Filtrations of right ideals related to projectivity of left ideals

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[This paper is in final form and no version of it will be submitted for publication else-where].

Let k be a field and A a finite-dimensional k-algebra. Since the endomorphism ring of the right A-module A_A is A itself, one must be able to describe all properties of A, for example properties of left ideals of A, in terms of the right A-module A_A. The aim of the present note is to show that the projectivity of certain left ideals can be characterized by the existence of suitable filtrations of right ideals.

As an application, we deal with quasi-hereditary rings. They have been defined by Scott **[S]** using heredity chains of ideals, thus using an inductive procedure of enlarging algebras. In this way one deals with a total ordering $e_1, \dots, e_n$ of a complete set of primitive idempotents, with e_n being added last. But there is a reverse procedure based on investigations of Mirollo and Vilonen **[MV]**, and described in **[DR2]**: there we construct A from $\varepsilon_2 A \varepsilon_2$ where $\varepsilon_2 = e_2 + e_3 + \cdots + e_n$. We characterize quasi-hereditary algebras such that the class of modules with Weyl filtrations is closed under submodules in terms of the two recursive procedures. And we show that algebras which satisfy this and the opposite condition have global dimension at most 2. It follows that the deep algebras introduced in **[DR3]**, as well as the peaked ones defined in this paper have global dimension at most 2.

1. The main results

Unless otherwise stated, modules will be (finitely generated) *right* A-modules. Let $\mathcal{M}$ be a set of A-modules. Given a module X_A, an *$\mathcal{M}$-filtration* of X_A is a chain of submodules $0 = X_0 \subset X_1 \subset \cdots \subset X_t = X$ such that for all $1 \le i \le t$, the module X_i/X_{i-1} is isomorphic to a module in $\mathcal{M}$.

Let N be the (Jacobson) radical of A. Let $e_1, \dots, e_n$ be a complete set of primitive (and orthogonal) idempotents. Let $E(i) = E(e_i)$ be the simple A-module not annihilated by e_i; thus $E_i \cong e_iA/e_iN$. Let $P(i) = P(e_i)$ be a projective cover of $E(i)$; thus $P(i) \cong e_iA$. Given a primitive idempotent e, we denote by $\hat{e}(i)$ the maximal quotient of $P(i)$ of Loewy length at most 2, whose radical is a direct sum of copies of $E(e)$. The set of modules $\hat{e}(i)$, with $1 \le i \le n$, is denoted by $\hat{e}$. The number of composition factors (in a composition series) of a module X which are isomorphic to $E(i)$ will be denoted by $\ell_i(X)$. We recall that a module is said to be *torsionless* provided it is isomorphic to a submodule of a projective module.

Theorem 1. *Let e be a primitive idempotent of A. The following statements are equivalent:*

(*i*) *The left ideal Ne is a projective left module.*

(*ii*) *A_A has an $\hat{e}$-filtration and $\mathrm{Ext}^1_A(E(e), E(e)) = 0$.*

(*ii'*) *Every right ideal has an $\hat{e}$-filtration.*

(*ii''*) *Every torsionless module has an $\hat{e}$–filtration.*

Let $\varepsilon_i = e_i + \cdots + e_n$ for $1 \le i \le n$, and $\varepsilon_{n+1} = 0$. We denote by $\Delta(i)$ the largest factor module of $P(i)$ with all composition factors of the form $E(j)$, with $j \le i$; thus $\Delta(i) = e_iA/e_iA\varepsilon_{i+1}A$. The set of modules $\Delta(i)$, with $1 \le i \le n$, is denoted by Δ, note that these modules $\Delta(i)$ depend on the chosen ordering $e_1, \ldots, e_n$. Let $I_i = A\varepsilon_{n-i+1}A$, thus $0 = I_0 \subset I_1 \subset \cdots \subset I_n = A$ is a saturated chain of idempotent ideals of A. Note that $(I_i)_i$ is a heredity chain if and only if first, A_A has a Δ–filtration, and second, $\ell_i(\Delta(i)) = 1$, for all $1 \le i \le n$: in this case, A is said to be quasi–hereditary. (In case that A is quasi–hereditary, the Δ–filtrations of a module X are also called "Weyl filtrations" **[PS]**. Also, X has a Δ–filtration if and only if its filtration $0 = XI_0 \subseteq XI_1 \subseteq \cdots \subseteq XI_n = X$ is "good" in the sense of **[DR2]**; this follows from Lemma 1* in section 2.)

Theorem 2. *Assume that $(I_i)_i$ is a heredity chain, where $I_i = A\varepsilon_{n-1+1}A$, and let $C_i = \varepsilon_iA\varepsilon_i$. Then the following conditions are equivalent:*

(*i*) ε_iNe_i *is a projective left C_i–module, for* $1 \le i \le n$,

(*i'*) $\varepsilon_{i+1}Ne_i$ *is a projective left C_{i+1}–module, for* $1 \le i \le n-1$,

(*ii*) *rad* $\Delta(i)$ *has a Δ–filtration, for* $1 \le i \le n$,

(*ii'*) *every right ideal has a Δ–filtration,*

(*ii''*) *every torsionless module has a Δ–filtration,*

(*ii'''*) *submodules of modules with a Δ–filtration have a Δ–filtration.*

The left modules $\Delta^*(i)$ and $\Delta^* = \{\Delta^*(i) | 1 \le i \le n\}$ are defined similarly as $\Delta(i)$ and Δ, namely: $\Delta^*(i)$ is the largest factor module of $P^*(i)$ with all composition factors of the form $E^*(j)$ with $j \le i$, thus $\Delta^*(i) = Ae_i/A\varepsilon_{i+1}Ae_i$. The fact that $(I_i)_i$ is a heredity chain may be expressed in a similar way in terms of Δ^*. In the next theorem we deal with those algebras A such that both A and its opposite satisfy the equivalent conditions of Theorem 2.

Theorem 3. *Let $(I_i)_i$ be a heredity chain. Assume that any right ideal of A has a Δ–filtration and that any left ideal of A has a Δ^*–filtration. Then gl.dim.$A \le 2$.*

Corollary 1. *Deep quasi–hereditary algebras have global dimension at most 2.*

We recall that the quasi–hereditary algebra A is said to be *deep* **[DR3]** if, for every $1 \le i \le n$, both the right A–module rad $\Delta(i)$ and the left A–module rad $\Delta^*(i)$ are projective.

The proofs of these results will be given in section 2, 3, and 4 of the paper. Section 5 contains a construction of a class of quasi–hereditary algebras of global dimension 2 which we call the *peaked* algebras. These are examples of algebras A such that both A and A^{opp} satisfy the conditions of Theorem 2.

2. Preliminaries on filtrations of modules.

First, let $\mathcal{M}$ be an arbitrary set of modules. We consider modules which have an $\mathcal{M}$–filtration. It is sometimes necessary to arrange the various quotients occurring in a filtration. In order to be able to do so, we will use the following well–known lemma.

Lemma 1. *Assume that some $M \in \mathcal{M}$ satisfies $Ext^1_A(M', M) = 0$ for all $M' \in \mathcal{M}$. Let $\mathcal{M}' = \mathcal{M}\backslash\{M\}$. If a module X has an $\mathcal{M}$-filtration, then it has a submodule X' with an $\mathcal{M}'$-filtration such that X/X' is a direct sum of copies of M.*

Proof. Let X'' be a submodule of X with an $\mathcal{M}$–filtration such that X/X'' belongs to $\mathcal{M}$. By induction, there is a submodule X''' of X'' with an $\mathcal{M}'$–filtration such that X''/X''' is a direct sum of copies of M. Since $\mathrm{Ext}^1_A(X/X'', X''/X''') = 0$, there is a submodule Y of X with $Y \cap X'' = X'''$ and $Y + X'' = X$. If X/X'' belongs to $\mathcal{M}'$, let $X' = Y$; otherwise, let $X' = X'''$.

Lemma 1*. *Assume that some $M \in \mathcal{M}$ satisfies $Ext^1_A(M, M') = 0$ for all $M' \in \mathcal{M}$. Let $\mathcal{M}' = M\backslash\{M\}$. If a module Y has an $\mathcal{M}$-filtration, then it has a submodule Y' which is a direct sum of copies of M such that Y/Y' has an $\mathcal{M}'$-filtration.*

Clearly, this is the dual assertion. Both results have been used by Cline–Parshall–Scott **[CPS]** for dealing with modules over quasi–hereditary rings, or, more generally, with objects in highest weight categories.

We will be interested to know whether submodules of modules with an $\mathcal{M}$–filtration again have $\mathcal{M}$–filtrations. The following is a useful criterion in this direction.

Lemma 2. *Assume that for any $M \in \mathcal{M}$, every maximal submodule of M has an $\mathcal{M}$-filtration. Then submodules of modules with an $\mathcal{M}$-filtration have an $\mathcal{M}$-filtration.*

Proof. Let $0 = X_0 \subset X_1 \subset \cdots \subset X_t = X$ be an $\mathcal{M}$–filtration of the module X, let Y be a submodule of X. We claim that Y has an $\mathcal{M}$–filtration. By induction on the length of X/Y, we may assume that Y is a maximal submodule of X. Choose i minimal with $X_i \not\subseteq Y$. Then $X_i \cap Y$ is a maximal submodule of X_i containing X_{i-1}. By assumption, $X_i \cap Y/X_{i-1}$ has an $\mathcal{M}$–filtration. Using it, we may refine the filtration $0 = X_0 \subset \ldots X_{i-1} \subseteq X_i \cap Y \subset \cdots \subset X_t \cap Y = Y$ in order to obtain an $\mathcal{M}$–filtration for Y.

We return to the complete set $e_1, \ldots, e_n$ of primitive idempotents of A, and we denote $e = e_1$. *We assume that* $\mathrm{Ext}^1_A(E(e), E(e)) = 0$. Let $\mathcal{M}(e) = \{\hat{e}(i) | 2 \leq i \leq n\}$, and let $\bar{\mathcal{M}}(e)$ be the set of non–zero quotient modules of modules in $\mathcal{M}(e)$.

Lemma 3. *A module X has an $\bar{\mathcal{M}}(e)$-filtration if and only if $Hom_A(X, E(e)) = 0$.*

Proof. If M is in $\bar{\mathcal{M}}(e)$, then $\mathrm{Hom}_A(M, E(1)) = 0$. Thus, if X has an $\bar{\mathcal{M}}(e)$–filtration, $\mathrm{Hom}_A(X, E(1)) = 0$. Conversely, assume $\mathrm{Hom}_A(X, E(1)) = 0$. We may assume $X \neq 0$, thus let X' be a maximal submodule of X. Then $X/X' \cong E(j)$ for some $2 \leq j \leq n$. Let $X'' = \mathrm{rad}\, X'$. There are (uniquely determined) submodules Y, Y' of X' containing

X'' such that $X'/X'' = Y/X'' \oplus Y'/X''$ with Y'/X'' a direct sum of copies of $E(1)$, and Y/X'' a direct sum of various $E(i)$, with $2 \leq i \leq n$. We claim that X/Y belongs to $\bar{\mathcal{M}}(e)$. For, the submodule X'/Y of X/Y is a direct sum of copies of $E(1)$, the quotient is $X/X' \cong E(j)$, and $\mathrm{Hom}_A(X/Y, E(1)) = 0$, thus $X'/Y = \mathrm{rad}(X/Y)$. On the other hand, $\mathrm{Hom}_A(Y, E(1)) = 0$, since otherwise $\mathrm{Ext}^1_A(E(1), E(1)) \neq 0$. By induction, Y has an $\bar{\mathcal{M}}(e)$-filtration and thus X has an $\bar{\mathcal{M}}(e)$-filtration.

The length of the module X will be denoted by $\ell(X)$; hence $\ell(X) = \sum_{i=1}^{n} \ell_i(X)$. Let $s_i = \ell(\hat{e}(i))$.

Lemma 4. *Assume that X has an $\bar{\mathcal{M}}(e)$-filtration. Then*

$$\ell(X) \leq \sum_{i=2}^{n} \ell_i(X) s_i \,;$$

moreover the following assertions are equivalent:

(*i*)
$$\ell(X) = \sum_{i=2}^{n} \ell_i(X) s_i \,,$$

(*ii*) *the module X has an $\mathcal{M}(e)$-filtration,*

(*iii*) *any $\bar{\mathcal{M}}(e)$-filtration of X is an $\mathcal{M}(e)$-filtration.*

Proof. Let $0 = X_0 \subset X_1 \subset \cdots \subset X_t = X$ be an $\bar{\mathcal{M}}(e)$-filtration, with $X_j/X_{j-1} \cong \hat{e}(\sigma(j))/U_j$, where $U_j \subseteq \mathrm{rad}\, \hat{e}(\sigma(j))$, and $2 \leq \sigma(j) \leq n$. Clearly, for $2 \leq i \leq n$, the number $\ell_i(X)$ is just the number of j's with $\sigma(j) = i$. Thus

$$\begin{aligned} \ell(X) &= \sum_{j=1}^{t} \ell(X_j/X_{j-1}) = \sum_{j=1}^{t} \ell(\hat{e}(\sigma(j))) - \sum_{j=1}^{t} \ell(U_j) \\ &= \sum_{i=2}^{n} \ell_i(X) s_i - \sum_{j=1}^{t} \ell(U_j) \leq \sum_{i=2}^{n} \ell_i(X) s_i \,, \end{aligned}$$

and we have equality if and only if all $U_j = 0$, that is if and only if the given filtration is an $\mathcal{M}(e)$-filtration.

Lemma 5. *Assume that X has an $\mathcal{M}(e)$-filtration, and let e' be an idempotent of A with $eAe' \subseteq N$. Then also $X/Xe'A$ has an $\mathcal{M}(e)$-filtration.*

Proof. Since $\mathrm{Hom}_A(Xe'A, E(e)) = 0$, the module $Xe'A$ has an $\bar{\mathcal{M}}(e)$–filtration according to Lemma 3. Since X has an $\bar{\mathcal{M}}(e)$–filtration, also $X/Xe'A$ has one, and therefore X has an $\bar{\mathcal{M}}(e)$–filtration passing through $Xe'A$. But by Lemma 4, any $\bar{\mathcal{M}}(e)$–filtration is an $\mathcal{M}(e)$–filtration.

Lemma 6. *Assume X has an $\hat{e}$–filtration. Then there is a submodule X' of X with an $\mathcal{M}(e)$–filtration such that X/X' is a direct sum of copies of $E(e)$.*

Proof. Since $\mathrm{Ext}^1(E(e), E(e)) = 0$, we have $\mathrm{Ext}^1(\hat{e}(i), E(1)) = 0$ for all $1 \le i \le n$. Now we apply Lemma 1.

3. Proof of Theorem 1.

As before, we deal with a complete set $e = e_1, e_2, \dots, e_n$ of primitive idempotents.

If the left ideal Ne is a projective left module, its indecomposable summands have to be of the form Ae_i, with $2 \le i \le n$. Since Ae cannot be embedded into Ne, but $\mathrm{Ext}^1_A(E(e), E(e)) = 0$.

We are going to establish the equivalence of assertions(i) and (ii) in Theorem 1, so we may assume from the beginning that $\mathrm{Ext}^1_A(E(e), E(e)) = 0$.

Recall that the species $\mathcal{S} = (D_i, {}_iM_j)_{i,j}$ of A is defined as follows: D_i is the division ring e_iAe_i/e_iNe_i, and ${}_iM_j$ is the D_i–D_j–bimodule $e_iNe_j/e_iN^2e_j$. Let $d_i = \dim_k D_i, d_{ij} = \dim({}_iD_j)_{D_j}, d'_{ij} = \dim_{D_i}({}_iM_j)$; thus $\dim_k({}_iM_j) = d_id'_{ij} = d_{ij}d_j$. We observe that $\mathrm{rad}\,\hat{e}(i) = d_{i1}E(1)$ (thus $s_i = d_{i1} + 1$).

The simple left A–modules will be denoted by $E^*(i) = Ae_i/Ne_i$, their projective covers by $P^*(i) = Ae_i$. The top of the left A–module Ne is isomorphic to $\bigoplus_{i=1}^{n} d'_{i1}E^*(i)$, and we consider the projective cover $p : {}_AP \longrightarrow {}_ANe$ of left A–modules: here, ${}_AP \cong \bigoplus_{i=1}^{n} d'_{i1}P^*(i)$. Actually, the assumption $\mathrm{Ext}^1_A(E(e), E(e)) = 0$ can be reformulated as ${}_1M_1 = 0$; thus $d_{11} = 0 = d'_{11}$. Let ${}_AY$ be the kernel of p.

We decompose $A_A = e'A \oplus e''A$, where $e'A$ is a direct sum of copies of eA, and $eAe'' \subseteq N$. Let $X_A = e'N \oplus e''A$, thus $Xe = Ne$, and $Xe_i = Ae_i = P^*(i)$ for $2 \le i \le n$. In particular, for $2 \le i \le n$, we have

$$\dim_k P^*(i) = \dim_k Xe_i = \ell_i(X)d_i\,;$$

therefore

$$\dim_k P = \sum_{i=2}^{n} d'_{i1}\dim_k P^*(i) = \sum_{i=2}^{n} \ell_i(X)d_id'_{i1} = \sum_{i=2}^{n} \ell_i(X)d_{i1}d_1\,.$$

Since $\mathrm{Ext}^1_A(E(e), E(e)) = 0$, we have $\mathrm{Hom}_A(X_A, E(e)) = 0$. Hence Lemma 3 asserts that X_A has an $\bar{\mathcal{M}}(e)$–filtration, say $0 = X_0 \subset X_1 \subset \cdots \subset X_t = X$ with $X_j/X_{j-1} \cong \hat{e}(\sigma(j))/U_j$

for some submodule U_j of $\hat{e}(\sigma(j))$ and $2 \le \sigma(j) \le n$. The number of j's with $\sigma(j) = i$ is $\ell_i(X)$. Since

$$\ell_1(X_j/X_{j-1}) = \ell_1(\hat{e}(\sigma(j))) - \ell_1(U_j) = d_{\sigma(j),1} - \ell(U_j),$$

we have

$$\begin{aligned}\dim_k Xe &= \sum_{j=1}^{t} \ell_1(X_j/X_{j-1})d_1 = \sum_{j=1}^{t}(d_{\sigma(j),1} - \ell(U_j))d_1 \\ &= \sum_{i=1}^{n} \ell_i(X)d_{i1}d_1 - \sum_{j=1}^{t} \ell(U_j)d_1 .\end{aligned}$$

Comparing the dimensions of P and $Ne = Xe$, we obtain the dimension for the kernel Y of p

$$\dim_k Y = \sum_{j=1}^{t} \ell(U_j)d_1 .$$

If we assume that ${}_ANe$ is a projective left A–module, then p is bijective, thus $Y = 0$. Therefore all $U_j = 0$, and our $\bar{\mathcal{M}}(e)$–filtration of X_A is an $\mathcal{M}(e)$–filtration. Since A_A/X_A is a direct sum of copies of $E(e) = \hat{e}(1)$, we conclude that A_A has an $\hat{e}$–filtration.

Conversely, assume that A_A has an $\hat{e}$–filtration. According to Lemma 6, we obtain a submodule $\bar{X}_A$ of A_A with an $\mathcal{M}(e)$–filtration such that $A_A/\bar{X}_A$ is a direct sum of copies of $E(e)$. Clearly, $\bar{X}_A = X_A$, so X_A has an $\mathcal{M}(e)$–filtration. It follows that $U_j = 0$ for all j, consequently p is bijective, and therefore ${}_ANe$ is a projective left A–module.

This shows the equivalence of assertions (i) and (ii). Every module $\hat{e}(i)$ in $\hat{e}$ has a unique maximal submodule, and this submodule is a direct sum of copies of $\hat{e}(1) = E(e)$. Hence, it has an $\hat{e}$–filtration. Lemma 1 asserts that submodules of modules with $\hat{e}$–filtrations have $\hat{e}$–filtrations. Under the assumption of (ii), any free module has an $\hat{e}$–filtration, thus any torsionless module has an $\hat{e}$–filtration. This shows (ii) $\Rightarrow$ (ii″), and trivially (ii″) $\Rightarrow$ (ii′).

Finally, we show the implication (ii′) $\Rightarrow$ (ii). Take a right ideal Y_A of minimal length having $E(e)$ as a composition factor.

Clearly, Y_A has a unique maximal submodule Y', and $Y/Y' \cong E(e)$, whereas Y' has no composition factor of the form $E(e)$. Take an $\hat{e}$–filtration $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_t = Y$ of Y. Then $Y_{t-1} \subseteq Y'$, and $Y'/Y_{t-1} = \operatorname{rad}(Y_t/Y_{t-1})$. Since $Y/Y' \cong E(e)$, we see that $Y_t/Y_{t-1} \cong \hat{e}(1)$. Since Y'/Y_{t-1} has no composition factor $E(1)$, it follows that $Y'/Y_{t-1} = 0$. Thus $\hat{e}(1) = E(e)$, and therefore $\operatorname{Ext}^1_A(E(e), E(e)) = 0$.

4. Proof of theorem 2.

We assume that $(I_i)_i$ is a heredity chain, where $I_i = A\varepsilon_{n-i+1}A$, with $\varepsilon_i = e_i + \cdots + e_n$, for $1 \le i \le n$, and $\varepsilon_{n+1} = 0$, and we denote $C_i = \varepsilon_i A \varepsilon_i$.

Lemma 7. *The left ideal ${}_ANe_1$ is a projective left A-module if and only if $\varepsilon_2 Ne_1$ is a projective left C_2-module.*

Proof. First, assume that ${}_ANe_1$ is projective. Then ${}_ANe_1$ is isomorphic to a module of the form $\bigoplus_{i=2}^{n} m_i Ae_i$, for some $m_i \in \mathbb{N}_0$, since Ae_1 cannot be embedded into Ne_1. Thus $\varepsilon_2 Ne_1 \cong \bigoplus_{i=2}^{n} m_i(\varepsilon_2 Ae_i)$, as a left C_2-module. But $\varepsilon_2 Ae_i$ is a projective left C_2-module for $2 \le i \le n$, since $\varepsilon_2 = e_2 + \cdots + e_n$ with orthogonal idempotents $e_2, \ldots, e_n$.

Conversely, assume $\varepsilon_2 Ne_1$ is a projective left C_2-module. Since $A\varepsilon_2 A$ belongs to a heredity chain, we know that the multiplication map

$$A\varepsilon_2 \otimes_{C_2} \varepsilon_2 A \longrightarrow A\varepsilon_2 A$$

is bijective (Prop. 7 of **[DR2]**). Multiplying from the right by e_1, we obtain an isomorphism $A\varepsilon_2 \bigotimes_{C_2} \varepsilon_2 Ae_1 \cong A\varepsilon_2 Ae_1$ of left A-modules. Since $A\varepsilon_2$ is a projective left A-module, and $\varepsilon_2 Ae_1 = \varepsilon_2 Ne_1$ is a projective left C_2-module, it follows that $A\varepsilon_2 Ae_1$ is a projective left A-module. It remains to be shown that $A\varepsilon_2 Ae_1 = Ne_1$. First of all, $\varepsilon_2 Ae_1 \subseteq N$, thus $A\varepsilon_2 Ae_1 = A\varepsilon_2 Ne_1$. Second, $e_1 Ne_1 = e_1 N^2 e_1$, thus the left A-module Ne_1 is generated by $A\varepsilon_2$, consequently $A\varepsilon_2 Ne_1 = Ne_1$.

Note that the left A-module Ne_1 is projective if and only if the left C_1-module $\varepsilon_1 Ne_1$ is projective. This an immediate consequence of the Morita equivalence of A and C_1.

The equivalence of the assertions (i) and (i′) in Theorem 1 is an immediate consequence of Lemma 7: we apply it to the rings C_i and their corresponding heredity chains (**[DR1]**, statement 10). The implication (ii) ⇒ (ii‴) is asserted in Lemma 2. Since A_A has a Δ-filtration, the same is true for any free A-module, thus (ii‴) ⇒ (ii″). The implications (ii″) ⇒ (ii′) is trivial. In order to prove the implication (ii′) ⇒ (ii), we assume that the right ideals $e_i N$ have Δ-filtrations. Then there are Δ-filtrations of $e_i N$ passing through $e_i N \varepsilon_{i+1} A$, and therefore also $\operatorname{rad}\Delta(i) = e_i N / e_i N \varepsilon_{i+1} A$ has a Δ-filtration.

It remains to verify the equivalence of the conditions (i) and (ii). We will use induction on n. The algebra C_2 has the heredity chain $0 = \varepsilon_2 I_0 \varepsilon_2 \subset \varepsilon_2 I_1 \varepsilon_2 \subset \cdots \subset \varepsilon_2 I_{n-1} \varepsilon_2 = C_2$, and for C_2, we deal with the modules $\Delta_2(i) = e_i A\varepsilon_2 / e_i A \varepsilon_{i+1} A \varepsilon_2 = \Delta(i)\varepsilon_2$, where $2 \le i \le n$.

First, we assume that $\operatorname{rad}\Delta(i)$ has a Δ-filtration, for $1 \le i \le n$. Then $\operatorname{rad}\Delta_2(i)$ has a Δ_2-filtration, for $2 \le i \le n$, thus, by induction, $\varepsilon_i Ne_i$ is a projective left C_i-module, for $2 \le i \le n$. We want to show that Ne_1 is a projective left A-module. According to

Theorem 1, it suffices to show that A_A has an $\hat{e}$–filtration where $e = e_1$. Now A_A has a Δ–filtration, so we use the following lemma.

Lemma 8. *Assume that* $\operatorname{rad}\Delta(i)$ *has a* Δ*-filtration, for all* $1 \leq i \leq n$. *Then any module with a* Δ*-filtration has an* $\hat{e}$*-filtration.*

Proof. Let X be a module with a Δ–filtration. We use induction on $\ell(X)$. We may assume $X = \Delta(i)$ for some i. If $\ell(\Delta(i)) = 1$, then $\operatorname{Ext}^1_A(E(i), E(j)) = 0$ for all $j \leq i$; in particular, $\operatorname{Ext}^1_A(E(i), E(1)) = 0$. Hence $\hat{e}(i) = E(i) = \Delta(i)$. Now assume $\ell(\Delta(i)) > 1$. Let $X = \operatorname{rad}\,\Delta(i)$. By induction, X has an $\hat{e}$–filtration, thus there is a submodule X' with an $\mathcal{M}(e)$–filtration such that X/X' is a direct sum of copies of $E(1)$. It follows that $X' = e_i N \varepsilon_2 A$, thus $\Delta(i)/X' = \hat{e}(i)$. Since X' has an $\hat{e}$–filtration, we see that $\Delta(i)$ has an $\hat{e}$–filtration.

Finally, we verify the implication (i) $\Rightarrow$ (ii). For $1 \leq i \leq n$, let $\varepsilon_i N e_i$ be a projective left C_i–module. By induction we know that $\operatorname{rad}\Delta_2(i)$ has a Δ_2–filtration, for $2 \leq i \leq n$. Since Ne_1 is a projective left A–module, Theorem 1 asserts that A_A has an $\hat{e}$–filtration. We are going to show that $\operatorname{rad}\Delta(j)$, with $1 \leq j \leq n$, has a Δ–filtration. Since $\Delta(1) = E(1)$, we may assume $2 \leq j \leq n$. Consider $Z_{jr} = (\operatorname{rad}\,\Delta(j))\varepsilon_r A/(\operatorname{rad}\,\Delta(j))\varepsilon_{r+1}A$, with $1 \leq r \leq n$. We claim that Z_{jr} is a direct sum of copies of $\Delta(r)$. Again the case $r = 1$ is trivial, so assume $2 \leq r \leq n$. First of all, $\operatorname{top} Z_{jr}$ is clearly a direct sum of copies of $E(r)$, say $\operatorname{top} Z_{jr} = m_{jr}E(r)$. Since $\Delta(r)$ is the projective $A/A\varepsilon_{r+1}A$–cover of $E(r)$, and Z_{jr} is annihilated by $A\varepsilon_{r+1}A$, it follows that there is a surjective map $Y \longrightarrow Z_{jr}$ with $Y = m_{jr}\Delta(r)$. In order to show that this is an isomorphism, we are going to prove that $\ell(Y) = \ell(Z_{jr})$. First, we claim that both Y and Z_{jr} have $\mathcal{M}(e)$–filtrations. For, $e_r A$ has an $\hat{e}$–filtration, and $\operatorname{Hom}_A(e_r A, E(1)) = 0$, since $r \geq 2$; thus $e_r A$ has an $\mathcal{M}(e)$–filtration by Lemma 6. According to Lemma 5, $\Delta(r) = e_r A/e_r A\varepsilon_{e+1}A$ has an $\mathcal{M}(e)$–filtration, thus the same is true for Y. Since $\Delta(j)$ has an $\hat{e}$–filtration, also $\Delta(j)e_r A$ has one, according to Lemma 2. Using again $r \geq 2$, Lemma 5 and Lemma 6, we see that Z_{jr} has an $\mathcal{M}(e)$–filtration. Given any A–module X, and $i \geq 2$, the number $\ell_i(X)$ coincides with the number $\ell^{(2)}(X\varepsilon_2)$ of composition factors of the C_2–module $X\varepsilon_2$ which are of the form $E(i)\varepsilon_2 = e_i A\varepsilon_2/e_i N\varepsilon_2$. We use Lemma 4 in order to express $\ell(Y)$ and $\ell(Z_{jr})$ as follows:

$$\ell(Y) = \sum_{i=2}^{n} \ell_i(Y)s_i = \sum_{i=2}^{n} \ell_i^{(2)}(Y\varepsilon_2)s_i\,,$$

$$\ell(Z_{jr}) = \sum_{i=2}^{n} \ell_i(Z_{jr})s_i = \sum_{i=2}^{n} \ell_i^{(2)}(Z_{jr}\varepsilon_2)s_i\,.$$

On the other hand,

$$\begin{aligned} Z_{jr}\varepsilon_2 &= (\operatorname{rad}\,\Delta(j))\varepsilon_r A\varepsilon_2/(\operatorname{rad}\,\Delta(j))\varepsilon_{r+1}A\varepsilon_2 = \\ &= (\operatorname{rad}\,\Delta_2(j))\varepsilon_r C_2/(\operatorname{rad}\,\Delta_2(j))\varepsilon_{r+1}C_2 \end{aligned}$$

is a direct sum of copies of $\Delta_2(r)$, since $\Delta_2(j)$ has a Δ_2–filtration. It follows that $Z_{jr}\varepsilon_2 \cong m_{jr}\Delta_2(r) = Y\varepsilon_2$. As a consequence, $\ell(Y) = \ell(Z_{jr})$. This completes the proof of the implication (i) $\Rightarrow$ (ii).

5. Algebras of global dimension 2.

We are going to present the proof of Theorem 3 as well as some related examples. As before let $e_1, \ldots, e_n$ be a complete set of primitive and orthogonal idempotents, and let $\varepsilon_i = e_1 + \cdots + e_n$ for $1 \leq i \leq n$. Again, we assume that $(I_i)_i$ is a heredity chain, where $I_i = A\varepsilon_{n-i+1}A$.

Lemma 9. *Let $\varepsilon = \varepsilon_2$. Let $C = \varepsilon A \varepsilon$. Assume that $\varepsilon N e_1$ is a projective left C-module and that $e_1 N$ is a projective right A-module. Then proj.dim.$E(1)_A \leq 1$, and proj.dim.$E(i)_A \leq$ max $\{2, \text{proj.dim.}(E(i)\varepsilon)_C\}$ for $2 \leq i \leq n$.*

Proof. Since $E(1) = e_1A/e_1N$, it follows that proj.dim.$E(1)_A \leq 1$. Consider now $E(i)$, where $2 \leq i \leq n$. We can assume that proj.dim. $(E(i)\varepsilon)_C$ is finite; let

$$0 \longrightarrow P^{(m)} \longrightarrow \ldots \longrightarrow P^{(1)} \longrightarrow P^{(0)} \longrightarrow E(i)\varepsilon \longrightarrow 0$$

be a projective resolution of the C-module $(E(i)\varepsilon)_C$. We tensor this sequence with ${}_C(\varepsilon A)$. Note that ${}_C(\varepsilon A)$ is a direct sum of copies of ${}_C(\varepsilon A e_j)$, with $1 \leq j \leq n$. For $2 \leq j \leq n$, the left C-module ${}_C(\varepsilon A e_j)$ is projective, since e_j is an idempotent of C, and ${}_C(\varepsilon A e_1) ={}_C (\varepsilon N e_1)$ is projective by assumption. Thus

$$0 \longrightarrow P^{(m)} \otimes_C \varepsilon A \longrightarrow \ldots \longrightarrow P^{(0)} \otimes_C \varepsilon A \longrightarrow E(i)\varepsilon \otimes_C \varepsilon A \longrightarrow 0$$

is exact. Since the A-modules $P^{(s)} \otimes {}_C(\varepsilon A)$ are projective, it follows that proj.dim. $E(i)\varepsilon \otimes {}_C(\varepsilon A)_A \leq m$. The exact sequence $0 \longrightarrow e_iN \longrightarrow e_iA \longrightarrow E(i) \longrightarrow 0$ yields first by multiplying with ε and then tensoring with ${}_C(\varepsilon A)$, the exact sequence

$$0 \longrightarrow e_iN\varepsilon \otimes_C \varepsilon A \longrightarrow e_iA\varepsilon \otimes_C \varepsilon A \longrightarrow E(i)\varepsilon \otimes_C \varepsilon A \longrightarrow 0\,.$$

Since $A\varepsilon A$ belongs to a heredity chain, we can identify $A\varepsilon \otimes_C \varepsilon A$ with $A\varepsilon A$ and therefore $e_iA\varepsilon \otimes_C \varepsilon A$ with $e_iA\varepsilon A = e_iA$. We see that $E(i)\varepsilon \otimes_C \varepsilon A \cong e_iA/e_iN\varepsilon A = \hat{e}(i)$. Thus proj.dim.$\hat{e}(i)_A \leq m$. There is the exact sequence

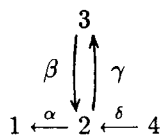
$$0 \longrightarrow d_{i1}E(1) \longrightarrow \hat{e}(i) \longrightarrow E(i) \longrightarrow 0.$$

Since proj.dim.$E(1) \leq 1$, it follows that

$$\text{proj.dim.}E(i) \leq \max\{2, \text{proj.dim.}\hat{e}(i)_A\} = \max\{2, m\}\,.$$

Proof of Theorem 3. We use induction on n. Condition (i) of Theorem 2 applied to A and to its opposite shows that $C = C_2$ satisfies the corresponding assumptions (every right ideal of C_2 has a Δ_2–filtration, every left ideal of C_2 has a Δ_2^*–filtration). Thus gl.dim.$C \leq 2$. Also, $\varepsilon_2 N e_1$ is a projective left C_2–module by condition (i′) of Theorem 2. And $e_1 N \varepsilon_1$ is a projective right C_1–module by condition (i) of Theorem 2, applied to the opposite of A, thus $e_1 N$ is a projective A–module. We apply Lemma 9 and conclude that gl.dim$A \leq 2$.

Let us remark that not all algebras of global dimension 2 satisfy the conditions of Theorem 2: A simple example is provided by the path algebra of the graph



modulo the ideal $\langle \beta\alpha, \beta\gamma, \delta\gamma \rangle$:

$$A_A = 1 \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \\ 3 \\ 2 \end{smallmatrix}.$$

Here,

$$\Delta(1) = 1, \quad \Delta(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \quad \Delta(3) = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}, \quad \Delta(4) = \begin{smallmatrix} 4 \\ 2 \\ 3 \\ 2 \end{smallmatrix},$$

thus rad$\,\Delta(4)$ has no Δ–filtration. On the other hand, the path algebra of

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-1}} n$$

modulo $\langle \alpha_{i-1}\alpha_i \,|\, 2 \leq i \leq n-1 \rangle$ satisfies the conditions of Theorem 2, but has global dimension $n-1$. Of course, for $n \geq 4$ this implies that its opposite algebra does not satisfy these conditions. Observe that, for $n = 3$ this is an example of an algebra of global dimension 2 whose dimension (namely 5) is less than the dimension of the corresponding peaked algebra (of dimension 6) as defined in the next section.

6. Peaked algebras

In this last section, we intend to give a construction of a new class of quasi–hereditary algebras of global dimension 2 which may be of further interest. Let $\mathcal{S} = (D_i, {}_iM_j)_{1\le i,j\le n}$ be a labelled species without loops **[DR3]**: thus ${}_iM_i = 0$ for all i, and the index set $\{1, 2, \ldots, n\}$ is considered with its natural ordering. As in **[DR3]**, let

$$T = T(n) = \{(t_o, t_1, \ldots, t_m) \,|\, 0 \le t_i \le n \text{ are integers, } m \ge 1, \text{ and } t_{i-1} \ne t_i \text{ for all } 1 \le i \le m\};$$

for every $t = (t_0, t_1, \ldots, t_m) \in T$, let

$$M(t) = {}_{t_0}M_{t_1} \otimes_{D_{t_1}} {}_{t_1}M_{t_2} \otimes_{D_{t_2}} \cdots \otimes_{D_{t_{m-1}}} {}_{t_{m-1}}M_{t_m},$$

and for $T' \subseteq T$, let

$$M(T') = \bigoplus_{t\in T'} M(t).$$

We define the ideal $M(W^0)$ of the tensor algebra $\mathcal{T}(\mathcal{S})$ by specifying the subset W^0 of T as follows:

$$W^0 = W^0(n) = \{(t_0, t_1, \ldots, t_m) \in T| \text{ there is } 0 < i < m \text{ such that } t_{i-1} > t_i < t_{i+1}\}.$$

Let W be the complement $T\backslash W^0$, thus

$$W = \{(t_0, t_1, \ldots, t_m) \in T \,|\, \text{ there is } 0 \le i \le m \text{ such that } t_0 < t_1 < \cdots < t_i > \cdots > t_{m-1} > t_m\}.$$

Hence

$$[M(T)]^{2n-1} \subseteq M(W^0) \subseteq M(T)$$

and thus $M(W^0)$ is an admissible ideal. Let

$$\mathcal{P}(\mathcal{S}) = \mathcal{T}(\mathcal{S})/M(W^0).$$

Observe that the Loewy length of $\mathcal{P}(\mathcal{S})$ is at most $2n - 1$, and that, as an abelian group, $\mathcal{P}(\mathcal{S})$ can be identified with

$$\prod_{i=1}^{n} D_i \oplus M(W).$$

We call $\mathcal{P}(\mathcal{S})$ the *peaked* algebra with labelled species $\mathcal{S}$.

Proposition. *Let $\mathcal{P}(\mathcal{S})$ be the peaked algebra with labelled species $\mathcal{S}$. Then $\mathcal{P}(\mathcal{S})$ is quasi-hereditary, every right ideal of $\mathcal{P}(\mathcal{S})$ has a Δ-filtration, every left ideal of $\mathcal{P}(\mathcal{S})$ has a Δ^*-filtration. In particular, gl.dim.$\mathcal{P}(\mathcal{S}) \leq 2$.*

Proof. For any $1 \leq i \leq n$, we claim that $\operatorname{rad}\Delta(i)$ is a direct sum of various $\Delta(j)$. Since $\Delta(1)$ is simple, we can assume $2 \leq i \leq n$. Let

$$T_i = \{(i, t_1, \ldots, t_m) \in T \mid i > t_1 > \cdots > t_m\}.$$

Then $\Delta(i)$ may be identified with $D_i \oplus M(T_i)$, thus

$$\operatorname{rad}\Delta(i) = M(T_i) = \bigoplus_{(i,j,t_2,\ldots,t_m)\in T_i} d_{ij}\Delta(j),$$

where, as before, $d_{ij} = \dim({}_iM_j)_{D_j}$.

In comparison with the deep algebras over a given labelled species (whose global dimension is also at most 2), the dimensions of the peaked algebras are considerably smaller. For instance, for $\mathcal{S}_n = (D_i, {}_iM_j)_{1\leq i,j\leq n}$, where $D_i = k$ for all i and ${}_iM_j = {}_kk_k$ for all $i \neq j$ and ${}_iM_i = 0$ for all i, the dimensions $p(n)$ of $\mathcal{P}(\mathcal{S}_n)$ clearly satisfy

$$p(n+1) = p(n) + 4^n,$$

and thus, for all n,

$$p(n) = \frac{1}{3}(4^n - 1).$$

On the other hand, let $d(n)$ be the dimension of a deep algebra over $\mathcal{S}_n$. We have $d(5) = 3263441$ while $p(5) = 341$, and $d(10) \approx 2.7 \times 10^{208}$ (!) while $p(10) = 349525$. Even $p(20)$ is "only" 366503875925.

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