ_s. Thus, instead of considering the Auslander-Reiten quiver Γ_A, it has been customary to consider the corresponding <u>stable Auslander-Reiten quiver</u> $(\Gamma_A)_s$, and to try to recover Γ_A from $(\Gamma_A)_s$. For example, Riedtmann's classification of the representation-finite selfinjective algebras was done along these lines. Of course, for a selfinjective algebra, the only vertices of Γ_A which do not belong to $(\Gamma_A)_s$ are those given by the indecomposable projective-injective modules, so there are only few such vertices, and their positions can be uniquely determined from the knowledge of $(\Gamma_A)_s$ and the restriction of the usual length function to $(\Gamma_A)_s$. On the other hand, there may be whole components Γ of Γ_A such that any τ-orbit in Γ contains a projective or an injective vertex, thus Γ_s is empty in this case, so it is impossible to recover Γ from Γ_s. For example, this is the case for any preprojective or preinjective component, but also for any component obtained from a regular tube by inserting rays. The latter example shows that there may be more subtle ways of deleting vertices in order to obtain a regular translation quiver from a non-regular one, however it always will be awkward to reinsert vertices. Let us outline a completely different strategy of investigating A-mod by considering solely regular components.

We will consider in the sequel (associative) algebras R (defined over k) which are not necessarily finite-dimensional and which are <u>not</u> required to have a unit element, however, we do require that

$R^2 = R$ (here, R^2 denotes the subspace of R generated by all products r_1r_2, with $r_1, r_2 \in R$). The algebras which we will encounter will at least have sufficiently many idempotents, since they may be thought of as being given by small preadditive categories. Any small k-preadditive category A gives rise to an algebra $\oplus A$, with underlying vectorspace $\bigoplus_{x,y} \mathrm{Hom}_A(x,y)$, where x,y range over all objects of A, and where the multiplication is given by the composition of maps in A whenever defined, and zero otherwise. Of course, in $\oplus A$ there are many idempotents, namely all the identity maps of the objects of A. However, $\oplus A$ has a unit element only in case A has only finitely many objects. Note that an algebra R is of the form $\oplus A$ if and only if there is a complete set $\{e_x \mid x \in I\}$ of pairwise orthogonal idempotents in R (the completeness means that $R = \bigoplus_{x,y} e_xRe_y$).

Given an algebra R, a (left) R-module M is always supposed to satisfy the condition $RM = M$ (as above, RM denotes the subspace of M generated by all elements of the form rm, with $r \in R$, $m \in M$). Note that in case R has a unit element, the condition $RM = M$ is equivalent to the usual one on M to be unital. If $\{e_x \mid x \in I\}$ is a complete set of pairwise orthogonal idempotents, and M is an R-module, then, as a vectorspace, M decomposes in the form $M = \oplus e_xM$. An R-module M is said to be finitely generated in case there are elements $m_1,\dots,m_n \in M$ with $M = \Sigma Rm_i$. [Note that in general ${}_RR$ itself is not finitely generated! If R has a complete set of pairwise orthogonal idempotents and ${}_RR$ is finitely generated, then R actually has a unit element.] We denote by R-Mod the category of all R-modules, by R-mod the full subcategory of all finitely generated ones. If A is a small k-preadditive category, then $\oplus A$-Mod is nothing else than the category of additive contravariant functors from $\oplus A$ into the category k-Mod of k-vectorspaces, and $\oplus A$-mod is just the full subcategory of all finitely generated functors. [If $F : A \longrightarrow$ k-Mod is a contravariant functor, the corresponding $\oplus A$-module is given by the vectorspace $\bigoplus_x F(x)$, where x ranges over all objects x of A, and $\oplus A$ operates on this vectorspace as follows: for $\alpha \in \mathrm{Hom}_A(x,y)$, $m \in F(z)$, let $\alpha m = F(\alpha)(m)$ in case $y = z$, and $= 0$ otherwise. Conversely, given an $\oplus A$-module M, the corresponding functor is defined as follows: any object x of A is sent to 1_xM, any map $\alpha : x \longrightarrow y$ to the map $1_yM \longrightarrow 1_xM$ given by left multiplication by α.]

The algebra R is said to be <u>locally bounded</u> provided there exists a complete set of pairwise orthogonal primitive idempotents $\{e_x \mid x \in I\}$ with Re_x and e_xR being finite-dimensional, for all $x \in I$. Assume that R is locally bounded, and let $\{e_x \mid x \in I\}$ be a complete set of pairwise orthogonal primitive idempotents in R. The R-module Re_x will be denoted by P(x). One obtains in this way all possible indecomposable projective R-modules (there may be some dublications: some of the P(x) may be isomorphic). Similarly, let $Q(x) = \mathrm{Hom}_k(e_xR,k)$, one obtains in this way all possible indecomposable injective R-modules (again, may be, with dublications). For a locally bounded algebra R, any finitely generated R-module is of finite length, thus R-mod is abelian, and the simple R-modules are of the form P(x)/rad P(x). Also in this case, any finitely generated R-module has both a projective cover and an injective envelope in R-mod. It follows that R-mod has Auslander-Reiten sequences, and we will denote by Γ_R the corresponding Auslander-Reiten quiver. From now on, given a locally bounded algebra R, the R-modules we will deal with, always will be supposed to be finitely generated. We call an algebra R a Frobenius-algebra provided it is locally bounded, and the indecomposable projective R-modules coincide with the indecomposable injective R-modules. (Observe that we deviate from the usual terminology, even in case R being finite dimensional; the usual name in this case is that R is "Quasi-Frobenius", or selfinjective; whereas Nakayama's "Frobenius algebras" are finite-dimensional Frobenius-algebras which satisfy some additional multiplicity condition).

Given a Frobenius algebra R, we denote by R-$\underline{\mathrm{mod}}$ the stable module category: its objects are the same as those of R-mod, namely the finitely generated R-modules, and given two finitely generated R-modules X,Y, the set of morphisms from X to Y in R-$\underline{\mathrm{mod}}$ is denoted by $\underline{\mathrm{Hom}}(X,Y)$, and $\underline{\mathrm{Hom}}(X,Y) = \mathrm{Hom}_R(X,Y)/\sim$, where $f \sim g$ iff $f - g$ factors through a projective R-module. The residue class of a map $f : X \longrightarrow Y$ in $\underline{\mathrm{Hom}}(X,Y)$ will be denoted by $\underline{f}$. Note that the isomorphism classes of indecomposable objects in R-$\underline{\mathrm{mod}}$ correspond naturally to the isomorphism classes of the indecomposable non-projective R-modules, thus to the vertices of the stable Auslander-Reiten quiver $(\Gamma_A)_s$. The stable module category R-$\underline{\mathrm{mod}}$ preserves much information concerning R-mod.

Lemma. Let R be a Frobenius algebra, and X,Y indecomposable non-projective R-modules satisfying $\underline{\mathrm{Hom}}_R(X,Y) \neq 0$. Then there exists an indecomposable (non-projective) R-module M such that $\underline{\mathrm{Hom}}_R(X,M) \neq 0$, $\underline{\mathrm{Hom}}_R(M,Y) \neq 0$.

Proof (Vossieck): First of all, assume there exists $f : X \longrightarrow Y$ which is epi. We claim that $\underline{f}$ itself is non-zero. For assume $f = f_1 f_2$, where $f_1 : X \longrightarrow P$, $f_2 : P \longrightarrow Y$ with P projective. Since f is epi, and P is projective, there exists $f_2' : P \longrightarrow X$ with $f_2' f = f_2$, and therefore $f = f_1 f_2 = (f_1 f_2') f = (f_1 f_2')^n f$ for arbitrary $n \in \mathbb{N}$. Since f_1 is not split mono, and X is indecomposable, it follows that $(f_1 f_2')^n = 0$ for large n, thus $f = 0$, a contradiction. Similarly, we see that any mono map $g : X \to Y$ satisfies $\underline{g} = 0$.

In general, take an arbitrary non-zero map $h : X \longrightarrow Y$, and let M be an indecomposable direct summand of the image of h. Let $f : X \to$ and $g : M \longrightarrow Y$ be the maps induced by h, thus f is epi, g is mono. As a consequence, we see $\underline{\mathrm{Hom}}_R(X,M) \neq 0$, $\underline{\mathrm{Hom}}_R(M,Y) \neq 0$.

It follows that for a Frobenius algebra R, a search for directing modules and for separating subcategories in R-mod can be carried out in R-$\underline{\mathrm{mod}}$. We will see below examples where we may use this technique.

Also, we should mention that for a general Frobenius algebra R, and Γ a component of Γ_R, the stable subquiver Γ_s is still connected. Of course, there is the following trivial exception: if R is a simple artinian ring, then Γ_R consists of a single vertex and no arrow, whereas $(\Gamma_R)_s$ is empty. [Also note the following slightly pathological case: Assume that R has an indecomposable projective R-module of length 2. If R is connected, then all the indecomposable projective R-modules are of length 2, the remaining indecomposable R-modules all are simple, and $(\Gamma_R)_s = \mathbb{Z}\mathbb{A}_1$ or $\mathbb{Z}\mathbb{A}_1/n$. Now, $\mathbb{Z}\mathbb{A}_1$ and $\mathbb{Z}\mathbb{A}_1/n$ are connected as translation quivers, but the only case which is connected as a mere quiver is $\mathbb{Z}\mathbb{A}_1/1$.

Of course, if R has no projective R-modules of length 2, and Γ is any component of Γ_R, then Γ_s will be connected not only as a translation quiver, but as a mere quiver.]

Given any finite dimensional algebra A, let us construct the corresponding <u>repetitive</u> algebra $\hat{A}$, as proposed by Hughes and Waschbüsch [HW]. It will be a Frobenius algebra and always infinite-dimensional (except in the trivial case $A = 0$ which we may exclude). Denote by Q the A-A-bimodule $Q = \mathrm{Hom}_k(A,k)$ (the bimodule actions are defined in the obvious way: given $a',a'' \in A$, $\varphi \in Q$, then $a'\varphi a''$ is the k-linear map which sends $a \in A$ to $\varphi(a''aa')$. The underlying vectorspace of $\hat{A}$ is given by

$$\hat{A} = \Big(\bigoplus_{i\in\mathbb{Z}} A\Big) \oplus \Big(\bigoplus_{i\in\mathbb{Z}} Q\Big),$$

we denote the elements of $\hat{A}$ by $(a_i,q_i)_i$, where $a_i \in A$, $q_i \in Q$, of course with almost all a_i,q_i being zero. The multiplication is defined by

$$(a_i,q_i)_i(a_i',q_i')_i = (a_ia_i', a_{i+1}q_i' + q_ia_i')_i.$$

We also may use the categorical description $\hat{A} = \oplus \hat{\mathcal{A}}$, which is derived as follows: Choose a complete set $\{e_x \mid x \in I\}$ of pairwise orthogonal primitive idempotents in A (of course, I is a finite set), let $\mathcal{A}$ be the category with object set I, with $\mathrm{Hom}_{\mathcal{A}}(x,y) = e_xAe_y$, for all $x,y \in I$, and with composition of morphisms being the multiplication in A. Thus, $A = \oplus \mathcal{A}$. Define the category $\hat{\mathcal{A}}$ as follows: as object set we take $I\times\mathbb{Z}$, instead of (x,n), where $x \in I$, $n \in \mathbb{Z}$, we write $x\langle n\rangle$, we define

$$\mathrm{Hom}_{\hat{\mathcal{A}}}(x\langle n\rangle,y\langle m\rangle) = \begin{cases} e_xAe_y & m = n, \\ \mathrm{Hom}_k(e_yAe_x,k) \quad \text{for} & m = n-1, \\ 0 & m \neq n,n-1. \end{cases}$$

and the composition of maps is derived from the multiplication in A and the A-A-bimodule structure of Q. [More precisely, the composition of maps $x\langle n\rangle \to y\langle n\rangle \to z\langle n\rangle$ is given by the multiplication in A, the compostion $x\langle n\rangle \to y\langle n\rangle \to z\langle n-1\rangle$ is given by the canonical map $e_xAe_y \otimes \mathrm{Hom}_k(e_zAe_y,k) \longrightarrow \mathrm{Hom}(e_zAe_x,k)$ which sends $e_xae_y \otimes \varphi$ to the linear form $e_zbe_x \longmapsto \varphi(e_zae_xae_y)$; similarly, the composition

$x<n> \longrightarrow y<n-1> \longrightarrow z<n-1>$ is given by the canonical map $\mathrm{Hom}_k(e_yAe_x,k) \otimes e_yAe_z \longrightarrow \mathrm{Hom}(e_zAe_x,k)$ which sends $\psi \otimes e_yae_z$ to the linear form $e_zbe_x \longmapsto \varphi(e_yae_zbe_x)$.] It follows that $\hat{A} = \oplus \hat{A}$. Note that there exists an infinite cyclic group of automorphisms of both $\hat{A}$ and $\hat{A}$, given by shifting the indices, we denote by ν a generator of this group, namely the automorphism of $\hat{A}$ which sends $(a_i,q_i)_i$ to $(a_i',q_i')_i$ with $a_i' = a_{i-1}, q_i' = q_{i-1}$, and also the corresponding automorphism of $\hat{A}$ which sends $x<n>$ to $x<n+1>$.

The $\hat{A}$-modules can be written in the following way: $M = (M_i,f_i)_i$, where the M_i are A-modules, all but finitely many being zero, the f_i are A-linear maps $f_i = f_i^M : Q \otimes_A M_i \longrightarrow M_{i+1}$ such that the conditions $(Q \otimes f_i)f_{i+1} = 0$ are satisfied for all i, where always $i \in \mathbb{Z}$. [To wit, given such an $M = (M_i,f_i)_i$, consider $\oplus M_i$ as an $\hat{A}$-module using the scalar multiplication $(a_i,q_i)_i(m_i)_i = (a_im_i+(q_{i-1} \otimes m_{i-1})f_{i-1})_i$, where $(a_i,q_i)_i \in \hat{A}$, and $(m_i)_i \in \oplus M_i$. For the converse, note that the element $1_j = (\delta_{ij},0)_i \in \hat{A}$ (where δ_{ij} is the Kronecker symbol) is an idempotent, and actually the family $\{1_j \mid j \in \mathbb{Z}\}$ is a complete set of pairwise orthogonal idempotents. Given an $\hat{A}$-module M, we decompose M with respect to this family, thus $M_j = 1_j \cdot M$.] It is easy to calculate the indecomposable projective modules. Given $x \in I$, $n \in \mathbb{Z}$, we obtain

$$P(x<n>)_i = \begin{cases} P_A(x) & i = n \\ Q_A(x) & \text{for } i = n-1 \\ 0 & i \neq n,n-1 \end{cases}$$

and $f_n^{P(x<n>)} : Q \otimes P_A(x) \longrightarrow Q_A(x)$ the canonical isomorphism. Of course, this $\hat{A}$-module is also the indecomposable injective module corresponding to the vertex $x<n-1>$, thus

$$P(x<n>) = Q(x<n-1>).$$

In particular, we see in this way that $\hat{A}$ is a Frobenius algebra.

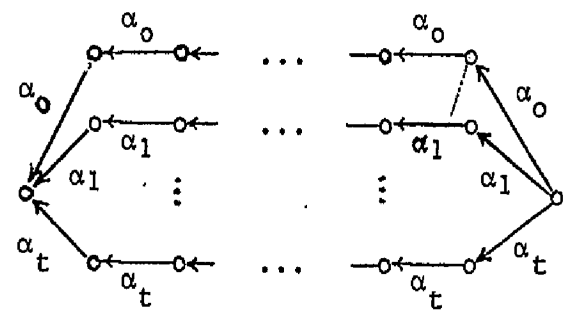
There are countably many obvious embeddings of A-mod into $\hat{A}$-mod, indexed over $\mathbb{Z}$. The $-<n>$ embedding with index n will send M to $M<n>$, where $M<n>_i$ for $i = n$, and $= 0$ otherwise. Observe that the composition of any of these embeddings $-<n>$ with the canonical functor $\hat{A}$-mod $\longrightarrow \hat{A}$-<u>mod</u> is still a full embedding. [The reason is the following: Given A-modules M_1,M_2 and a map $f : M_1 \longrightarrow M_2$, and suppose the corresponding map $f\langle n\rangle : M_1\langle n\rangle \longrightarrow M_2\langle n\rangle$ factors through a projective module. This implies that $f\langle n\rangle$ actually factors through a projective cover P of $M_2\langle n\rangle$. The indecomposable projective summands of P are of the form $P(a\langle n\rangle) = Q(a\langle n-1\rangle)$, and clearly $\mathrm{Hom}(M_1\langle n\rangle, Q(a\langle n-1\rangle)) = 0$.] The embedding $-\langle 0\rangle$ will be called the canonical embedding, and we will identify A-mod with its image under $-\langle 0\rangle$. Note that in this way we have achieved a full embedding of A-mod into the rather well-behaved category $\hat{A}$-mod.

We should observe that the category $\hat{A}$-mod may be defined alternatively as the category of graded modules over some graded algebra. To wit, let $T(A)$ be the trivial extension algebra of A. The underlying vectorspace of $T(A)$ is $A \oplus Q$ (recall that $Q = \mathrm{Hom}_k(A,k)$), and the multiplication is given by

$$(a,q)(a',q') = (aa', aq'+qa')$$

for $a,a' \in A$; and $q,q' \in Q$. The algebra $T(A)$ with the displayed decomposition $T(A) = A \oplus Q$ is a $\mathbb{Z}$-graded algebra, where $A \oplus 0$ are the elements of degree 0, those of $0 \oplus Q$ the elements of degree 1. We denote by $T(A)$-grmod the category of finitely generated $\mathbb{Z}$-graded modules over $T(A)$ and morphisms of degree zero. Obviously, a finitely generated $\mathbb{Z}$-graded module is of the form (M_i, f_i), where the M_i are A-modules, all but finitely many being zero, and the f_i are A-linear maps $Q \otimes M_i \longrightarrow M_{i+1}$ satisfying $(Q \otimes f_i)f_{i+1} = 0$ for all i. Thus $T(A)\text{-grmod} = \hat{A}\text{-mod}$.

We want to consider one example in detail. First, let us introduce the canonical algebras $C(\lambda,p)$ where $\lambda = (\lambda_0,\lambda_1,\ldots,\lambda_t)$ is an $(t+1)$-tupel of pairwise different elements in $\mathbb{P}_1 k$ (and we may suppose $\lambda_0 = \infty$, $\lambda_1 = 0$), and $p = (p_0,p_1,\ldots,p_t)$ an $(t+1)$-tupel of positive integers, and $t \geq 1$. Take the disjoint union of linearly ordered quivers of types $\mathbb{A}_{p_0+1},\ldots,\mathbb{A}_{p_t+1}$, the arrows in the i-th quiver will all be denoted by the letter α_i, and identify all sinks to a single vertex, and identify all sources again to a single vertex. Thus, we deal with the following quiver $\Delta(\lambda,p)$



with paths from the source to the sink of length $p_o,\dots,p_t$, respectively. The canonical algebra $C(\lambda,p)$ is the opposite of the path algebra of this quiver modulo the relations

$$\lambda_i\alpha_o^{p_o} + \alpha_1^{p_1} + \alpha_i^{p_i} = o, \quad \text{for } 2 \le i \le t.$$

[It may look fancy that we insist to deal with the opposite, since obviously the opposite of a canonical algebra is a canonical algebra again. The reason is that we want to identify the category $C(\lambda,p)$-mod with a category of representations of $\Delta(\lambda,p)$.] The Frobenius algebra $F(\lambda,p,q)$ which we want to consider is defined as follows: We start with an $(n+1)$-tupel $\lambda = (\infty,o,\lambda_2,\dots,\lambda_t)$ of pairwise different elements of $\mathbb{P}_1k$, two $(t+1)$-tuples $p = (p_o,\dots,p_t)$, $q = (q_o,\dots,q_t)$ of positive integers, take countably many copies $\Delta\langle n\rangle$, $n \in \mathbb{Z}$ of the quiver $\Delta(\lambda,p)$, countably many copies $\Delta'\langle n\rangle$, $n \in \mathbb{Z}$ of the quiver $\Delta(\lambda,q)$, and identify the sink of $\Delta\langle n\rangle$ with the source of $\Delta'\langle n-1\rangle$ and call this vertex $a\langle n\rangle$, and the sink of $\Delta'\langle n\rangle$ with the source of $\Delta\langle n\rangle$ and call this vertex $b\langle n\rangle$. As relations on this quiver $\Delta(\lambda,p,q)$ we use first of all those which make all $\Delta\langle n\rangle$, $\Delta'\langle n\rangle$ into canonical algebras, and, in addition the following ones: Denote as above the arrows of any $\Delta\langle n\rangle$ by α_i $(o \le i \le t)$, and denote those of $\Delta'\langle n\rangle$ in the same way by β_i $(o \le i \le t)$. The additional relations are the following ones (whenever they make sense):

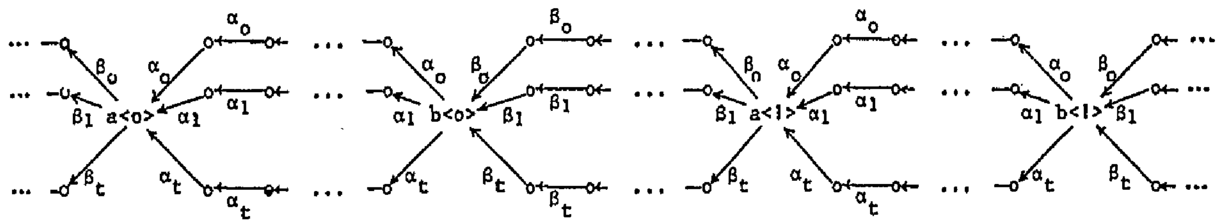
$$\alpha_i\beta_i = o, \quad \beta_i\alpha_i = o \qquad \text{for all } o \le i \le t,$$

$$\alpha_o^{p_o}\beta_1^{q_1} = \alpha_1^{p_1}\beta_o^{q_o}, \qquad \beta_o^{q_o}\alpha_1^{p_1} = \beta_1^{q_1}\alpha_o^{q_o}$$

$$\alpha_i^u\beta_j^{q_j}\alpha_i^{p_i-u+1} = 0 \qquad \text{for all } i \neq j,\ 1 \le u \le p_i$$

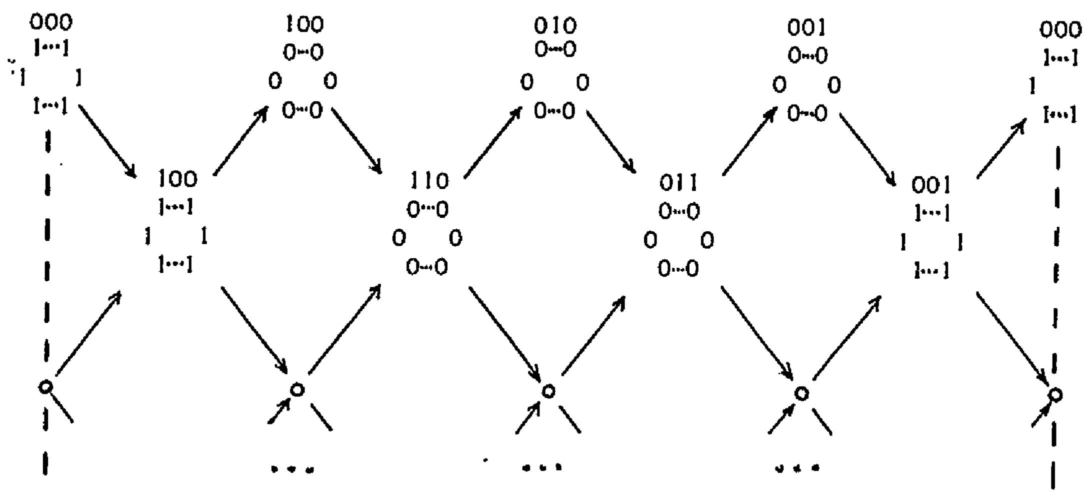
$$\alpha_i^u\beta_j^{q_j}\alpha_i^{p_i-u+1} = 0 \qquad \text{for all } i \neq j,\ 1 \le u \le q_i$$

and in this way, we obtain $F(\lambda,p,q)$ (again as the opposite of the path algebra modulo the listed relations).



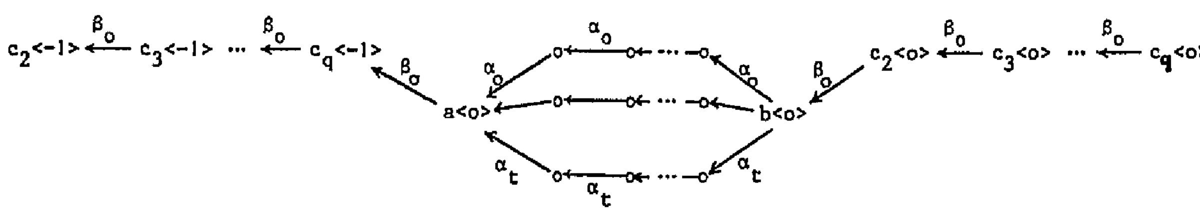
It is easy to check that $F(\lambda,p,q)$ is a Frobenius algebra, and it may be considered as the repetitive algebra $\hat{A}$ for many algebras A defined by full subquivers of $\Delta(\lambda,p,q)$ and the corresponding relations. For example, as the notation suggest, we may take the full subquiver of all vertices in $\Delta\langle o\rangle \cup \Delta'\langle o\rangle$ different from $a\langle 1\rangle$.

We are going to show in which way we can use the techniques presented in the last lecture in order to determine at least parts of the category $F(\lambda,p,q)$-mod, for arbitrary λ,p,q. First, we consider the canonical algebra $C(\lambda,p)$. Denote by $\mathcal{P}$ the additive subcategory of $C(\lambda,p)$-mod whose indecomposable objects are those representations of $\Delta(\lambda,p)$ for which all maps α_i are injective, and at least one α_i is not surjective, similarly, let $\mathcal{Q}$ be the additive subcategory of $C(\lambda,p)$-mod with indecomposables those representations for which all maps α_i are surjective and at least one α_i is not injective. Finally, let $\mathcal{T}$ be the additive subcategory whose indecomposables are the remaining ones (those for which either all α_i are bijective, or else some α_i is not injective and some α_j, may be the same, is not surjective). We observe that $\mathcal{T}$ is a separating tubular family of type p separating $\mathcal{P}$ from $\mathcal{Q}$. [For, we can apply the construction theorem for separating tubular families as presented in the last lecture: $C(\lambda,p)$ is the one-point extension of a hereditary algebra A_o with quiver a star, by a certain A_o-module R_o (the dimension vector of R_o is of the form $\begin{smallmatrix} & 11\ldots1 \\ 2 & 11\ldots1 \\ & 11\ldots1 \end{smallmatrix}$); the only non-serial indecomposable injective A_o-module is a wing A_o-module which is dominated by R_o, and it is not difficult to show that the module classes $\mathcal{P},\mathcal{T},\mathcal{Q}$ established there have the description given above.] Let us write down explicitely the mouth of one of the exceptional tubes $\mathcal{T}(\rho)$ in $\mathcal{T}$, say, the one with index $\rho = \infty$ and to be specific, in the case where $p_o = 4$:



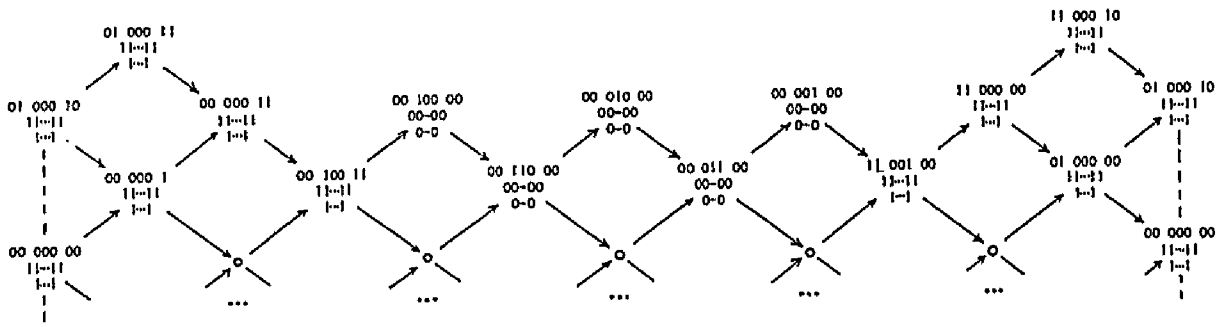
here, we have to identify the dashed vertical boundaries. What happens with this tubular family $\mathcal{T}$ when we consider $C(\lambda,p)$-mod as a full subcategory of $F(\lambda,p,q)$-mod, identifying $\Delta(\lambda,p)$ with $\Delta\langle o\rangle$?

First, consider the full subquiver



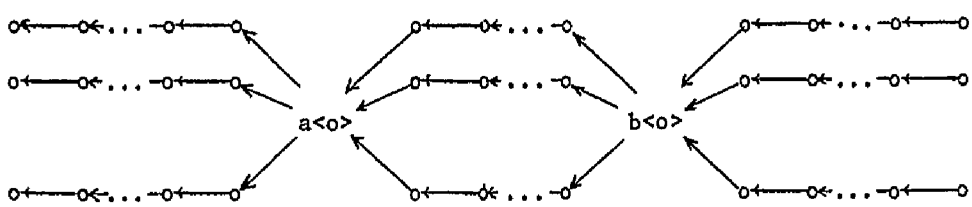
taking into account besides $\Delta\langle o\rangle$ also those arrows β_o of $\Delta\langle -1\rangle$ which do not end in the sink of $\Delta'\langle -1\rangle$, and those arrows β_o of $\Delta\langle o\rangle$ which do not start in the source of $\Delta\langle o\rangle$ (and the relations which live on this subquiver). We obtain this algebra from $C(\lambda,p)$ by successive tubular extensions, followed by successive tubular coextensions, and these extensions and coextensions change the tube $\mathcal{T}(\infty)$ with index ∞, but leave untouched the remaining tubes in $\mathcal{T}$. The new tube with index ∞ is obtained from $\mathcal{T}(\infty)$ by first inserting $q-1$ rays, and then inserting the same number of corays; there are q_o-1 indecomposable modules which are both projective and injective, namely the modules $P(c_i\langle o\rangle) = Q(c_i\langle -1\rangle)$, $2 \le i \le q_o$. Deleting these modules, we obtain a stable tube of the form $\mathbb{Z}\mathbb{A}_\infty/(p_o+q_o-1)$. Let us indicate the mouth of the new tube with index ∞

for the case $q_o = 3$



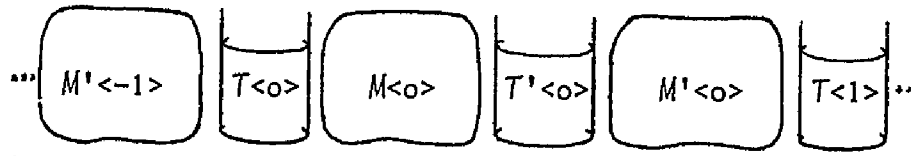
here again, the vertical dashed boundary lines have to be identified.

If we consider the full subquiver of $\Delta(\lambda,p,q)$ given by all vertices in $\Delta'\langle -1\rangle \cup \Delta\langle o\rangle \cup \Delta\langle o\rangle$ different from $b\langle -1\rangle$ and $a\langle 1\rangle$:



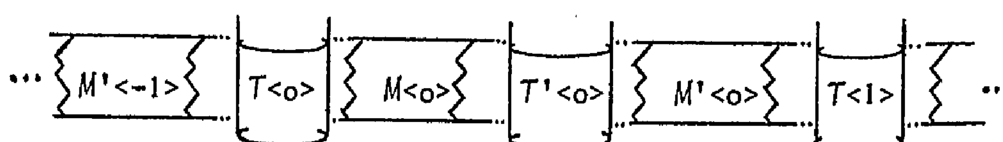
and all relations which live on this subquiver, we see that we obtain a tubular family $\mathcal{T}\langle o\rangle$ which contains the tubular family $\mathcal{T}$ of $C(\lambda,p)$ and which is obtained by inserting q_i-1 rays and also q_i-1 corays into the tube $\mathcal{T}(\lambda_i)$, $0 \le i \le t$. This tubular family $\mathcal{T}\langle o\rangle$ again is separating. If we continue to form one-point extensions and one-point coextensions in order to arrive finally at $F(\lambda,p,q)$, we see that the tubular family $\mathcal{T}\langle o\rangle$ remains unchanged, and still is separating. So denote by $\mathcal{P}\langle o\rangle$, $\mathcal{Q}\langle o\rangle$ the module classes in $F(\lambda,p,q)$ which are separated by $\mathcal{T}\langle o\rangle$. An $F(\lambda,p,q)$-module P belongs to $\mathcal{P}\langle o\rangle$ if and only if it satisfies the following two properties: First of all, P has to live on the union of all $\Delta'\langle 1\rangle$, $n \le -1$, and all $\Delta\langle m\rangle$, $m \le o$, and second, the indecomposable summands of the restriction of P to $\Delta\langle o\rangle$ all have to belong to $\mathcal{P}$.

What we have done starting with $\Delta\langle o\rangle$, we can do for any $\Delta\langle n\rangle$ and any $\Delta'\langle n\rangle$. We obtain tubular families $\mathcal{T}\langle n\rangle$ and $\mathcal{T}'\langle n\rangle$ whose stable type is $(p_o+q_o-1,\dots,p_t+q_t-1)$, and corresponding module classes $\mathcal{P}\langle n\rangle$, $\mathcal{Q}\langle n\rangle$ and $\mathcal{P}'\langle n\rangle$, $\mathcal{Q}'\langle n\rangle$. Let $\mathcal{M}\langle n\rangle = \mathcal{Q}\langle n\rangle \cap \mathcal{P}'\langle n\rangle$, and $\mathcal{M}'\langle n\rangle = \mathcal{Q}'\langle n\rangle \cap \mathcal{P}\langle n+1\rangle$. Then the following module classes

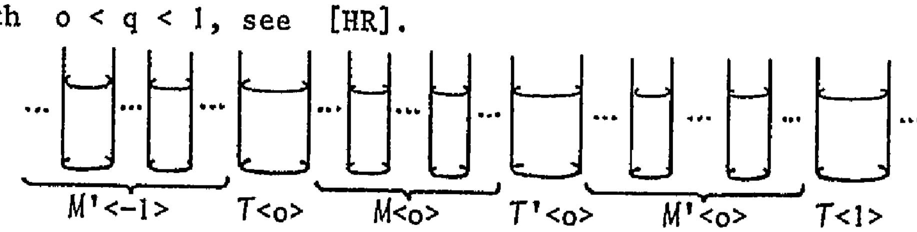


exhaust $F(\lambda,p,q)$-mod, and they have the following separation property: any of these classes separates the additive subcategory given by the module classes to its left from the additive subcategory given by the module classes to its right. Note that the modules in $M\langle n\rangle$ live on $\Delta\langle n\rangle \cup \Delta'\langle n\rangle$, those in $M'\langle n\rangle$ live on $\Delta'\langle n\rangle \cup \Delta\langle n+1\rangle$, the modules $T\langle n\rangle$ live on $\Delta'\langle n-1\rangle \cup \Delta\langle n\rangle \cup \Delta'\langle n\rangle$, those in $T'\langle n\rangle$ live on $\Delta\langle n\rangle \cup \Delta'\langle n\rangle \cup \Delta\langle n+1\rangle$. Altogether, we see that the indecomposable $F(\lambda,p,q)$-modules have bounded support.

The structure of the module classes $M\langle n\rangle$ and $M'\langle n\rangle$ is known only in particular cases. Consider the star $\mathbb{T} = \mathbb{T}_{p_o+q_o-1,\dots,p_t+q_t-1}$. In case $\mathbb{T}$ is one of the Dynkin graphs $\mathbb{A}_m$, $\mathbb{D}_m$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, then any $M\langle n\rangle$ and $M'\langle n\rangle$ is a single component, and it is of the form $\mathbb{Z}\tilde{\mathbb{T}}$ (in case $\mathbb{T} = \mathbb{A}_m$, say $p_i+q_i = 2$ for all $i \geq 2$, so that $\tilde{\mathbb{T}}$ is not a tree, we have to specify which orientation of $\tilde{\mathbb{T}}$ has to be taken: it is just that of $\tilde{\mathbb{A}}_{p_o+q_o-1,p_1+q_1-1}$). Thus, in this case the category $F(\lambda,p,q)$-mod has the following shape: there are the tubular families $T\langle n\rangle$ and $T'\langle n\rangle$, and in between there are sort of connecting components of the form $\mathbb{Z}\tilde{\mathbb{T}}$.



Next consider the case of $\mathbb{T}$ being a Euclidean graph, thus one of $\tilde{\mathbb{D}}_4$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ and $\tilde{\mathbb{E}}_8$. In this case, all components in $M\langle n\rangle$ and in $M'\langle n\rangle$ are again tubes, all even regular, these tubes form again $\mathbb{P}_1k$-families of type $(p_o+q_o-1,\dots,p_t+q_t-1)$ and the set of families in any $M\langle n\rangle$ and in any $M'\langle n\rangle$ may be indexed in a rather natural way by the rational numbers q with $o < q < 1$, see [HR].



We return now to a general Frobenius category. The category R-mod usually is no longer an abelian category, but it carries some additional structure which seems to be similarly useful: it is the underlying category of a triangulated category. The notion of a triangulated category was introduced by Verdier [V]. The relevant features of R-mod as a triangulated category were observed by Heller [He], however the system of axioms as proposed by Verdier was not yet available; the fact that R-mod actually satisfies all the axioms required by Verdier was noted only recently by Happel [Ha]. We will not repeat all the axioms of a triangulated category and refer directly to Verdier [V]. But let us sketch some of the basic principles of triangulated categories in general, and the way they are used when dealing with R-mod. First of all, a triangulated category is an additive category $\mathcal{A}$ together with an automorphism T of $\mathcal{A}$ and a set $\mathcal{T}$ of sextuples of the form (X,Y,Z,u,v,w) where $u : X \longrightarrow Y$, $v : Y \longrightarrow Z$, $w : Z \longrightarrow T(X)$ are maps in $\mathcal{A}$, the elements of $\mathcal{T}$ being called triangles. Of course, the set $\mathcal{T}$ is supposed to be closed under isomorphisms (a map from (X,Y,Z,u,v,w) to (X',Y',Z',u',v',w') is of the form (f,g,h), where $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$, $h : Z \longrightarrow Z'$ are maps in $\mathcal{A}$ satisfying $ug = fu'$, $vh = gv'$ and $wT(f) = hw'$). In the case of $\mathcal{A}$ = R-mod, one takes for T the suspension functor of Heller: for any object X, choose an injective module $I(X)$ with submodule X, and let $T(X) = I(X)/X$. (There are some set-theoretical sutleties in order to ensure that T actually is an automorphism of R-mod as required and not only a self-equivalence). Given a map $\alpha : X \longrightarrow Y$ in R-mod, consider the induced exact sequence

$$\begin{array}{ccccccccc} o & \longrightarrow & X & \xrightarrow{\iota} & I(X) & \xrightarrow{\pi} & T(X) & \longrightarrow & o \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\bar{\alpha}} & & \Big\| & & \\ o & \longrightarrow & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & T(X) & \longrightarrow & o \end{array}$$

where $\iota = \iota_X$ denotes the inclusion, $\pi = \pi_X$ the projection map. Then $(X,Y,Z,\underline{\alpha},\underline{\beta},\underline{\gamma})$ is called a standard triangle, and, by definition $\mathcal{T}$ is the class of sextuples which are isomorphic to standard triangles. For example, with the standard triangle $(X,Y,Z,\underline{\alpha},\underline{\beta},\underline{\gamma})$ above, also the isomorphic one $\left(X,Y \oplus I(X),Z,\underline{[\alpha\iota]},\underline{\begin{bmatrix}\beta\\ -\bar{\alpha}\end{bmatrix}},\underline{\gamma}\right)$ is a triangle (in R-mod, there is the isomorphism $\left(1,\begin{bmatrix}1\\0\end{bmatrix},1\right)$ from the first to the second sextuple!) Given any exact sequence

$$E = (o \longrightarrow U \xrightarrow{\mu} V \xrightarrow{\varepsilon} W \longrightarrow o)$$

in R-mod, there is $\bar{\mu} : V \longrightarrow I(U)$ with $\mu\bar{\mu} = \iota_U$, and therefore a commutative diagram

$$\begin{array}{ccccccccc}
o & \longrightarrow & U & \xrightarrow{\mu} & V & \xrightarrow{\varepsilon} & W & \longrightarrow & o \\
 & & \| & & \downarrow{\scriptstyle \bar{\mu}} & & \downarrow{\scriptstyle \nu} & & \\
o & \longrightarrow & U & \xrightarrow{\iota} & I(U) & \xrightarrow{\pi} & T(U) & \longrightarrow & o \;.
\end{array}$$

Note that $E \longmapsto w(E) = -\underline{\nu}$ yields a map $w : \mathrm{Ext}^1(W,U) \longrightarrow \underline{\mathrm{Hom}}(W,T(U))$ which is known to be bijective. On the other hand, we can rearrange these maps to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
o & \longrightarrow & U & \xrightarrow{\iota} & I(U) & \xrightarrow{\pi} & T(U) & \longrightarrow & o \\
 & & {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle [01]} & & \| & & \\
o & \longrightarrow & V & \xrightarrow{[\varepsilon\bar{\mu}]} & W \oplus I(U) & \xrightarrow{\left[\begin{smallmatrix}-\nu \\ \pi\end{smallmatrix}\right]} & T(U) & \longrightarrow & o
\end{array}$$

which shows that $(U,V,W \oplus I(U),\underline{\mu}, \underline{\varepsilon\bar{\mu}}, \underline{\left[\begin{smallmatrix}-\nu \\ \pi\end{smallmatrix}\right]})$ is a standard triangle. But this standard triangle is isomorphic to $(U,V,W,\underline{\mu},\underline{\varepsilon},\underline{-\nu}) = (U,V,W,\underline{\mu},\underline{\varepsilon},w(E))$, thus also the latter is a triangle. Let us return to the standard triangle $(X,Y,Z,\underline{\alpha},\underline{\beta},\underline{\gamma})$ considered above, and the isomorphic triangle $(X,Y \oplus I(X),Z,\underline{[\alpha\iota]},\underline{\left[\begin{smallmatrix}\beta \\ -\bar{\alpha}\end{smallmatrix}\right]},\underline{\gamma})$. We observe that

$$E_o := (o \longrightarrow X \xrightarrow{[\alpha\iota]} Y \oplus I(X) \xrightarrow{\left[\begin{smallmatrix}\beta \\ -\bar{\alpha}\end{smallmatrix}\right]} Z \longrightarrow o)$$

is an exact sequence, and the commutative diagram

$$\begin{array}{ccccccccc}
o & \longrightarrow & X & \xrightarrow{[\alpha\iota]} & Y \oplus I(X) & \xrightarrow{\left[\begin{smallmatrix}\beta \\ -\bar{\alpha}\end{smallmatrix}\right]} & Z & \longrightarrow & o \\
 & & \| & & \downarrow{\scriptstyle \left[\begin{smallmatrix}o \\ 1\end{smallmatrix}\right]} & & \downarrow{\scriptstyle -\gamma} & & \\
o & \longrightarrow & X & \xrightarrow{\iota} & I(X) & \xrightarrow{\pi} & T(X) & \longrightarrow & o
\end{array}$$

shows that $w(E_o) = \underline{\gamma}$. This demonstrates that any triangle in R-$\underline{\mathrm{mod}}$ is isomorphic to a triangle of the form $(U,V,W,\underline{\mu},\underline{\varepsilon},w(E))$, where

$$E = (o \longrightarrow U \xrightarrow{\mu} V \xrightarrow{\varepsilon} W \longrightarrow o)$$

is an exact sequence.

By the construction above, we see that any map u in R-mod occurs as a first map in some triangle (X,Y,Z,u,v,w), and in case u is an identity map, then Z is zero (in R-mod). this is one of the conditions required in a triangulated category. Note that Z is determined by u up to an isomorphism, but given two triangles (X,Y,Z,u,v,w) and (X,Y,Z',u,v',w'), there may be different maps $h_1,h_2 : Z \longrightarrow Z'$ in R-mod with $(1,1,h_1)$, $(1,1,h_2)$ both being isomorphisms of triangles. The non-unicity of maps when working inside a triangulated category is one of the phenomena, which makes such a category completely different to an abelian category! Given two triangles (X,Y,Z,u,v,w), (X',Y',Z',u',v',w') and $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ with $ug = fu'$, then one of the axioms of a triangulated category requires the existence of some $h : Z \longrightarrow Z'$ (again not necessarily a unique one) such that (f,g,h) is a map of triangles. There are two other axioms which have to be checked. One of these, the socalled octahedral axiom, starts with two composable maps, say $u_1 : X_1 \longrightarrow X_2$, $u_2 : X_2 \longrightarrow X_3$, and triangles which have u_1,u_2, and u_1u_2 as first map, respectively, and shows the relation between these triangles. The other one is the most surprising: it provides a rotation-symmetry: given a triangle (X,Y,Z,u,v,w), then also $(Y,Z,T(X),v,w,-T(u))$ is a triangle (there is also the converse, but this follows from the axioms). This shows that the properties mentioned with respect to the first map of a triangle have corresponding counterparts for the remaining maps. On the other hand, we also see that the properties mentioned with respect to the second or the third map of a triangle hold true also for the first map. In particular, given a triangle (X,Y,Z,u,v,w), up to isomorphism, we may represent u by a monomorphism, or an epimorphism of R-mod, or we may consider it as an element in $\mathrm{Ext}^1(X,T^{-1}Y)$.

In any triangulated category A, we may speak of Auslander-Reiten triangles, these are triangles (X,Y,Z,u,v,w) with both X, and Z indecomposable, $w \neq o$, and the following equivalent conditions are satisfied: (i) for all $f : X \longrightarrow V$, f not split mono, there exists $f' : Y \longrightarrow V$ with $uf' = f$; (ii) for all $g : W \longrightarrow Z$, g not split epi, there exists $g' : W \longrightarrow Y$ with $g'v = g$; (iii) if $h_1 : U_1 \longrightarrow Z$, h_1 not split epi, then $h_1w = o$; (iv) if $h_2 : TX \longrightarrow U_2$, h_2 not split mono, then $wh_2 = o$. In case (X,Y,Z,u,v,w) is an Auslander-Reiten triangle, the objects X and Z determine each other (up to isomorphism), and we write $X = \tau Z$, $Z = \tau^{-}X$, and τ is called the Auslander-Reiten translation; also, if we decompose $Y = \oplus Y_i$ with indecomposable objects Y_i, then the induced morphisms $X \longrightarrow Y_i$ are irreducible, and we obtain in this way sufficiently many irreducible morphisms starting in X; similarly, the induced morphisms $Y_i \longrightarrow Z$ are irreducible, and we obtain sufficiently many irreducible morphisms ending in Z. We say that $\mathcal{A}$ has Auslander-Reiten triangles, provided for any indecomposable object of $\mathcal{A}$ there is an Auslander-Reiten triangle where it occurs in the first position, and one where it occurs in the third position. For a triangulated category $\mathcal{A}$ with Auslander-Reiten triangles, we can introduce its Auslander-Reiten quiver in the same way as for module categories, note however that here we obtain a regular translation quiver!

Let us display the Auslander-Reiten triangles in R-$\underline{\text{mod}}$, where R is a Frobenius algebra. Given any indecomposable non-injective R-module X, let $E = (o \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow o)$ be an Auslander-Reiten sequence starting in X. Then $(X,Y,Z,f,g,w(E))$ is an Auslander-Reiten triangle in R-$\underline{\text{mod}}$, and any Auslander-Reiten triangle is isomorphic to such a triangle. In particular, we see that R-$\underline{\text{mod}}$ has Auslander-Reiten triangles. In case $R = \hat{A}$, where A is a finite-dimensional algebra, some of the Auslander-Reiten sequences in A-mod are still Auslander-Reiten sequences in $\hat{A}$-mod, thus give rise to an Auslander-Reiten triangle in $\hat{A}$-$\underline{\text{mod}}$:

Lemma. Let A be a finite-dimensional algebra, and $E := (o \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow o)$ an Auslander-Reiten sequence in A-mod. Then the following conditions are equivalent:

(i) E is an Auslander-Reiten sequence in $\hat{A}$-mod.

(ii) inj.dim.X = 1, proj.dim.Z = 1.

(iii) $\mathrm{Hom}_A(I,X) = o$ for any injective A-module I, and $\mathrm{Hom}_A(Z,P) = o$ for any projective A-module P.

Auslander-Reiten sequences with these equivalent properties may be said to be conservative. The equivalence of the conditions (ii) and (iii) is quite well-known. For the equivalence of (i) and (iii), one may use a generalized version of [Ri2], 2.5.5. We note that for a conservative Auslander-Reiten sequence $E = (o \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow o)$ in A-mod, we have $\underline{\dim} X = -(\underline{\dim}\, Z) C_A^{-T} C_A$, provided C_A is invertible.

(The transformation $\Phi_A = -C_A^{-T} C_A$ which yields $\underline{\dim}\, X = (\underline{\dim}\, Z)\Phi_A$ is usually called the Coxeter transformation for A).

In particular, we see that the Auslander-Reiten sequences inside a sincere separating regular tubular family $\mathcal{T}$ of A-mod remain Auslander-Reiten sequences in $\hat{A}$-mod. Thus $\mathcal{T}$ remains a set of components of $\hat{A}$-mod. Actually, $\mathcal{T}$ also remains to be separating, but, of course, not sincere. With $\mathcal{T}$ also the module classes of $\hat{A}$-mod obtained from $\mathcal{T}$ by applying powers of the shift ν are separating regular tubular families, thus in this case $\hat{A}$-mod has countably many separating regular tubular families.

Similarly, assume that there is a slice $\mathcal{S}$ in A-mod, let ${}_AS$ be a slice module in $\mathcal{S}$, and $k\Delta = \mathrm{End}({}_AS)$. As a module class in $\hat{A}$-mod, $\mathcal{S}$ still will be path closed and will satisfy the following property: given any Auslander-Reiten sequence $o \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow o$ in $\hat{A}$-mod, at most one of X,Z belongs to $\mathcal{S}$, and one of X,Z belongs to $\mathcal{S}$ in case an indecomposable direct summand of Y is in $\mathcal{S}$. It follows that $\mathcal{S}$ belongs to one component Γ of $\hat{A}$-mod, and that the corresponding stable translation quiver Γ_s is of the form $\Gamma_s = \mathbb{Z}\Delta$.

We have seen above that for any Frobenius algebra R, the stable category R-$\underline{\text{mod}}$ can be made into a triangulated category. There is a more general result which includes this as a special case: Instead of dealing with the abelian category R-mod (and assuming that the projective objects and the injective objects coincide), one may start with an arbitrary exact category $(\mathcal{A},\mathcal{S})$. (Here, $\mathcal{A}$ is an additive category, which is embedded into some abelian category $\mathcal{A}'$ as a full and extension-closed subcategory, and $\mathcal{S}$ is the set of all sequences $o \longrightarrow X \xrightarrow{u} Y \xrightarrow{e} Z \longrightarrow o$ in $\mathcal{A}$ which are exact when considered as sequences in $\mathcal{A}'$; given such an element of $\mathcal{S}$, the map u is called a proper mono, the map e a proper epi.) Given an exact category $(\mathcal{A},\mathcal{S})$, we say that P is $\mathcal{S}$-projective provided P has the usual lifing property with respect to all proper epis, and we say that Q is $\mathcal{S}$-injective provided Q has the usual extension property with respect to all proper monos. We call $\mathcal{S}$ a <u>Frobenius structure</u> on $\mathcal{A}$ provided $(\mathcal{A},\mathcal{S})$ is an exact category, the $\mathcal{S}$-projective objects in $\mathcal{A}$ coincide with the $\mathcal{S}$-injective objects, and there are enough $\mathcal{S}$-projective and enough $\mathcal{S}$-injective objects (this means that for any object X in $\mathcal{A}$, there is a proper epi $P \longrightarrow X$ with P $\mathcal{S}$-projective, and a proper mono $X \longrightarrow Q$ with Q S-injective). The corresponding stable category $\underline{A}$ has the same objects as A, and $\mathrm{Hom}_{\underline{A}}(X,Y) = \mathrm{Hom}_A(X,Y)/\sim$, with $f \sim g$ iff $f-g$ factors through an S-projective object. There is the following general result [Ha]: if S is a Frobenius structure on A, then $\underline{A}$ is the underlying category of a triangulated category.

A special case of this result is the following one: We start with any additive category A. A sequence of the form $o \longrightarrow X \xrightarrow{[1o]} X \oplus Z \xrightarrow{\left[\begin{smallmatrix} o \\ 1 \end{smallmatrix}\right]} Z \longrightarrow o$, and isomorphic ones, are said to be split exact. Denote by $C^b(A)$ the category of all bounded complexes over A (a bounded complex is of the form $X^\cdot = (X^i, d^i)_i$ where X^i are objects of A, indexed over $\mathbb{Z}$, all but only finitely many being non-zero, and $d^i_{X^\cdot} = d^i : X^i \longrightarrow X^{i+1}$ are maps in A satisfying $d^i d^{i+1} = o$ for all $i \in \mathbb{Z}$. Let S be the set of all sequences $o \longrightarrow X^\cdot \xrightarrow{f^\cdot} Y^\cdot \xrightarrow{g^\cdot} Z^\cdot \longrightarrow o$ in $C^b(A)$ such that the sequences $o \longrightarrow X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \longrightarrow o$ are split exact. Then S is a Frobenius structure on $C^b(A)$, and the usual homotopy category $K^b(A)$ of $C^b(A)$ is just the stable category $\underline{C^b(A)}$ of $C^b(A)$ with respect to S. The automorphism T required in the definition of a triangulated category is given by the shift T wich sends the complex $X^\cdot = (X^i, d^i)_i$ to $T(X^\cdot)$ where $T(X^\cdot)^i = X^{i+1}$ and $d^i_{T(X^\cdot)} = -d^{i+1}_{X^\cdot}$. Thus, the well-known fact that the homotopy category $K^b(A)$ of bounded complexes over A can be made into a triangulated category is a special case of the general result concerning Frobenius structures.

In particular, we are interested in $K^b(A\text{-proj})$, where A is a finite-dimensional algebra (and A-proj the full subcategory of A-mod of all projective modules). In case A has finite global dimension, we may define the <u>derived category</u> $D^b(A)$ to be just $K^b(A\text{-proj})$. More precisely, $D^b(A)$ is usually defined as the triangulated category obtained from $K^b(A\text{-mod})$ by formally inverting all maps which induce an isomorphism in cohomology, thus there is a functor $\Phi : K^b(A\text{-mod}) \longrightarrow D^b(A)$ with the following properties: first of all, if $X^\cdot, Y^\cdot$ are bounded complexes, $f^\cdot : X^\cdot \longrightarrow Y^\cdot$ a map with $H^i(f^\cdot)$ an isomorphism for all i, then $\Phi(\underline{f^\cdot})$ of the homotopy class $\underline{f^\cdot}$ of $f^\cdot$ under Φ is invertible in $D^b(A)$. Second, any other functor $\Phi' : K^b(A\text{-mod}) \longrightarrow \mathcal{D}$ having this property can be factored through Φ. Now, $K^b(A\text{-proj})$ is a full subcategory of

K^b(A-mod), and, in case A has finite global dimension, the composition of functors

$$K^b(A\text{-proj}) \hookrightarrow K^b(A\text{-mod}) \xrightarrow{\Phi} D^b(A)$$

is an equivalence of triangulated categors. Similarly, for A of finite global dimension, also the composition of functors

$$K^b(A\text{-inj}) \hookrightarrow K^b(A\text{-mod}) \xrightarrow{\Phi} D^b(A)$$

is an equivalence of categories (here, A-inj is the full subcategory of A-mod of all injective modules). Note that there are obvious embeddings of A-mod into K^b(A-mod), and using then Φ, into $D^b(A)$ which we may denote by $-[n]$, where $n \in \mathbb{Z}$. Given an A-module M, let $M[n]$ be the complex with $M[n]^i = M$ for $i = -n$, and $= o$ otherwise, or better, the corresponding image under Φ in $D^b(A)$. We will identify A-mod with the image of the functor $-[o]$. Note that if

$$o \longrightarrow M \longrightarrow I^o \longrightarrow I^1 \longrightarrow \dots \longrightarrow I^r \longrightarrow o$$

is an injective resolution of M, then $M = M[o]$ is isomorphic in $D^b(A)$ to the image of the complex

$$\dots \longrightarrow o \longrightarrow I^o \longrightarrow I^1 \longrightarrow \dots \longrightarrow I^r \longrightarrow o \quad \rightarrow \dots$$

under Φ. Similarly, if

$$o \longrightarrow P^{-r} \longrightarrow \dots \longrightarrow P^{-1} \longrightarrow P^o \longrightarrow M \longrightarrow o$$

is a projective resolution of M, then $M = M[o]$ is isomorphic in $D^b(A)$ to the image of the complex

$$\dots \longrightarrow o \longrightarrow P^{-r} \longrightarrow \dots \longrightarrow P^{-1} \longrightarrow P^o \longrightarrow o \quad \rightarrow \dots$$

under Φ. Of course, the different embeddings $-[n]$ of A-mod into $D^b(A)$ are related by the powers of T operating on $D^b(A)$, we have $T(M[n]) = M[n+1]$. For given A-modules M,N, we may identify $\mathrm{Hom}_{D^b(A)}(M[m],N[n])$ with $\mathrm{Ext}_A^{n-m}(M,N)$ (again, $\mathrm{Ext}^o = \mathrm{Hom}$, and $\mathrm{Ext}^i = o$ for $i < o$).

Let us exhibit the Auslander-Reiten triangles in D^b(A-mod). We denote by $D = \mathrm{Hom}_k(-,k)$ the duality with respect to the base field k.

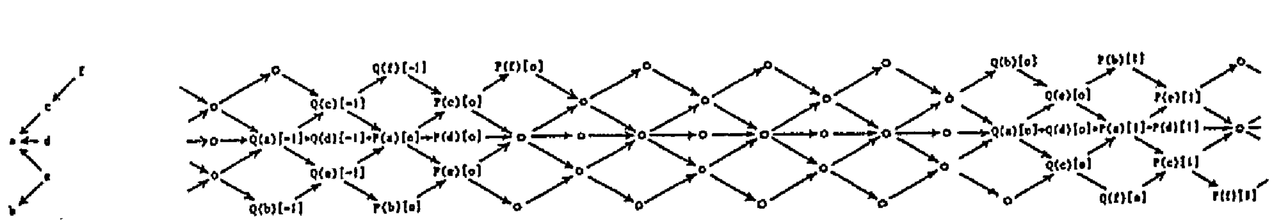
The endofunctor $\nu = D\,\mathrm{Hom}_A(-,{}_AA)$ of A-mod is called the Nakayama functor, it defines an equivalence from A-proj to A-inj. For $X,Y \in$ A-mod, there is a natural map $\alpha_{XY} : D\,\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(Y,\nu X)$ which is the composition $\alpha_{XY} = \alpha'_{XY}\alpha''_{XY}$, where α'_{XY} is the dual of the map $\mathrm{Hom}(X,{}_AA) \otimes Y \longrightarrow \mathrm{Hom}(X,Y)$ which sends $f \otimes y$ to $(x \longmapsto (xf)y)$, for $f : X \longrightarrow {}_AA$, $y \in Y$, $x \in X$, and where α''_{XY} is the adjunction map. Note that for X a projective A-module, the map α'_{XY}, and therefore α_{XY} itself, is bijective. We extend ν to an endofunctor of C^b(A-mod), and of K^b(A-mod), and we see that its restriction to K^b(A-proj) provides an equivalence $\nu : K^b(\text{A-proj}) \longrightarrow K^b(\text{A-inj})$. Also, given $X^\cdot, Y^\cdot \in C^b$(A-mod), there is the corresponding natural map $\alpha_{X^\cdot Y^\cdot} : D\,\mathrm{Hom}(X^\cdot,Y^\cdot) \to \mathrm{Hom}(Y^\cdot,\nu X^\cdot)$. Assume now that A has finite global dimension, so that any object in $D^b(A)$ can be written in the form $P^\cdot$, where $P^\cdot$ is a bounded complex of projective A-modules. Assume that $P^\cdot$ is indecomposable, and let $\varphi \in D\,\mathrm{Hom}(P^\cdot,P^\cdot)$ be a non-zero linear form on $\mathrm{Hom}(P^\cdot,P^\cdot) = \mathrm{End}(P^\cdot)$ which vanishes on rad $\mathrm{End}(P^\cdot)$. Consider $\alpha_{P^\cdot P^\cdot}(\varphi)$, this is a non-zero map $P^\cdot \longrightarrow \nu P^\cdot$ which has the following properties: if $X^\cdot$ is an indecomposable object of $D^b(A)$ and $\xi : X^\cdot \longrightarrow P^\cdot$ is non-invertible, or $\eta : \nu P^\cdot \longrightarrow X^\cdot$ is non-invertible, then $\xi\alpha_{P^\cdot P^\cdot}(\varphi) = 0$, or $\alpha_{P^\cdot P^\cdot}(\varphi)\eta = 0$, respectively. Take a triangle in $D^b(A)$ whose third map is $\alpha_{P^\cdot P^\cdot}(\varphi)$, say $(T^{-1}\nu P^\cdot, Y, P^\cdot, u, v, \alpha_{P^\cdot P^\cdot}(\varphi))$, then this is an Auslander-Reiten triangle, and, up to isomorphism, one obtains all Auslander-Reiten triangles in $D^b(A)$ in this way. Note that we see that the Auslander-Reiten translation on K^b(A-proj) ($\approx D^b(A)$) is given by the functor $\tau := T^{-1}\nu$.

Following Happel [Ha], we consider one example in detail. Let A be a hereditary algebra. In this case (and only in this case), the images of the various full embeddings $-[n] : \text{A-mod} \to D^b(A)$ exhaust the category $D^b(A)$: any indecomposable object of $D^b(A)$ belongs to one (and only one) such subcategory. This is of course straightforward to see since obviously any complex in C^b(A-proj) can be written as a direct sum of complexes of the form $\ldots 0 \longrightarrow P^{n-1} \xrightarrow{d} P^n \longrightarrow 0 \ldots$ with d a monomorphism, and in $D^b(A)$ this complex is isomorphic to the complex (Cok d)[-n]. Note that (for A hereditary!) the embedding $-[n]$ preserves irreducibility of maps and that any Auslander-Reiten sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in A-mod gives rise to Auslander-Reiten triangles $(X[n],Y[n],Z[n],f[n],g[n],h_n)$ where $h_n : Z[n] \to X[n+1]$.

Consequently, we may visualize $D^b(A)$ as follows: Take copies $\Gamma_A[n]$ of Γ_A, indexed by $n \in \mathbb{Z}$, and connect these copies together using additional arrows and extending the translation τ to all vertices. In fact, each arrow $a \xrightarrow{\alpha} b$ in Δ gives rise to an irreducible morphism $Q(a)[n] \longrightarrow P(b)[n+1]$, in the following way: instead of writing down a map $Q(a)[n] \longrightarrow P(b)[n+1]$, we may as well exhibit an element in $\mathrm{Ext}^1(Q(a),P(b))$. Let $V(\alpha)$ be the representation of Δ which first of all coincides on the support of $P(b)$ with $P(b)$, on the support of $Q(a)$ with $Q(a)$, which second has $V(\alpha)_\alpha = 1_k$, and finally, which elsewhere is given by zero spaces and zero maps. Then there is a non-split exact sequence

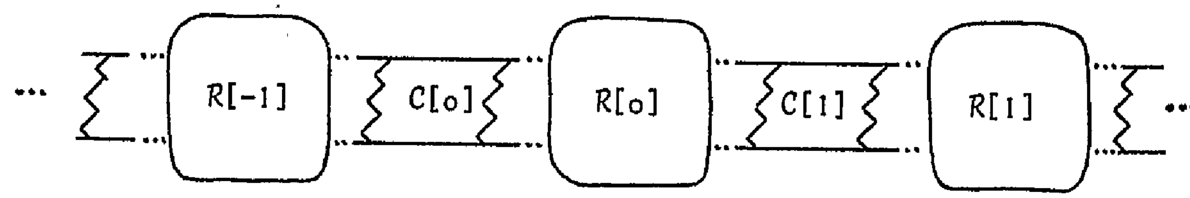
$$o \longrightarrow P(b) \longrightarrow V(\alpha) \longrightarrow Q(a) \longrightarrow o$$

and this, in fact, corresponds to an irreducible morphism $Q(a)[n] \longrightarrow P(b)[n+1]$, for every $n \in \mathbb{Z}$. Also, for every $n \in \mathbb{Z}$, and any vertex c of Δ, there is a triangle of the form $(Q(c)[n],Y,P(c)[n+1],u_n,v_n,w[n+1])$, where $w \in \mathrm{Hom}_A(P(c),Q(c))$ is a non-zero map, and one easily checks that this indeed is an Auslander-Reiten triangle, and that $Y = (\bigoplus_{a\to c} Q(a)) \oplus (\bigoplus_{c\to b} P(b))$. This shows that the additional arrows which we need in order to connect the various $\Gamma_A[n]$, are of the form $[Q(a)[n]] \longrightarrow [P(b)[n+1]]$, one for each arrow $a \longrightarrow b$, and that $\tau[P(c)[n+1]] = [Q(c)[n]]$. In case Δ is of Dynkin type, so that Γ_A is finite, the various copies $\Gamma_A[n]$ all are connected together, and we obtain just $\mathbb{Z}\Delta^*$ (and, actually, $D^b(A)$ is equivalent, as a category, to $k(\mathbb{Z}\Delta^*)$). For example, for the following quiver Δ of type $\mathbb{E}_6$, and $A = k\Delta^*$, the copies $\Gamma_A[-1]$, $\Gamma_A[o]$ and $\Gamma_A[1]$ are connected as indicated:



Δ $\Gamma_A[-1]$ $\Gamma_A[o]$ $\Gamma_A[1]$

In case Δ is connected and not of Dynkin type, the preinjective component of $\Gamma_A[n-1]$ is connected with the preprojective component $\Gamma_A[n]$ and together they form a component of the form $\mathbb{Z}\Delta^*$. We denote by $\mathcal{C}[n]$ the full subcategory whose indecomposable objects are those in the preinjective component of $\Gamma_A[n-1]$ and those in the preprojective component of $\Gamma_A[n]$, (then $\mathcal{C}[n]$ is actually equivalent, as a category, to $k(\mathbb{Z}\Delta^*)$), and by $\mathcal{R}[n]$ the image of the full subcategory of all regular A-modules under $-[n]$. With these notations, $D^b(A)$ can be visualised as follows:



Note that if X belongs to $\mathcal{C}[n]$, and Y is indecomposable with $\mathrm{Hom}(X,Y) \neq o$, then Y belongs to $\mathcal{C}[n]$ or $\mathcal{R}[n]$ or $\mathcal{C}[n+1]$. Similarly, if X belongs to $\mathcal{R}[n]$ and Y is indecomposable with $\mathrm{Hom}(X,Y) \neq o$, then Y belongs to $\mathcal{R}[n]$ or $\mathcal{C}[n+1]$ or $\mathcal{R}[n+1]$.

Given any finite dimensional algebra A, there are two triangulated categories obtained from A-mod, namely $\hat{A}$-mod and $D^b(A)$. In case A has finite global dimension, there is the following theorem of Happel which asserts that these categories are equivalent, not only as categories, but even as triangulated categories. [Given two triangulated categories $\mathcal{A},\mathcal{B}$, an equivalence of triangulated categories is given by an equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ of categories and an equivalence $\eta : FT \longrightarrow TF$ of functors such that for any triangle (X,Y,Z,u,v,w) in $\mathcal{A}$, the sextuple $(FX,FY,FZ,Fu,Fv,Fw\cdot\eta_X)$ is a triangle in $\mathcal{B}$].

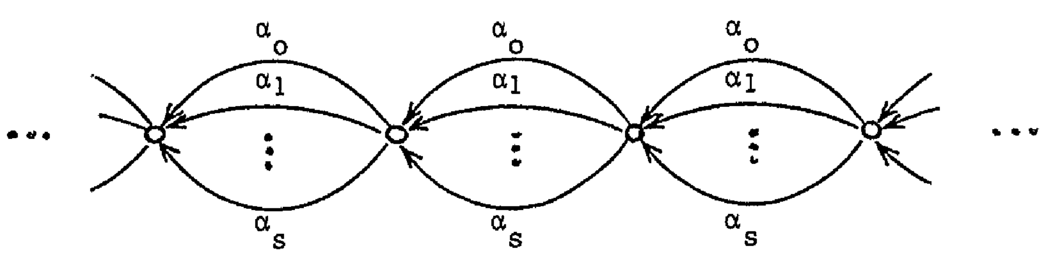
<u>Theorem</u> (Happel). Assume A is a finite dimensional algebra of finite global dimension. Then $\hat{A}$-mod and $D^b(A)$ are equivalent as triangulated categories.

The use of this theorem is twofold: if interested in $\hat{A}$-mod or in T(A)-mod, we may use it in order to transform information on $D^b(A)$ to $\hat{A}$-mod, thus the report above on the structure of $D^b(A)$ for A being hereditary yields the same information for $\hat{A}$-mod. Similarly, if we are interested in $D^b(A)$, we may work instead with the categories $\hat{A}$-mod and $\hat{A}$-mod; for example, for A a canonical algebra, our investigation of $\hat{A}$-mod yields a clear description of $D^b(A)$ in this case. Note that one advantage of considering $\hat{A}$-mod instead of $D^b(A)$ lies in the fact that

$\hat{A}$-$\underline{\text{mod}}$ is obtained from the category $\hat{A}$-mod by a rather easy modification, and $\hat{A}$-mod is abelian!

We should add the following warning: there are given countably many embeddings of A-mod both into $\hat{A}$-$\underline{\text{mod}}$ (denoted by $-\langle n\rangle$, $n \in \mathbb{Z}$) and into $D^b(A)$ (denoted by $-[n]$, $n \in \mathbb{Z}$, and Happel's equivalence $D^b(A) \longrightarrow \hat{A}$-$\underline{\text{mod}}$ is constructed by first identifying the subcategory $(A\text{-mod})[o]$ of $D^b(A)$ with the subcategory $(A\text{-mod})\langle o\rangle$ of $\hat{A}$-$\underline{\text{mod}}$ and extending this to an equivalence $D^b(A)$ and $\hat{A}$-$\underline{\text{mod}}$. We should warn that the remaining full subcategories $(A\text{-mod})[n]$ and $(A\text{-mod})\langle n\rangle$, $n \neq o$ will <u>not</u> coincide, in general. In fact, the two shift functors ν and T (which are both defined on $\hat{A}$-mod) are related by the Auslander-Reiten translation τ: we have $\nu = T^2\tau$.

A special case of Happel's theorem has been known before and is of particular interest. Investigations by Beilinson [Bei] and Bernstein-Gelfand-Gelfand [BGG] have related the category $V = V_n$ of vector bundles over the projective space $\mathbb{P}_n k$ to problems in linear algebra. Both papers provide a description of the derived category $D^b(\mathrm{Coh}\ \mathbb{P}_n k)$ of bounded complexes of coherent sheaves over $\mathbb{P}_n k$, and the theorem above yields a direct interrelation between these two descriptions. Consider the following quiver $\tilde{\Delta}$



with set of vertices the integers $\mathbb{Z}$; for any vertex $a \in \mathbb{Z}$, there are $n+1$ arrows $\alpha_s = \alpha_s^{(a)} : a \longrightarrow a+1$, $o \leq s \leq n$. Denote $\tilde{\Lambda}$ the opposite of the path algebra of $\tilde{\Delta}$ over the field k modulo the relations

$$\alpha_s\alpha_s = o, \quad \alpha_s\alpha_t + \alpha_t\alpha_s = o, \quad \text{for all } s,t \text{ with } s < t.$$

Note that $\tilde{\Lambda}$-mod is just the category Λ-grmod of graded Λ-modules, where Λ is the exterior algebra on the vectorspace $\Lambda_1 = k^{n+1}$, with the usual grading, thus $\Lambda_i = \Lambda^i(k^{n+1})$ [one may observe that $\tilde{\Lambda}$ is a Galois covering of Λ; on $\tilde{\Lambda}$ there is an obvious $\mathbb{Z}$-action by shifting the quiver, and $\Lambda = \tilde{\Lambda}/\mathbb{Z}$]. It is easy to see that $\tilde{\Lambda}$ is a Frobenius algebra. Indeed, we have $P_{\tilde{\Lambda}}(a) = Q_{\tilde{\Lambda}}(a-n-1)$, for any $a \in \mathbb{Z}$.

Given $a \leq b$ in $\mathbb{Z}$, denote by $\tilde{\Lambda}_{ab}$ the restriction of $\tilde{\Lambda}$ to the full subquiver of $\tilde{\Delta}$ with vertices x satisfying $a \leq x \leq b$, and let $A = \tilde{\Lambda}_{on}$. We claim that $\hat{A} = \tilde{\Lambda}$. Note that A has a unique simple projective module, namely $P_A(o)$, and we may identify the one-point extension $A[Q_A(o)]$ with $\tilde{\Lambda}_{o,n+1}$. Inductively, we see that $\tilde{\Lambda}_{ob}$, for $b > n$, is obtained from $\tilde{\Lambda}_{o,b-1}$ as one-point extension, using the indecomposable injective $\tilde{\Lambda}_{o,b-1}$-module with socle at $b-n-1$. The dual process of forming successively one-point coextensions finally shows that $\hat{A} = \tilde{\Lambda}$.
The category

$$\Lambda\text{-grmod} = \tilde{\Lambda}\text{-mod} = \hat{A}\text{-mod}$$

is used by Bernstein-Gelfand-Gelfand in order to describe $D^b(\mathrm{Coh}\ \mathbb{P}_n k)$ they construct an equivalence

$$\hat{A}\text{-}\underline{\mathrm{mod}} \approx D^b(\mathrm{Coh}\ \mathbb{P}_n k).$$

On the other hand, the description of $D^b(\mathrm{Coh}\ \mathbb{P}_n k)$ given by Beilinson is

$$D^b(\mathrm{Coh}\ \mathbb{P}_n k) \approx D^b(A).$$

[The actual statement of Beilinson is $D^b(\mathrm{Coh}\ \mathbb{P}_n k) \approx K^b(\mathcal{A})$, where $\mathcal{A}$ is the additive category of all finite direct sums of copies $P_{\tilde{\Lambda}}(a)$, $o \leq a \leq n$, but, of course, $\mathcal{A} \approx A\text{-proj}$, thus $K^b(\mathcal{A}) \approx K^b(A\text{-proj}) \approx D^b(A)$.] Combining both assertions, we obtain the equivalence $\hat{A}\text{-}\underline{\mathrm{mod}} \approx D^b(A)$ for this special A.

We return now to the consideration of tilting modules.

<u>Theorem</u> (Happel). Let A be a finite-dimensional algebra of finite global dimension, and ${}_AT$ a tilting module, with $B = \mathrm{End}({}_AT)$. Then $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories.

An equivalence $D^b(A) \longrightarrow D^b(B)$ is obtained as follows: the functor $F = \mathrm{Hom}_A({}_AT_B,-) : A\text{-mod} \longrightarrow B\text{-mod}$ is left exact, thus there is the right derived functor $\underline{\underline{R}}F : D^b(A) \longrightarrow D^b(B)$ [on the full and dense subcategory $K^b(A\text{-inj})$ of $D^b(A)$, the functor $\underline{\underline{R}}F$ is defined as follows: $I^\cdot \in K^b(A\text{-inj})$ is sent to the complex $(\underline{\underline{R}}F)(I^\cdot)$ with $((\underline{\underline{R}}F)(I^\cdot))^i = FI^i$], and $\underline{\underline{R}}F$ is an equivalence of triangulated categories.

Actually, Happel [Ha] shows that one may consider instead of tilting modules more generally modules ${}_AT$ of finite projective dimension such that $\mathrm{Ext}^i({}_AT, {}_AT) = o$ for all $i \geq 1$, and such that the T-codimension of ${}_AA$ is finite, and $B = \mathrm{End}({}_AT)$. Note that Miyashita [M] recently has shown that under these assumptions also B is of finite global dimension and that T_B satisfies the corresponding conditions (as a right B-module). [In by-passing, we should stress that the investigations of Miyashita are not restricted to finite-dimensional algebras, but to arbitrary rings (with 1). In this way, the generalized tilting theory established by him also incorporates (as case $r = o$) the Morita equivalence for arbitrary rings].

Combining the two theorems of Happel, we know the structure of $D^b(A)$ for quite a number of different algebras. First of all, if A is a tilted algebra, say $A = \mathrm{End}({}_{k\Delta}T)$, where Δ is a finite quiver without oriented cycles, and ${}_{k\Delta}T$ a tilting module, then $D^b(A) \approx D^b(k\Delta)$, and we have outlined above the structure of $D^b(k\Delta)$. If A is the endomorphism ring of a tilting module over a canonical algebra B, then $D^b(A) \approx D^b(B)$, and again we know much about the structure of $D^b(B)$, this time using the equivalence $D^b(B) \approx \hat{B}$-mod. In case $C = C(\lambda, p)$ is a canonical algebra, where $p = (p_o, \ldots, p_t)$ gives rise to a Dynkin graph $\Delta = \mathbb{T}_{p_o, \ldots, p_t}$, then our description above of $D^b(C)$ coincides with the description of $D^b(k\tilde{\Delta})$. This is now no longer surprising, since one easily shows that C is, in fact, a tilted algebra, which has a slice module with endomorphism ring $k\tilde{\Delta}$.

Let us demonstrate in which way the concepts presented in this lecture can be used in order to obtain insight into tilted algebras. We want to outline the proof of a result announced in the second lecture: that an algebra with slices in two different components has to be concealed. Thus, let A be a (necessarily connected) algebra with slices S and S', which belong to different components of A-mod. Let ${}_AS$ be a slice module for S and $k\Delta = \mathrm{End}({}_AS)$. Thus, $D^b(A)$ and $D^b(k\Delta)$ are equivalent categories and we may use some fixed identification. Also, we may identify $D^b(A)$ and $\hat{A}$-mod, identifying A-mod both with A-mod[o] $\subseteq D^b(A)$ and with A-mod<o> $\subseteq \hat{A}$-mod. We know that S belongs to some component of $\hat{A}$-mod of the form $\mathbb{Z}\Delta$, thus without loss of generality to C[o]. Similarly, S' belongs to some C[m], and since S and S' belong to different components of A-mod, one easily sees that we must have $m \neq o$. Now, $\mathcal{S}$ is a separating subcategory of A-mod, say separating $\mathcal{P}$ from $\mathcal{Q}$. Let S_1', be an indecomposable module in $\mathcal{S}'$. Assume S_1' belongs to $\mathcal{Q}$, then clearly $\mathrm{Hom}(\mathcal{S},S_1') \neq o$. [For, take an indecomposable projective A-module P with $\mathrm{Hom}(P,S_1') \neq o$, then P in $\mathcal{P}$ or $\mathcal{S}$, and factor some non-trivial map from P to S_1' through $\mathcal{S}$]. In this case we must have $m = 1$. Similarly, if S_1' belongs to $\mathcal{P}$, then $m = -1$. Without loss of generality, we assume $m = 1$. Since $\mathcal{S}$ is sincere, any indecomposable projective A-module belongs to $\mathcal{C}[-1]$ or $\mathcal{R}[-1]$ or $\mathcal{C}[o]$, since $\mathcal{S}'$ is sincere, any indecomposable projective A-module belongs to $\mathcal{C}[o]$ or $\mathcal{R}[o]$ or $\mathcal{C}[1]$, thus the indecomposable projective A-modules all lie in $\mathcal{C}[o]$. Now, $\mathcal{C}[o]$ is of the from $k(\mathbb{Z}\Delta)$, thus we see that A can be written as the endomorphism ring of a preprojective tilting $k\Delta$-module. This shows that A is a concealed algebra.

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