HAMMOCKS

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Introduction

Hammocks have been considered by Brenner [5] in order to give a numerical criterion for a finite translation quiver to be the Auslander–Reiten quiver of some representation-finite algebra over an algebraically closed field. The main purpose of the present paper is to show that the hammocks considered by Brenner are of the form $\Gamma\ell(S)$, where $\ell(S)$ is the category of S-spaces of a finite partially ordered set S and $\Gamma\ell(S)$ is its Auslander–Reiten quiver. This result may be interpreted in the following way: given any representation-finite algebra A over an algebraically closed field and any simple A-module E, there is defined a finite partially ordered set S_E such that $\ell(S_E)$ yields a rather large amount of information concerning the indecomposable A-modules having composition factors of the form E. Our Theorem 2 provides an axiomatic description of the translation quivers $\Gamma_\infty\ell(S)$, where ${}_\infty\ell(S)$ is the preprojective component of some $\ell(S)$, with S an arbitrary finite partially ordered set, as the 'thin left hammocks with finitely many projective vertices'. The left hammocks are introduced in § 1. Our Theorem 3 gives an intrinsic characterization of these left hammocks. The left hammocks are suitable translation quivers with a uniquely determined additive function, the so-called hammock function. Theorem 1 is concerned with the growth of the hammock function. The power of Theorem 1 can be seen by the large number of consequences exhibited in § 1. Finite left hammocks are called hammocks, this generalizes the concept as considered by Brenner.

In § 1, the three main results of the paper, Theorems 1, 2, and 3, are stated and several consequences are derived. The proofs of the main theorems are given in §§ 2–7. In §§ 9 and 10, we give some applications of these results to the representation theory of finite-dimensional algebras. We should stress that the assertion of the first main theorem is purely combinatorial; however our proof uses some representation theory. A consequence of this combinatorial result is an estimate of multiplicities of Jordan–Hölder factors which is due to v. Höhne. This estimate plays an important role in our investigation, and the appendix gives an alternative proof. We will use the terminology of [12]. The authors are indebted to S. Brenner for helpful advice in the preparation of the paper.

1. *The main results, and some consequences*

Let $H = (H_0, H_1, \tau)$ be a proper translation quiver. Note that we allow multiple arrows, with the number of arrows from x to y denoted by $m(y, x)$. The function $m(x, -)$ on H_0 will be denoted by $x^{(+)}$, the function $m(-, y)$ by $y^{(-)}$. Note that x^+ is the support of $x^{(+)}$, and y^- the support of $y^{(-)}$. If z is not projective, then $z^{(-)} = (\tau z)^{(+)}$.

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Given functions $f, g\colon H_0 \to \mathbb{Z}$, with the support of g finite, then we define

$$(\Sigma f)(g) = \sum_{x \in H_0} f(x)g(x).$$

In particular, for any $f\colon H_0 \to \mathbb{Z}$, and any $x \in H_0$, the numbers $(\Sigma f)(x^{(+)})$ and $(\Sigma f)(x^{(-)})$ are defined.

Also recall the inductively defined full subquivers ${}_dH$ of H. First of all, ${}_{-1}H$ is the empty quiver, and z belongs to ${}_dH$ if and only if $z^- \subseteq {}_{d-1}H$. Also, ${}_\infty H = \bigcup_{d \in \mathbb{N}} {}_dH$. Thus, for all $d \in \mathbb{N} \cup \{\infty\}$, we see that ${}_dH$ is a predecessor closed subquiver, and we may consider it as a translation quiver, using the restriction of τ.

Suppose H has a unique source ω, and $H = {}_\infty H$. Then we define $h_H\colon H_0 \to \mathbb{Z}$ inductively as follows. By abuse of notation, let $h_H(\tau x) = 0$ for x projective (note that in this case, τx is not defined!). Now, let

$$h_H(\omega) = 1$$

and, for $x \neq \omega$, with h_H already defined on all proper predecessors of x, let

$$h_H(x) = (\Sigma h_H)(x^{(-)}) - h_H(\tau x).$$

(If x is injective, so that $\tau^- x$ is not defined, we define $h_H(\tau^- x) = 0$, again abusing the notation!)

With these preparations, we are able to give the main definition: the translation quiver H is said to be a *left hammock* provided

(1) $H = {}_\infty H$,

(2) H has a unique source ω,

(3) h_H takes values in the set $\mathbb{N}_1$ of positive integers,

(4) if q is an injective vertex then

$$h_H(q) \geq (\Sigma h_H)(q^{(+)}).$$

When H is a left hammock, the function h_H is said to be its *hammock function.*

Let us mention some obvious properties of a left hammock H. Condition (1) implies that for any vertex x of H, there is a path starting at a source, and ending at x; thus, according to (2), the path starts at ω. It follows that a left hammock is connected. Also, for any $d \in \mathbb{N}$, the subquiver ${}_dH$ is finite. Our first result is concerned with the growth of the hammock function along a τ-orbit.

THEOREM 1. *Let H be a left hammock. Then, for any $x \in H_0$,*

$$h_H(x) - 1 \leq h_H(\tau^- x).$$

The proof of the three main theorems (Theorems 1, 2, 3) will be given in the next sections. Here, we are going to derive some corollaries. There are several consequences of Theorem 1.

COROLLARY 1. *Let q be an injective vertex of a left hammock. Then $h_H(q) = 1$. As a consequence, there is at most one arrow starting at q, and if $q \to y$, then also $h_H(y) = 1$. Also, $|q^-| \leq 3$, and if $q \to y$, then $|y^-| \leq 3$.*

Proof. By definition, $h_H(\tau^- q) = 0$. Since $h_H(q) \geq 1$, the theorem implies that

$h_H(q)=1$. Again using Condition (3), now for the elements in q^+, and Condition (4), we see that $|q^+|\leq 1$, and that for $q^+=\{y\}$, we have both $m(q,y)=1$ and $h_H(y)=1$. Any vertex x with $h_H(x)=1$ satisfies $|x^-|\leq 3$. If x is projective, then in fact $|x^-|\leq 1$. Otherwise, $h_H(\tau x)\leq 2$ by the theorem; therefore $|x^-|\leq 3$.

A left hammock H will be said to be *thin* provided $h_H(p)=1$ for any projective vertex p of H. If H is a thin left hammock, and p is a projective vertex of H, then there is at most one arrow ending in p, and if $y\to p$, then also $h_H(y)=1$. (For the proof, we use Condition (3) and the defining equality $h_H(p)=(\Sigma h_H)(p^{(-)})$.) As a consequence, a thin left hammock has no multiple arrows. (For the proof, we only have to observe that the τ-orbit of any vertex contains a projective vertex, since $H={}_\infty H$.) Also note that a thin left hammock is always 'simply connected'.

COROLLARY 2. *Let H be a finite left hammock. Then the opposite translation quiver H^* is also a left hammock, both H and H^* are thin, and the hammock functions on $H_0=H_0^*$ coincide.*

Proof. Condition (1) for H together with the finiteness show that H^* also satisfies (1). According to Corollary 1, given an injective vertex of H, there is at most one arrow starting in it. Thus, given a projective vertex of H^*, there is at most one arrow ending in it. Since H^* is connected, it follows easily that H^* has a unique source, say ω'. Later we will consider h_H as a function on H_0^* $(=H_0)$. In preparation, we note the following: if x is an injective vertex of H, or the immediate successor of an injective vertex of H, then $h_H(x)=1$, according to Corollary 1. Also, we again use the fact that an injective vertex $x\neq\omega'$ of H is the starting point for precisely one arrow. Thus, for $x\neq\omega'$ and x injective, we have

$$h_H(x)=(\Sigma h_H)(x^{(+)}),$$

and

$$h_H(\omega')=1.$$

If x is not injective in H, say $x=\tau z$, then the defining condition for h_H,

$$h_H(z)=(\Sigma h_H)(z^{(-)})-h_H(\tau z),$$

can be rewritten in the form

$$h_H(x)=(\Sigma h_H)(x^{(+)})-h_H(\tau^- x).$$

Altogether, we see that h_H, as a function on H_0^*, satisfies the defining conditions for h_{H^*}; thus $h_{H^*}(x)=h_H(x)$ for all $x\in H_0^*=H_0$. In particular, Condition (3) is satisfied for H^*, since it is satisfied for H. For a projective vertex $p\neq\omega$ of H, we have

$$h_H(p)=(\Sigma h_H)(p^{(-)}),$$

by definition of h_H. This shows that Condition (4) is satisfied for H^* and all injective vertices of H^* different from ω (even with equality). This condition is trivially satisfied for the sink ω of H^*. Thus, H^* is a left hammock. Of course, H^* is thin, since we know from Corollary 1 that $h_H(q)=1$ for any injective vertex q of H. We have shown that given a finite left hammock H, the opposite H^* is a thin left hammock. Now since H^* is again a finite left hammock, it follows that $H=H^{**}$ is thin.

Finite left hammocks will be called *hammocks*. Note that a left hammock H is a hammock if and only if H^* is also a left hammock. (One direction has been shown above. For the converse, we note the following. Let H be a left hammock. If we assume only that H has a sink, then H already has to be finite!)

COROLLARY 3. *Let H be a hammock, and x a vertex of H. Then $h_H(x)=1$ for x projective or injective. If x is not projective, then $|h_H(\tau x)-h_H(x)|\leqslant 1$. Always, $|x^+|\leqslant 3$, and $|x^-|\leqslant 3$.*

Proof. The first assertions are direct consequences of Theorem 1 and Corollary 1. In order to show that $|x^-|\leqslant 3$, we can assume that x or an element of x^- is injective, otherwise replace x by some τ^--translate. Now use Corollary 1.

There is a strong relationship between thin left hammocks and the representation theory of partially ordered sets. We will restrict to finite partially ordered sets (but one could equally well extend the following results to some kinds of noetherian partially ordered sets). Fix some field k. Given a partially ordered set S, an S-space $V=(V_\omega\,;V_s)_{s\in S}$ is given by a vector space V_ω over k, and subspaces V_s of V_ω, where $s\in S$, with $V_s\subseteq V_t$ for $s\leqslant t$. We call V_ω the total space of V, and denote its k-dimension by $\underline{\dim}_\omega V=\dim V_\omega$. Given two S-spaces V, W, a map $f\colon V\to W$ is given by a k-linear map $f_\omega\colon V_\omega\to W_\omega$ satisfying $(V_s)f_\omega\subseteq W_s$ for all $s\in S$; the induced map $V_s\to W_s$ will be denoted by f_s. The category of all S-spaces is denoted by $\mathfrak{L}(S,k)$. Let S be finite. Then we denote the full subcategory of $\mathfrak{L}(S,k)$ given by all S-spaces V with V_ω finite dimensional by $\ell(S,k)$. Note that $\ell(S,k)$ may be considered as an exact category, the exact sequences in $\ell(S,k)$ being of the form

$$0\longrightarrow V'\xrightarrow{f}V\xrightarrow{g}V''\longrightarrow 0$$

where f, g are maps of S-spaces, such that all the sequences

$$0\longrightarrow V'_s\xrightarrow{f_s}V_s\xrightarrow{g_s}V''_s\longrightarrow 0$$

with $s\in S\cup\{\omega\}$, are exact sequences of vector spaces. It is well known that $\ell(S,k)$ has Auslander–Reiten sequences, and $\Gamma\ell(S,k)$ denotes the corresponding Auslander–Reiten quiver.

Given a Krull–Schmidt category $\mathfrak{K}$, let us recall the definition of the full subcategories ${}_d\mathfrak{K}$. First of all, ${}_{-1}\mathfrak{K}$ contains only the zero object. Second, an indecomposable object X of $\mathfrak{K}$ belongs to ${}_d\mathfrak{K}$ if and only if any indecomposable object Y of $\mathfrak{K}$ with $\operatorname{rad}(Y,X)\neq 0$ belongs to ${}_{d-1}\mathfrak{K}$. Finally, ${}_\infty\mathfrak{K}=\bigcup_{d\in\mathbb{N}}{}_d\mathfrak{K}$.

THEOREM 2. *Let S be a finite partially ordered set, and k a field. Then ${}_\infty\ell(S,k)$ is closed under irreducible maps in $\ell(S,k)$, and $H(S):=\Gamma_\infty\ell(S,k)$ is a thin left hammock with finitely many projective vertices. The hammock function on $H(S)$ is $\underline{\dim}_\omega$. Also, $H(S)$ is independent of k.*

Conversely, given a thin left hammock H with n projective vertices, there exists a partially ordered set $S(H)$ with $n-1$ elements such that $H\approx H(S(H))$ as translation quivers, and ${}_\infty\ell(S(H),k)\approx k(H)$ as categories. If H is a hammock, $S(H)$ is (up to isomorphism) the only partially ordered set S with $H(S)\approx H$.

Here, $k(H)$ denotes the mesh category for H.

For hammocks, we can derive the following consequences. A partially ordered set S is said to be *representation-finite* provided there is only a finite number of isomorphism classes of indecomposable S-spaces with finite-dimensional total space. (Actually, for a representation-finite partially ordered set, the total space of any indecomposable S-space is finite-dimensional.)

COROLLARY 4. *The isomorphism classes of representation-finite partially ordered sets correspond bijectively to the isomorphism classes of hammocks, as follows: given the representation-finite partially ordered set S, then $H(S)=\Gamma\ell(S,k)$ is a hammock which is independent of the chosen field k.*

Proof. Besides the assertions of Theorem 2, we need the following two rather well-known facts (or see the remarks in § 7). First, if S is a partially ordered set with ${}_\infty\ell(S,k)$ finite, then ${}_\infty\ell(S,k)=\ell(S,k)$, and thus S is representation-finite. Second, if S is representation-finite, then always ${}_\infty\ell(S,k)=\ell(S,k)$.

COROLLARY 5. *Let k be any field. Let H be a hammock with source ω and sink ω'. Then $\mathrm{Hom}_{k(H)}(\omega,\omega')$ is a one-dimensional k-vector space, and, for any $x\in H$, the composition map*

$$\mathrm{Hom}_{k(H)}(\omega,x)\times\mathrm{Hom}_{k(H)}(x,\omega')\to\mathrm{Hom}_{k(H)}(\omega,\omega')$$

is a non-degenerate bilinear form.

Proof. According to Corollary 4, there is a representation-finite partially ordered set S with $H=H(S)$. Let $P(\omega)$, $Q(\omega')$ be the following S-spaces: $P(\omega)_\omega=k$, $P(\omega)_s=0$ for $s\in S$, and $Q(\omega')_\omega=Q(\omega')_s=k$ for all $s\in S$. Then $[P(\omega)]=\omega$, $[Q(\omega')]=\omega'$, as one verifies easily (or see the general discussion in § 7).

Fix a generator $\xi\in P(\omega)_\omega=k$. Given an S-space V, we can identify $\mathrm{Hom}(P(\omega),V)$ with V_ω, sending $f\in\mathrm{Hom}(P(\omega),V)$ to ξf_ω. Similarly, we identify $\mathrm{Hom}(V,Q(\omega'))$ with the dual space V_ω^*, by sending $g\in\mathrm{Hom}(V,Q(\omega'))$ to g_ω. Under these identifications, the composition map

$$\mathrm{Hom}(P(\omega),V)\times\mathrm{Hom}(V,Q(\omega'))\to\mathrm{Hom}(P(\omega),Q(\omega'))$$

just corresponds to the evaluation map $V_\omega\times V_\omega^*\to k$; thus it is a non-degenerate bilinear form. Since $\ell(S,k)\approx k(H)$, according to Theorem 2, we obtain the corresponding result for $k(H)$.

Famous theorems of Nazarova, Rojter, and Klejner assert that a finite partially ordered set S is representation-finite if and only if the dimension of the total spaces of the indecomposable S-spaces is bounded, and that in this case it is bounded by 6. Thus, we also have the following criterion:

COROLLARY 6. *Let H be a thin left hammock. Then H is a hammock if and only if h_H is bounded. If h_H is bounded, then it is bounded by* 6.

The main assertion of Theorem 2 is that it is possible to realize a thin left hammock in the form $\Gamma_\infty\ell(S,k)$. There is a general result providing a realization of any left hammock in the form $\Gamma_\infty\mathfrak{K}$, where $\mathfrak{K}$ is a Krull–Schmidt category such that all indecomposable objects in $_\infty\mathfrak{K}$ have sink maps and source maps (also called, respectively, minimal left and right almost split maps) in $_\infty\mathfrak{K}$. In order to be able to formulate it, we need some further definitions. Given a Krull–Schmidt category Λ, we denote by Λ-mod the category of finitely presented functors $\Lambda^{\mathrm{op}}\to Ab$ (Ab is the category of abelian groups), and Λ-spmod is the full subcategory of Λ-mod given by all functors $F\in\Lambda$-mod which have a projective socle. Recall that in order to define the mesh category $k(H)$ for a left hammock H, we have to choose a polarization σ of H (see [**12**]), but (as for a preprojective translation quiver) we obtain equivalent categories when choosing different polarizations.

THEOREM 3. *Let H be a left hammock with source ω, and k a field. Let $\mathfrak{P}(H, k)$ be the full additive subcategory of $k(H)$ whose indecomposable objects are just the projective vertices of H. The functor*

$$M\colon k(H)\to\mathfrak{P}(H,k)\text{-mod}$$

given by

$$M(x)=\mathrm{Hom}_{k(H)}(-,x)\mid\mathfrak{P}(H,k)$$

is a full embedding of $k(H)$ into $\mathfrak{P}(H, k)$-mod; *its image is $_\infty\mathfrak{F}$, with $\mathfrak{F}=\mathfrak{P}(H,k)$*-spmod. *There is a unique simple projective object in* $\mathfrak{P}(H, k)$-mod, *namely $M(\omega)$. Thus an object X of* $\mathfrak{P}(H, k)$-mod *belongs to $\mathfrak{F}$ if and only if its socle is generated by $M(\omega)$. Any indecomposable object X of $_\infty\mathfrak{F}$ has a source map $X\to X'$ and a sink map $'X\to X$ in $\mathfrak{F}$, and both X' and $'X$ again belong to $_\infty\mathfrak{F}$. We obtain an isomorphism*

$$H\to\Gamma_\infty\mathfrak{F}$$

of translation quivers, sending x to $[M(x)]$. Thus $\Gamma_\infty\mathfrak{F}$ is a hammock isomorphic to H, and the hammock function on $\Gamma_\infty\mathfrak{F}$ is $N\mapsto\dim N(\omega)=\dim\mathrm{Hom}(M(\omega),N)$.

When the left hammock H has only finitely many projective vertices, $\mathfrak{P}(H, k)$ is a finite k-category; therefore $\mathfrak{P}(H,k)\text{-mod}\approx A\text{-mod}$ for some finite-dimensional k-algebra A, and $\mathfrak{P}(H,k)\text{-spmod}\approx A\text{-spmod}$, the category of finitely generated A-modules with projective socle. It is well known that for any finite-dimensional algebra, the category of finitely generated modules with projective socle has source maps and sink maps. Thus Theorem 3 asserts that $H\approx\Gamma_\infty(A\text{-spmod})$ as translation quivers, and $k(H)\approx{}_\infty(A\text{-spmod})$ as categories.

The relationship between Theorems 2 and 3 is as follows. Given a finite partially ordered set S, let S^+ be the partially ordered set obtained from S by adding an element ω with $s<\omega$ for all $s\in S$. If H is a thin left hammock with finitely many projective vertices, then $\mathfrak{P}(H, k)$ is the incidence category of the partially ordered set $(S(H)^+)^*$, and $\mathfrak{P}(H,k)\text{-spmod}\approx\ell(S(H),k)$. This will be shown in § 7.

2. *Partial left hammocks*

A translation quiver H is said to be a *partial left hammock* provided

(1) $H={}_\infty H$,

(2) H has a unique source ω,

(3) h_H takes values in $\mathbb{N}_1$,

(4′) if q is an injective vertex, and

$$h_H(q) < (\Sigma h_H)(q^{(+)}),$$

then any path in H starting at q, is sectional.

As in the case of a hammock, the function h_H is called the hammock function of the partial left hammock H.

Given a partial left hammock H, any non-empty full translation subquiver H' which is closed under predecessors is also a partial left hammock, and $h_{H'}$ is the restriction of h_H to H'. In particular, the non-empty, predecessor closed, full translation subquivers of left hammocks are partial left hammocks. Also the converse is true, as we are going to show. If H is a left hammock, and H' a full translation subquiver which is closed under predecessors and contains all projective vertices of H, then H will be called a *completion* of H'.

PROPOSITION. *Any partial left hammock has a completion, and such a completion is unique up to isomorphism.*

Proof. Let H be a partial left hammock which is not a left hammock. Choose d minimal such that there is an injective vertex $q \in {}_dH$ with $h_H(q) < (\Sigma h_H)(q^{(+)})$. For any such q, we add a new vertex q' to H, and for any arrow $q \to y$ an arrow $y \to q'$ (note that always $q^+ \neq \varnothing$), and define $\tau q' = q$. We denote the translation quiver obtained in this way by H'; it contains H as a predecessor closed full translation subquiver, and the vertices of H' which do not belong to H are sinks. Since ${}_\infty H = H$, and since q^+ is finite, for any q, we see that ${}_\infty H' = H'$. Also, ω is the only source of H'. Thus Conditions (1) and (2) are satisfied for H'. By definition of $h_{H'}$, its restriction to H will be h_H, since the new vertices are sinks, and for each new vertex q', with $\tau q' = q$, we have

$$\begin{aligned} h_{H'}(q') &= (\Sigma h_{H'})((q')^{(-)}) - h_{H'}(\tau q') \\ &= (\Sigma h_H)(q^{(+)}) - h_H(q) > 0. \end{aligned}$$

Thus Condition (3) is also satisfied for H'. It remains to consider (4′). The new vertices q' are sinks; thus $h_{H'}(q') > 0 = (\Sigma h_{H'})((q')^{(+)})$, trivially. Let x be an injective vertex of H' such that $h_{H'}(x) < (\Sigma h_{H'})(x^{(+)})$, and assume there is a non-sectional path

$$x = x_0 \to x_1 \to \ldots \to x_n$$

in H', say with $\tau x_{i+1} = x_{i-1}$ for some $1 \leqslant i < n$; in particular, $n \geqslant 2$. Note that either the whole path is inside H, or else x_n is a new vertex, whereas $x = x_0 \to x_1 \to \ldots \to x_{n-1}$ is in H. Thus, if every path in H starting in x is sectional, then x_n is a new vertex, $x_{n-2} = \tau x_n$, and x_{n-2} belongs to ${}_dH$. First, assume x^+ is contained in H, so that $h_H(x) \leqslant (\Sigma h_H)(x^{(+)})$. According to (4′) for H, every path in H starting at x is sectional. Hence $x_{n-2} \in {}_dH$, and is not injective in H'. This shows that $n \geqslant 3$, and therefore $x \in {}_{d-1}H$. But this contradicts the minimality of d. Thus, we see that x^+ contains some new vertex q', and therefore $q = \tau q' \to x$. But, using again Condition (4′) for H, we see that every path in H starting at x is sectional. Thus again x_{n-2} belongs to ${}_dH$, and therefore $q \to x$ shows $q \in {}_{d-1}H$; again we have a contradiction.

The partial left hammock H' obtained in this way satisfies $h_{H'}(q) \geqslant (\Sigma h_{H'})(q^{(+)})$ for every injective vertex $q \in {}_dH'$. Iterating this construction, we obtain a left hammock $\bar{H}$ which is a completion of H. On the other hand, given any completion $\bar{\bar{H}}$ of H, we can extend the identity map on H to an embedding of H' into $\bar{\bar{H}}$ (this extension is uniquely defined on vertices; there are choices in the case of multiple arrows), and therefore to an embedding φ of $\bar{H}$ into $\bar{\bar{H}}$. But clearly, φ also will be surjective. This shows the uniqueness of $\bar{H}$.

3. *Relative Auslander–Reiten sequences*

Let k be a field, and A a finite-dimensional k-algebra. Let $(\mathfrak{F}, \mathfrak{T})$ be a torsion pair in A-mod, with $\mathfrak{F}$ the class of torsion-free modules, $\mathfrak{T}$ the class of torsion modules. Given an A-module M, let $M_{\mathfrak{T}}$ be its maximal torsion submodule, $M_{\mathfrak{F}} = M/M_{\mathfrak{T}}$ its maximal torsion-free factor module; denote by $\pi_M\colon M \to M_{\mathfrak{F}}$ the canonical projection. Assume now that M is an indecomposable A-module belonging to $\mathfrak{F}$.

(1) If $f_M\colon M \to M'$ is the source map for M in A-mod, then

$$f_M^{\mathfrak{F}} = f_M \pi_{M'}\colon M \to M'_{\mathfrak{F}}$$

is the (relative) source map for M in $\mathfrak{F}$.

(2) If $\mathrm{Ext}^1(\mathfrak{F}, M) = 0$, then $f_M^{\mathfrak{F}}$ is surjective, and $(\tau^- M)_{\mathfrak{F}} = 0$.

(3) Assume $\mathrm{Ext}^1(\mathfrak{F}, M) \neq 0$. Then $f_M^{\mathfrak{F}}$ is injective, its cokernel is $(\tau^- M)_{\mathfrak{F}}$, and there is the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f_M} & M' & \longrightarrow & \tau^- M & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \pi_{M'}} & & \downarrow{\scriptstyle \pi_{\tau^- M}} & & \\
0 & \longrightarrow & M & \xrightarrow[f_M^{\mathfrak{F}}]{} & M'_{\mathfrak{F}} & \longrightarrow & (\tau^- M)_{\mathfrak{F}} & \longrightarrow & 0
\end{array}$$

and the lower sequence is a (relative) Auslander–Reiten sequence in $\mathfrak{F}$. Thus we denote $(\tau^- M)_{\mathfrak{F}}$ by $\tau_{\mathfrak{F}}^- M$.

For a proof, we refer to Hoshino [**9**].

Next, let us consider special torsion pairs. Let P be a simple projective A-module. Let $\mathfrak{F}_P$ be the set of modules with socle generated by P, and $\mathfrak{T}_P$ the set of modules X with $\mathrm{Hom}(P, X) = 0$. Then $(\mathfrak{F}_P, \mathfrak{T}_P)$ is a torsion pair. We know from above that $\mathfrak{F}_P$ has source maps.

(4) $\mathfrak{F}_P$ has sink maps.

For a proof, we may refer to Simson [**13**]. The argument is as follows. We may assume that A is basic. Let $P = Ae$ for some idempotent e, let $B = eAe$, $N = eA(1-e)$, $C = (1-e)A(1-e)$. Then N is a $B-C$-bimodule, B a division ring, and A-mod can be identified with the category $\mathfrak{L}(N) = \mathfrak{L}({}_BN_C)$ of representations of the bimodule ${}_BN_C$, a representation being a triple $X = (X_1, X_2, \gamma_X)$, where X_1 is a B-module, X_2 a C-module, and $\gamma_X\colon N \otimes_C X_2 \to X_1$ is B-linear. Note that X belongs to $\mathfrak{F}_P$ if and only if the adjoint map

$$\bar{\gamma}_X\colon X_2 \to \mathrm{Hom}_B(N, X_1) \approx N^* \otimes_B X_1$$

is a monomorphism. There is the functor $F\colon \mathfrak{L}(N) \to \mathfrak{L}(N^*)$ sending (X_1, X_2, γ_X)

to (X_2', X_1, γ_X'), where $\gamma_X'\colon N^* \otimes_B X_1 \to X_2'$ is the cokernel of $\bar{\gamma}_X$, and its restriction to $\mathfrak{F}_P$ is an equivalence from $\mathfrak{F}_P$ onto the image of F. This image is the category of torsion modules for some torsion pair in $\mathfrak{L}(N^*)$; therefore it has sink maps. Thus $\mathfrak{F}_P$ has sink maps.

(5) Assume that ${}_AA$ belongs to $\mathfrak{F}_P$. If Z is an indecomposable non-projective module in $\mathfrak{F}_P$, then there exists a (relative) Auslander–Reiten sequence in $\mathfrak{F}_P$,

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

and we write $X = \tau_{\mathfrak{F}} Z$.

Proof. Let $g\colon Y \to Z$ be a sink map for Z in $\mathfrak{F}_P$. Let $\varphi\colon P \to Z$ be a projective cover of Z. Since Z is not projective, there is $\varphi'\colon P \to Y$ with $\varphi = \varphi' g$, and thus g is surjective. Let $f\colon X \to Y$ be the kernel of g. Since $\mathfrak{F}_P$ is closed under submodules, X belongs to $\mathfrak{F}_P$. Thus we have constructed an exact sequence with all terms in $\mathfrak{F}_P$, and the right map g being a sink map. It follows from [**12**, 2.3.2] that f is a source map.

4. *Mesh categories*

Let Γ be a translation quiver, and k a field. We fix a polarization σ, and denote the mesh category $k(\Gamma, \sigma)$ just be $k(\Gamma)$. Recall that we denote by $s\alpha$ the starting point, by $e\alpha$ the endpoint of an arrow α. Given a path $w = (x \mid \alpha_1, \ldots, \alpha_l \mid y)$ in Γ, we denote its residue class in $k(\Gamma)$ by $\bar{w}$; a path $(x \mid \alpha \mid y)$ of length 1 will just be identified with the corresponding arrow α. Given a vertex x, we denote by f_x and g_x the maps

$$f_x = (\bar{\beta})_\beta\colon x \to \bigoplus_{\substack{\beta \\ s\beta = x}} e\beta, \quad g_x = (\bar{\alpha})_\alpha\colon \bigoplus_{\substack{\alpha \\ e\alpha = x}} s\alpha \to x$$

in $k(\Gamma)$; actually, in case x is not injective, it will be more convenient to use as index set for f_x the set of all $\sigma\alpha$, where $e\alpha = \tau^- x$. Thus

$$f_x = (\overline{\sigma\alpha})_\alpha\colon x \to \bigoplus_{\substack{\alpha \\ e\alpha = \tau^- x}} s\alpha.$$

In this way, we have $f_{\tau z} g_z = 0$, for any non-projective vertex z.

We denote by $\mathfrak{P}(\Gamma, k)$ the full subcategory of $k(\Gamma)$ with objects the direct sums of projective vertices of Γ. This is a Krull–Schmidt category. We consider the functor category $\mathfrak{P}(\Gamma, k)$-mod (given by all finitely presented functors $\mathfrak{P}(\Gamma, k)^{\mathrm{op}} \to Ab$). Let us assume from now on that $\Gamma = {}_\infty\Gamma$. In this case, given $x \in \Gamma$, the functor

$$M(x) = \mathrm{Hom}_{k(\Gamma)}(-, x) \mid \mathfrak{P}(\Gamma, k)$$

is obviously of finite length. In particular, the indecomposable projective objects in $\mathfrak{P}(\Gamma, k)$-mod are of finite length, since they are of the form $M(p)$, with p a projective vertex of Γ. Consequently $\mathfrak{P}(\Gamma, k)$-mod is the category of all finitely generated functors $\mathfrak{P}(\Gamma, k)^{\mathrm{op}} \to Ab$, and all $M(x)$, for $x \in \Gamma_0$, belong to $\mathfrak{P}(\Gamma, k)$-mod. (However note that the functor $M\colon k(\Gamma) \to \mathfrak{P}(\Gamma, k)$-mod is not necessarily fully faithful: for example, take $\Gamma = \mathbb{N}\Delta$, with Δ an oriented Dynkin diagram.) Given a projective vertex p of Γ, we denote by $E(p)$ the corresponding simple object of $\mathfrak{P}(\Gamma, k)$-mod; it has $M(p)$ as projective cover.

There are the following exact sequences:

(1) If p is a projective vertex of Γ, there is the exact sequence

$$0 \longrightarrow \bigoplus_{\substack{\alpha \\ e\alpha=p}} M(s\alpha) \xrightarrow{M(g_p)} M(p) \longrightarrow E(p) \longrightarrow 0.$$

(2) If z is a non-projective vertex of Γ, there is the exact sequence

$$M(\tau z) \xrightarrow{M(f_{\tau z})} \bigoplus_{\substack{\alpha \\ e\alpha=z}} M(s\alpha) \xrightarrow{M(g_z)} M(z) \longrightarrow 0,$$

but in general, the map $M(f_{\tau z})$ does not have to be a monomorphism.

Proof. This is a modification of a rather well-known result presented in [4] as Lemma 2.6. Note that our translation quivers may have multiple arrows and that we do not assume that Γ is locally bounded (and, in the main applications, it will not be). However, the fact that Γ is locally bounded is used in [4] only in order to ensure that the functors $\mathrm{Hom}_{k(\Gamma)}(-, x)$ are of finite length, and this is the case under our assumption $\Gamma = {}_\infty\Gamma$. The minimal projective resolutions in $k(\Gamma)$-mod,

$$0 \longrightarrow \bigoplus_{\substack{\alpha \\ e\alpha=p}} \mathrm{Hom}_{k(\Gamma)}(-, s\alpha) \xrightarrow{\mathrm{Hom}(-,g_p)} \mathrm{Hom}_{k(\Gamma)}(-, p) \longrightarrow E(p) \longrightarrow 0$$

for p a projective vertex, and

$$\mathrm{Hom}_{k(\Gamma)}(-, \tau z) \xrightarrow{\mathrm{Hom}(-,f_{\tau z})} \bigoplus_{\substack{\alpha \\ e\alpha=z}} \mathrm{Hom}_{k(\Gamma)}(-, s\alpha) \xrightarrow{\mathrm{Hom}(-,g_z)} \mathrm{Hom}_{k(\Gamma)}(-, z) \longrightarrow E(z) \longrightarrow 0$$

for z a non-projective vertex (with $E(z)$ the corresponding simple functor), remain exact sequences, when restricted to $\mathfrak{P}(\Gamma, k)$. Since for non-projective z, the restriction of $E(z)$ to $\mathfrak{P}(\Gamma, k)$ is zero, we obtain the exact sequences asserted above.

5. *Realization of left hammocks* (*proof of Theorem* 3)

Let H be a left hammock with source ω, and let k be a field. For any $d \in \mathbb{N}$, we will consider the full translation subquiver ${}_dH$ of H, and we note that $k({}_dH)$ is just the full subcategory of $k(H)$ given by the vertices which belong to ${}_dH$.

Assume first that H contains only finitely many, say n, projective vertices. Thus $\mathfrak{P}(H, k)$-mod is equivalent to A-mod for some finite-dimensional k-algebra A. We therefore call the elements in $\mathfrak{P}(H, k)$-mod (they are functors) modules. Since ω is a source, $M(\omega)$ is simple projective, and it will turn out that this is the only simple projective module. Let $\mathfrak{F}$ be the set of modules with socle generated by $M(\omega)$, and $\mathfrak{T}$ the set of modules X with $\mathrm{Hom}(M(\omega), X) = 0$. Then, $(\mathfrak{F}, \mathfrak{T})$ is a torsion pair, and we know from § 3 that $\mathfrak{F}$ has both source maps and sink maps.

By induction first on n, and then on d, we are going to show the following.

(a) If p is a projective vertex, then $M(p)$ is indecomposable, belongs to $\mathfrak{F}$, and satisfies $h_H(p) = \dim M(p)(\omega)$, and $M(g_p)$ is a sink map.

(b) Let x be a vertex in ${}_dH$. Then $M(f_x)$ is a source map in $\mathfrak{F}$. If x is injective, then the cokernel of $M(f_x)$ is in $\mathfrak{T}$, and $\tau_{\mathfrak{F}}^{-}M(x)=0$. If x is not injective, then $M(\tau^{-}x)$ is indecomposable, belongs to $\mathfrak{F}$, and satisfies $h_H(\tau^{-}x)=\dim M(\tau^{-}x)(\omega)$. Further, in this case $M(g_{\tau^{-}x})$ is a sink map in $\mathfrak{F}$ and $M(\tau^{-}x)\approx\tau_{\mathfrak{F}}^{-}M(x)$.

(c) The functor M gives a full embedding of $k({}_{d+1}H)$ into ${}_{d+1}\mathfrak{F}$, and induces a full embedding of translation quivers from ${}_{d+1}H$ into $\Gamma\mathfrak{F}$.

Proof. Note that the indecomposable projective modules are of the form $M(p)$ with p a projective vertex. If p is a projective vertex, it follows from § 4 that $M(g_p)$ is the sink map of $M(p)$ in $\mathfrak{P}(H,k)$-mod. In particular, $M(g_\omega)$: $0\to M(\omega)$ is a sink map in $\mathfrak{P}(H,k)$-mod, and since $M(\omega)$ is in $\mathfrak{F}$, this is a sink map in $\mathfrak{F}$. Also, $h_H(\omega)=1=\dim M(\omega)(\omega)$.

Consider the case where $n=1$. Then H has only one vertex, namely ω, and $\mathfrak{P}(H,k)$-mod is equivalent to k-mod, the only indecomposable object in $\mathfrak{P}(H,k)$-mod being $M(\omega)$. In this case, the map $M(f_\omega)$: $M(\omega)\to 0$ is a source map. Thus all assertions are satisfied.

Assume now that $n\geqslant 2$. Using induction on n, we shall show that the following two conditions are satisfied.

(d) Let x be a predecessor of a projective vertex of H. Then $M(x)$ is indecomposable, belongs to $\mathfrak{F}$, and satisfies $h_H(x)=\dim M(x)(\omega)$. If x is in ${}_{e+1}H\setminus{}_eH$, then $M(x)$ is in ${}_{e+1}\mathfrak{F}\setminus{}_e\mathfrak{F}$.

(e) If u is a vertex of H, and $M(u)$ is a direct summand of the radical of some $M(p)$, where p is a projective vertex, then either $u\to p$ or else u is a successor of p. (However, the last possibility cannot occur, as we shall see later.)

For the proof, let t be a projective vertex of H with no proper successor of H being projective. (Such a vertex exists, since there are only finitely many projective vertices in H and no cyclic paths. Actually, we may assume in addition that t is successor of any given projective vertex; this will be needed later.) Let H' be the full translation subquiver of H given by the vertices u which are not successors of t. Note that H' is a predecessor closed translation subquiver (if u is a vertex in H', and u' is a predecessor of u, then $t\leqslant u'$ would imply $t\leqslant u'\leqslant u$, which is impossible), and t does not belong to H', whereas all other projective vertices of H belong to H'. It follows that $\mathfrak{P}(H',k)$ is a full subcategory of $\mathfrak{P}(H,k)$, and we may, and will, identify $\mathfrak{P}(H',k)$-mod with the full subcategory of $\mathfrak{P}(H,k)$-mod given by all functors X with $X(t)=0$. Note that H' is a partial left hammock, and let H'' be a completion of H'. Thus $\mathfrak{P}(H'',k)=\mathfrak{P}(H',k)$. Since H'' is a left hammock with $n-1$ projective vertices, we can use induction. Given a vertex x of H'', we denote by $M''(x)$ the functor

$$M''(x)=\operatorname{Hom}_{k(H'')}(-,x)\mid\mathfrak{P}(H'',k).$$

Note that for a vertex x of H', both $M(x)$ and $M''(x)$ are defined, and they coincide under our identification. In particular, $M(\omega)=M''(\omega)$; thus the set $\mathfrak{F}''$ of objects in $\mathfrak{P}(H'',k)$-mod with socle generated by $M''(\omega)$ is just $\mathfrak{F}\cap\mathfrak{P}(H'',k)$-mod. Using (b), applied to H'', we see that all $M(x)$, with $x\in H'$, are indecomposable, in $\mathfrak{F}$, and satisfy $\dim M(x)(\omega)=h_{H''}(x)=h_H(x)$. Now, consider $M(t)$ itself. Since $n\geqslant 2$, we know that t is not a source. Given an arrow

$\alpha: y \to t$, the vertex y belongs to H'. Thus $M(y)$ is non-zero and in $\mathfrak{F}$. Now

$$M(g_t): \bigoplus_{\substack{\alpha \\ e\alpha=t}} M(s\alpha) \to M(t)$$

is the sink map for $M(t)$. Therefore

$$\operatorname{rad} M(t) \approx \bigoplus_{\substack{\alpha \\ e\alpha=t}} M(s\alpha)$$

is non-zero and belongs to $\mathfrak{F}$, and so $M(t)$ belongs to $\mathfrak{F}$. Also,

$$\begin{aligned}\dim M(t)(\omega) &= \dim \operatorname{Hom}(M(\omega), M(t)) \\ &= \dim \operatorname{Hom}(M(\omega), \operatorname{rad} M(t)) \\ &= \sum_{\substack{\alpha \\ e\alpha=t}} \dim \operatorname{Hom}(M(\omega), M(s\alpha)) \\ &= \sum_{\substack{\alpha \\ e\alpha=t}} h_H(s\alpha) \\ &= h_H(t).\end{aligned}$$

Thus, if x is a predecessor of a projective vertex of H, then either $x = t$ or else x belongs to H'. Thus $M(x)$ is always indecomposable, belongs to $\mathfrak{F}$, and satisfies $h_H(x) = \dim M(x)(\omega)$. Also, if x is, in addition, in ${}_{e+1}H$ and not in ${}_eH$, then $M(x)$ is in ${}_{e+1}\mathfrak{F}$, and not in ${}_e\mathscr{F}$. This gives (d).

For the proof of (e), let p be a projective vertex, and u any vertex of H, with $M(u)$ a direct summand of $\operatorname{rad} M(p)$. We may assume that we have chosen t as a successor of p. We want to show that either u is a successor of p or else $u \to p$. Thus, assume u is not a successor of p. Therefore u is not a successor of t, so that $u \in H'$. The indecomposable direct summands of $\operatorname{rad} M(p)$ are of the form $M(v)$, with $v \to p$, and all such v belong to H'. But using Condition (c) for H'', we see that $M(u) \approx M(v)$ only in the case where $u = v$; thus $u \to p$.

We have shown, in this way, Property (a). Now, we use induction on d in order to obtain (b) and (c). Consider first the case where $d = -1$. Since ${}_0H$ contains as only vertex ω, we obtain (c) without difficulty, whereas (b) is empty.

Now assume that some $d \geq 0$. Let x be a vertex in ${}_dH$. Let $x \to y$. We claim that $M(y)$ is indecomposable, in $\mathfrak{F}$, and satisfies $h_H(y) = \dim M(y)(\omega)$, and that $M(g_y)$ is a sink map in $\mathscr{F}$. If y is projective, then use (a) and the fact that in this case, $M(g_y)$ is a sink map even in $\mathfrak{P}(H, k)$-mod, as follows from § 4. If y is not projective, then $\tau y \in {}_{d-1}H$, so we can use induction in order to obtain these assertions. In addition, for y not projective (and $x \to y$), we obtain that $M(\tau y) \approx \tau M(y)$. We use the dual of [**12**, 2.2.2] in order to see that the residue classes of the maps $M(\beta)$: $M(x) \to M(y)$ in $\operatorname{Irr}_{\mathfrak{F}}(M(x), M(y))$, where β runs through the arrows $x \to y$, form a basis of $\operatorname{Irr}_{\mathfrak{F}}(M(x), M(y))$. Now conversely, let N be an indecomposable module and assume $\operatorname{Irr}_{\mathfrak{F}}(M(x), N) \neq 0$. First, let $N \approx M(p)$ for some projective vertex p. From (e), either $x \to p$, or else x is a successor of p. But, if x is a successor of p, then both x and p belong to ${}_dH$, and then $\operatorname{Irr}_{\mathfrak{F}}(M(x), M(p)) \neq 0$ implies $x \to p$, according to the inductive assumption of (c). Of course, the latter is impossible, since x as a successor of p cannot be a proper predecessor of p. Next, consider the case where N is not projective. Then $\tau_{\mathfrak{F}} N \neq 0$, as shown in § 3, and $\operatorname{Irr}_{\mathfrak{F}}(\tau_{\mathfrak{F}} N, M(x)) \neq 0$ shows that $\tau_{\mathfrak{F}} N \approx M(v)$ for some v with $v \to x$. (If x is projective, we use (a). Otherwise, we use (b) for τx; this is possible since $\tau x \in {}_{d-2}H$.) But then $N \approx \tau_{\mathfrak{F}}^{-} M(v)$. Since $v \in {}_{d-1}H$, we conclude from (b) that v cannot be injective (otherwise, $\tau_{\mathfrak{F}}^{-} M(v) = 0$), and that $\tau_{\mathfrak{F}}^{-} M(v) \approx M(\tau^{-} v)$. Thus $N \approx M(\tau^{-} v)$ and there is an arrow $x \to \tau^{-} v$. Altogether, we have shown that $\operatorname{Irr}_{\mathfrak{F}}(M(x), N) \neq 0$ implies $N \approx M(y)$, with $y \in x^{+}$. It follows from [**12**, 2.2.2] that $M(f_x)$ is a source map.

Since $\mathfrak{F}$ is closed under submodules, an irreducible map in $\mathfrak{F}$ is either injective or surjective. Thus $M(f_x)$ is injective or surjective.

First, suppose that $M(f_x)$ is injective, and let C be its cokernel. Thus we deal with the exact sequence

$$0 \longrightarrow M(x) \xrightarrow{M(f_x)} \bigoplus_{\substack{\beta \\ s\beta = x}} M(e\beta) \longrightarrow C \longrightarrow 0,$$

and its evaluation at ω is the exact sequence

$$0 \to M(x)(\omega) \to \bigoplus_{\substack{\beta \\ s\beta = x}} M(e\beta)(\omega) \to C(\omega) \to 0.$$

We use this exact sequence in order to determine $\dim C(\omega)$; here, we also take into account that $h_H(x) = \dim M(x)(\omega)$, and $h_H(y) = \dim M(y)(\omega)$ for all $y \in x^{+}$. Thus

$$\dim C(\omega) = -\dim M(x)(\omega) + \sum_{\substack{\beta \\ s\beta = x}} \dim M(e\beta)(\omega) = -h_H(x) + (\Sigma h_H)(x^{(+)}).$$

When x is injective, $\dim C(\omega) \geq 0$, while $-h_H(x) + (\Sigma h_H)(x^{(+)}) \leq 0$. Thus in this case, $C(\omega) = 0$, and therefore C is in $\mathfrak{T}$ and $\tau_{\mathfrak{F}}^{-} M(x) = 0$. When x is not injective, we have $C \approx M(\tau^{-} x)$, from § 4. Thus in this case

$$\dim M(\tau^{-} x)(\omega) = -h_H(x) + (\Sigma h_H)(x^{(+)}) = h_H(\tau^{-} x).$$

It follows from § 3 that the exact sequence

$$0 \longrightarrow M(x) \xrightarrow{M(f_x)} \bigoplus_{\substack{\beta \\ s\beta = x}} M(e\beta) \xrightarrow{M(g_{\tau^{-} x})} M(\tau^{-} x) \longrightarrow 0$$

is a (relative) Auslander–Reiten sequence in $\mathfrak{F}$. In particular, $M(\tau^{-} x)$ is indecomposable and in $\mathfrak{F}$. We see that $M(g_{\tau^{-} x})$ is a sink map in $\mathfrak{F}$, and $\tau_{\mathfrak{F}}^{-} M(x) = M(\tau^{-} x)$.

If $M(f_x)$ is surjective, then its cokernel is zero, and thus in $\mathfrak{T}$. Therefore $\tau_{\mathfrak{F}}^{-} M(x) = 0$. We claim that in this case x has to be injective. Otherwise,

$$h_H(\tau^{-} x) = -h_H(x) + (\Sigma h_H)(x^{(+)}) = -\dim M(x)(\omega) + \sum_{\substack{\beta \\ s\beta = x}} \dim M(e\beta)(\omega) \leq 0$$

which contradicts the fact that h_H takes only positive values on H_0. This finishes the proof of (b).

In order to establish (c), consider the functor

$$M\colon\ k({}_{d+1}H) \to \mathfrak{P}(H, k)\text{-mod}.$$

Let z be a vertex of ${}_{d+1}H$. Then $M(z)$ belongs to ${}_{d+1}\mathfrak{F}$. (If z is projective, use (a), otherwise use (b) for $x = \tau z \in {}_{d-1}H$. In both cases the sink map for $M(z)$ has a left term which by induction belongs to ${}_d\mathfrak{F}$.) Also, if z belongs to ${}_{d+1}H$ but not to ${}_dH$, then $M(z)$ is in ${}_{d+1}\mathfrak{F}$, but not in ${}_d\mathfrak{F}$ (for, there is an arrow $y \to z$ with y in ${}_dH$, but not in ${}_{d-1}H$, and by induction, $M(y)$ is in ${}_d\mathfrak{F}$ and not in ${}_{d-1}\mathfrak{F}$. Since $\mathrm{rad}(M(y), M(z)) \neq 0$, we see that $M(z)$ is not in ${}_d\mathfrak{F}$.)

We want to show that M is fully faithful, so we have to consider the maps

$$M_{uv}\colon \mathrm{Hom}_{k(H)}(u, v) \to \mathrm{Hom}_{\mathfrak{F}}(M(u), M(v))$$

given by applying M. By induction, M_{uv} is bijective for u, v both in ${}_dH$. In the case where $u = v$, we know that $\mathrm{Hom}_{k(H)}(v, v) = k$, and we have $\mathrm{Hom}_{\mathfrak{F}}(M(v), M(v)) = k$, since $M(v)$ is directing. Of course, M_{vv} maps 1_v onto $1_{M(v)}$, and so M_{vv} is bijective. So consider the case where $u \neq v$. We claim that $M(u)$ and $M(v)$ are not isomorphic. This is clear if both u, v are projective vertices, since the subcategory of $\mathfrak{P}(H, k)$-mod given by the projective modules is equivalent to $\mathfrak{P}(H, k)$. If, say, u is a projective vertex, and v is not, then $M(u)$ is a projective module, and $M(v)$ is not (we know that $\tau_{\mathfrak{F}}M(v) = M(\tau v) \neq 0$). Finally, if both u, v are non-projective, then $\tau_{\mathfrak{F}}M(v) = M(\tau v)$, $\tau_{\mathfrak{F}}M(u) = M(\tau u)$, and $\tau v \neq \tau u$ belong to ${}_{d-1}H$. Thus by induction $M(\tau u)$ and $M(\tau v)$ are not isomorphic. Therefore, given $\varphi\colon M(u) \to M(v)$, we can factor it through $M(g_v)$, say $\varphi = \varphi' M(g_v)$. If $u \in {}_dH$, we know by induction that $\varphi' = M(\psi)$ for some ψ in $k({}_dH)$; thus M_{uv} is surjective in this case. On the other hand, if $u \notin {}_dH$, then $M(u)$ is not in ${}_d\mathfrak{F}$, and so $\mathrm{Hom}(M(u), M(v)) = \mathrm{rad}(M(u), M(v)) = 0$. Thus, M_{uv} is always surjective. Now assume $M_{uv}(\gamma) = 0$ for some $\gamma \in \mathrm{Hom}_{k(H)}(u, v)$. Since $u \neq v$, we can write $\gamma = \gamma' g_v$ and we can assume that u is a proper predecessor of v (since otherwise $\gamma' = 0$, trivially). Therefore $\gamma' \in k({}_dH)$. We have $0 = M(\gamma) = M(\gamma')M(g_v)$. If v is projective, then we know that $M(g_v)$ is injective. Thus $M(\gamma') = 0$, and so by induction, $\gamma' = 0$. If v is not projective, the kernel of $M(g_v)$ is $M(f_{\tau v})$. Thus $M(\gamma') = \delta M(f_{\tau v})$, for some δ. However $\delta = M(\gamma'')$ for some $\gamma'' \in k({}_dH)$, and therefore $M(\gamma' - \gamma'' f_{\tau v}) = 0$. Note that $\gamma' - \gamma'' f_{\tau v}$ belongs to $k({}_dH)$. Thus, by induction, it is equal to 0, so $\gamma' = \gamma'' f_{\tau v}$, and therefore $\gamma = \gamma' g_v = \gamma'' f_{\tau v} g_v = 0$. This shows that M_{uv} is also injective. Thus

$$M\colon k({}_{d+1}H) \to {}_{d+1}\mathfrak{F}$$

is a full embedding. Of course, our knowledge of sink maps and source maps shows that M induces a full embedding of translation quivers from ${}_{d+1}H$ into $\Gamma\mathfrak{F}$.

This finishes the proof of the assertions (a), (b), (c) for H a left hammock with finitely many projective vertices. In order to derive Theorem 3 for such a left hammock, we only have to observe that the functor $M\colon k(H) \to {}_\infty\mathfrak{F}$ is actually dense. Given an indecomposable projective module, it is of the form $M(p)$, and is thus in the image of M. We use induction on d in order to show that any indecomposable module N in ${}_d\mathfrak{F}$ is in the image of M. If $d = 0$, then N is simple projective. The only simple projective module is $M(\omega)$, and thus is in the image of M. Now assume N is non-projective and in ${}_d\mathfrak{F}$. Then $\tau_{\mathfrak{F}}N$ is non-zero, and in ${}_{d-2}\mathfrak{F}$, and so of the form $M(x)$ for some x. Note that x cannot be injective, since otherwise $\tau_{\mathfrak{F}}^{-}M(x) = 0$. Thus $\tau^{-}x$ exists, and $N \approx \tau_{\mathfrak{F}}^{-}M(x) \approx M(\tau^{-}x)$ is in the image of M. This finishes the proof of Theorem 3 in the case where H has only finitely many projective vertices.

It remains to consider the case where H contains infinitely many projective vertices. Let $\mathfrak{P}(d)$ be the full subcategory of $\mathfrak{P}(H, k)$ given by all projective vertices p with $p \in {}_dH$. We may, and will, identify $\mathfrak{P}(d)$-mod with the full subcategory of $\mathfrak{P}(H, k)$-mod given by all functors X with $X(p')=0$ for all projective vertices p' with $p' \notin {}_dH$. Note that we have the inclusions

$$\mathfrak{P}(0)\text{-mod} \subseteq \mathfrak{P}(1)\text{-mod} \subseteq \dots \subseteq \mathfrak{P}(d)\text{-mod} \subseteq \dots$$

and we have

$$\mathfrak{P}(H, k)\text{-mod} = \bigcup_{d \in \mathbb{N}} \mathfrak{P}(d)\text{-mod},$$

since any object in $\mathfrak{P}(H, k)$-mod is of finite length. Note that $M(\omega)$ belongs to $\mathfrak{P}(0)$-mod, and thus to all $\mathfrak{P}(d)$-mod, and we denote by $\mathfrak{F}(d)$ the set of all objects in $\mathfrak{P}(d)$-mod with socle generated by $M(\omega)$, and let $\mathfrak{F}$ be the set of all objects in $\mathfrak{P}(H, k)$-mod with socle generated by $M(\omega)$; thus we also have $\mathfrak{F}(d) = \mathfrak{F} \cap \mathfrak{P}(d)$-mod. For any $d \in \mathbb{N}$, the translation quiver ${}_dH$ is a partial left hammock. Thus we may consider a completion $H(d)$ of ${}_dH$. Note that now $H(d)$ is a left hammock with only finitely many projective vertices, and $\mathfrak{P}(H(d), k) = \mathfrak{P}(d)$. For any vertex x of H, let

$$M(x) = \operatorname{Hom}_{k(H)}(-, x) \mid \mathfrak{P}(H, k).$$

It is an object in $\mathfrak{P}(H, k)$-mod. If x belongs to ${}_dH$, then clearly $M(x)$ belongs to $\mathfrak{P}(d)$-mod (and it coincides with the object $\operatorname{Hom}_{k(H(d))}(-, x) \mid \mathfrak{P}(H(d), k)$, which in $\mathfrak{P}(H(d), k)$-mod would also be denoted by $M(x)$, so no confusion need occur!)

Now consider a vertex x of H. We claim that $M(g_x)$ is a sink map for $M(x)$ in $\mathfrak{F}$, and $M(f_x)$ is a source map for $M(x)$ in $\mathfrak{F}$. This means that we have to verify the corresponding lifting properties for all maps $N \to M(x)$ which are not split epimorphisms, and all maps $M(x) \to N$ which are not split monomorphisms. However, any such N belongs to some $\mathfrak{F}(e)$, with e sufficiently large, so it is sufficient to establish the lifting property in $\mathfrak{P}(H(e), k)$-mod (where we assume that e is chosen in such a way that both $N \in \mathfrak{F}(e)$, and $x^+ \subseteq {}_eH$), but this has been established above. This finishes the proof of Theorem 3.

6. *The main estimate and proof of Theorem* 1

As a preparation for the proof of Theorem 1, we are going to derive an inequality concerning suitable Jordan–Hölder multiplicities of indecomposable A-modules, where A is a finite-dimensional k-algebra, and k is algebraically closed. The proof of Theorem 1 given in this section will rely, in addition, on Theorem 3.

PROPOSITION. *Let k be an algebraically closed field, and A a finite-dimensional k-algebra. Let $P(\omega)$ be a simple projective A-module, and $\mathfrak{F}$ the full subcategory of A-mod given by all (finitely generated) A-modules with socle generated by $P(\omega)$. Let X be a indecomposable A-module in ${}_\infty\mathfrak{F}$. Then*

$$\dim \operatorname{Hom}(P(\omega), X) - 1 \leq \dim \operatorname{Hom}(P(\omega), \tau_A^- X) = \dim \operatorname{Hom}(P(\omega), \tau_{\mathfrak{F}}^- X).$$

Proof. The assertion is trivial in the case where $\dim \operatorname{Hom}(P(\omega), X) \leq 1$. So assume $\dim \operatorname{Hom}(P(\omega), X) \geq 2$. Since $X \in {}_\infty\mathfrak{F}$, there are only finitely many indecomposable modules in Y in $\mathfrak{F}$ with $\operatorname{Hom}(Y, X) \neq 0$. Since every submodule of X again belongs to $\mathfrak{F}$, there are only finitely many isomorphism classes of submodules of X. (For, let $m = \dim_k X$, and let $Y_1, \dots, Y_r$ be the indecomposable A-modules in $\mathfrak{F}$, with $\operatorname{Hom}(Y_i, X) \neq 0$. Then any submodule of X is of the form $\bigoplus_{i=1}^{r} Y_i^{m_i}$, with all $m_i \leq m$.) Let $Q_A(\omega)$ be the injective envelop of $P(\omega)$. Given a map ψ: $X \to Q_A(\omega)$, let $K(\psi)$ be its kernel. Since there are only finitely many isomorphism classes of possible $K(\psi)$, we claim that there is a dense open subset U of $\mathrm{Hom}(X, Q_A(\omega))$ (in the Zariski topology), with $K(\psi)$ being isomorphic, for all $\psi \in U$, say isomorphic to K and $K \neq 0$. Consider pairwise non-isomorphic A-modules $K_1, \ldots, K_s$ such that any submodule of X is isomorphic to some K_i. By Chevalley's theorem, the projection map

$$\mathrm{Hom}(K_i, X) \times \mathrm{Hom}(X, Q_A(\omega)) \to \mathrm{Hom}(X, Q_A(\omega))$$

maps the locally closed set $\{(\varphi, \psi) \mid \varphi \in \mathrm{Hom}(K_i, X),\ \psi \in \mathrm{Hom}(X, Q_A(\omega)),\ \varphi$ a monomorphism, $\varphi\psi = 0\}$ onto a constructible subset, say W_i' of $\mathrm{Hom}(X, Q_A(\omega))$, and W_i' is the set of all maps ψ: $X \to Q_A(\omega)$ with kernel containing a submodule isomorphic to K_i. Let

$$W_i = W_i' \setminus \bigcup_{j \in J(i)} W_j',$$

where $J(i)$ is the set of all j with $\dim K_j > \dim K_i$. The W_i are again constructible. Since $\mathrm{Hom}(X, Q_A(\omega))$ is the disjoint union of the finitely many constructible subsets W_i, one of these subsets W_i, say W_1, has to be dense, and therefore contains a dense open subset U of $\mathrm{Hom}(X, Q_A(\omega))$. Of course, W_1 is the set of all maps ψ: $X \to Q_A(\omega)$ with kernel isomorphic to K_1. Thus let $K = K_1$. Since $\dim \mathrm{Hom}(P(\omega), X) \geq 2$, and $\dim \mathrm{Hom}(P(\omega), Q(\omega)) = 1$, we see that $K \neq 0$.

We choose for every $\psi \in U$ some monomorphism $\mu(\psi)$: $K \to X$ with $\mu(\psi)\psi = 0$. Let V be the set of all pairs $(\varphi, \psi) \in \mathrm{Hom}(K, X) \times \mathrm{Hom}(X, Q_A(\omega))$ with φ a monomorphism, $\psi \in U$, and $\varphi\psi = 0$. This is a locally closed subset of $\mathrm{Hom}(K, X) \times \mathrm{Hom}(X, Q_A(\omega))$, and we may consider the two projections

$$\pi_1\colon V \to \mathrm{Hom}(K, X), \quad \pi_1(\varphi, \psi) = \varphi;$$
$$\pi_2\colon V \to U, \quad \pi_2(\varphi, \psi) = \psi.$$

First, we consider π_2. Given $\psi \in U$, we claim that

$$\pi_2^{-1}(\psi) = \{(\alpha\mu(\psi), \psi) \mid \alpha \in \mathrm{Aut}(K)\},$$

where $\mathrm{Aut}(K)$ denotes the set of automorphisms of K. For, if $\alpha \in \mathrm{Aut}(K)$, clearly $\alpha\mu(\psi)$ is a monomorphism and $\alpha\mu(\psi)\psi = 0$, and so $(\alpha\mu(\psi), \psi) \in V$. On the other hand, if $(\varphi, \psi) \in V$, there is β: $K \to K$ with $\varphi = \beta\mu(\psi)$, since $\mu(\psi)$ is a kernel of ψ, and $\varphi\psi = 0$. Since φ is a monomorphism, so is β, and thus β is an automorphism. It follows that $\dim \pi_2^{-1}(\psi) = \dim \mathrm{Aut}(K) = \dim \mathrm{End}(K)$, for all $\psi \in U$. Therefore,

$$\dim V = \dim U + \dim \mathrm{End}(K) = \dim \mathrm{Hom}(X, Q_A(\omega)) + \dim \mathrm{End}(K).$$

Next, consider π_1. We claim that for $\varphi \in \mathrm{Hom}(K, X)$, the fibre $\pi_1^{-1}(\varphi)$ is either empty or one-dimensional. Thus, assume $\pi_1^{-1}(\varphi)$ is non-empty and take $(\varphi, \nu(\varphi)) \in V$. Let us show that

$$\pi_1^{-1}(\varphi) = \{(\varphi, c\nu(\varphi)) \mid 0 \neq c \in k\}.$$

Of course, for non-zero $c \in k$, we have $(\varphi, c\nu(\varphi)) \in V$. Conversely, let $(\varphi, \psi) \in V$. We write $\nu(\varphi) = \psi_0'\psi_0''$, $\psi = \psi'\psi''$ with ψ_0', ψ' epimorphisms, and ψ_0'', ψ'' monomorphisms. Since both ψ_0', ψ' are cokernels of φ, there is an isomorphism γ with $\psi' = \psi_0'\gamma$. Since $Q_A(\omega)$ is injective, there is $\delta \in \mathrm{End}(Q_A(\omega))$ with

$\gamma\psi'' = \psi_0''\delta$, and $\delta \neq 0$. Altogether,

$$\psi = \psi'\psi'' = \psi_0'\gamma\psi'' = \psi_0'\psi_0''\delta = \nu(\varphi)\delta.$$

Note that $\operatorname{End}(Q_A(\omega)) = k$, since the socle of $Q_A(\omega)$ is simple projective. Thus $\psi = c\nu(\varphi)$ for some non-zero $c \in k$. It follows that

$$\dim V \leqslant \dim \operatorname{Hom}(K, X) + 1.$$

Combining the two assertions concerning $\dim V$, and using the fact that $\dim \operatorname{End}(K) \geqslant 1$, we obtain

$$\begin{aligned} \dim \operatorname{Hom}(K, X) &\geqslant \dim V - 1 \\ &= \dim \operatorname{Hom}(X, Q_A(\omega)) + \dim \operatorname{End}(K) - 1 \\ &\geqslant \dim \operatorname{Hom}(X, Q_A(\omega)). \end{aligned}$$

Choose some pair $(\varphi, \psi) \in V$, and a factorization $\psi = \psi'\psi''$, with $\psi'\colon X \to Y$ an epimorphism, $\psi''\colon Y \to Q_A(\omega)$ a monomorphism. The exact sequence

$$0 \longrightarrow K \xrightarrow{\varphi} X \xrightarrow{\psi'} Y \longrightarrow 0$$

induces an exact sequence

$$\operatorname{Hom}(X, X) \to \operatorname{Hom}(K, X) \to \operatorname{Ext}^1(Y, X) \to \operatorname{Ext}^1(X, X).$$

Here, the first term is one-dimensional and the last term is zero, since $X \in {}_\infty\mathfrak{F}$ is directing in $\mathfrak{F}$, and $\mathfrak{F}$ is closed under extensions. Thus,

$$\dim \operatorname{Ext}^1(Y, X) \geqslant \dim \operatorname{Hom}(K, X) - 1.$$

(Actually, we have equality: we may extend the long exact sequence to the left by $\operatorname{Hom}(Y, X)$. But $\operatorname{Hom}(Y, X) = 0$. For, Y belongs to $\mathfrak{F}$, since it is a submodule of $Q_A(\omega)$. Thus $\operatorname{Hom}(Y, X) \neq 0$ would produce a cycle $Y \preccurlyeq X \prec Y$.) In addition, we use the Auslander–Reiten formula

$$\operatorname{Ext}^1(Y, X) \approx D\, \underline{\operatorname{Hom}}(\tau_A^- X, Y),$$

and so obtain

$$\dim \operatorname{Ext}^1(Y, X) \leqslant \dim \operatorname{Hom}(\tau_A^- X, Y).$$

Also ψ'' induces a monomorphism

$$\operatorname{Hom}(\tau_A^- X, Y) \to \operatorname{Hom}(\tau_A^- X, Q_A(\omega)).$$

Thus

$$\dim \operatorname{Hom}(\tau_A^- X, Y) \leqslant \dim \operatorname{Hom}(\tau_A^- X, Q_A(\omega)).$$

Altogether we have

$$\begin{aligned} \dim \operatorname{Hom}(\tau_A^- X, Q_A(\omega)) &\geqslant \dim \operatorname{Hom}(\tau_A^- X, Y) \\ &\geqslant \dim \operatorname{Ext}^1(Y, X) \\ &\geqslant \dim \operatorname{Hom}(K, X) - 1 \\ &\geqslant \dim \operatorname{Hom}(X, Q_A(\omega)) - 1. \end{aligned}$$

Finally, we note that $\tau_{\mathfrak{F}}^- X = (\tau_A^- X)_{\mathfrak{F}}$, and therefore

$$\dim \operatorname{Hom}(P_A(\omega), \tau_A^- X) = \dim \operatorname{Hom}(P_A(\omega), \tau_{\mathfrak{F}}^- X).$$

This finishes the proof.

Proof of Theorem 1. Let H be a left hammock, and x a vertex of H. We may assume that H has only finitely many projective vertices. (For, we choose d so that x^+, and, if x is not injective, $\tau^- x$ also, belong to ${}_dH$. Let $H(d)$ be the completion of the partial left hammock ${}_dH$. Then h_H and $h_{H(d)}$ coincide on x, x^+, and, if x is not injective in H, on $\tau^- x$. Also, x is injective in H if and only if it is injective in $H(d)$. Thus, we may replace H by $H(d)$.)

Let k be an algebraically closed field. Since $\mathfrak{P}(H, k)$ is a finite k-category, $\mathfrak{P}(H, k)$-mod is equivalent to A-mod, for some finite-dimensional k-algebra. We use Theorem 3. Let $P(\omega)$ be the unique simple projective A-module, and $\mathfrak{F} = A$-spmod. Then $\mathfrak{F} = A\text{-spmod} \approx \mathfrak{P}(H, k)\text{-spmod} \approx k(H)$. We can identify $\Gamma_\infty \mathfrak{F}$ with H, and then $h_H([X]) = \dim \operatorname{Hom}(P(\omega), X)$, for any indecomposable A-module X in $\mathfrak{F}$. We have $\tau^-[X] = [\tau_{\mathfrak{F}}^- X]$, provided $\tau_{\mathfrak{F}}^- X \neq 0$. In the case where $\tau_{\mathfrak{F}}^- X = 0$, the vertex $[X]$ of $H = \Gamma_\infty \mathfrak{F}$ is injective, and so $h_H(\tau^-[X]) = 0 = \dim \operatorname{Hom}(P(\omega), \tau_{\mathfrak{F}}^- X)$. Thus, for all indecomposable A-modules in $\mathfrak{F}$, we have $h_H(\tau^-[X]) = \dim \operatorname{Hom}(P(\omega), \tau_{\mathfrak{F}}^- X)$. Therefore Theorem 1 is a direct consequence of the proposition above.

7. *Representations of finite partially ordered sets* (*proof of Theorem* 2)

Let S be a finite partially ordered set, and fix some field k. We write $\ell(S) := \ell(S, k)$. Recall that we denote by S^+ the partially ordered set obtained from S by adding an element ω with $s < \omega$ for all $s \in S$. Similarly, we denote by S_+ the partially ordered set obtained from S by adding an element ω' with $\omega' < s$ for all $s \in S$.

Let us exhibit some useful S-spaces, $P(t) = P_S(t)$, $R(t) = R_S(t)$, where $t \in S^+$, by putting

$$P(t)_s = \begin{cases} k & \text{for } s \geqslant t, \\ 0 & \text{for } s \ngeqslant t, \end{cases} \qquad R(t)_s = \begin{cases} k & \text{for } s > t, \\ 0 & \text{for } s \not> t, \end{cases}$$

where $s \in S^+$. Note that $R(\omega) = 0$, whereas the remaining $R(t)$, for $t \in S$, and all the $P(t)$, for $t \in S^+$, have one-dimensional total spaces. We will also consider the S-spaces $Q(t) = Q_S(t)$, where $t \in S_+$, and $N(t) = N_S(t)$, where $t \in S$,

$$Q(t)_s = \begin{cases} k & \text{for } s \nleqslant t, \\ 0 & \text{for } s \leqslant t, \end{cases} \qquad N(t)_s = \begin{cases} k & \text{for } s \nless t, \\ 0 & \text{for } s < t, \end{cases}$$

with $s \in S^+$. Also, let $N(\omega') = 0$; the remaining $N(t)$, for $t \in S$, and all $Q(t)$, for $t \in S_+$ have one-dimensional total spaces. We have

$$\operatorname{Hom}(P(t_1), P(t_2)) = \begin{cases} k & \text{for } t_1 \geqslant t_2, \\ 0 & \text{for } t_1 \ngeqslant t_2, \end{cases}$$

where $t_1, t_2 \in S^+$, and similarly

$$\operatorname{Hom}(Q(t_1), Q(t_2)) = \begin{cases} k & \text{for } t_1 \geqslant t_2, \\ 0 & \text{for } t_1 \ngeqslant t_2, \end{cases}$$

where $t_1, t_2 \in S_+$. We denote by $\mathfrak{P}(S)$ the full subcategory of $\ell(S)$ given by the S-spaces $P(t)$, with $t \in S^+$, and by $\mathfrak{Q}(S)$ the full subcategory of $\ell(S)$ given by the S-spaces $Q(t)$, with $t \in S_+$. Then $S(\mathfrak{P}(S)) = (S^+)^*$, $S(\mathfrak{Q}(S)) = (S_+)^*$; thus we can recover S from $\mathfrak{P}(S)$ as well as from $\mathfrak{Q}(S)$.

Let us consider

$$A_S = A((S^+)^*),$$

the incidence algebra of the partially ordered set opposite to S^+. We may consider, in the obvious way, any S-space $V = (V_\omega, V_s)_{s\in S}$ as an A_S-module. Recall that the quiver of A_S has as vertices the elements of S^+, and there is an arrow $a \leftarrow b$, provided a covers b (that means $a > b$, and $a > b' \geqslant b$ implies $b' = b$). We consider V as a representation of this quiver by using the vector spaces V_ω and V_s, and the inclusion maps $V_a \hookleftarrow V_b$, for $a \leftarrow b$. In this way, we may and will consider $\ell(S) = \ell(S, k)$ as a full subcategory of A_S-mod. The S-spaces $P(t)$, with $t \in S^+$, considered as A_S-modules, are just the indecomposable projective A_S-modules. In particular, there is precisely one simple projective A_S-module, namely $P(\omega)$. Note that any S-space V, considered as an A_S-module, has socle generated by $P(\omega)$, and thus belongs to A_S-spmod. Conversely, any module in A_S-spmod is isomorphic to an object of $\ell(S)$. Thus, we can identify $\ell(S)$ with the category A_S-spmod. Of course, the exact sequences in $\ell(S)$ as introduced in § 1, are just the exact sequences in A_S-mod with all terms belonging to A_S-spmod. Since A_S-spmod is closed under extensions in A_S-mod, it follows that $\ell(S)$ is an exact category.

We want to consider the Auslander–Reiten quiver $\Gamma\ell(S)$. We can use all the results mentioned in § 3, since $\ell(S) = A_S$-spmod is of the form $\mathfrak{F}_{P(\omega)}$ and the indecomposable projective A_S-modules all belong to A_S-spmod. Thus $\ell(S)$ has source maps and sink maps and, given an indecomposable S-space Z, not of the form $P(t)$ for any $t \in S^+$, then there is a (relative) Auslander–Reiten sequence in $\ell(S)$ ending in Z. Thus the only projective vertices in $\Gamma\ell(S)$ are those of the form $p(t) = [P(t)]$, with $t \in S^+$. Since $\ell(S)^{\text{op}} \approx \ell(S^*)$, we see similarly that, for any indecomposable S-space X not of the form $Q(t)$ for any $t \in S$, there is a (relative) Auslander–Reiten sequence in $\ell(S)$ starting in X. Therefore the only injective vertices in $\Gamma\ell(S)$ are those of the form $q(t) = [Q(t)]$ with $t \in S_+$. For any $t \in S^+$, the canonical map $R(t) \to P(t)$ is a sink map for $P(t)$ in $\ell(S)$. Similarly, for any $t \in S_+$, the canonical map $Q(t) \to N(t)$ is a source map for $Q(t)$ in $\ell(S)$. It follows that $p(\omega)$ is the unique source, and $q(\omega')$ the unique sink, of $\Gamma\ell(S)$. Also given $t \in S$, we see that $p(t)$ is endpoint of precisely one arrow, and its starting point is $[R(t)]$, whereas $q(t)$ is starting point of precisely one arrow, and its endpoint is $[N(t)]$. Now it is easy to see that $H(S) := {}_\infty\Gamma\ell(S) = \Gamma_\infty\ell(S)$ is a component of $\Gamma\ell(S)$, and a preprojective translation quiver:

LEMMA. *Let Γ be a translation quiver, with only finitely many projective vertices, and a unique source. Assume that $|p^-| \leqslant 1$ for any projective vertex p of Γ. Then ${}_\infty\Gamma$ is a component of Γ, it is a preprojective translation quiver, and $x \in {}_d\Gamma$ implies $x^+ \subseteq {}_{d+1}\Gamma$.*

Proof. We show by induction on d that $x \in {}_d\Gamma$ implies $x^+ \subseteq {}_{d+1}\Gamma$. If $p \in x^+$ is a projective vertex, then by assumption $p^- = \{x\} \subseteq {}_d\Gamma$, and so $p \in {}_{d+1}\Gamma$. In particular, this gives the assertion for $d = 0$, since the immediate successors of any source are projective. It remains to consider $x \in {}_d\Gamma$, $y \in x^+$, with y non-projective. Then $\tau y \in x^-$, and thus $\tau y \in {}_{d-1}\Gamma$. By induction, $y^- = (\tau y^+) \subseteq {}_d\Gamma$. Thus $y \in {}_{d+1}\Gamma$.

In particular, this shows that ${}_\infty\Gamma$ is closed under neighbours, and therefore a union of components of Γ. Since by assumption ${}_\infty\Gamma$ has only finitely many projective vertices, it follows that ${}_\infty\Gamma$ is a preprojective translation quiver. Thus ${}_\infty\Gamma$ is connected, since any component of a preprojective translation quiver contains a source.

Since for any $t \in S^+$, the canonical map $R(t) \to P(t)$ is a monomorphism in $\ell(S)$, we can use Lemma 2.3.3 of **[12]**, and conclude that ${}_\infty\ell(S)$ is standard. Recall that we denote the dimension of the total space of V by $\underline{\dim}_\omega V$. Since $\underline{\dim}_\omega$ is additive on exact sequences of $\ell(S)$ and since $\underline{\dim}_\omega P(t) = 1$ for $t \in S^+$, and $\underline{\dim}_\omega R(t) = 1$ for $t \in S$, it follows that $\underline{\dim}_\omega$ coincides on $H(S)$ with $h_{H(S)}$. Thus, $h_{H(S)}$ satisfies Condition (3) for a left hammock. It also satisfies Condition (4), since the injective vertices are of the form $q(t)$ and $\underline{\dim}_\omega Q(t) = 1$, for all $t \in S_+$, and $\underline{\dim}_\omega N(t) = 1$, for all $t \in S$.

Note that usually not all $P(t)$ belong to ${}_\infty\ell(S)$. We denote by S' the set of all $t \in S$ with $P(t) \in {}_\infty\ell(S)$. Then S' is a filter in S, that is, $s \in S'$ and $s < t$ implies $t \in S'$ (for, if $s < t$, then $\mathrm{Hom}(P(t), P(s)) \neq 0$), and S' contains all maximal elements of S (for, if t is maximal in S, then $R(t) = P(\omega)$, and thus $P(t) \in {}_1\ell(S)$). We can recover S' from $H(S)$ as follows. Since ${}_\infty\ell(S)$ is standard, we have

$$\mathrm{Hom}_{k(H(S))}(p(s), p(t)) \approx \mathrm{Hom}_{\ell(S)}(P(s), P(t)),$$

for all $s, t \in S'$, and thus $\mathrm{Hom}_{k(H(S))}(p(s), p(t)) \neq 0$ if and only if $t \leq s$.

Let us show that given a projective vertex p of $H(S)$, we can describe combinatorially, without reference to k, the k-dimension of $\mathrm{Hom}_{k(H(S))}(p, x)$, for any vertex x of $H(S)$. Thus, fix some projective vertex $p = p(t)$. Let $H(S)[p]$ be the full translation subquiver of $H(S)$ with vertices the isomorphism classes $[Z]$ of the S-spaces Z with $Z_t \neq 0$. Clearly, this is a left hammock with hammock function $h_{H(S)[p]}$ given by $h_{H(S)[p]}([Z]) = \dim Z_t$. For, if $Z = P(t)$, then $\dim V_t = 1$; if $[Z]$ is a projective vertex of $H(S)$ different from p, say $Z = P(s)$ for some $s \neq t$, then $\dim P(s)_t = \dim R(s)_t$, whereas for $[Z]$ a non-projective vertex of $H(S)$, with Auslander–Reiten sequence

$$(*) \qquad 0 \to X \to Y \to Z \to 0$$

in $\ell(S)$, we have the exact sequence

$$0 \to X_t \to Y_t \to Z_t \to 0.$$

Note that the projective vertices of $H(S)[p]$ are of two kinds. First of all, there are the projective vertices $p(s)$ of $H(S)$, with $s \leq t$. Second, there are the non-projective vertices $z = [Z]$ of $H(S)$, with Auslander–Reiten sequence $(*)$, such that $X_t = 0$, $Y_t \neq 0$. Thus the latter are non-projective vertices z of $H(S)$ such that $\tau_{H(S)}(z)$ does not belong to $H(S)[p]$, whereas at least one of the elements $y_i \in z^-$ does belong to $H(S)[p]$. Inductively, we see that $H(S)[p]$ is uniquely determined by $H(S)$ and p, without reference to k, and so therefore is its hammock function $h_{H(S)[p]}$. But, as we have seen,

$$\dim \mathrm{Hom}_{k(H)}(p, z) = \dim Z_t = h_{H(S)[p]}(z),$$

for $z = [Z]$ a vertex of $H(S)[p]$, and

$$\dim \mathrm{Hom}_{k(H)}(p, z) = \dim Z_t = 0,$$

for $z = [Z]$ a vertex outside $H(S)[p]$.

It follows that $H(S)$ is independent of k, by induction on the number of elements of S. If S is empty, then $H(S)$ consists of a single vertex, so nothing has to be shown. If S is non-empty, choose a minimal element u of S such that either $P(u)$ does not belong to ${}_\infty\ell(S)$, or else it does not belong to some ${}_{d-1}\ell(S)$, whereas all $P(s)$, with $s \in S$, belong to ${}_d\ell(S)$. Let S_u be obtained from S by deleting u. We consider the S-spaces V with $V_u = 0$ as S_u-spaces. In particular, $R(u)$ can be considered as an S_u-space. By induction, $H(S_u)$ is independent of k, and we can decide combinatorially whether $[R(u)]$ belongs to $H(S_u)$, and if it belongs to $H(S_u)$, its precise position. If $[R(u)]$ does not belong to $H(S_u)$, we have $H(S) = H(S_u)$, for any field k. Otherwise $H(S)$ is the completion of ${}_dH(S)$, and ${}_dH(S)$ is obtained from ${}_dH(S_u)$ by adding the vertex $p(t)$ and an arrow $[R(u)] \to p(t)$.

This finishes the proof of the first part of Theorem 2. Let us consider now the second part of Theorem 2. Thus, let H be a thin left hammock with n projective vertices. Theorem 3 implies that $k(H) \approx {}_\infty(\mathfrak{P}(H, k)\text{-spmod})$ as categories, and $H \approx \Gamma_\infty(\mathfrak{P}(H, k)\text{-spmod})$ as translation quivers. Let us show that we can identify $\mathfrak{P}(H, k)$ with the category of finitely generated projective $A_{S(H)}$-modules for the partially ordered set $S(H)$ with

$$S(H)^+ = S(\mathfrak{P}(H, k))^*.$$

We use induction on n, the case where $n = 1$ being trivial. If $n \geq 2$, choose d minimal such that all projective vertices of H belong to ${}_dH$. Let p be a projective vertex of H not belonging to ${}_{d-1}H$, let y be the immediate predecessor of p, and denote the arrow $y \to p$ by α. Delete from ${}_dH$ the vertex p, and consider the completion H'. This again is a thin left hammock. Thus $\mathfrak{P}(H', k)$ may be considered as the category of finitely generated projective $A_{S(H')}$-modules. We consider the $\mathfrak{P}(H, k)$-module $M(y)$, and its restriction $M'(y)$ to $\mathfrak{P}(H', k)$. Clearly, we can consider $\mathfrak{P}(H, k)$ as the category of finitely generated projective B-modules, where B is the one-point extension of $A_{S(H')}$ by $M'(y)$. But $M'(y)$ is in $A_{S(H')}$-spmod; therefore we may consider $M'(y)$ as an $S(H')$-space. Note that $\dim M(y)(\omega) = h_H(y) = 1$, since H is thin. Now $M'(y)(\omega)$ is just the total space of $M'(y)$, considered as an $S(H')$-space. Let U be the set of projective vertices p' of H' with $M(y)(p') \neq 0$. Since $M'(y)$ is an $S(H')$-space with one-dimensional total space, we see that U is a filter on $S(H')$ and we have $p' < p$ in $S(H)$ if and only if p' belongs to U. (For, if $p' \in U$, then $M(y)(p') = M'(y)(p') \neq 0$, and $M(\alpha)$ is a monomorphism, from § 4. Thus $\operatorname{Hom}_{k(H)}(p', p) = M(y)(p) \neq 0$, and so $p' < p$. On the other hand, if $p_1 < p$, for some p_1 in $S(H)$, there is p_2, with $p_1 \leq p_2 < p$, in $S(H)$, with $\operatorname{Hom}_{k(H)}(p_2, p) \neq 0$. Thus also $M'(y)(p_2) = M(y)(p_2) = \operatorname{Hom}_{k(H)}(p_2, y) \neq 0$. Therefore $p_2 \in U$, and so $p_1 \in U$.) It follows that $M'(y)$, as an $S(H')$-space, is nothing other than $R(p)$, also considered as an $S(H')$-space. Since $A_{S(H)}$ is the one-point extension of $A_{S(H')}$ by the $S(H')$-module $R(p)$, we see that $B = A_{S(H)}$. This shows that

$$k(H) \approx {}_\infty(\mathfrak{P}(H, k)\text{-spmod}) \approx {}_\infty(A_{S(H)}\text{-spmod}) \approx {}_\infty\ell(S(H), k)$$

as categories, and

$$H \approx \Gamma_\infty(\mathfrak{P}(H, k)\text{-spmod}) \approx \Gamma_\infty\ell(S(H), k) = H(S(H)),$$

as translation quivers.

Finally, assume that H is a hammock, and that S is a partially ordered set with $H(S) \approx H$. Since ${}_\infty\ell(S, k)$ is finite, we must have $\ell(S, k) = {}_\infty\ell(S, k)$. Thus all $P(s)$, for $s \in S$, belong to ${}_\infty\ell(S, k)$, and therefore $s \mapsto [P(s)]$ gives an isomorphism $S \to S(H(S))$ of partially ordered sets. Thus S and $S(H)$ are isomorphic. This finishes the proof of Theorem 2.

For later reference, let us formulate the *recipe* for obtaining the partially ordered set $S(H)$ from the thin left hammock H. The elements of $S(H)$ are the projective vertices p, different from the source ω, and $p_1 \geqslant p_2$ in $S(H)$ if and only if $\mathrm{Hom}_{k(H)}(p_1, p_2) \neq 0$.

8. *Hammocks of Type* $\mathbb{A}_n$

Given a hammock H with source p and sink q, the length of any path from p to q in H will be called the *length* of H, and denoted by $\|H\|$. Note that the length of H is d if and only if $H = {}_dH \neq {}_{d-1}H$.

Given two partially ordered sets S and S', with a bijective, order-preserving map $S' \to S$, then S is called a *refinement* of S'. Note that if S is a refinement of S', we may consider $\ell(S)$ as a full subcategory of $\ell(S')$. (Let φ: $S' \to S$ be a bijective order-preserving map. If $V = (V_\omega, V_s)_{s \in S}$ is an S-space, then V', with $V'_\omega = V_\omega$, $V'_s = V_{\varphi(s)}$, is an S'-space, and in this way, we obtain a full exact embedding of $\ell(S)$ into $\ell(S')$.) Note that if S is a refinement of S', and S' is representation-finite, then S is also representation-finite, and $\|H(S)\| \leqslant \|H(S')\|$.

If S is a finite chain, say with n elements, then $H(S)$ is a linearly oriented quiver of Type $\mathbb{A}_{n+1}$, with all vertices projective (and thus also injective). It follows that in this case, the length of $H(S)$ is equal to n.

Any partially ordered set S has a refinement which is a chain. It follows that for a representation-finite partially ordered set S, we have $\|H(S)\| \geqslant |S|$, where $|S|$ denotes the number of elements of S. The representation-finite partially ordered sets S with $\|H(S)\| = |S|$ are characterized by the next proposition.

Proposition. *Let S be a finite partially ordered set. The following conditions are equivalent:*

(i) $\mathrm{width}(S) \leqslant 2$;

(ii) $h_{H(S)}(x) = 1$ *for any* $x \in H(S)_0$;

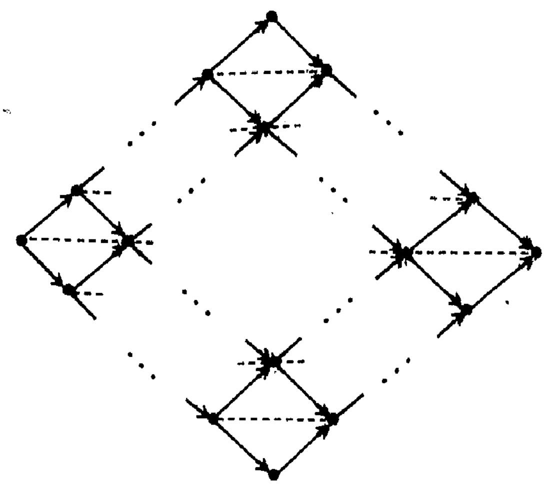
(iii) *the length of* $H(S)$ *is equal to the cardinality of* S;

(iv) $|z^{-1}| \leqslant 2$, *and* $|z^{+}| \leqslant 2$, *for any* $z \in H(S)_0$.

When these conditions are satisfied, we call $H(S)$ a hammock of Type $\mathbb{A}_n$.

Remark. Note that we may reformulate (ii) as follows: the total space of any indecomposable S-space is one-dimensional. In this way, we see that the equivalence of (i) and (ii) is well known. If S contains three pairwise incomparable elements, then it is easy to construct an indecomposable S-space with 2-dimensional total space (see the proof of the implication (ii)$\Rightarrow$(i) below). On the other hand, if S has width at most 2, then any S-space may be decomposed as a direct sum of S-spaces with one-dimensional total spaces. The usual proof providing such a decomposition uses filtrations of vector spaces and their refinements. The obvious proof of the implication (i)$\Rightarrow$(ii) using Auslander–Reiten theory will be exhibited below.

Proof of the proposition. (i) $\Rightarrow$ (ii), (iii). First, let S be the disjoint union of two chains. Then $H(S)$ can be constructed without difficulty step by step, and $H(S)$ is seen to be of the form



It follows easily that $h_{H(S)}(x)=1$ for all $x \in H(S)_0$. Thus any indecomposable S-space has a one-dimensional total space. Also, $\|H(S)\|=|S|$.

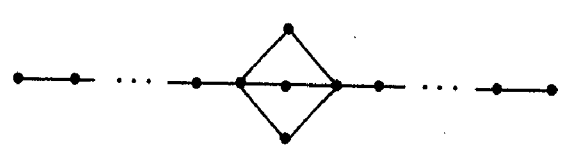
If now S is an arbitrary representation-finite partially ordered set of width 2, then S may be considered as a refinement of some partially ordered set S' which is the disjoint union of two chains. Since $\ell(S)$ may be considered as a full subcategory of $\ell(S')$, we see that the total space of any indecomposable S-space is one-dimensional and that

$$\|H(S)\| \leqslant \|H(S')\| = |S'| = |S|.$$

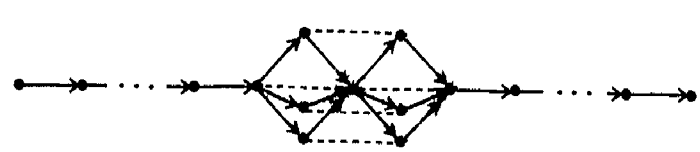
Of course, we also have $\|H(S)\| \geqslant |S|$, and thus equality.

(ii) $\Rightarrow$ (i). Assume S contains three pairwise incomparable elements s_1, s_2, s_3. Let $V=(V_\omega, V_s)_{s\in S}$ be defined by $V_\omega = k^2$, $V_{s_1}=k\times 0$, $V_{s_2}=0\times k$, $V_{s_3}=\{(\alpha,\alpha)\mid \alpha\in k\}$, and $V_t=0$ if $t<s_i$ for some i, whereas $V_t=V_\omega$ for the remaining t. Clearly, V is an S-space, V is indecomposable, and its total space is 2-dimensional.

(iii) $\Rightarrow$ (i). Again, assume S contains three pairwise incomparable elements s_1, s_2, s_3. Let S'' be a refinement of S such that (the images of) s_1, s_2, s_3 in S'' are still pairwise incomparable, whereas all other pairs of elements of S'' are comparable. Thus, S'' is of the form



and $\|H(S'')\| \leqslant \|H(S)\|$. However, it is easy to see that $H(S'')$ is of the form



and $\|H(S'')\| = |S''|+1$. Therefore, $\|H(S)\| \geqslant |S|+1$.

(ii) $\Rightarrow$ (iv). Let $h_{H(S)}(x)=1$ for all $x\in H(S)_0$. If z is projective, then $|z^-|\leqslant 1$. If z is not projective, then

$$1 = h_{H(S)}(z) = (\Sigma h_{H(S)})(z^{(-)}) - h_{H(S)}(\tau z) = |z^-| - 1,$$

and therefore $|z^-|=2$. If z is injective, then Corollary 1 shows that $|z^+|\leqslant 1$. If z is not injective, then $|z^+|=|(\tau^- z)^-|=2$.

(iv)$\Rightarrow$(ii). Assume there exists $x\in H(S)_0$ with $h_{H(S)}(x)\geqslant 2$. Take such an $x\in {}_dH(S)$, with d minimal. Thus $h_{H(S)}(y)=1$ for all proper predecessors y of x. Since $H(S)$ is a hammock, x cannot be projective. Therefore

$$h_{H(S)}(x)=(\Sigma h_{H(S)})(x^{(-)})-h_{H(S)}(\tau x)=|x^-|-1,$$

and so $|x^-|\geqslant 3$.

9. *Application*

Let k be an algebraically closed field, and A a finite-dimensional k-algebra. Let E be a simple A-module with projective cover $P(E)$ and injective envelope $Q(E)$. Recall that the dimension of the vector space $\mathrm{Hom}(P(E), X)$ is the multiplicity of E as a factor in any composition series of the A-module X. There is a filtration of $\mathrm{Hom}(P(E), X)$ given by the subspaces $X(d)=\mathrm{rad}^d(P(E), X)$, and we denote by $\bar{X}(d)$ the corresponding factor-spaces $X(d)/X(d+1)$. By definition, $X(d)=\mathrm{Hom}(P(E), X)$ for $d\leqslant 0$; thus $\bar{X}(d)=0$ for $d<0$. The dimension of $\bar{X}(d)$ counts the multiplicity of those Jordan–Hölder factors of X which can be reached from $P(E)$ by maps in rad^d, but not in rad^{d+1}. Now we define a translation quiver $H(E)$ as follows. Its vertices are the pairs $([X], d)$, where X is an indecomposable A-module with $\bar{X}(d)\neq 0$. There are arrows $([X], d)\to([Y], e)$ only for $e=d+1$, and the number of arrows is equal to $\dim_k \mathrm{Irr}(X, Y)$ (recall that $\mathrm{Irr}(X, Y)=\mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$ measures the set of irreducible maps from X to Y). Finally, $([X], d)$ is projective when $(\tau_A X)(d-2)=0$; otherwise let $\tau([X], d)=([\tau_A X], d-2)$.

PROPOSITION. *$H(E)$ is a left hammock, with hammock function $h_{H(E)}(([X], d))=\dim_k \bar{X}(d)$. Also, $H(E)$ is a hammock if and only if there is some $r\in\mathbb{N}$ such that the canonical map $P(E)\to Q(E)$ with image E does not belong to* $\mathrm{rad}^r(P(E), Q(E))$.

Proof. Any $\varphi\in X(d)$, with $d\geqslant 1$, can be written in the form $\varphi=\sum\alpha_i\beta_i$ with $\alpha_i\in Y_i(d-1)$, $\beta_i\in\mathrm{rad}(Y_i, X)$ and all Y_i indecomposable. If φ, in addition, does not belong to $X(d+1)$, then there is some i with $\alpha_i\notin Y_i(d)$, $\beta_i\notin\mathrm{rad}^2(Y_i, X)$. Then $([Y_i], d-1)$ is a vertex of $H(E)$ and there is an arrow $([Y_i], d-1)\to([X], d)$ in $H(E)$. It follows that no $([X], d)$, with $d\geqslant 1$, is a source. On the other hand, $\bar{X}(0)\neq 0$ is only possible for $X\approx P(E)$, and actually $\overline{P(E)}(0)\neq 0$. Thus $H(E)$ has a unique source, namely $([P(E)], 0)$. It follows that any $([X], d)$ belongs to ${}_dH(E)\setminus{}_{d-1}H(E)$. (For, by definition, the immediate predecessors of $([X], d)$ are of the form $([Y], d-1)$, and, as we have seen above, there is an immediate predecessor, provided $d>0$.) In particular, $H(E)={}_\infty H(E)$, and therefore $h_{H(E)}$ is defined, and we have

$$h_{H(E)}(([P(E)], 0))=1=\dim_k \overline{P(E)}(0).$$

In order to see that $h_{H(E)}(([X], d))=\dim_k \bar{X}(d)$ always (so that $h_{H(E)}$ takes only positive values) and that $h_{H(E)}$ satisfies Condition (4) for a left hammock, we use results of Igusa and Todorov **[10]**. First of all, given an indecomposable projective module P with radical $\bigoplus Y_i^{m_i}$, all Y_i being indecomposable, Lemma 4.2

of **[10]** asserts that the inclusion map induces an isomorphism

$$\bigoplus \bar{Y}_i(d)^{m_i} \to \bar{P}(d+1),$$

for every $d \geqslant 0$. Thus

$$\dim_k \bar{P}(d+1) = \sum_i m_i \dim_k \bar{Y}_i(d).$$

Also, the main theorem of **[10]** asserts that for every $d \geqslant -1$, and every Auslander–Reiten sequence

$$0 \to X \to \bigoplus_i Y_i^{m_i} \to Z \to 0,$$

with all Y_i indecomposable, we obtain an exact sequence

$$0 \to \bar{X}(d) \to \bigoplus \bar{Y}_i(d+1)^{m_i} \to \bar{Z}(d+2) \to 0.$$

Thus

$$\dim_k \bar{Z}(d+2) = \sum_i m_i \dim_k \bar{Y}(d+1) - \dim_k \bar{X}(d).$$

Using induction on d, we see that $([X], d) \mapsto \dim_k \bar{X}(d)$ is the hammock function on $H(E)$, and that

$$h_{H(E)}(([X], d)) = (\Sigma h_{H(E)})(([X], d)^{(+)})$$

if X is an indecomposable, non-injective module and $\bar{X}(d) \neq 0$, but $([X], d)$ is an injective vertex of $H(E)$. It remans to verify Condition (4) for a left hammock for vertices of $H(E)$ of the form $([Q], d)$ with Q an indecomposable injective A-module. First, consider the case where $Q = Q(E)$, the injective envelope of E, and assume the inclusion map μ: $E \to Q(E)$ belongs to

$$\operatorname{rad}^{e'}(E, Q(E)) \backslash \operatorname{rad}^{e'+1}(E, Q(E))$$

(or, equivalently, that e' is maximal with the property that the image of $\operatorname{Hom}(-, \mu)$ lies in $\operatorname{rad}^{e'}(-, Q(E))$, and that the canonical projection π: $P(E) \to E$ lies in $E(e) \backslash E(e+1)$. Then according to the main theorem of **[10]**, μ induces a monomorphism

$$\bar{E}(e) \to \overline{Q(E)}(e+e'),$$

and so $\pi\mu \in Q(E)(e+e') \backslash Q(E)(e+e'+1)$. Since every non-zero map φ: $P(E) \to X$ can be composed with a map ψ: $X \to Q(E)$ in order to have $\varphi\psi = \pi\mu$, we see that $X(e+e'+1) = 0$ for all X. Thus $([Q(E)], e+e')$ is a sink of $H(E)$. We always denote by ε: $Q \to Y = Q/\operatorname{soc} Q$ the canonical projection, and we decompose $Y = \bigoplus_i Y_i^{m_i}$, with all Y_i indecomposable. Consider again the case where $Q = Q(E)$, and take some $\overline{Q(E)}(d)$, where $\pi\mu$ belongs to $Q(E)(d+1)$ (thus either $d \leqslant e+e'$, or one of e, e' or both are not defined). Then, ε induces a monomorphism

$$(*) \qquad Q(d) \to \bigoplus Y_i(d+1)^{m_i},$$

and, similarly, if $Q \neq Q(E)$, then μ always induces a monomorphism $(*)$. Using the fact that $P(E)$ is projective and that ε is surjective, we easily see that these maps $(*)$ are bijective. This shows that

$$h_{H(E)}(([Q], d)) = (\Sigma h_{H(E)})(([Q], d)^{(+)}),$$

except when $([Q], d)$ is a sink.

Of course, a left hammock is a hammock if and only if it contains a sink. If there is some r with $\pi\mu \notin Q(E)(r)$, there is e with $\pi \in E(e)\backslash E(e+1)$, and e' with $\mu \in \mathrm{rad}^{e'}(E, Q(E))\backslash \mathrm{rad}^{e'+1}(E, Q(E))$. Thus $([Q(E)], e+e')$ is a sink. On the other hand, if $\pi\mu \in Q(E)(r)$ for all r, then at least one of e, e' cannot exist, since otherwise we would have $Q(E)(e+e'+1)=0$. Thus $H(E)$ does not have a sink.

10. *Directed algebras*

Given an A-module M, we denote by $\mathrm{Hom}(M, A\text{-mod})$ the category defined as follows: its objects are the A-modules, and

$$\mathrm{Hom}_{\mathrm{Hom}(M,A\text{-mod})}(X, Y) = \mathrm{Hom}_A(X, Y)/\sim,$$

where for f, g: $X \to Y$ we have $f \sim g$ if and only if $\mathrm{Hom}(M, f) = \mathrm{Hom}(M, g)$. Given a map f in A-mod, we denote by $\tilde{f} = \mathrm{Hom}(M, f)$ the residue class of f in $\mathrm{Hom}(M, A\text{-mod})$. Similarly we may write $\tilde{X} = \mathrm{Hom}(M, X)$ instead of X, when we consider X as an object of $\mathrm{Hom}(M, A\text{-mod})$.

In the following discussion, we shall usually deal with the case when M is projective. Given an object class $\mathfrak{M}$ in a Krull–Schmidt category $\mathfrak{K}$, we denote by $\mathfrak{K}/\mathfrak{M}$ the factor category of $\mathfrak{K}$ modulo the ideal of maps which factor through objects in $\mathfrak{M}$. (Thus, the objects of $\mathfrak{K}/\mathfrak{M}$ are the objects in $\mathfrak{K}$ and

$$\mathrm{Hom}_{\mathfrak{K}/\mathfrak{M}}(X, Y) = \mathrm{Hom}_{\mathfrak{K}}(X, Y)/\mathrm{Hom}_{\mathfrak{K}}(X, Y)_{\mathfrak{M}},$$

where $\mathrm{Hom}_{\mathfrak{K}}(X, Y)_{\mathfrak{M}}$ is the subspace of all maps in $\mathrm{Hom}_{\mathfrak{K}}(X, Y)$ which factor through an object in $\mathfrak{M}$.) The module classes $\mathfrak{M}$ we are interested in will be of the following form. There is a given projective A-module P, and $\mathfrak{M}$ is the module class of all A-modules M with $\mathrm{Hom}(P, M) = 0$. Now, given a projective module P, there exists an idempotent e in A such that for any A-module M, we have $\mathrm{Hom}(P, M) = 0$ if and only if $eM = 0$, and thus if and only if M is an $A/\langle e \rangle$-module. This shows that $\mathfrak{M}$ is the class of all $A/\langle e \rangle$-modules.

LEMMA 1. *Let P be a projective A-module, and $\mathfrak{M}$ the module class of all A-modules M with* $\mathrm{Hom}(P, M) = 0$. *Then* $\mathrm{Hom}(P, A\text{-mod}) = A\text{-mod}/\mathfrak{M}$.

Proof. Clearly, given $\varphi \in \mathrm{Hom}_A(X, Y)_{\mathfrak{M}}$, we have $\mathrm{Hom}(P, \varphi) = 0$. On the other hand, assume there is given ψ: $X \to Y$ with $\tilde{\psi} = \mathrm{Hom}(P, \psi) = 0$. We factor $\psi = \psi_1\psi_2$ with ψ_1: $X \to I$ an epimorphism, ψ_2: $I \to Y$ a monomorphism. We claim that $I \in \mathfrak{M}$. For, given α: $P \to I$, there is α' with $\alpha = \alpha'\psi_1$, since P is projective. Thus $\alpha\psi_2 = \alpha'\psi_1\psi_2 = 0$, since it is the image of α' under $\tilde{\psi}$. Since ψ_2 is a monomorphism, $\alpha = 0$.

We consider now the special case of A being a directed algebra. Again, let k be an algebraically closed field, A a finite-dimensional k-algebra, E a simple A-module, and $P(E)$ its projective cover. Let $\mathfrak{M}_E$ be the module class of all A-modules M with $\mathrm{Hom}(P(E), M) = 0$, and thus of all A-modules M which have no composition factor of the form E.

According to the lemma, $\mathrm{Hom}(P(E), A\text{-mod}) = A\text{-mod}/\mathfrak{M}_E$. Assume now, in addition, that A is representation-finite and that the Auslander–Reiten quiver Γ_A of A is directed (that is, A is a directed algebra). Let H_E be the full translation subquiver of Γ_A given by all vertices $[X]$ of Γ_A, with X an indecomposable A-module outside $\mathfrak{M}_E$.

LEMMA 2. *The categories* Hom$(P(E), A\text{-mod})$ *and* $k(H_E)$ *are equivalent.*

Proof. Since A is directed, we know that A-mod is equivalent to $k(\Gamma_A)$ (see [**12**, 2.4.11]), with an indecomposable module X corresponding to the object $[X]$ in $k(\Gamma_A)$. Let $\mathfrak{M}'_E$ be the full additive subcategory of $k(\Gamma_A)$ whose indecomposable objects are of the form $[X]$ with X an indecomposable A-module in $\mathfrak{M}_E$. Of course, $\mathfrak{M}_E$ and $\mathfrak{M}'_E$ correspond to each other under the equivalence $A\text{-mod} \approx k(\Gamma_A)$. Since $k(H_E) \approx k(\Gamma_A)/\mathfrak{M}'_E$, we obtain the required equivalence

$$\mathrm{Hom}(P(E), A\text{-mod}) \approx A\text{-mod}/\mathfrak{M}_E \approx k(\Gamma_A)/\mathfrak{M}'_E \approx k(H_E).$$

On the vertices of Γ_A, we define the function

$$\underline{\dim}_E([X]) = \dim_k \mathrm{Hom}(P(E), X),$$

which, as we have mentioned, counts the multiplicity of E as Jordan–Hölder factor of X, and H_E is just the support of this function.

PROPOSITION. *H_E is a hammock, with hammock function* $\underline{\dim}_E$.

Proof. With Γ_A, also H_E is finite and directed. Also, H_E has a unique source, namely $[P(E)]$. For, if $[X]$ belongs to H_E, then $\mathrm{Hom}(P(E), X) \neq 0$; thus there is a chain of irreducible maps

$$P(E) = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n = X$$

with non-zero composition f. But then $\mathrm{Hom}(P(E), X_i) \neq 0$, for all i, and therefore the path

$$[P(E)] = [X_0] \to [X_1] \to \dots \to [X_n] = [X]$$

of Γ_A is completely contained in H_E. Thus, $[P(E)]$ is the unique source of H_E, and obviously $\underline{\dim}_E([P(E)]) = 1$. The function $\dim \mathrm{Hom}(P(E), -)$ is additive on exact sequences, and so given an Auslander–Reiten sequence

$$0 \to X \to \bigoplus_i Y_i^{m_i} \to Z \to 0$$

in A-mod, we have

$$\underline{\dim}_E([Z]) = (\Sigma\, \underline{\dim}_E)([Z]^{(-)}) - \underline{\dim}_E(\tau[X]).$$

Also, given an indecomposable projective module P, not isomorphic to $P(E)$, with radical Y, we have

$$\dim_k \mathrm{Hom}(P(E), P) = \dim_k \mathrm{Hom}(P(E), Y).$$

Thus

$$\underline{\dim}_E([P]) = (\Sigma\, \underline{\dim}_E)([P]^{(-)}).$$

Altogether, this shows that we have $h_{H_E} = \underline{\dim}_E$. Finally, if Q is an indecomposable injective module, not isomorphic to the injective envelope $Q(E)$ of E, and $Y = Q/\mathrm{soc}\, Q$, then

$$\dim_k \mathrm{Hom}(P(E), Q) = \dim_k \mathrm{Hom}(P(E), Y).$$

Thus

$$\underline{\dim}_E([Q]) = (\Sigma\, \underline{\dim}_E)([Q]^{(+)}).$$

However, $Q(E)$ is a sink in H_E. This finishes the proof that H_E satisfies all the conditions of a left hammock.

COROLLARY. *Let A be a directed algebra, and E a simple A-module. The hammocks $H(E)$ and H_E may be identified under the correspondence*

$$([X], d) \mapsto [X].$$

Proof. We obtain a covering of translation quivers $H(E) \to H_E$ if we define $([X], d) \mapsto [X]$. However, H_E is simply connected, and thus this covering is an isomorphism of translation quivers.

In the following, we write $\underline{\dim}_E X$ instead of $\underline{\dim}_E([X])$.

COROLLARY (v. Höhne). *Let A be a directed algebra, and E a simple A-module. If P is an indecomposable projective A-module, then $\underline{\dim}_E P \leqslant 1$. If Q is an indecomposable injective A-module, then $\underline{\dim}_E Q \leqslant 1$. If Z is indecomposable and not projective, then*

$$|\underline{\dim}_E Z - \underline{\dim}_E \tau Z| \leqslant 1.$$

Proof. This is just Corollary 3 of § 1.

The original proof by v. Höhne [**8**] of this result is based on a factorization of the corresponding Coxeter transformation as a product of reflections. For the convenience of the reader, we give an additional proof (without reference to hammocks or reflections) in the appendix: we use quadratic forms and roots. The result may be used for a different approach to hammocks (which actually was our first one). One does not presuppose the knowledge given by Theorem 1 (or Corollary 2 to Theorem 1), and one defines a hammock as a finite, *thin*, left hammock. v. Höhne's result is then used in order to establish that H_E is a hammock, for E a simple module over a directed algebra.

A special case of this corollary plays the essential role, namely the following: if Y is an indecomposable A-module with $\underline{\dim}_E Y \neq 0$ and $\underline{\dim}_E \tau Y = 0$, then $\underline{\dim}_E Y = 1$. The set of isomorphism classes $[Y]$ of indecomposable A-modules Y different from $P(E)$ with $\underline{\dim}_E Y \neq 0$ and $\underline{\dim}_E \tau Y = 0$ will be denoted by S_E. We define a relation $\geqslant$ on S_E by

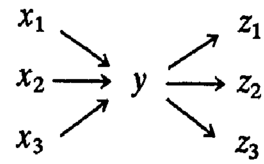
$$[X] \geqslant [Y] \quad \text{if and only if} \quad \operatorname{Hom}(P(E), X)\operatorname{Hom}(X, Y) \neq 0,$$

and we obtain in this way a partially ordered set. (The relation is anti-symmetric, since A is directed. In order to see that the relation is transitive, we have to use that $\underline{\dim}_E X = 1$ for $[X] \in S_E$.) Combining the observations above with Theorem 2, we obtain

THEOREM. $\operatorname{Hom}(P(E), A\text{-mod}) \approx \ell(S_E)$, *as categories, and* $H_E \approx \Gamma\ell(S_E)$, *as translation quivers.*

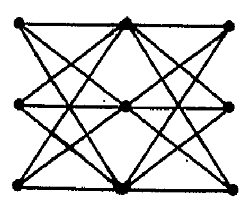
Proof. From Lemma 2, we have $k(H_E) \approx \operatorname{Hom}(P(E), A\text{-mod})$. The recipe at the end of § 7 shows that $S_E = S(H_E)$. From Theorem 2, we obtain $k(H_E) \approx \ell(S(H_E)) = \ell(S_E)$ and $H_E \approx \Gamma\ell(S(H_E)) = \Gamma\ell(S_E)$.

REMARK. We stress that not all hammocks can be realized in the form $\Gamma\,\mathrm{Hom}(P, A\text{-mod})$, with P indecomposable projective, and A directed. Equivalently, not all representation-finite partially ordered sets S are of the form S_E, with E a simple A-module, A directed. To see this, assume there is a hammock H containing the following subquiver

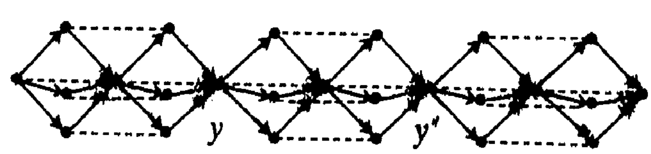


with x_1, x_2, x_3 injective vertices in H, and z_1, z_2, z_3 projective vertices in H. Assume that $H=\Gamma\,\mathrm{Hom}(P, A\text{-mod})$ for some indecomposable projective A-module P, and A directed. We claim that in this case $y=[P/\mathrm{rad}\,P]$. For, let $x_i=[\tilde{X}_i]$, $y=[\tilde{Y}]$, $z_i=[\tilde{Z}_i]$, $1 \leqslant i \leqslant 3$, with indecomposable A-modules X_i, Y, Z_i. Now, Y has at least the three immediate successors Z_1, Z_2, Z_3 in $\Gamma(A)$. Assume there exists an additional immediate successor Z_4 in $\Gamma(A)$. Then, the source map for Y is the direct sum of at least four indecomposable summands. Thus three of the modules $Z_1, \ldots, Z_4$ are not projective, by a result of Bautista and Brenner (see **[2]** or **[12**, 6.4.2]). We can assume that Z_1, Z_2 are not projective, but then X_1, X_2, X_3, τZ_1, τZ_2 are five pair-wise different immediate predecessors of Y in $\Gamma(A)$, contradicting the Bautista–Brenner theorem. Dually, we see that X_1, X_2, X_3 are the only immediate predecessors of Y in $\Gamma(A)$. It follows that all the Z_i are projective, since if one of them, say Z_1, is not projective, then τZ_1 is an additional immediate predecessor of Y in $\Gamma(A)$. Thus, the source map for Y ends in a projective module, and therefore Y is simple **[1]**. Since $\tilde{Y} \neq 0$, it follows that $Y = P/\mathrm{rad}\,P$.

Now consider the following partially ordered set S:



Then $H(S)$ is of the form



If $S=S_E$ for some simple A-module E, then both y, y' are equal to $[\tilde{E}]$, which is impossible.

Appendix

Let A be a directed algebra over an algebraically closed field k, with simple modules $E(i)$, for $1 \leqslant i \leqslant n$. We identify $K_0(A)$ with $\mathbb{Z}^n$, using as basis the elements $e(i)=[E(i)]$. Given an A-module M, we denote the corresponding element in $K_0(A)$ by $\underline{\dim}\,M$.

PROPOSITION (v. Höhne). *Let* $0 \to X \to Y \to Z \to 0$ *be an Auslander–Reiten sequence in* A-mod. *Let* $x=\underline{\dim}\,X$, $z=\dim Z$. *Then* $|x_i - z_i| \leqslant 1$, *for all* $1 \leqslant i \leqslant n$.

The proof will rely on an investigation concerning positive roots of the quadratic form χ_A which we recall. We denote by C_A the Cartan matrix of A, by $\phi_A = -C_A^{-T}C_A$ its Coxeter matrix. There is a symmetric bilinear form on $K_0(A)$ given by $(x, y) = \frac{1}{2}x(C^{-1} + C^{-T})x^T$, and χ_A denotes the corresponding quadratic form.

Lemma. *Assume χ_A is weakly positive, and let $x \in K_0(A)$ be a positive root for χ_A. Then $(x\phi_A)_i \leq x_i + 1$, $(x\phi_A^{-1})_i \leq x_i + 1$, for all $1 \leq i \leq n$.*

Proof. Let $p(i) = e(i)C^T$. Then $p(i)$ is the dimension vector of an indecomposable (projective) A-module. Thus $p(i)$ is positive, and $\chi_A(p(i)) = 1$. Now

$$\begin{aligned} x_i - (x\phi_A)_i &= (x - x\phi)e(i)^T = x(I + C^{-T}C)e(i)^T \\ &= x(C^{-1} + C^{-T})Ce(i)^T \\ &= x(C^{-1} + C^{-T})p(i)^T \\ &= 2(x, p(i)) \\ &= \chi_A(x, p(i)) - \chi_A(x) - \chi_A(p(i)). \end{aligned}$$

Since $x + p(i)$ is positive, $\chi_A(x + p(i)) \geq 1$. Since both x and $p(i)$ are roots for χ_A, it follows that $x_i - (x\phi_A)_i \geq -1$.

For the second assertion, we may argue in the same way. Alternatively, note that with A its opposite A^{op} is also directed. Of course, the Cartan matrix for A^{op} is C_A^T. Therefore, its Coxeter matrix is $-C_A^{-1}C_A^T = \phi_A^{-1}$, and $\chi_{(A^{\mathrm{op}})} = \chi_A$.

Corollary. *Assume χ_A is weakly positive, and let x and $x\phi_A$ be positive roots for χ_A. Then $|x_i - (x\phi_A)_i| \leq 1$.*

Proof. Applying the second assertion of the lemma to χ_A, we see that $x_i = (x\Phi_A\Phi_A^{-1})_i \leq (x\Phi_A)_i + 1$. Of course, we also have $(x\Phi_A)_i \leq x_i + 1$.

Remark. Let χ_A be weakly positive, and x a positive root. Of course, $x\phi_A$ is again a root, but not necessarily positive. If $x\phi_A$ is not positive, then we may have $x_i \geq (x\phi_A)_i + 2$. For example,

$$C_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the Cartan matrix of a directed algebra B with global dimension 2; thus χ_B is weakly positive (by [**12**, 2.3.9] or direct calculation). Both $(0\ 0\ 0\ 1)$ and $(0\ 0\ 1\ 1)$ are positive roots for χ_B, but

$$(0\ 0\ 0\ 1)\Phi_B = (-2\ \ -1\ \ -1\ \ 1),$$
$$(0\ 0\ 1\ 1)\Phi_B = (-1\ \ 0\ \ -1\ \ 0).$$

Proof of the proposition. Let $0 \to X \to Y \to Z \to 0$ be an Auslander–Reiten sequence and let $x = \underline{\dim}\, X$, $z = \underline{\dim}\, Z$. Replacing, if necessary, A by the restriction of A to the support of Y, we can assume that Y is sincere. According to [**12**, 6.4.1], one of the modules X, Z or an indecomposable direct summand of Y is sincere. Thus A is sincere, and therefore χ_A is weakly positive [**12**, 2.4.9]. Also, proj. $\dim Z = 1$ (otherwise, we have $\mathrm{Hom}(Q, X) \neq 0$ for some indecomposable injective module Q, see [**12**, 2.4.1]). Then $\mathrm{Hom}(Y_i, Q) \neq 0$ for some indecomposable direct summand Y_i of Y, since Y is sincere. But then $Q \leqslant X < Y_i \leqslant Q$ is a cycle in A-mod. Similarly, $\mathrm{Hom}(X, {}_A A) = 0$. Therefore, $x = z\Phi_A$, see [**12**, 2.4.4]. On the other hand, both x and z are positive roots for χ_A, see [**12**, 2.4.8]. We can apply the corollary above and obtain that $|x_i - z_i| \leqslant 1$, for all $1 \leqslant i \leqslant n$. This finishes the proof.

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