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EVERY SEMIPRIMARY RING IS THE ENDOMORPHISM RING OF A PROJECTIVE MODULE OVER A QUASI-HEREDITARY RING

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ABSTRACT. The paper provides a proof of the following statement: Given a semiprimary ring R, there is a quasi-hereditary ring A and an idempotent $e \in A$ such that $R \simeq eAe$.

In his Queen Mary College lectures [A], M. Auslander has shown that any left artinian ring R occurs as the endomorphism ring of a projective A-module P_A where A is a semiprimary ring of finite global dimension. This ring A is constructed as follows: let N be the radical of R, let n be its nilpotency index, and $M_R = \bigoplus_{i=1}^{n}(R/N^i)_R$; take $A = \mathrm{End}(M_R)$. Clearly, there is an idempotent e of A with $R = eAe$; thus R is the endomorphism ring of the projective A-module eA. Hence the main step of the proof is the verification that A has global dimension at most n. In case that R is an Artin algebra, also A is an Artin algebra. As Auslander has stressed, his result asserts that the Artin algebras of finite global dimension determine all Artin algebras.

The aim of this note is to show, that the so constructed ring A is, in fact, quasi-hereditary in the sense of Cline–Parshall–Scott [S], [CPS]. For the definition and properties of quasi-hereditary rings we refer to [CPS], [PS], and also to [DR$_1$]; we point out, in particular, that they always have finite global dimension. We recall that the notion of a quasi-hereditary ring was introduced in order to have available a class of rings whose bounded derived categories can be built up from the bounded derived categories of division rings using "recollement".

Actually, the ring R we will start with may be an arbitrary semiprimary ring. Let N be its radical, n the nilpotency index of N, and $M_R = \bigoplus_{i=1}^{n}(R/N^i)_R$. We consider $A = \mathrm{End}(M_R)$; this is again a semiprimary ring. Let J_t be the set of all endomorphisms φ of M_R which factor through a module in $\mathrm{add}(\bigoplus_{i=1}^{t} R/N^i)_R$; this is obviously an ideal of A.

Theorem. *The chain of ideals* $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A$ *is a hereditary chain, thus* A *is quasi-hereditary.*

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As a consequence, we see that any semiprimary ring is the endomorphism ring of a projective module over a quasi-hereditary ring. In this sense, the quasi-hereditary rings determine all semiprimary rings.

The proof of the theorem is similar to our construction of a heredity chain in an Auslander algebra [DR$_2$]. The common criterion for the construction of heredity ideals in factor rings of endomorphism rings is formulated in §1, and §2 gives the proof of the theorem.

1. Heredity ideals in factor rings of endomorphism rings

In this section we consider modules over an arbitrary ring R. Given a module M, the direct sum of c copies of M will be denoted by cM, and $\operatorname{add} M$ denotes the category of all direct summands of finite direct sums of M. A homomorphism $\alpha : X \to Y$ is said to factor through $\operatorname{add} M$ provided it factors through a module in $\operatorname{add} M$.

Proposition. *Let M', M, M'' be modules with semiprimary endomorphism rings. Assume the following two conditions are satisfied:*

(a) *If X, Y are indecomposable direct summands of M, and $\gamma: Y \to X$ is a noninvertible map, then γ factors through* $\operatorname{add} M'$.

(b) *If X, Y are indecomposable direct summands of $M \oplus M''$, and $\delta : Y \to C$, $\gamma : C \to X$ are maps with $C \in \operatorname{add} M$ such that $\gamma\delta$ factors through $\operatorname{add} M'$, then there exists a decomposition*

$$C = C_1 \oplus C_2, \gamma = [\gamma_1, \gamma_2], \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

such that both γ_1 and δ_2 factor through $\operatorname{add} M'$.

Let $A = \operatorname{End}(M' \oplus M \oplus M'')$, let J' be the set of elements of A which factor through $\operatorname{add} M'$, and J be the set of elements of A which factor through $\operatorname{add}(M' \oplus M)$. Then J/J' is a heredity ideal of A/J'.

Proof. Let $\bigoplus_{i=1}^{p} M_i$ be a direct summand of M such that $M_1, \dots, M_p$ are pairwise nonisomorphic indecomposable modules and not isomorphic to direct summands of M' and such that $\operatorname{add}(M' \oplus \bigoplus_{i=1}^{p} M_i) = \operatorname{add}(M' \oplus M)$. We may assume that $M = \bigoplus_{i=1}^{p} M_i$ by replacing M'' by $M'' \oplus \widetilde{M}$, where $\widetilde{M}$ is a direct complement of $\bigoplus_{i=1}^{p} M_i$ in M. Let e_i be the projection of $M' \oplus M \oplus M'' = M' \oplus (\bigoplus_{i=1}^{p} M_i) \oplus M''$ onto M_i, and $e = \sum_{i=1}^{p} e_i$; furthermore, e' be the projection of $M' \oplus M \oplus M''$ onto M'. Note that $e', e_1, \dots, e_p$ are orthogonal idempotents, and that $e_1, \dots, e_p$ are pairwise inequivalent and primitive. We have $J' = Ae'A$, $J = A(e' + e)A$. Consider the residue ring $\overline{A} = A/J'$, and denote the residue class of $\alpha \in A$ in $\overline{A}$ by $\bar{\alpha}$; a similar notation will be used for subsets of A. Clearly, $\overline{J}$ is idempotent, and $\overline{J} = \overline{A}\bar{e}\overline{A}$. It follows from condition (a) that $\bar{e}\overline{A}\bar{e}$ is the product of the division rings $\bar{e}_i\overline{A}\bar{e}_i \simeq End(M_i)/\operatorname{rad}\operatorname{End}(M_i)$ for $1 \le i \le p$; thus $\bar{e}\overline{A}\bar{e} \simeq \operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M)$ and $\bar{e}\operatorname{rad}\overline{A}\bar{e} = 0$. It remains to be seen that the multiplication map $\overline{A}\bar{e} \otimes_{\bar{e}\overline{A}\bar{e}} \bar{e}\overline{A} \to \overline{A}\bar{e}\overline{A}$ is injective.

Let X, Y be indecomposable direct summands of $M \oplus M''$ and e_X, e_Y the corresponding idempotents of A. It is sufficient to show that the multiplication map

$$\bar{e}_X \overline{A} \bar{e} \bigotimes_{\bar{e}\overline{A}\bar{e}} \bar{e} \overline{A} \bar{e}_Y \to \bar{e}_X \overline{A} \bar{e} \overline{A} \bar{e}_Y$$

is injective. The elements in $\bar{e}_X \overline{A} \bar{e} \otimes_{\bar{e}\overline{A}\bar{e}} \bar{e}\overline{A}\bar{e}_Y$ are of the form $u = \sum_{j=1}^{c} \overline{\gamma}_j \otimes \overline{\delta}_j$ where $\gamma_j \in e_X A e = \mathrm{Hom}(M, X)$ and $\delta_j \in eAe_Y \in \mathrm{Hom}(Y, M)$. Let $C = cM$,

$$\gamma = [\gamma_1, \dots, \gamma_c] : C \to X \qquad \text{and} \qquad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_c \end{bmatrix} : Y \to C.$$

The multiplication map sends $u = \sum \overline{\gamma}_j \otimes \overline{\delta}_j$ to $\sum \overline{\gamma}_j \overline{\delta}_j = \overline{\gamma\delta}$. Assume now that $\overline{\gamma\delta} = 0$, thus $\gamma\delta$ factors through $\mathrm{add}\, M'$. Therefore the condition (b) asserts that there is a decomposition $C = C_1 \oplus C_2$, $\gamma = [\gamma_1, \gamma_2]$, $\delta = \left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$ such that both γ_1 and δ_2 factor through $\mathrm{add}\, M'$. Let ω_1 be the projection of C onto C_1, thus $\omega_2 = 1 - \omega_1$ is the projection of C onto C_2. Let $\iota_j : M \to cM$ be the jth inclusion, and $\varepsilon_j : cM \to M$ the jth projection. Thus $\gamma_j = \gamma\iota_j$, and $\delta_j = \varepsilon_j \delta$. By the lemma in [DR$_2$]

$$\sum_{j=1}^{c} \gamma_j \otimes \delta_j = \sum_{j=1}^{c} \gamma\omega_1\iota_j \otimes \varepsilon_j\omega_1\delta + \sum_{j=1}^{c} \gamma\omega_2\iota_j \otimes \varepsilon_j\omega_2\delta$$

in $\mathrm{Hom}(M, X) \otimes_{\mathrm{End}(M)} \mathrm{Hom}(Y, M) = e_X A e \otimes_{eAe} eAe_Y$. Since $\gamma\omega_1$ factors through $\mathrm{add}\, M'$, it follows that $\gamma\omega_1\iota_j$ belongs to $e_X J' e$, thus the first summand on the right becomes zero in $\bar{e}_X \overline{A} \bar{e} \otimes_{\bar{e}\overline{A}\bar{e}} \bar{e}\overline{A}\bar{e}_Y$. Similarly, $\omega_2\delta$ factors through $\mathrm{add}\, M'$, and therefore also the second summand becomes zero in $\bar{e}_X \overline{A} \bar{e} \otimes_{\bar{e}\overline{A}\bar{e}} \bar{e}\overline{A}\bar{e}_Y$. Consequently $u = 0$. This completes the proof.

2. Proof of the theorem

Let R be semiprimary ring, N its radical, and $N^n = 0$. Note that for any i, the endomorphism ring $\mathrm{End}(R/N^i)_R$ is just R/N^i, thus semiprimary. It follows that the endomorphism ring A of $\bigoplus_{i=1}^{n} (R/N^i)_R$ is semiprimary. Let J_t be the ideal of all elements of A which factor through $\bigoplus_{i=1}^{t} (R/N^i)_R$. We claim that J_t/J_{t-1} is a hereditary ideal of A, for any $1 \le t \le n$. We want to apply the criterion of §1. Let

$$M' = \bigoplus_{i=1}^{t-1} (R/N^i)_R, M = (R/N^t)_R, M'' = \bigoplus_{i=t+1}^{n} (R/N^i)_R.$$

The indecomposable direct summands of M are of the form eR/eN^t with a primitive idempotent e of R, those of M'' are of the form eR/eN^i with a primitive idempotent e of R, and $i > t$. The modules in $\mathrm{add}\, M$ are just the projective R/N^t-modules. Also, note that M' is annihilated by N^{t-1}. We are going to verify the two conditions (a) and (b).

(a) Let e_1, e_2 be primitive idempotents of R, and $\gamma : e_1R/e_1N^t \to e_2R/e_2N^t$ a noninvertible map. Note that γ cannot be surjective, since both e_iR/e_iN^t, $i = 1, 2$, are indecomposable projective R/N^t-modules. Thus the image of γ is contained in e_2N/e_2N^t and therefore annihilated by N^{t-1}. It follows that γ can be factored through e_1R/e_1N^{t-1} which belongs to $\operatorname{add} M'$.

(b) Again, let e_1, e_2 be primitive idempotents of R, and $X = e_1R/e_1N^i$, $Y = e_2R/e_2N^j$ for some $i, j \geq t$. Let C be a projective R/N^t-module. Let $\delta : Y \to C$, $\gamma : C \to X$ be maps such that $\gamma\delta$ factors through $\operatorname{add} M'$. First, assume that the image $\delta(e_2)$ of e_2 under δ is annihilated by N^{t-1}. Then e_2N^{t-1} is in the kernel of δ, thus δ factors through the module $e_2R/e_2N^{t-1} \in \operatorname{add} M'$. In this case, take $C_1 = 0$, and $C_2 = C$.

It remains to consider the case that $\delta(e_2)N^{t-1} \neq 0$. In this case, $C_1 = \delta(e_2)R$ is a projective R/N^t-module which is a direct summand of C. For, let $C = \bigoplus U_i$ with indecomposable modules U_i, and with projections $\pi_i : C \to U_i$. Then, there exists i with $\pi_i\delta(e_2)N^{t-1} \neq 0$. Now, U_i is an indecomposable projective R/N^t-module, thus U_iN is its unique maximal submodule. Since $\pi_i\delta(e_2)$ is not contained in U_iN, it follows that $\pi_i\delta : Y \to U_i$ is surjective. The image $C_1 = \delta(e_2)R$ of δ is a local R-module which is annihilated by N^t and which is mapped under π_i onto the indecomposable projective R/N^t-module U_i. This shows that C_1 itself is projective and that the inclusion map $C_1 \to C$ is a split monomorphism.

Let C_2 be a direct complement of C_1 in C; write $\gamma = [\gamma_1, \gamma_2]$, $\delta = \left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right]$. Then $\delta_2 = 0$, thus it factors through $\operatorname{add} M'$. Since $\gamma\delta$ factors through $\operatorname{add} M'$, the image of $\gamma\delta$ is annihilated by N^{t-1}. But $\gamma\delta(e_2)\cdot N^{t-1} = 0$ implies that $\delta(e_2)N^{t-1}$ is contained in the kernel of γ. Now, $\delta(e_2)R$ is isomorphic to e_2R/e_2N^t, thus γ_1 factors through $e_2R/e_2N^{t-1} \in \operatorname{add} M'$.

This completes the proof of the theorem.

Remark. The ring $A = \operatorname{End}(M_R)$ is usually not basic, even if the ring R is basic. The reason is the following one: In case that R is basic, each non-projective indecomposable summand occurs in a direct decomposition of M_R with multiplicity 1, whereas each indecomposable projective R-module of Loewy length l occurs with multiplicity $n - l + 1$. Deleting the repeated copies of the indecomposable projective summands of M_R one obtains a module whose endomorphism ring is basic and Morita equivalent to A.

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