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Hall Polynomials for the Representation-Finite Hereditary Algebras

CLAUS MICHAEL RINGEL

Fakultät für Mathematik, Universität Bielefeld, D-4800 Bielefeld 1, West Germany

Let k be a field. Let R be a finite-dimensional k-algebra with centre k which is representation-finite and hereditary; thus R is Morita equivalent to the tensor algebra of a k-species with underlying graph Δ a disjoint union of Dynkin diagrams, and the set of isomorphism classes of indecomposable R-modules corresponds bijectively to the set Φ^+ of positive roots of the corresponding semisimple complex Lie algebra $\mathbf{g}$ (see [G] and [DR1]). Consequently, the Grothendieck group $K(R-\mathrm{mod})$ of all finitely generated R-modules modulo split exact sequences is the free abelian group with basis indexed by Φ^+. Let $\mathbf{h}$ be a Cartan subalgebra of $\mathbf{g}$ and $\mathbf{g} = \mathbf{n}_+ \oplus \mathbf{h} \oplus \mathbf{n}_-$ the corresponding triangular decomposition. Note that $\mathbf{n}_+$ is the direct sum of one-dimensional complex vectorspaces indexed by the elements of Φ^+, so we may identify $K(R-\mathrm{mod}) \otimes \mathbb{C}$ and $\mathbf{n}_+$ as vectorspaces, and we deal with the problem of how to recover the Lie multiplication of $\mathbf{n}_+$ on $K(R-\mathrm{mod})$.

We have shown in [R2] that the Grothendieck group $K(R-\mathrm{mod})$ may be considered in a natural way as a Lie algebra by using as structure constants the evaluations of Hall polynomials at 1.

The aim of this paper is to show that this Lie algebra $K(R-\mathrm{mod})$ can be identified with a Chevalley $\mathbb{Z}$-form of $\mathbf{n}_+$; in particular $K(R-\mathrm{mod}) \otimes \mathbb{C}$ and $\mathbf{n}_+$ are isomorphic as Lie algebras.

We are going to determine all possible polynomials which occur as Hall polynomials φ^y_{zx}, where $x, y, z \in \Phi^+$. There are precisely 16 different polynomials (including the zero polynomial φ_0), and the absolute value of their evaluations at 1 is bounded by 3. One easily observes that $\varphi^y_{zx} = 0$ in case $y \neq z + x$; thus let us assume $y = z + x$. In this case, precisely one of the two polynomials φ^y_{zx} and φ^y_{xz} is non-zero. The non-zero polynomials φ^y_{zx} can be written in the form $\zeta_r \varphi_i$, where $\zeta_r = \sum_{i=0}^{r-1} T^i$, with $1 \leqslant r \leqslant 3$, and φ_i is one of the following 12 integral polynomials:

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$\varphi_1 = 1$		
$\varphi_2 = T-2$	$\varphi_8 = T$	
$\varphi_3 = (T-2)^2$	$\varphi_9 = T^2 - T - 1$ $\varphi_{10} = T^2 - 2$	
$\varphi_4 = T^3 - 5T^2 + 10T - 7$ $\varphi_5 = (T-2)^3$		$\varphi_{12} = T(T^2 - T - 1)$
$\varphi_6 = (T-2)(T^3 - 4T^2 + 8T - 6)$	$\varphi_{11} = T(T^3 - 2T^2 - 2T^2 - T + 3)$	
$\varphi_7 = T^5 - 6T^4 + 15T^3 - 23T^2 + 25T - 13$		

Note that $|\varphi_i(1)| = 1$ for all $1 \leqslant i \leqslant 12$. The number $r = r_{zx}$ is the number of roots of the form $z - tx$ with $t \in \mathbb{N}_0$. This number r_{zx} only depends on the relative lengths of the roots x, y, z and on Δ. Namely, for $\Delta = B_n, C_n, F_4$, we have $r_{zx} = 2$ provided x, z are small roots, and y is a long root; for $\Delta = G_2$, we have $r_{zx} = 3$ provided x, z are small roots, y is a long root, and $r_{zx} = 2$ provided all x, y, z are small roots; in all other cases, $r_{zx} = 1$. [This can easily be verified by a case-by-case inspection of the possible rank 2 root systems generated by x and z.]

Our investigations lead to a convenient way of exhibiting structure constants for the semisimple complex Lie algebra $\mathbf{g}$. We recall that Chevalley [C] has shown that there exists a basis of $\mathbf{g}$ so that the structure constants are integers. The bases which he exhibited are compatible with the triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$; the basis elements in $\mathbf{n}_+$ are indexed by the positive roots, say v_x, with $x \in \Phi^+$, and for $x, z \in \Phi^+$, we have $[v_z, v_x] = 0$ in case $x + z \notin \Phi^+$, and $[v_z, v_x] = N_{zx} v_y$ in case $y = x + z \in \Phi^+$, with $|N_{zx}| = r_{zx}$. In this way the structure constants N_{zx} are determined up to sign, and it is obvious that there are several ways of choosing signs consistently. In [T], Tits gave a complete description of all possible choices. An algorithm for obtaining a consistent choice of signs is also given in [Sa]. For every non-simple positive root y, one chooses a pair $x(y)$, $z(y) \in \Phi^+$ with $x(y) + z(y) = y$, one prescribes arbitrarily signs for $N_{z(y), x(y)}$, and now one has to determine the remaining signs. But there is a canonical way of choosing signs once we have chosen pairs $x(y)$, $z(y) \in \Phi^+$ with $x(y) + z(y) = y$ for the positive roots y of height 2. Note that the choice of such a decomposition for the roots of height 2 is the same as choosing an orientation Ω on Δ. Having fixed Δ and Ω, there are defined the Hall polynomials φ^y_{zx}, φ^y_{xz} for any triple x, z, $y = x + z \in \Phi^+$, and we may take

$$N_{zx} = \varphi^y_{zx}(1) - \varphi^y_{xz}(1).$$

Let us write down the recipe for this choice of structure constants explicitly, and without reference to the algebra R or the Hall polynomials. We choose an orientation Ω on Δ, and obtain in this way a (usually non-symmetric) bilinear form $(-,-)_\Omega$ on the root lattice as follows: Let $e_1, ..., e_n$ be the simple positive roots ordered with respect to the orientation (say if there is an edge $i \leftarrow j$ in (Δ, Ω), then $i<j$). As usual, let $[a_{ij}]$ be the corresponding Cartan matrix (with $a_{ij}=2(e_i, e_j)(e_j, e_j)^{-1}$, (see [Hu])). Let $[f_1, ..., f_n]$ be the minimal symmetrization for the Cartan matrix (thus $a_{ij}f_j = a_{ji}f_i$ for all i, j, and $f_1, ..., f_n$ are relative prime positive integers). Then $(-,-)_\Omega$ is defined by

$$(e_i, e_j)_\Omega = \begin{cases} f_i & \text{for } i=j \\ a_{ij}f_j & \text{for } i<j \\ 0 & \text{for } i>j. \end{cases}$$

Now, assume $x, z, x+z \in \Phi^+$. Then precisely one of $(x,z)_\Omega$, $(z,x)_\Omega$ is negative, say $(z,x)_\Omega < 0$. We choose the sign of N_{zx} according to the following table:

$(x,z)_\Omega$	$\Delta = A_n, D_n, E_m$	$\Delta = B_n, C_n, F_4$	$\Delta = G_2$
0	+	+	+
1	−	+	+
2	+	−	+
3	−	−	−
4	+	+	
5	−		

The remaining signs are obtained from these signs by the usual rules (see [Sa]).

For the proof, we only note that this table is obtained from the corresponding one in Section 10 by taking the sign of the evaluation at 1. For the cases $\Delta = A_n$, D_n, E_m, this choice of signs was first exhibited by Frenkel and Kac [FK].

1. First Reduction

Without loss of generality, we assume in addition that R is basic and connected; thus R is the tensor algebra of a species with underlying graph Δ a Dynkin diagram.

1.1. If $x, y, z \in \Phi^+$ with $x+z \neq y$, then $\varphi^y_{zx} = 0$. This follows from the fact that $F^M_{N_1 N_2} \neq 0$ only for those modules M, N_1, N_2 which satisfy $\dim M = \dim N_1 + \dim N_2$.

As a consequence, we only have to consider the polynomials φ_{zx}^{y} with $y = x + z$, so we may drop the upper index and just write φ_{zx}. Actually, we are happy to have the new free slot available for specifying the ring R, or, at least, the orientation Ω on Δ with which we work. We recall that the Hall polynomial φ_{zx} depends on Ω; thus we write φ_{zx}^{Ω} whenever there may occur some indeterminacy.

1.2. Let $x_1, x_2 \in \Phi^+$, let i be a sink of (Δ, Ω), and assume x_2 is not the simple root e_i corresponding to i. Let s be the reflection corresponding to i. Then $\varphi_{x_1 x_2}^{\Omega} = \varphi_{sx_1, sx_2}^{s\Omega}$ and $r_{x_1 x_2} = r_{sx_1, sx_2}$.

For the proof, let $S = S_i^+$ be the reflection functor corresponding to i (see [BGP, DR2]); it is a functor from R − mod to, say, sR − mod. Let $y = x_1 + x_2 \in \Phi^+$, so we consider the R-modules $M(x_1)$, $M(x_2)$, and $M(y)$. (The reader should be warned not to write $M(x_1 + x_2)$ instead of $M(y)$: using the convention introduced in [R3], this may be interpreted as $M(x_1) \oplus M(x_2)$, whereas $M(y)$ is the indecomposable module with the same dimension vector.) Clearly, any submodule U of $M(y)$ with $U \cong M(x_2)$ and $M(y)/U \cong M(x_1)$ gives a submodule SU of $SM(y)$ with $SU \cong SM(x_2)$, $SM(y)/SU \cong SM(x_1)$, since S is an exact functor on the full subcategory of all R-modules without direct summands $M(e_i)$, and $SM(x_i) = M(sR, sx_i)$, $SM(y) = M(sR, sy)$. Thus $F_{M(x_1), M(x_2)}^{M(y)} = F_{M(sR, sx_1), M(sR, sx_2)}^{M(sR, sy)}$. Since the root lengths do not change under s, we also have $r_{x_1, x_2} = r_{sx_1, sx_2}$.

1.3. It follows that we only have to consider $\varphi_{x_1, x_2}^{\Omega}$, where $x_2 = e_i$ for some sink i of Ω. Equivalently, we want to calculate F_{ZE}^{Y}, where Y, Z are indecomposable R-modules, and E is a simple projective R-module.

More generally, we will consider a representation-directed algebra A, with a simple projective A-module E, and indecomposable A-modules Y and Z such that $\dim Y = \dim E + \dim Z$. Here, given an A-module X, we denote by $\dim X$ the corresponding element in the Grothendieck group $K(A)$ of all (finitely generated) A-modules modulo all exact sequences. We denote by $r_{[Z][E]}$ the number of non-negative integers t such that $\dim Z - t \dim E$ is the dimension vector of an indecomposable A-module.

2. The Main Results

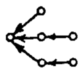
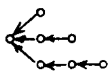
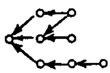
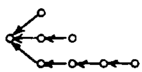
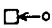
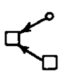
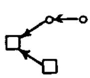
THEOREM 1. *Let A be a representation-directed k-algebra, let E be a simple projective A-module, and let Y, Z be indecomposable A-modules such that* $\dim Y = \dim E + \dim Z$. *Then there exists a representation-directed k-algebra A' with symmetrization index* 1, *a simple projective A'-module E',* *and indecomposable A'-modules Y', Z' with* $\dim Y' = \dim E' + \dim Z'$, *and, moreover, with* $E' \oplus \operatorname{top} Y'$ *sincere, such that*

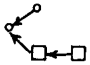
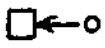
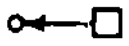
$$\varphi^{[Y]}_{[Z][E]}(T^t) = \varphi^{[Y']}_{[Z'][E']}(T) \qquad \textit{and} \qquad r_{[Z][E]} = r_{[Z'][E']}.$$

for some $t \in \mathbb{N}_1$.

We may call the passage from A to A' in Theorem 1 tightening of the support. Theorem 2 will present a list of all possible cases with tight support. The first six algebras $\mathscr{F}(i)$, $1 \leqslant i \leqslant 6$, which we have to consider will be given by a quiver: in the first occurrence we exhibit the underlying graph, and note that all arrows are supposed to point to the left (so that the far left vertex is a sink and the only one, and therefore corresponds to the unique simple projective module). In the case $\mathscr{F}(5)$, the algebra to be considered is the factor algebra of the corresponding path algebra modulo the unique commutativity relation; in all other cases $\mathscr{F}(j)$, we deal with the path algebra itself. The remaining algebras will be tensor algebras of a species of type B_2, B_3, C_3, F_4, or G_2, with a suitable orientation. Again, in the first occurrence we exhibit the underlying graph of the species; the vertices represented by a circle ∘ are endowed with the field k, those represented by a square □ are endowed with a field K, where $[K:k]=2$ or 3. As before, the orientation is supposed to be chosen in such a way that all arrows point to the left. The indecomposable module Y is given by its dimension vector $\dim Y$; it will be displayed in accordance to the graph. Since $\dim Y = \dim Z + \dim E$, one obtains immediately also the dimension vector of Z. The fifth column $(*)$ shows the endomorphism rings End E, End Y, End Z, in cases where different fields k and K are involved in the algebra; here s stands for "small," and l for "large"; thus , for example ssl means $\operatorname{End}(E) = \operatorname{End}(Y) = k$, $\operatorname{End}(Z) = K$. We denote by $e = [E]$, $y = [Y]$, $z = [Z]$ the corresponding vertices in the Auslander–Reiten quiver, and the table lists the Hall polynomial φ^y_{ze} and its evaluation $\varphi^y_{ze}(1)$. For later reference, it will be convenient to have available two other pieces of information. The minimal projective resolution of Z is of the form $0 \to dE \to \bigoplus d_i P(i) \to Z \to 0$ with $P(i)$ the indecomposable projective module corresponding to the vertex i, and we arrange these numbers in the form $d[\cdots d_i \cdots$, again in accordance to the graph, but with the additional square bracket [. (This may be interpreted also as the dimension vector $\dim C^{-1}(Z)$ of a corresponding object in the subspace category of a vectorspace category.) Finally, we exhibit the rotational equivalence classes which will be mentioned in Section 5: whenever the shift to the right yields another case, this case is listed in the last column $(**)$.

THEOREM 2. *Let A be a representation-directed basic k-algebra with* *symmetrization index* 1. *Let* E *be a simple projective* A-*module, and* Y, Z *indecomposable* A-*modules with* $\dim Y = \dim E + \dim Z$ *and* $E \oplus \operatorname{top} Y$ *sincere. Then we deal with one of the following cases*:

Case		A	$\dim Y$	*	φ^y_{ze}	$\varphi^y_{ze}(1)$	$\dim C^{-1}(Z)$	**
1	$\mathscr{F}(1)$		1 1		φ_1	1	1 [1	
2	$\mathscr{F}(2)$		1 2 1 1		φ_2	−1	1 2 [1 1	
3	$\mathscr{F}(3)$		1 3 2 1 2 1		φ_3	1	1 3 [1 1 1 1	
4	$\mathscr{F}(4)$		2 4 2 1 3 2 1		φ_4	−1	2 4 [1 1 1 1 1	
5	$\mathscr{F}(5)$		2 1 4 3 1 3 2 1		φ_5	−1	1 1 4 [1 1 1 1 1	
6	$\mathscr{F}(5)$		3 1 5 4 2 3 2 1		φ_6	1	2 1 5 [1 2 1 1 1	7
7	$\mathscr{F}(6)$		2 5 3 1 4 3 2 1		φ_6	1	2 5 [2 1 1 1 1 1	8
8	$\mathscr{F}(6)$		2 5 3 2 4 3 2 1		φ_6	1	2 5 [1 2 1 1 1 1	6
9	$\mathscr{F}(6)$		3 6 4 2 4 3 2 1		φ_7	1	3 6 [2 2 1 1 1 1	
10	B_2		1 1	lss	φ_1	1	1 [1	11
11	B_2'		1 1	ssl	φ_1	1	1 [1	12
12	B_2'		2 1	sls	$\zeta_2\varphi_1$	2	2 [1	10
13	B_3		2 2 1	lll	φ_9	−1	2 2 [1	
14	C_3		1 2 1	sss	φ_8	1	1 2 [1	
15	F_4		2 1 2 1	lss	φ_9	−1	1 1 2 [1	18

Case		A	dim Y	*	φ_{ze}^{y}	$\varphi_{ze}^{y}(1)$	dim $C^{-1}(Z)$	**
16	F_4		4 2 3 1	lll	φ_{11}	1	2 2 3 [1	
17	F_4'		1 3 2 1	sss	φ_{10}	-1	1 3 [1 1	
18	F_4'		2 3 2 1	ssl	φ_9	-1	2 3 [1 1	19
19	F_4'		2 1 4 2	sls	$\zeta_2\varphi_9$	-2	2 4 [1 1	15
20	G_2		1 1	lss	φ_1	1	1 [1	22
21	G_2		2 3	lll	φ_{12}	-1	2 [3	
22	G_2'		1 1	ssl	φ_1	1	1 [1	24
23	G_2'		2 1	sss	$\zeta_2\varphi_1$	2	2 [1	
24	G_2'		3 1	sls	$\zeta_3\varphi_1$	3	3 [1	20

3. Preliminaries on Representation-Directed Algebras

We denote by A a representation-directed algebra, and E will be a simple projective A-module with endomorphism ring D. We denote by $\langle -, - \rangle$ the bilinear form on $K(A)$ given by $\langle \dim X_1, \dim X_2 \rangle = \sum_i (-1)^i \dim_k \operatorname{Ext}^i(X_1, X_2)$ for arbitrary A-modules X_1, X_2. The following lemma is well known in case the base field is algebraically closed, or in case we deal with indecomposable modules.

3.1. *Let X, Y be A-modules with* $\operatorname{Ext}^1(X, X) = 0 = \operatorname{Ext}^1(Y, Y)$ *and* $\dim X = \dim Y$. *Then* $X \cong Y$.

Proof. Let $X = \bigoplus X_i$, $Y = \bigoplus Y_j$ with indecomposable modules X_i, Y_j. We can assume $X \neq 0$, and choose X_i maximal with respect to the ordering $\preccurlyeq$. Since $\langle \dim X_i, \dim Y \rangle = \langle \dim X_i, \dim X \rangle > 0$, we have $\operatorname{Hom}(X_i, Y) \neq 0$. Choose Y_j with $\operatorname{Hom}(X_i, Y_j) \neq 0$. Since $\langle \dim Y_j, \dim X \rangle = \langle \dim Y_j, \dim Y \rangle > 0$, we have $\operatorname{Hom}(Y_j, X) \neq 0$; thus there is X_r with $\operatorname{Hom}(Y_j, X_r) \neq 0$. The maximality of X_i implies $X_i \cong X_r$, and consequently $X_i \cong Y_j$. Let $X = X_i \oplus X'$, $Y = Y_j \oplus Y'$. By induction $X' \cong Y'$; thus $X \cong Y$.

Recall that the projective dimension of any sincere indecomposable

A-module Z is at most 1. In case Z is not necessarily sincere, we obtain the same conclusion under some additional assumptions:

3.2. *Let Y, Z be indecomposable A-modules with* $\dim Y = \dim E + \dim Z$. *Assume that Y is sincere. Then* proj.dim. $Z \leqslant 1$.

Proof. If $\operatorname{Hom}(E, Z) \neq 0$ then Z itself is sincere; thus proj.dim $Z \leqslant 1$. We assume now $\operatorname{Hom}(E, Z) = 0$. Since E is simple projective, and $\dim Y = \dim E + \dim Z$, there is a unique submodule U of Y isomorphic to E. We consider the exact sequence $0 \to U \to Y \to Y/U \to 0$. Since Y is sincere, proj.dim. $Y \leqslant 1$; thus the induced map $\operatorname{Ext}^1(Y, Y) \to \operatorname{Ext}^1(Y, Y/U)$ is surjective. But $\operatorname{Ext}^1(Y, Y) = 0$; therefore $\operatorname{Ext}^1(Y, Y/U) = 0$. Since $\operatorname{Hom}(U, Y/U) = 0$, it follows that the induced map $\operatorname{Ext}^1(Y/U, Y/U) \to \operatorname{Ext}^1(Y, Y/U)$ is injective; therefore $\operatorname{Ext}^1(Y/U, Y/U) = 0$. According to 3.1, we see that $Y/U \cong Z$. As a consequence, $\operatorname{Ext}^1(Z, E) \neq 0$; thus $\operatorname{Hom}(E, \tau Z) \neq 0$, and therefore $\operatorname{Hom}(\tau Z, I(E)) \neq 0$, with $I(E)$ the injective envelope of E. Note that τZ cannot be injective; thus $\tau Z \prec I(E)$, and therefore $\operatorname{Hom}(I(E), \tau Z) = 0$. On the other hand, if I is indecomposable injective, and $I \not\cong I(E)$, then $\operatorname{Hom}(Z, I) \neq 0$; therefore $\operatorname{Hom}(I, \tau Z) = 0$. This shows that $\operatorname{Hom}(J, \tau Z) = 0$, for any injective A-module J; thus proj.dim. $Z \leqslant 1$.

3.3. *Let Y, Z be indecomposable A-modules with* $\dim Y = \dim E + \dim Z$. *Then* $\dim \operatorname{Ext}^1(Z, E)_D = 1 + \dim \operatorname{Ext}^1(Y, E)_D$.

Proof. Clearly, neither Y nor Z can be isomorphic to E; thus $\operatorname{Hom}(Y, E) = 0$, $\operatorname{Hom}(Z, E) = 0$. We may assume that Y is sincere; thus proj.dim $.Y \leqslant 1$. Also, proj.dim. $Z \leqslant 1$, according to 3.2. It follows from

$$\langle \dim Y, \dim E \rangle = \langle \dim E, \dim E \rangle + \langle \dim Z, \dim E \rangle$$

that

$$\dim_k \operatorname{Ext}^1(Y, E) = \dim_k D - \dim_k \operatorname{Ext}^1(Z, E)$$

since $\operatorname{Hom}(Y, E) = 0$, $\operatorname{Hom}(Z, E) = 0$, and $\operatorname{Ext}^1(E, E) = 0$. Dividing by $\dim_k D$, we obtain the desired equality.

3.4. Let Y be an indecomposable A-module, not isomorphic to E. Let $d(Y) = \dim \operatorname{Ext}^1(Y, E)_D$. There exists an exact sequence

$$0 \to d(Y)E \xrightarrow{\gamma(Y)} \tilde{Y} \xrightarrow{\pi(Y)} Y \to 0$$

such that the induced map $\operatorname{Hom}(d(Y)E, E) \to \operatorname{Ext}^1(Y, E)$ is bijective. Up to isomorphism, the module $\tilde{Y}$ is uniquely determined by Y, and we call the sequence the *canonical exact sequence for Y*.

LEMMA. $\operatorname{Hom}(\tilde{Y}, E)=0$, $\operatorname{Ext}^1(E\oplus \tilde{Y}, E\oplus \tilde{Y})=0$.

Proof. Since $Y \ncong E$, we have $\operatorname{Hom}(Y, E)=0$. The bijectivity of $\operatorname{Hom}(d(Y)E, E) \to \operatorname{Ext}^1(Y, E)$ and $\operatorname{Ext}^1(E, E)=0$ imply that $\operatorname{Hom}(\tilde{Y}, E)=0$ and $\operatorname{Ext}^1(\tilde{Y}, E)=0$. Applying $\operatorname{Ext}^1(-, Y)$ to the canonical exact sequence for Y, we conclude that $\operatorname{Ext}^1(\tilde{Y}, Y)=0$, since $\operatorname{Ext}^1(Y, Y)=0$ and $\operatorname{Ext}^1(E, Y)=0$. Application of $\operatorname{Ext}^1(\tilde{Y}, -)$ to the same sequence yields $\operatorname{Ext}^1(\tilde{Y}, \tilde{Y})=0$. since $\operatorname{Ext}^1(\tilde{Y}, Y)=0$ and $\operatorname{Ext}^1(\tilde{Y}, E)=0$. Since also $\operatorname{Ext}^1(E, E\oplus \tilde{Y})=0$, it follows that $\operatorname{Ext}^1(E\oplus \tilde{Y}, E\oplus \tilde{Y})=0$.

3.5. *Let* Y, Z *be indecomposable* A*-modules, with* $\dim Y=\dim E+\dim Z$. *Then* $\tilde{Y}\cong \tilde{Z}$.

Proof. According to 3.3, we have $d(Z)=d(Y)+1$; thus $\dim \tilde{Z}= d(Z)\dim E+\dim Z = (d(Y)+1)\dim E+\dim Z = d(Y)\dim E+\dim Y= \dim \tilde{Y}$. According to 3.4, we have both $\operatorname{Ext}^1(\tilde{Y}, \tilde{Y})=0$, $\operatorname{Ext}^1(\tilde{Z}, \tilde{Z})=0$. The assertion now follows from 3.1.

4. VECTORSPACE CATEGORIES

We want to use vectorspace categories in order to describe certain full subcategories of module categories. This procedure is due to Nazarova and Rojter, and has been used extensively, and in various ways, by several authors. We follow closely the presentation by Müller [Mü] and Dräxler [Dx]. For the convenience of the reader, we outline the essential details.

Let D be a finite-dimensional division k-algebra. Let $\mathscr{K}$ be a k-additive category with only finitely many indecomposable objects and we assume that $\mathscr{K}$ is a Krull–Schmidt category. Let $|-|$ be an additive functor from $\mathscr{K}$ to $D-\operatorname{mod}$. The pair $(\mathscr{K}, |-|)$ is called a *vectorspace category*. A vectorspace category $(\mathscr{K}, |-|)$ is said to be *faithful* provided $|-|$ is a faithful functor, and *directed*, provided the indecomposable objects $X_1, ..., X_n$ of $\mathscr{K}$ can be indexed in such a way that $\operatorname{rad}(X_i, X_j) = 0$ for $i \geqslant j$. Given a vectorspace category $(\mathscr{K}, |-|)$, we denote by $\mathscr{U}=\mathscr{U}(\mathscr{K}, |-|)$ the *subspace category* of $(\mathscr{K}, |-|)$. The objects of $\mathscr{U}$ are triples $V=(V_0, V_\omega, \gamma_V)$, where V_0 is an object of $\mathscr{K}$, V_ω is a finite-dimensional D-space, and $\gamma_V\colon V_\omega \to |V_0|$ is a D-linear monomorphism. The maps $V \to V'$ are pairs (f_0, f_ω) with $f_0\colon V_0 \to V_0'$ a map in $\mathscr{K}$, and $f_\omega\colon V_\omega \to V'_\omega$ D-linear such that $\gamma_V|f_0| = f_\omega\gamma_V$. In case $\mathscr{U}$ has only finitely many indecomposable objects, the vectorspace category $(\mathscr{K}, |-|)$ is said to be *subspace-finite*. An object $V\in\mathscr{U}$ with add $V_0=\mathscr{K}$ is said to be *sincere*, and $(\mathscr{K}, |-|)$ is said to be *sincere* provided there exists an indecomposable sincere object in $\mathscr{U}$.

As before, let A be representation-directed, and let E be a simple projective A-module with endomorphism ring D. Let $\mathscr{K}$ be a finite module class in A − mod such that $\mathrm{Hom}(\mathscr{K}, E)=0$, $\mathrm{Ext}^1(\mathscr{K}, E)=0$, and let $|-|$ be the restriction of $\mathrm{Hom}(E,-)$ to $\mathscr{K}$. Given an object in $\mathscr{U}=\mathscr{U}(\mathscr{K}, |-|)$, let $\tilde{\gamma}_V: E \otimes_D V_\omega \to V_0$ be the adjoint map of $\gamma_V: V_\omega \to \mathrm{Hom}(E, V_0)$, and let $\pi_V: V_0 \to C(V)$ be its cokernel. Observe that with γ_V also $\tilde{\gamma}_V$ is a monomorphism; thus we deal with the exact sequence

$$0 \to E \otimes_D V_\omega \xrightarrow{\tilde{\gamma}_V} V_0 \xrightarrow{\pi_V} C(V) \to 0.$$

4.1. *The functor* $C: \mathscr{U} \to A-\mathrm{mod}$ *is a full embedding.*

Proof. Let V, V' be objects in $\mathscr{U}$, and $f: C(V) \to C(V')$. Since $\mathrm{Ext}^1(V_0, E \otimes_D V'_\omega)=0$, we obtain $f_0: V_0 \to V'_0$ with $f_0 \pi_{V'} = \pi_V f$; thus also $f'_\omega: E \otimes_D V_\omega \to E \otimes_D V'_\omega$ and $f'_\omega \tilde{\gamma}_V f_0$. Since $D=\mathrm{End}(E)$, there is $f_\omega: V_\omega \to V'_\omega$ with $f'_\omega = 1_E \otimes f_\omega$. It follows that (f_0, f_ω) is a map in $\mathscr{U}$, and $C(f_0, f_\omega)=f$; on the other hand, let $(g_0, g_\omega): V \to V'$ be a map in $\mathscr{U}$ with $C(g_0, g_\omega)=0$. Since $g_0 \pi_{V'}=0$, there is h: $V_0 \to E \otimes_D V'_\omega$ with $h \tilde{\gamma}_{V'} = g_0$; thus $\tilde{\gamma}_V h = g_\omega$, since g_0 is a monomorphism. However , $\mathrm{Hom}(V_0, E)=0$; thus $h=0$, and therefore $g_0=0$, $g_\omega=0$.

If $\mathscr{K}$ is fixed, we write C^{-1} for the inverse functor to C, it is defined on a full subcategory, and $C^{-1}(X)=V$, provided $V \in \mathscr{U}$ and $C(V)=X$.

4.2. Fix some indecomposable module X, not isomorphic to E, but with $\mathrm{Hom}(E, X) \neq 0$ or $\mathrm{Ext}^1(X, E) \neq 0$.

LEMMA. *The category* add $\tilde{X}$ *is directed,* $\mathrm{Hom}(\mathrm{add}\, \tilde{X}, E) = 0$, $\mathrm{Ext}^1(\mathrm{add}\, \tilde{X}, E)=0$, *and the restriction of* $\mathrm{Hom}(E,-)$ *to* add $\tilde{X}$ *is faithful.*

Proof. Clearly, add $\tilde{X}$ is directed, since A − mod is directed. The assertions $\mathrm{Hom}(\tilde{X}, E)=0$, $\mathrm{Ext}^1(\tilde{X}, E)=0$ have been shown in 3.4. It remains to show that the restriction of $\mathrm{Hom}(E,-)$ to add $\tilde{X}$ is faithful. Let X_1, X_2 be indecomposable direct summands of $\tilde{X}$ aind assume there is a non-zero map $f: X_1 \to X_2$ with $\mathrm{Hom}(E, f)=0$. Write $\tilde{X}=X_1 \oplus X'_1$ and consider $[f\ 0]: \tilde{X} \to X_2$. We consider the map $\gamma(X)$: $d(X)E \to \tilde{X}$. Since $\mathrm{Hom}(E, f)=0$, we have $\gamma(X)[f\ 0]=0$; thus there is $f': \tilde{X} \to X_2$ with $[f\ 0]=\pi(X) f'$. With f also f' is non-zero. However, the restriction π_2 of $\pi(X)$ to X_1 is non-zero; thus $X \preccurlyeq X_2 \preccurlyeq X$. It follows that π_2 is an isomorphism, and therefore $\tilde{X}=X_2 \cong X$. Thus $\mathrm{Ext}^1(Y, E)=0$. But then, by assumption, $\mathrm{Hom}(E, Y) \neq 0$. Let $0 \neq g: E \to X$. Since $X_1=X_2=X$, the map f is an automorphism of X; thus $gf \neq 0$. But this contradicts the assumption $\mathrm{Hom}(E, f)=0$.

4.3. It follows that we can apply 4.1. We denote by $\mathscr{K}(E, X)$ the vectorspace category $\mathscr{K}(E, X)=(\mathrm{add}\, \tilde{X}, |-|)$, with $|-|$ the restriction of $\mathrm{Hom}(E, -)$ to add $\tilde{X}$.

PROPOSITION. *$\mathcal{K}(E, X)$ is a faithful and directed vectorspace category; it is subspace finite and sincere. In fact, $C^{-1}X$ is an indecomposable sincere object in $\mathcal{U}(\mathcal{K}(E, X))$, with $(C^{-1}X)_0 = \tilde{X}$ and $(C^{-1}X)_\omega = d(X)D$.*

Proof. Since C: $\mathcal{U}(\mathcal{K}(E, X)) \to A - \text{mod}$ is a full embedding, $\mathcal{U}(\mathcal{K}(E, X))$ is a category with finitely many indecomposable objects; thus $\mathcal{K}(E, X)$ is subspace finite. Consider the canonical exact sequence for X

$$0 \longrightarrow d(X)E \xrightarrow{\gamma(X)} \tilde{X} \xrightarrow{\pi(X)} X \longrightarrow 0.$$

We have $d(X)E = E \otimes_D d(X)D$; thus $\gamma(X)$: $E \otimes_D d(X)D \to \tilde{X}$ is the adjoint of a map $\gamma_X: d(X)D \to \text{Hom}(E, \tilde{X})$, and $C^{-1}(X) = (\tilde{X}, d(X)D, \gamma_X)$. In particular, $C^{-1}(X)$ is a sincere object of $\mathcal{U}(\mathcal{K}(E, X))$.

4.4. Let Y, Z be indecomposable A-modules with $\dim Y = \dim E + \dim Z$. Then neither Y nor Z is isomorphic to E. Since we have $\text{Hom}(E, Y) \neq 0$, we can use the considerations above for $X = Y$. Also, $\text{Ext}^1(Z, E) \neq 0$ by 3.3; thus we can use these considerations also for $X = Z$. Finally, we recall from 3.5 that $\tilde{Y} \cong \tilde{Z}$. Thus, we deal with a subspace-finite, faithful, directed vectorspace category $\mathcal{K}(E, Y) = \mathcal{K}(E, Z)$ for which there exist two non-isomorphic indecomposable sincere objects in $\mathcal{U}(\mathcal{K}(E, Y))$, namely $C^{-1}(Y)$ and $C^{-1}(Z)$. Here,

$$C^{-1}(Z)_0 = \tilde{Z} \cong \tilde{Y} = C^{-1}(Y)_0,$$

and

$$C^{-1}(Z)_\omega = d(Z)D \cong d(Y)D \oplus D = C^{-1}(Y)_\omega \oplus D.$$

4.5. The faithful directed vectorspace categories which are subspace finite and sincere have been classified by Klejner [K] and Klemp and Simson [KS1, KS2]. Klejner has considered the case $k = D$; for a recent account of his result, including, in particular, a presentation of the corresponding Auslander–Reiten quivers, we refer the reader to [R2]. The treatment of the general case by Klemp and Simson is phrased in terms of socle-projective modules, but it may be reformulated in terms of vectorspace categories without difficulties. From the list of all possible faithfull directed vectorspace categories $(\mathcal{K}, |-|)$ which are subspace finite and sincere, we have to single out those with at least two different indecomposable sincere objects in $\mathcal{U}(\mathcal{K}, |-|)$. The cases singled out may be indexed by $\mathcal{F}(i)$, $1 \leqslant i \leqslant 6$, and B_2, B_2', B_3, C_3, F_4, F_4', G_2, G_2', similar to the algebras exhibited in Theorem 2, but here we have to admit arbitrary division rings which are finite-dimensional over k. In the cases different from $\mathcal{F}(5)$, we deal with a hereditary finite-dimensional k-algebra B with the oriented valued graph given in the table; in case $\mathscr{F}(5)$ we deal with the algebra B obtained from a path algebra of the quiver modulo the commutativity relation. (A non-trivial modulation of the graph would not allow one to form the corresponding factor algebra.) There is always a unique sink, and thus a unique simple B-module E', and $\mathscr{K}$ is the category of all projective B-modules without direct summand isomorphic to E', and $|-|$ is the restriction of $\mathrm{Hom}(E', -)$ to $\mathscr{K}$. Note that $\mathscr{U}(\mathscr{K}, |-|)$ may be identified with a full subcategory of $B-\mathrm{mod}$, using the functor $C: \mathscr{U}(\mathscr{K}, |-|) \to B-\mathrm{mod}$ of 4.1; the image of this functor is just the full subcategory of socle-projective B-modules. Given an indecomposable socle-projective B-module X, not isomorphic to E', let us consider its canonical exact sequence (with respect to E')

$$0 \to d(Y)\, E' \to \tilde{X} \to X \to 0.$$

Then $\tilde{X}$ is a projective B-module; thus the sequence is a minimal projective resolution of X. In this way, we see that the column in the table of Theorem 2 which exhibits the minimal projective resolution of Y, in fact gives the dimension vector $\dim C^{-1}(Y)$ of the object $C^{-1}(Y)$ in $\mathscr{U}(\mathscr{K}, |-|)$, namely, first the dimension $d(Y)$ of $C^{-1}(Y)_\omega$, and then, after the bracket [, the multiplicities of the indecomposable objects of $\mathscr{K}$ in a direct decomposition of $C^{-1}(Y)_0 = \tilde{Y}$.

Finally, we observe that B is always a tilted algebra of Dynkin type (with the exception of $\mathscr{F}(5)$, the algebra B itself is hereditary, and $\mathscr{F}(6)$ tilts to $\mathscr{F}(5)$); in particular, B is representation directed.

4.6. Let us complete the proof of Theorem 1 and of that part of Theorem 2 which asserts that there are only the 24 listed possibilities.

So assume that A is a representation-directed k-algebra, E is a simple projective A-module, and Y and Z are indecomposable A-modules such that $\dim Y = \dim E + \dim Z$. According to 4.4, the vectorspace category $\mathscr{K}(E, Y)$ is subspace-finite, faithful and directed, and $C^{-1}(Y)$ and $C^{-1}(Z)$ are non-isomorphic indecomposable sincere objects in $\mathscr{U}(\mathscr{K}(E, Y))$. Let $\tilde{Y} = \bigoplus_{i=1}^m d_i Y_i$ with pairwise non-isomorphic indecomposable A-modules Y_i. Let $T = E \oplus \bigoplus_{i=1}^m Y_i$, and let $A'' = \mathrm{End}(T)$. According to 4.5, the algebra A'' is representation-directed with a unique simple projective module E''. If we denote by Σ the functor $\Sigma = \mathrm{Hom}({}_A T, -)$: $A - \mathrm{mod} \to A'' - \mathrm{mod}$, then $E'' = \Sigma(E)$, and we identify $D = \mathrm{End}(E)$ with $\mathrm{End}(E'')$ under Σ. The image of the canonical exact sequence $0 \to dE \to \tilde{Y} \to Y \to 0$ is mapped under Σ to the canonical exact sequence for $Y'' = \Sigma(Y)$; in particular, the projective cover of Y'' is isomorphic to $\bigoplus_{i=1}^m d_i \Sigma(Y_i)$, and thus $E'' \oplus \mathrm{top}\, Y''$ is a sincere A''-module. Clearly, Σ

yields an equivalence add $\tilde{Y} \to$ add $\tilde{Y}''$ so that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{add} \tilde{Y} & \xrightarrow{\Sigma} & \operatorname{add} \tilde{Y}'' \\ & {\scriptstyle \operatorname{Hom}(E,-)} \searrow \quad \swarrow {\scriptstyle \operatorname{Hom}(E'',\ )} & \\ & D-\operatorname{mod} & \end{array}$$

Thus, the vectorspace categories $\mathscr{K}(E, Y)$ and $\mathscr{K}(E'', Y'')$ are equivalent. As a consequence, we obtain an equivalence of the corresponding subspace categories so that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{U}(\mathscr{K}(E, Y)) & \xrightarrow{\approx} & \mathscr{U}(\mathscr{K}(E'', Y'')) \\ \downarrow {\scriptstyle C} & & \downarrow {\scriptstyle C} \\ A-\operatorname{mod} & \xrightarrow{\Sigma} & A''-\operatorname{mod}. \end{array}$$

For the proof of Theorem 2, we suppose in addition that $E \oplus \operatorname{top} Y$ is sincere. Let X be a simple A-module, not isomorphic to E. Then Y maps onto X, since $E \oplus \operatorname{top} Y$ is sincere. Therefore, $\operatorname{Ext}^1(Y, X) = 0$, since A is representation-directed. Consequently, $\operatorname{Ext}^1(\tilde{Y}, X) = 0$, since also $\operatorname{Ext}^1(E, X) = 0$. According to (3.4), we also have $\operatorname{Ext}^1(\tilde{Y}, E) = 0$; therefore $\tilde{Y}$ is projective. On the other hand, top $\tilde{Y} \cong$ top Y, and thus any indecomposable projective A-module different from E is a direct summand of $\tilde{Y}$; thus T is a progenerator and Σ is an equivalence of categories. Since A is even basic, A and A'' are isomorphic, so we know the structure of A from 4.5. Finally, we use the assumption on the symmetrization index of A to be 1. In the cases $\mathscr{F}(i)$, $1 \leqslant i \leqslant 6$, we see that k is the endomorphism ring of any indecomposable A-module; in the remaining cases, the endomorphism rings of the indecomposable A-modules are k or $\mathscr{K}$, where $k \subset \mathscr{K}$ is a field extension of degree 2 or 3. This shows that A and Y are as described in Theorem 2.

Let us return to the general situation in order to complete the proof of Theorem 1. Let $Z'' = \Sigma(Z)$. The submodules U of Y with $U \cong E$ and $Y/U \cong Z$ correspond bijectively to the submodules U'' of Y with $U'' \cong E''$ and $Y''/U'' \cong Z''$; thus, in case k is a finite field, $F^Y_{ZE} = F^{Y''}_{Z''E''}$. Also, we observe that $r_{[z][E]} = r_{[Z''][E'']}$. Consider the vertices $e = [E]$, $y = [Y]$, $z = [Z]$ of Γ_A, and $e'' = [E'']$, $y'' = [Y'']$, $z'' = [Z'']$ of $\Gamma_{A''}$. It remains to be shown that $\Gamma_{A''}$ and the position of e'', y'', z'' in $\Gamma_{A''}$ only depend on the position of e, y, z in Γ_A. Now, e and y determine $d = \dim \operatorname{Ext}^1(Y, E)_D = \dim_D \operatorname{Hom}(E, \tau Y)$ (with the notation of [R3], we have $d = h([e], \tau[y])$ for Y non-projective, and $d = 0$ otherwise), and the vertices $y_i = [Y_i]$ with

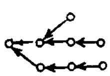
$1 \leqslant i \leqslant m$ are uniquely determined by the two properties $\operatorname{Ext}^1(Y_i, Y_j) = 0$ for all i, j, and $\sum_{i=1}^{m} d_i \cdot \dim Y_i = \dim Y + d \cdot \dim E$, both properties being determined by Γ_A. A glance at the last column of the table in Theorem 2 shows that the dimension vector $\dim C^{-1}(Y)$ together with the quotients $(\dim_k \operatorname{End} Y_i)(\dim_k \operatorname{End} Y_j)^{-1}$ for $0 \leqslant i, j \leqslant m$. where $Y_0 = E$, determine the type of the vectorspace category $\mathscr{K}(E, Y)$, thus $\Gamma_{A''}$, and again all the information needed is given by Γ_A. Finally, we obtain from the dimension vector $\dim C^{-1}(Y)$ of the object $C^{-1}(Y)$ in $\mathscr{U}(\mathscr{K}(E, Y)) \cong \mathscr{U}(\mathscr{K}(E'', Y''))$ the dimension vector $\dim Y''$ of Y'', and therefore y'' and z'' (and of course e'') in $\Gamma_{A''}$. If r is the symmetrization index of A and r'' is the symmetrization index of A'', we obtain $\varphi^{y}_{ze}(|k|^r) = \varphi^{y''}_{z''e''}(|k|^{r''})$. If we choose an algebra A' with $\Gamma_{A'} = \Gamma_{A''}$ and symmetrization index 1, and denote by E', Y', Z' the A'-modules with $e'' = [E']$, $y'' = [Y']$, and $z'' = [Z']$, then $\varphi^{y}_{ze}(T^r) = \varphi^{y'}_{z'e'}(T)$. This completes the proof of Theorem 1.

4.7. Corollary. *Let k be a finite field. Let A be a basic representation-directed k-algebra with centre k. Let E be a simple projective A-module, and let Y, Z be indecomposable A-modules with* $\dim Y = \dim E + \dim Z$ *and* $E \oplus \operatorname{top} Y$ *sincere. Then the symmetrization index of A is* 1.

Proof. Since finite division rings are commutative, we can determine the centre of A in any of the cases mentioned in 4.5. In the cases $\mathscr{F}(i)$, $1 \leqslant i \leqslant 6$, the centre will be the field over which the path algebra is formed; in the remaining cases, it will be the smaller of the two fields given by the species. This completes the proof.

5. Rotational Equivalence

5.1. Almost all algebras occurring in Theorem 2 are hereditary; the only exception is $\mathscr{F}(5)$. Insteed of $\mathscr{F}(5)$ we may consider the hereditary algebra $A = \mathscr{F}(5)'$ given by the quiver



We have to deal with the simple projective A-module E and with the indecomposable A-modules Y with

$$\dim Y = \begin{matrix} & & 2 & \\ 4 & 4 & 3 & 1 \\ & 3 & 2 & 1 \end{matrix}$$

for the case 5, and

$$\dim Y = \begin{array}{cccc} & 3 & & \\ 5 & 6 & 4 & 2 \\ & 3 & 2 & 1 \end{array}$$

for the case 6. For, in both cases, the indecomposable A-modules in add $\tilde{Y}$ have the following dimension vectors; we exhibit them together with the partially ordering:

$$\begin{array}{ccccc} & \begin{array}{cccc} & 1 & & \\ 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{array} & \longrightarrow & \begin{array}{cccc} & 1 & & \\ 1 & 1 & 1 & 0 \\ & 0 & 0 & 0 \end{array} & \\ & & \nearrow & & \\ & \begin{array}{cccc} & 0 & & \\ 1 & 1 & 1 & 0 \\ & 0 & 0 & 0 \end{array} & \longrightarrow & \begin{array}{cccc} & 0 & & \\ 1 & 1 & 1 & 1 \\ & 0 & 0 & 0 \end{array} & \end{array}$$

$$\begin{array}{cccc} & 0 & & \\ 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \end{array} \longrightarrow \begin{array}{cccc} & 0 & & \\ 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 \end{array} \longrightarrow \begin{array}{cccc} & 0 & & \\ 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 \end{array}.$$

The dimension vector for $C^{-1}(Y)$ is

$$\begin{array}{cccc} & & 1 & 1 \\ 3 & [& 1 & 1 \\ & & 1 & 1 & 1 \end{array} \quad \text{in the case } \dim Y = \begin{array}{cccc} & 2 & & \\ 4 & 4 & 3 & 1 \\ & 3 & 2 & 1 \end{array},$$

and

$$\begin{array}{cccc} & & 2 & 1 \\ 4 & [& 1 & 2 \\ & & 1 & 1 & 1 \end{array} \quad \text{in the case } \dim Y = \begin{array}{cccc} & 3 & & \\ 5 & 6 & 4 & 2 \\ & 3 & 2 & 1 \end{array}.$$

This shows that for calculating the Hall polynomials φ^{y}_{ze} for the cases occurring in Theorem 2, we may consider a hereditary algebra R, a simple projective R-module E, and indecomposable R-modules Y, Z with $y = e + z$ where $e = \dim E$, $y = \dim Y$, $y = \dim Z$.

5.2. Denote by α_x the polynomial for $x \in \Gamma_R$ which counts the number of automorphisms of any indecomposable module with dimension vector x

(see [R3]). As in 1.2, let s be the reflection corresponding to the vertex i with $e = e_i$. Then

$$\varphi^{s\Omega}_{e,sy} = \frac{\alpha_z}{\alpha_y} \varphi^{\Omega}_{ze}.$$

Proof. We denote again by $S = S_i^+$ the corresponding reflection functor. Consider any epimorphism $f\colon Y \to Z$. Since $y = e + z$, we see that $\operatorname{Ker} f \cong E$. Applying the left exact functor S to the exact sequence $0 \to E \to Y \xrightarrow{f} Z \to 0$, we obtain the exact sequence $0 \to 0 \to SY \xrightarrow{Sf} SZ$; thus Sf is a monomorphism. Since $sy = se + sz = -e + sz$, the cokernel of Sf is isomorphic to the simple sR-module E' with $\dim E' = e$. Since we also may use the reflection functor S_i^-, it follows that the epimorphisms $Y \to Z$ correspond bijectively to the monomorphism $SY \to SZ$. On the other hand, we obtain the number of epimorphisms $Y \to Z$ by multiplying F^Y_{ZE} with the number of automorphisms of Z, and we obtain the number of monomorphisms $SY \to SZ$ by multiplying $F^{SZ}_{E',SY}$ with the number of automorphisms of SY. Note that the automorphisms of Y correspond bijectively to the automorphisms of SY, under S. This completes the proof.

5.3. Starting with R, E, Y, Z, we have obtained the algebra sR, and indecomposable sR-modules SY, SZ, E' with $\dim SZ = \dim SY + \dim E'$. We may use the reduction of Section 1, in order to replace SY by a simple projective R'-module, where R' is derived from sR by a change of orientation, and we may ask which of the cases listed in Theorem 2 we deal with. The answer is listed in the last column of the table in Theorem 2 provided the new case differs from the original one.

The calculation may conveniently be carried out in the derived category $D^b(R)$ of R; note that the Auslander–Reiten quiver of $D^b(R)$ is precisely $\mathbb{Z}\Delta$ (see Happel [Ha]). For all the cases, we present part of $\mathbb{Z}\Delta$, namely a complete fundamental region with respect to the shift in $D^b(R)$. Here, vertices of the form $\circ$ are isomorphism classes of objects with endomorphism ring k, while those of the form $\square$ have endomorphism ring K. For the first case mentioned at the left, we have marked in $\mathbb{Z}\Delta$ the position of $[E]$, $[Y]$, $[Z]$, and $[E[1]]$ (from left to right). If three cases are mentioned at the left (for example 6, 7, 8), the left three marked vertices give the relative position of E, Y, Z for the first case (here 6), the right three give the relative position for the second case (here 7), and shifting further to the right, we obtain the relative position for the third case (here 8); we call these cases a rotational equivalence class (see Fig. 1).

We may derive the Hall polynomials φ^y_{ze} for the cases in one rotational equivalence class from each other. Since $\alpha_e = \alpha_y = \alpha_z$ for the case 6, we see that the Hall polynomials φ^y_{ze} coincide for the cases 6, 7, 8. In the cases 10,

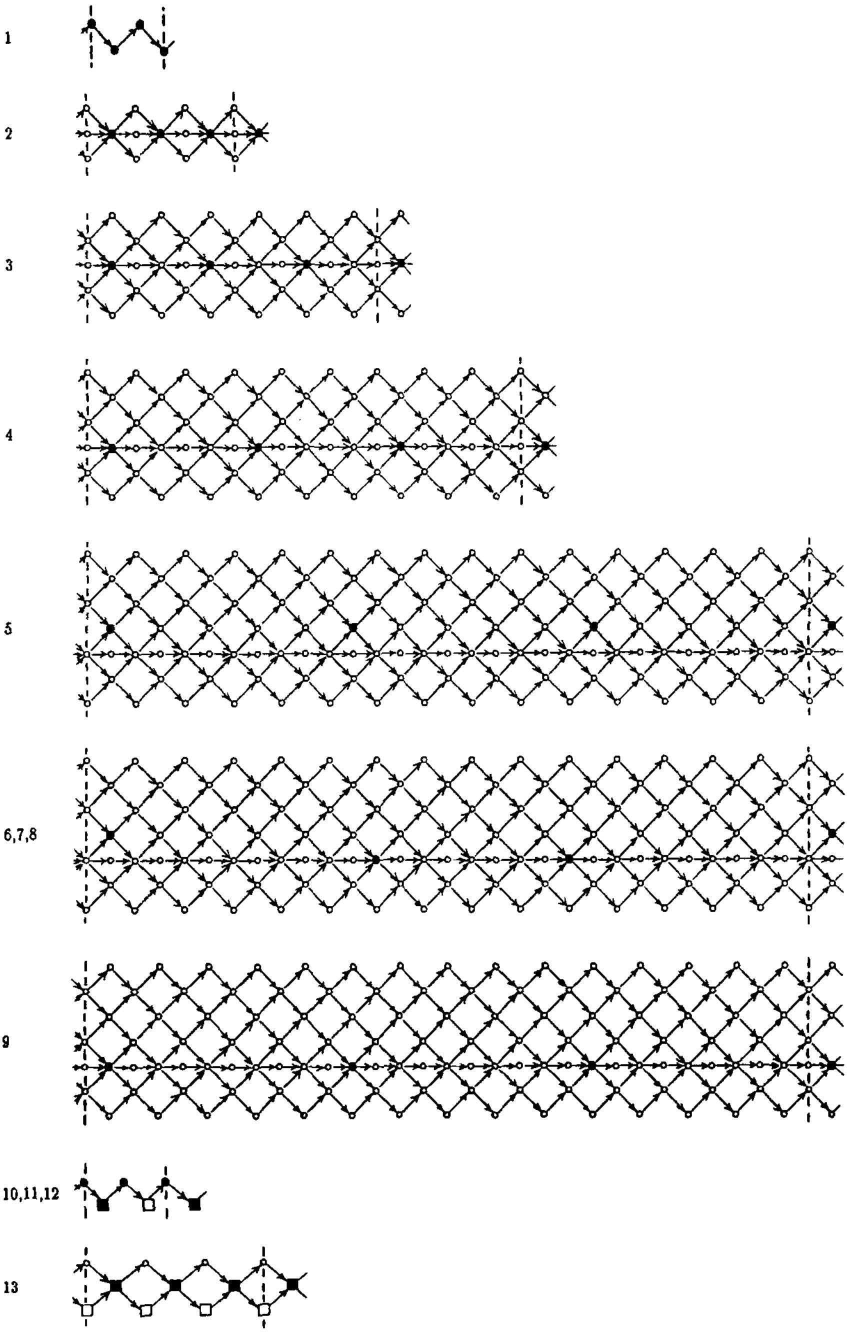


FIGURE 1

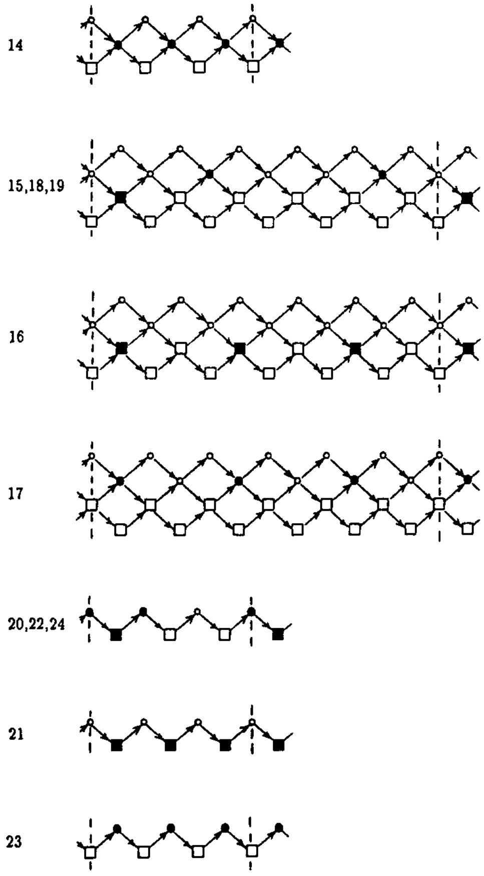


FIG. 1—*Continued*

15, and 20, we have $\alpha_e = T^r - 1$, where $r = [K:k]$, and $\alpha_y = \alpha_z = T - 1$. Note that $(T^r - 1)(T-1)^{-1} = \zeta_r$. It follows from 5.2 that the Hall polynomials φ^y_{ze} coincide for 10, 11, for 15, 18, and for 20, 22, and that the Hall polynomials φ^y_{ze} for 12, 19, and 24 are obtained from the corresponding polynomials for 11, 18, and 22, respectively, by multiplying with ζ_r.

6. TEST MODULES FOR INDECOMPOSABILITY

Let A be representation-directed, E a simple projective A-module, and Y, Z indecomposable A-modules with $\dim Y = \dim E + \dim Z$. We are looking for sets $\mathcal{N} = \{N_1, ..., N_t\}$ of indecomposable A-modules such that for any monomorphism $w: E \to Y$, we have Cok w indecomposable if and only if $\mathrm{Hom}(N_i, \mathrm{Cok}\, w) = 0$, for all $1 \leqslant i \leqslant t$. Such a set $\mathcal{N}$ will be called a test set. We first show that test sets do exist. Let $\mathcal{Q}$ be the set of indecomposable A-modules Q with $W \not\leqslant Z$.

6.1. *Let $\mathcal{N}$ be a set of indecomposable A-modules from $\mathcal{Q}$ such that for any indecomposable module V with* $\mathrm{Hom}(Z, V) \neq 0$ *and* $\dim V < \dim Z$, *there is $N \in \mathcal{N}$ with* $\mathrm{Hom}(N, V) \neq 0$. *Then $\mathcal{N}$ is a test set. In particular, $\mathcal{Q}$ itself is a test set.*

Proof. Let w: $E \to Y$ be a monomorphism, and let $C = \mathrm{Cok}\, w$. If C is indecomposable, then $C \simeq Z$, and then $\mathrm{Hom}(Q, C) = 0$ for all $Q \in \mathcal{Q}$. On the other hand, suppose C is decomposable. Assume we have $\mathrm{Hom}(N, C) = 0$ for all $N \in \mathcal{N}$. Let $\mathcal{P}$ be the set of A-modules P with $\mathrm{Hom}(Z, P) = 0$. Let C' be an indecomposable direct summand of C; thus $\mathrm{Hom}(N, C') = 0$. Since $\dim C' < \dim C = \dim Z$, we must have $\mathrm{Hom}(Z, C') = 0$. Consequently, $C' \in \mathcal{P}$; thus $C \in \mathcal{P}$. Note that C is a module in $\mathcal{P}$ with $\dim C = \dim Z$. Choose a module X in $\mathcal{P}$ with $\dim X = \dim Z$, and $\dim_k \mathrm{End}\, X$ minimal. According to 3.1, we have $\mathrm{Ext}^1(X, X) \neq 0$. Let $X = \bigoplus_{i=1}^m X_i$ with all X_i indecomposable. Without loss of generality, we can assume $\mathrm{Ext}^1(X_1, X_2) \neq 0$. Let $0 \to X_2 \to X_{m+1} \to X_1 \to 0$ be a non-split exact sequence. Since $X_1, X_2 \in \mathcal{P}$, we have $X_{m+1} \in \mathcal{P}$; thus $X' = \bigoplus_{i=3}^{m+1} X_i$ belongs to $\mathcal{P}$ and satisfies $\dim X' = \dim X = \dim Z$, but $\dim_k \mathrm{End}(X') < \dim_k \mathrm{End}(X)$. This is a contradiction to the minimality of $\dim_k \mathrm{End}(X)$. Thus there exists $N \in \mathcal{N}$ with $\mathrm{Hom}(N, C) \neq 0$.

6.2. *Let $Z \to \bigoplus_{i=1}^s Z_i$ be the source map for Z, and suppose Z_i has a filtration*

$$0 = Z_{i0} \subset Z_{i1} \subset \cdots \subset Z_{it_i} = Z_i$$

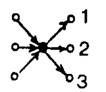
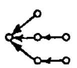
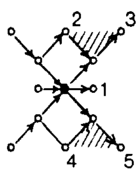
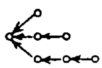
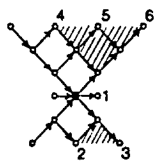
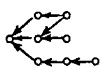
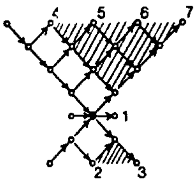
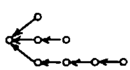
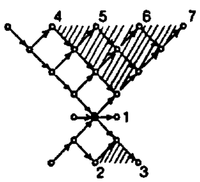
with indecomposable modules $N_{ij} = Z_{ij}/Z_{i,j-1}$ in $\mathcal{Q}$, where $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t_i$. Then $\mathcal{N} = \{N_{ij} \mid i, j\}$ is a test set.

Proof. We have to consider an indecomposable module V with $\mathrm{Hom}(Z, V) \neq 0$ and $\dim V < \dim Z$. Let $0 \neq w: Z \to V$. Since w is not a split monomorphism, it factors through the source map for Z; thus $\mathrm{Hom}(Z_i, V) \neq 0$ for some i. But then $\mathrm{Hom}(N_{ij}, V) \neq 0$ for some j.

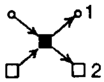
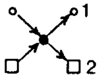
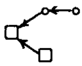
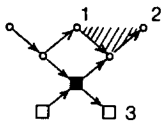
6.3. For many cases of interest we will obtain a test set using 6.2 in the following way: the modules Z_i will have a wing, and $N_{i1}, \ldots, N_{it_i}$ will be the modules on the boundary of the wing. We exhibit the corresponding part of Γ_A, with $[Z]$ being marked in black, and $[N_i]$ marked by i; the shaded areas are the non-trivial wings we deal with. See Schemes 1 and 2.

6.4. Similar considerations yield test sets $\mathcal{N} = \{N_1, ..., N_t\}$ for Cases 5, 15, and 17. See Scheme 3.

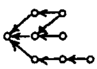
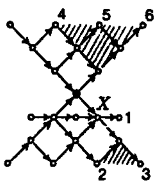
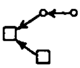
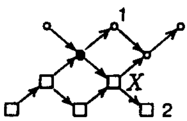
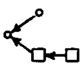
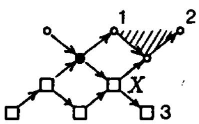
In all cases, there is an arrow $[Z] \to [X]$ with $\dim Z < \dim X$ (the vertex $[X]$ is marked by X). If $g: X \to V$ is a non-zero map, where V is indecomposable and $\dim V < \dim Z$, then g factors through the source map $X \to X'$ for X. Let $X' = \bigoplus X_i$ with X_i indecomposable. In Case 17, we take as test

Case	A	dim Z	part of Γ_A	dim $N_1, \ldots, \dim N_t$
2		1 11 1		1 0 0 00 ; 01 ; 00 0 0 1
3		1 221 21		0 1 0 111 ; 100 , 010 ; 11 11 00 1 0 111 , 000 00 10
4		2 321 321		1 0 1 110 ; 111 , 100 ; 110 110 111 0 1 0 110 , 111 , 000 111 000 100
6		31 442 321		21 11 10 221 ; 221 , 111 ; 110 111 100 11 00 10 00 110 , 111 , 100 , 010 100 110 111 000
9		3 542 4321		1 1 1 221 ; 211 , 110 ; 2110 2111 1100 0 1 0 1 111 , 100 , 110 , 111 1100 1110 1111 0000

SCHEME 1

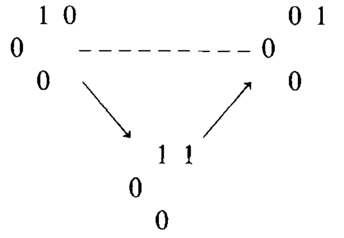
13		$1\,^{2}_{1}$		0^{1}_{0} ; 0^{0}_{1}
14		$1\,^{1}_{1}$		0^{1}_{0} ; 0^{0}_{1}
16		$2\,^{42}_{1}$		1^{11}_{1} , 0^{10}_{0} ; 1^{22}_{0}
21		13		01

SCHEME 2

Case	A	dim Z	Γ_A	dim N_1, ... , dim N_t
5		21 331 321		11 10 11 221 ; 111 , 110 ; 211 100 110 00 10 00 111 , 100 , 010 110 111 000
15		$1\,^{21}_{1}$		0^{10}_{0} ; 0^{00}_{1}
17		$2\,^{1}_{21}$		1^{0}_{11} , 0^{1}_{00} ; 0^{0}_{10}

SCHEME 3

module set the modules on the boundary of wings for the X_i. In Case 15, one of these wings is of the form



and we may delete

$$\begin{matrix} & 0\ 1 \\ 0 & \\ & 0 \end{matrix}$$

from our test module set: the only indecomposable module it maps to is the module itself, and it cannot be a direct summand of the cokernel of any map $E \to Y$. So, finally consider Case 5. Here, three X_i occur, with wings of rank 1, 2, and 4. So assume we have an indecomposable module V with $\dim V < \dim Z$, and a non-zero map $w: Z \to V$. We factor it through the source map $Z \to Z_1 \oplus Z_2$ of Z, and we can suppose $Z_1 = X$. If $\operatorname{Hom}(Z_2, V) \neq 0$, then $\operatorname{Hom}(N_j, V) \neq 0$ for one of $j = 4, 5, 6$; thus assume $\operatorname{Hom}(Z_2, V) = 0$. Thus w factors through X, say $w = fg$, with $f: Z \to X$, and $g: X \to V$. As above, we factor g through the source map $X \to X' = \bigoplus_{i=1}^{3} X_i$. If we assume that $\operatorname{Hom}(N_j, V) = 0$ for $j = 1, 2, 3$, then g factors through the unique module X_j, say $j = 3$, which has a wing of rank 4, say $g = g'h$ with $g': X \to X_3$ and $h: X_3 \to V$. But $Z \xrightarrow{f'} Z_1 = X \xrightarrow{g'} X_3$ factors through Z_2, and we have assumed $\operatorname{Hom}(Z_2, V) = 0$. This contradiction shows that $N_1, \ldots, N_6$ is a test set.

7. Some Hypersurfaces in Hom(E, Y)

Let Δ be a finite directed quiver with unique sink 0, and let B be the factor algebra of the path algebra of Δ over k modulo all commutativity relations. Let E be the unique simple projective B-module (it corresponds to the vertex 0). The set of vertices of Δ different from 0 will be denoted by Δ_0'. Given $t \in \Delta_0'$, we denote by $P(t)$ the corresponding indecomposable projective B-module, and $u_t: E \to P(t)$ will be a fixed non-zero map; note that $\dim_k \operatorname{Hom}(E, P(t)) = 1$.

Let Y be a B-module, and consider $W = \operatorname{Hom}(E, Y)$. We are interested in various subsets of W, and we assume $W \neq 0$. It would be more appropriate to consider instead of Hom(E, Y) the projective space **P** Hom(E, Y). However, the actual calculations which we have to do will be carried out in terms of maps.

First of all, given $t \in \Delta'_0$, let W_t be the set of maps $w: E \to Y$ which factor through $P(t)$. Since Hom(E, $P(t)$) is one-dimensional, W_t is a linear subspace of W. [Observe that in case the socle of Y is a direct sum of copies of E, then Y may be reconstructed from W and the subspaces W_t. Namely, the vectorspaces W and W_t, $t \in \Delta'_0$, together with inclusion maps, yield a representation of Δ which is a B-module, and this B-module is isomorphic to Y.] If $\operatorname{Hom}(P(t), P(t')) \neq 0$, then $u_{t'}$ factors through u_t; thus $W_{t'} \subseteq W_t$. Also, conversely, $W_{t'} \subseteq W_t$ implies that $u_{t'}$ factors throug u_t; thus $\operatorname{Hom}(P(t), P(t')) \neq 0$.

Given any B-module N, we denote by W_N the set of maps $w: E \to Y$ such that either $w = 0$ or else $\operatorname{Hom}(N, \operatorname{Cok} w) \neq 0$. We assume that $\operatorname{Ext}^1(N, P(t)) = 0$ for $t \in \Delta'_0$ and that $\operatorname{Hom}(N, Y) = 0$. In particular, E is not a direct summand of N. Then the minimal projective resolution of N is of the form

$$0 \longrightarrow dE \xrightarrow{[\gamma_{ij}\, u_j]} \bigoplus_j P_j \longrightarrow N \longrightarrow 0,$$

with $P_j = P(t_j)$, $u_j = u_{t_j}$ for some $t_j \in \Delta'_0$, and $\gamma_{ij} \in k$, where $1 \leqslant i \leqslant d$.

7.1. Lemma. *A non-zero element $w \in W$ belongs to W_N if and only if there are elements $w_j \in W_{t_j}$, not all zero, such that $\sum_j \gamma_{ij} w_j \in kw$, for all $1 \leqslant i \leqslant d$.*

Proof. Let $0 \neq w \in W$. First, assume there is $0 \neq f: N \to \operatorname{Cok} w$. Using projectivity, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & dE & \xrightarrow{[\gamma_{ij}\, u_j]} & \bigoplus P_j & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle [f_j]} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & E & \xrightarrow{w} & Y & \longrightarrow & \operatorname{Cok} w & \longrightarrow & 0 \end{array}$$

Let $w_j = u_j f_j$. This is an element of W_{t_j}. If we assume that all w_j are zero, then $[f_j]$ factors through N; thus f factors through Y. But $\operatorname{Hom}(N, Y) = 0$ gives a contradiction. The left commutative square shows that $\sum \gamma_{ij} w_j \in kw$. Conversely, assume there are elements $w_j \in W_{t_j}$, not all zero, such that $\sum_j \gamma_{ij} w_j = \lambda_i w$ for some $\lambda_i \in k$, and all $1 \leqslant i \leqslant d$. Write $w_j = u_j f_j$ for some $f_j: P_j \to Y$. Let $f' = [\lambda_i]_i$, so we obtain a commutative square as the left one above; thus we obtain a map $f: N \to \operatorname{Cok} w$. But $f \neq 0$, since otherwise we obtain a factorization of $[f_j]_j$ through w, but $\operatorname{Hom}(P_j, E) = 0$ for all j. This completes the proof. We consider now some special cases:

7.2. *Let J be a non-empty subset Δ'_0, and let N be the cokernel of* $[u_j]_j$: $E \to \bigoplus_{j \in J} P(j)$. *If* $\operatorname{Hom}(N, Y) = 0$, *then* $W_N = \sum_{j \in J} W_j$.

Proof. We have $d = 1$, and $\gamma_{1j} = 1$ for all j; so W_N consists precisely of the sums of elements of the W_j, $j \in J$.

7.3. Let $d \geqslant 1$, and we assume that some of the vertices in Δ'_0 have been labelled $1, 2, ..., d$. Let N be the cokernel of the map

$$E^{d-1} \xrightarrow{\begin{bmatrix} u_1 & -u_2 & & & \\ & u_2 & -u_3 & & \\ & & & \cdots & \\ & & & u_{d-1} & -u_d \end{bmatrix}} \bigoplus_{j=1}^{d} P(j),$$

and we assume again that $\operatorname{Hom}(N, Y) = 0$. Let $D(W_1, ..., W_d)$ be the set of all elements $w \in W$ which are either zero, or else there are elements $w_j \in W_j$, not all zero, with $w_i - w_j \in kw$, for all $1 \leqslant i, j \leqslant d$. Then $W_N = D(W_1, ..., W_d)$.

7.4. *Let N be the cokernel of*

$$E^2 \xrightarrow{\begin{bmatrix} u_1 & u_2 & 0 & u_4 \\ 0 & u_2 & u_3 & u_4 \end{bmatrix}} \bigoplus_{i=1}^{4} P(i),$$

and assume $\operatorname{Hom}(N, Y) \neq 0$. *Then* $W_N = D(W_1, W_3, W_2 + W_4)$.

Proof. A non-zero element $w \in W$ belongs to W_N if and only if there are elements $w_i \in W_i$, $1 \leqslant i \leqslant 4$, not all zero, such that $w_1 - (w_2 + w_4) \in kw$, and $(w_2 + w_4) - w_3 \in kw$, if and only if $w \in D(W_1, W_3, W_2 + W_4)$.

Subsets of the form $D(W_1, W_2, W_3)$ will be of further interest, so we are going to study them. In 7.5 and 7.6, we consider an arbitrary finite-dimensional vectorspace W with subspaces W_i, and $\mathscr{R}$ will be the ring of regular (=polynomial) functions on W. Given elements $r_1, ..., r_t$ of $\mathscr{R}$, we denote by $\mathscr{Z}(r_1, ..., r_t)$ the set of common zeros.

7.5. *Let $W = W_1 \oplus W_2$, let l_i, l'_i be linear forms on W_i, for $1 \leqslant i \leqslant 2$, and consider $r = l_1 - l_2$, $r' = l'_1 - l'_2$ as elements of $\mathscr{R}$. Let $W_3 = \mathscr{Z}(r, r')$. Then $D(W_1, W_2, W_3) = \mathscr{Z}(l_1 l'_2 - l'_1 l_2)$.*

Proof. Let $q = l_1 l'_2 - l'_1 l_2$. We show that q vanishes on $D(W_1, W_2, W_3)$. Let w be a non-zero element of $D(W_1, W_2, W_3)$; thus there are elements $w_i \in W_i$, $1 \leqslant i \leqslant 3$, not all zero, with $w_1 - w_2 = \lambda w$, $w_2 - w_3 = \mu w$ for some λ, $\mu \in k$. If $\lambda = 0$, then $w_1 = w_2 = 0$; thus $w_3 \neq 0$. Therefore $\mu \neq 0$; thus $w = -(1/\mu) w_3 \in W_3$. Write $w = v_1 + v_2$ with $v_1 \in W_1$, $v_2 \in W_2$. Then

$l_1(v_1)\,l_2'(v_2)=l_2(v_2)\,l_1'(v_1)$; thus q vanishes on w. Thus we can assume $\lambda\neq 0$, and even $\lambda=1$. Therefore $w_3=w_2-\mu w=-\mu w_1+(\mu+1)\,w_2$ shows that

$$\begin{aligned}0=q(w_3)&=-\mu(\mu+1)(l_1(w_1)\,l_2'(w_2)-l_1'(w_1)\,l_2(w_2))\\&=\mu(\mu+1)\,q(w_1-w_2).\end{aligned}$$

Consequently, $q(w_1-w_2)=0$ provided $\mu(\mu+1)=0$. So assume now $\mu(\mu+1)=0$. If $\mu=0$, then $w_3=w_2$; thus $0=r(w_2)=-l_2(w_2)$, and $0=r'(w_2)=-l_2'(w_2)$. Therefore $q(w_1-w_2)=0$. If $\mu=-1$, then $w_3=w_1$, and we obtain in the same way $q(w_1-w_2)=0$.

Conversely, let w be a non-zero element of W with $q(w)=0$. Write $w=x_1+x_2$ with $x_1\in W_1$, $x_2\in W_2$. First, assume $l_1(x_1)=0=l_1'(x_1)$; thus $x_1\in W_3$, so we choose $w_1=x_1$, $w_2=-x_2$, $w_3=x_1$, and obtain $w_i-w_j\in kw$, for $1\leqslant i,\ j\leqslant 3$. Now, we may assume $\alpha=l_1(x_1)\neq 0$, and not both $l_2(x_2)$, $l_2'(x_2)$ being zero. But $q(w)=0$ and $l_2(x_2)=0$ imply $l_2'(x_2)=0$. Therefore $\beta=l_2(x_2)\neq 0$, and $l_2'(x_2)=(\alpha/\beta)\,l_1'(x_1)$. Let $w_3=\beta x_1+\alpha x_2$. Then clearly $r(w_3)=0=r'(w_3)$; thus $w_3\in W_3$ and note that $w_3\neq 0$. Let $w_1=(\beta-\alpha)\,x_1$, $w_2=(\alpha-\beta)\,x_2$. Then

$$w_3-w_1=\alpha(x_1+x_2)=\alpha w \qquad \text{and} \qquad w_3-w_2=\beta(x_1+x_2)=\beta w;$$

thus $w\in D(W_1, W_2, W_3)$.

7.6. *Let $W=W_1\oplus W_2\oplus W_3$. For $1\leqslant i\leqslant 3$, let l_i, l_i' be linear forms on W_i, and consider $r_1=l_1-l_2'$, $r_2=l_2-l_3'$, $r_3=l_3-l_1'$, and $c=l_1l_2l_3-l_1'l_2'l_3'\in \mathscr{R}$. Let $W_4=\mathscr{Z}(r_1, r_2, r_3)$. Let U be the subset of W which contains besides 0 all elements w for which there are elements w_i, $w_i'\in W_i$, with $1\leqslant i\leqslant 3$, and $w_4\in W_4$, not all zero, such that*

$$w_1+w_2+w_3,\quad w_1'+w_2'+w_3' \qquad \textit{and} \qquad w_1'+w_3+w_4$$

belong to kw. Let W_4 be the set of elements of the form $x_1+x_2+x_3$ with $x_i\in W_i$, $1\leqslant i\leqslant 3$, such that $\lambda_1x_1+\lambda_2x_2+\lambda_3x_3\in W_4$ for some non-zero triple $(\lambda_1,\lambda_2,\lambda_3)$ in k^3. Then $U\subseteq \tilde{W}_4\subseteq \mathscr{Z}(c)$. Any element $w\in\mathscr{Z}(c)$, with $w=x_1+x_2+x_3$, where $x_i\in W_i-\mathscr{Z}(l_i,l_i')$ for $1\leqslant i\leqslant 3$, belongs to U.

Proof. Let u be a non-zero element of U; thus there are w_i, $w_i'\in W_i$, $1\leqslant i\leqslant 3$, and $w_4\in W_4$, not all zero, such that $w_1+w_2+w_3=\alpha w$, $w_1'+w_2'+w_3'=\beta w$, $w_1'+w_3+w_4=\gamma w$ for some $\alpha,\beta,\gamma\in k$. Assume $\alpha=\beta=0$. Then $w_i=w_i'=0$ for $1\leqslant i\leqslant 3$; thus $w_4\neq 0$, and therefore $\gamma\neq 0$. Thus $w=(1/\gamma)\,w_4\in\tilde{W}_4$. Thus $\alpha\neq 0$ or $\beta\neq 0$, and without loss of generality we can assume $\alpha=1$. Thus

$$w_4=(\gamma-\beta)\,w_1+\gamma w_2+(\gamma-1)\,w_3\in W_4,$$

and the triple $(\gamma-\beta,\gamma,\gamma-1)\in k^3$ is non-zero. Thus $w\in\tilde{W}_4$.

Let $x \in \tilde{W}_4$, say $x = x_1 + x_2 + x_3$ with $x_i \in W_i$; thus there is a non-zero triple $(\lambda_1, \lambda_2, \lambda_3) \in k^3$ with $\sum \lambda_i x_i \in W_4$. If all $\lambda_i \neq 0$, then

$$\begin{aligned} 0 &= c\left(\sum \lambda_i x_i\right) \\ &= l_1(\lambda_1 x_1)\, l_2(\lambda_2 x_2)\, l_3(\lambda_3 x_3) - l_1'(\lambda_1 x_1)\, l_2'(\lambda_2 x_2)\, l_3'(\lambda_3 x_3) \\ &= \lambda_1 \lambda_2 \lambda_3 (l_1(x_1)\, l_2(x_2)\, l_3(x_3) - l_1'(x_1)\, l_2'(x_2)\, l_3'(x_3)) \\ &= \lambda_1 \lambda_2 \lambda_3 c(x) \end{aligned}$$

shows that $c(x) = 0$. If two of the λ_i's are zero, say $\lambda_2 = \lambda_3 = 0$, then $\lambda_1 x_1 \in W_4$; thus

$$0 = r_1(\lambda_1 x_1) = \lambda_1 l_1(x_1), \qquad 0 = r_3(\lambda_1 x_1) = -\lambda_1 l_1'(x_1),$$

and thus $l_1(x_1) = l_1'(x_1) = 0$, since $\lambda_1 \neq 0$. Therefore $c(x) = 0$. If only one of the λ_i's is zero, say $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$, then $\lambda_1 x_1 + \lambda_2 x_2 \in W_2$ yields

$$0 = r_2(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_2 l_2(x_2), \qquad 0 = r_3(\lambda_1 x_1 + \lambda_2 x_2) = -\lambda_1 l_1'(x_1);$$

thus again $c(x) = 0$. This shows that $\tilde{W}_4 \subseteq \mathscr{Z}(c)$.

Conversely, let $w = x_1 + x_2 + x_3$ with $x_i \in W_i - \mathscr{Z}(l_i, l_i')$ for $1 \leqslant i \leqslant 3$, and assume w belongs to $\mathscr{Z}(c)$. Let $\alpha_i = l_i(x_i)$, $\alpha_i' = l_i'(x_i)$, for $1 \leqslant i \leqslant 3$. Since $\alpha_1 \alpha_2 \alpha_3 = \alpha_1' \alpha_2' \alpha_3'$, one checks immediately that the three elements

$$\begin{aligned} y &= \alpha_2 \alpha_3 x_1 + \alpha_1' \alpha_3' x_2 + \alpha_1' \alpha_2 x_3, \\ y' &= \alpha_2' \alpha_3' x_1 + \alpha_1 \alpha_3' x_2 + \alpha_1 \alpha_2 x_3, \\ y'' &= \alpha_2' \alpha_3 x_1 + \alpha_1 \alpha_3 x_2 + \alpha_1' \alpha_2' x_3 \end{aligned}$$

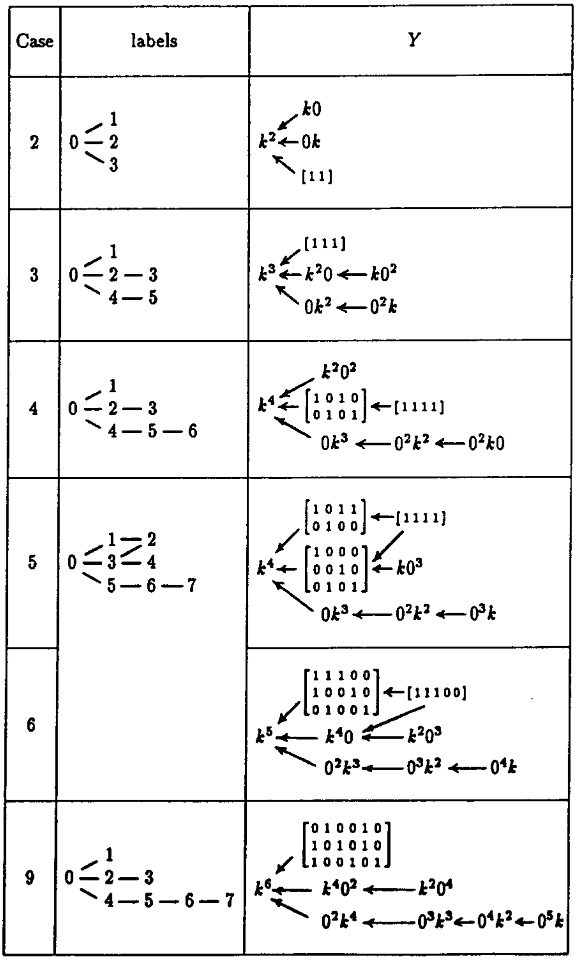
belong to W_4. We claim that at least one of these elements is non-zero. So assume $y = y' = y'' = 0$. Since x_1, x_2, x_3 are non-zero elements, it follows that all the coefficients are zero. Since $x_i \notin \mathscr{Z}(l_i, l_i')$ it follows from $\alpha_i = 0$ that $\alpha_i' \neq 0$. So assume $\alpha_1 = 0$. Then $\alpha_1' \neq 0$, but $\alpha_1' \alpha_2$ is one of the coefficients; thus $\alpha_2 = 0$, and therefore $\alpha_2' \neq 0$, but $\alpha_1' \alpha_2'$ is a coefficient; thus $\alpha_1' = 0$, a contradiction. Thus we see that $\alpha_1 \neq 0$, and similarly, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$. Since $\alpha_1 \alpha_2 \alpha_3 = \alpha_1' \alpha_2' \alpha_3'$, it follows that also $\alpha_1', \alpha_2', \alpha_3'$ are non-zero. This contradiction establishes that there is a non-zero linear combination $w_4 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$ in W_4. Let $w_i' = (\lambda_2 - \lambda_1) x_i$ and $w_i = (\lambda_2 - \lambda_3) x_i$. Then

$$\begin{aligned} w_1 + w_2 + w_3 &= (\lambda_2 - \lambda_3) w, \\ w_1' + w_2' + w_3' &= (\lambda_2 - \lambda_1) w, \\ w_1' + w_3 + w_4 &= \lambda_2 w; \end{aligned}$$

thus $w \in U$. This completes the proof.

8. The Open Stratum of Hom(E, Y)

We consider the cases $a=2, 3, 4, 5, 6$, and 9, so we are in the situation investigated in the last section. We need a specific realisation of the module Y by a vectorspace $W=W^{(a)}$, subspaces W_t, $t \in \Delta_0'$, and inclusion maps. Subspaces of $W=k^{n+1}$ which are not generated by some of the canonical basis vectors, say with a basis $[a_{i0}, ..., a_{in}]$, $1 \leqslant i \leqslant m$, will be exhibited by the matrix $[a_{ij}]$. Since $W=\mathrm{Hom}(E, Y)=k^{n+1}$, we have $\mathscr{R}=\mathscr{R}^{(a)}=k[T_0, ..., T_n]$. We also fix some labels for the vertices in Δ_0'. For $t \in \Delta_0'$, we write P_t instead of $P(t)$ (see Scheme 4).



Scheme 4

We use the test modules exhibited in Section 6, the jth test module in case a will be denoted by N_j or $N_j^{(a)}$. For every test module N, we associate a polynomial $r_N \in \mathscr{R}$, as explained in Section 7. Thus we obtain the following lists which give for every test module N its minimal projective presentation $0 \to d_N E \to P_N \to N \to 0$, and W_N and r_N.

	Case 2		
N	1 0 0 0	0 0 1 0	0 0 0 1
d_N	1	1	1
P_N	P_1	P_2	P_3
W_N	W_1	W_2	W_3
r_N	T_1	T_0	$T_0 - T_1$

	Case 3				
N	0 1 1 1 1 1	1 1 0 0 1 1	0 0 1 0 0 0	1 1 1 1 0 0	0 0 0 0 1 0
d_N	1	1	1	1	1
P_N	$P_3 \oplus P_5$	$P_1 \oplus P_5$	P_2	$P_1 \oplus P_3$	P_4
W_N	$W_3 + W_5$	$W_1 + W_5$	W_2	$W_1 + W_3$	W_4
r_N	T_1	$T_0 - T_1$	T_2	$T_1 - T_2$	T_0

Case 4

N	1 1 1 0 1 1 0	0 1 1 1 1 1 0	1 1 0 0 1 1 1	0 1 1 0 1 1 1	1 1 1 1 0 0 0	0 0 0 0 1 0 0
d_N	2	1	1	1	1	1
P_N	$P_1 \oplus P_2 \oplus P_5$	$P_3 \oplus P_5$	$P_1 \oplus P_6$	$P_2 \oplus P_6$	$P_1 \oplus P_3$	P_4
W_N	$D(W_1, W_5, W_2)$	$W_3 + W_5$	$W_1 + W_6$	$W_2 + W_6$	$W_1 + W_3$	W_4
r_N	$T_0 T_3 - T_1 T_2$	$T_0 - T_1$	T_3	$T_1 - T_3$	$T_2 - T_3$	T_0

Case 5

N	1 1 2 2 1 2 1 1	1 0 1 1 1 1 0 0	1 1 1 1 0 1 1 0	0 0 1 1 1 1 1 0	1 0 1 0 0 1 1 1	0 0 0 1 0 0 0 0
d_N	2	2	1	1	1	1
P_N	$P_2 \oplus P_4 \oplus P_5 \oplus P_7$	$P_1 \oplus P_4 \oplus P_5$	$P_2 \oplus P_6$	$P_4 \oplus P_6$	$P_1 \oplus P_7$	P_3
W_N	$D(W_4, W_5, W_2 \oplus W_7)$	$D(W_4, W_5, W_1)$	$W_2 + W_6$	$W_4 + W_6$	$W_1 + W_7$	W_3
r_N	$T_0(T_1 - T_2)$	$T_0(T_2 - T_3)$	$T_0 - T_1$	T_1	$T_0 - T_2$	$T_1 - T_3$

Case 6

N	2 1 2 2 1 1 1 0	1 1 2 2 1 1 1 1	1 0 1 1 1 1 0 0	1 1 1 1 0 1 0 0	0 0 1 1 1 1 1 0	1 0 1 0 0 1 1 1	0 0 1 1 0 0 0 0
d_N	2	1	2	1	1	1	1
P_N	$P_1 \oplus P_2 \oplus P_4 \oplus P_6$	$P_2 \oplus P_4 \oplus P_7$	$P_1 \oplus P_4 \oplus P_5$	$P_2 \oplus P_5$	$P_4 \oplus P_6$	$P_1 \oplus P_7$	P_3
W_N	$D(W_2 \oplus W_4, W_6, W_1)$	$W_2 + W_4 + W_7$	$D(W_4, W_5, W_1)$	$W_2 + W_5$	$W_4 + W_6$	$W_1 + W_7$	W_3
τ_N	$(T_0 - T_2)\, T_4 - (T_1 - T_2)\, T_3$	T_3	$T_0(T_2 + T_4) - T_1(T_2 + T_3)$	$T_0 - T_1$	T_2	$T_0 - T_2 - T_3$	T_4

Case 9

N	1 2 2 1 2 1 1 0	1 2 1 1 2 1 1 1	1 1 1 0 1 1 0 0	0 1 1 1 1 1 0 0	1 1 0 0 1 1 1 0	0 1 1 0 1 1 1 1	1 1 1 1 0 0 0 0
d_N	3	2	2	1	1	1	1
P_N	$P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_6$	$P_1 \oplus P_3 \oplus P_4 \oplus P_7$	$P_1 \oplus P_2 \oplus P_5$	$P_3 + P_5$	$P_1 \oplus P_6$	$P_2 \oplus P_7$	$P_1 \oplus P_3$
W_N	U	$D(W_3, W_4, W_1 \oplus W_7)$	$D(W_1, W_5, W_2)$	$W_3 + W_5$	$W_1 + W_6$	$W_2 + W_7$	$W_1 + W_3$
r_N	$T_0 T_3(T_4 + T_5) - (T_0 + T_1)(T_2 + T_3)\, T_5$	$T_0(T_4 - T_2) - T_1(T_2 + T_3)$	$(T_0 - T_2)\, T_4 - (T_1 + T_2)\, T_5$	T_2	$T_0 - T_2 - T_3$	T_4	$T_3 - T_5$

PROPOSITION. *For $a=2, 3, 4, 5, 6, 9$, we have*

$$\bigcup_j W_{N_j^{(a)}} = \bigcup_j \mathscr{X}(r_{N_j}^{(a)}).$$

Proof. Let $N = N_j^{(a)}$, and let $m = \dim_k \operatorname{Ext}^1(N, E)$. Then $1 \leqslant m \leqslant 3$, and $m=3$ only in case $(a, j) = (9, 1)$. We claim that for $m \leqslant 2$, we have $W_N = \mathscr{X}(r_N)$. For $m=1$, this follows immediately from 7.2. Consider the cases with $m=2$. For $(a, j) = (4.1)$, $(5, 2)$, $(6, 3)$, and $(9, 3)$, we use 7.3 and 7.5. There are obvious choices for $l_1, l_1', l_2 l_2'$ in the first three cases; in the case $(9, 3)$, we use

$$l_1 = T_1 + T_2, \quad l_1' = T_0 - T_2, \quad l_2 = -T_4 + T_1 + T_2, \quad l_2' = -T_5 + T_0 - T_2.$$

For the remaining cases $(a, j) = (5, 1)$, $(6,1)$, and $(9, 2)$, we use 7.4 and 7.5.
Finally, consider the case $(a, j) = (9, 1)$. Here we use 7.6 with $l_1 = T_0$, $l_1' = T_0 + T_1$, $l_2 = T_3$, $l_2' = T_2 + T_3$, $l_3 = T_4 + T_5$, $l_3' = T_5$. We conclude that

$$\mathscr{X}(r_N) - \mathscr{X}(T_0 T_2 T_4, T_1 T_2 T_4) \subseteq W_N \subseteq \mathscr{X}(r_N).$$

However,

$$\begin{aligned}\mathscr{X}(T_0 T_2 T_4, T_1 T_2 T_4) &= \mathscr{X}(T_0, T_1) \cup \mathscr{X}(T_2) \cup \mathscr{X}(T_4)\\ &\subseteq W_{N_2^{(9)}} \cup W_{N_4^{(9)}} \cup W_{N_6^{(9)}}.\end{aligned}$$

This completes the proof.

COROLLARY. *The set $\mathscr{O}^{(a)} := W^{(a)} - \bigcup_j \mathscr{X}(r_{N_j}^{(a)})$ is the set of non-zero maps $w\colon E \to Y$ with indecomposable cokernels.*

A straightforward, but tedious calculation shows the following: Let k be a finite field, and $q = |k|$. Then

$$|\mathscr{O}^{(a)}| = (q-1)\, \varphi^{(a)}(q),$$

where $\varphi^{(a)}$ is the polynomial listed in Theorem 2. These calculations may be done by hand. Alternatively, we may use a computer as follows: We know from [R3] that the function $|\mathscr{O}^{(a)}|$ is a polynomial in q which is divisible by $q-1$. Also, it is easy to see that the degree of this polynomial can be at most $n+1$. Thus, we ask a computer to count the number of elements of $\mathscr{O}^{(a)}$, where the base field k has p elements, with p one of the first $n+1$ primes, and use Lagrange interpolation. The author is indebted to I. Janiszczak for checking in this way the polynomials he had calculated by hand, and actually correcting three coefficients.

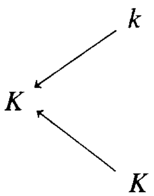
We finally remark that $|\mathscr{O}^{(a)}| = (q-1)\, \varphi_{ze}^{y}(q)$; therefore $\varphi_{ze}^{y} = \varphi^{(a)}$.

9. The Remaining Hall Polynomials

9.1. Assume Y is an indecomposable R-module with simple top. This condition is satisfied in Cases 1, 10, 11, 12, 20, 22, 23, and 24 of Theorem 2. Under this assumption, any non-zero factor mode of Y is indecomposable. So assume E is simple, and $\dim_{\mathrm{End}(E)} \mathrm{Hom}(E, Y) = r$. The set of submodules of Y which are isomorphic to E is the projective $(r-1)$-space over End (E); thus, if $\mathrm{End}(E)$ is finite, the number of such submodules is just $\zeta_r(|\mathrm{End}(E)|)$. This shows that in the cases mentioned above, the Hall polynomials φ^{y}_{ze} are just given by ζ_r.

We are going to investigate the remaining cases listed in Theorem 2. Always, there is given a field extension $k \subset K$ of degree 2 or 3, and we assume that k is the finite field with q elements. In case K is of degree 2 over k, let $a \in K - k$. We start with those cases where $\mathrm{End}(E) = k$.

9.2. *Case* 14. Here, Y is given by



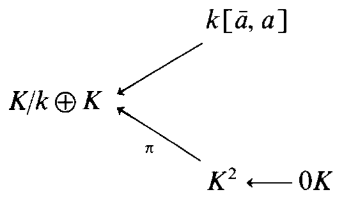
where the lower map is the identity map, and the upper one is the canonical inclusion. There are $q+1$ simple submodules U, and they correspond to the one-dimensional k-submodules of the 2-dimensional k-space K. If U is given by the canonical inclusion of k into K, then Y/U decomposes into

$$\begin{array}{ccc} & 1 & \\ 0 & & \\ & 0 & \end{array} \oplus \begin{array}{ccc} & 0 & \\ 1 & & \\ & 1 & \end{array} ;$$

otherwise Y/U will be indecomposable. This shows that $\varphi^{y}_{ze} = T$.

Remark. Observe that we have used the first of our test modules, but not the second one. It follows that in this case, the test module set exhibited in section 6 is not minimal.

9.3. *Case* 17. We use the following realisation of Y



where $\bar{a}$ is the residue class of a in K/k, and π is the canonical k-linear map with kernel $k0$. The elements in $W = K/k \oplus K$ are of the form $[w_0\bar{a}, w_1 + w_2 a]$ with $w = [w_0, w_1, w_2] \in k^3$, and we identify in this way $W = K/k \oplus K$ with k^3. Let $\mathcal{R} = k[T_0, T_1, T_2]$ be the ring of regular functions on W. Any element $w \in W$ gives rise to a map $E \to Y$ which again is denoted by w; it is a monomorphism if and only if $w \neq 0$. According to Section 6, there are three test modules N_1, N_2, and N_3, with dimension vectors

$$\begin{array}{ccc} & 0 & \\ 1 & & \\ & 1\ 1 & \end{array}, \quad \begin{array}{ccc} & 1 & \\ 0 & & \\ & 0\ 0 & \end{array}, \quad \text{and} \quad \begin{array}{ccc} & 0 & \\ 0 & & \\ & 1\ 0 & \end{array},$$

respectively. Let w be a non-zero element of W.

We have $\operatorname{Hom}(N_1, \operatorname{Cok} w) \neq 0$ if and only if $w \in 0K$, this if and only if $w \in \mathcal{Z}(T_0)$. We have $\operatorname{Hom}(N_2, \operatorname{Cok} w) \neq 0$ if and only if $w \in k[\bar{a}, a]$, thus if and only if $w \in \mathcal{Z}(T_1, T_0 - T_2)$. Finally, $\operatorname{Hom}(N_3, \operatorname{Cok} w) \neq 0$ if and only if $\pi^{-1}(kw)$ is a K-subspace of K^2, thus the K-subspace $K0$ generated by $[10] \in K^2$. Consequently, $\operatorname{Hom}(N_3, \operatorname{Cok} w) \neq 0$ if and only if $w \in \mathcal{Z}(T_1, T_2)$. We see that

$$\mathcal{O} = W - (\mathcal{Z}(T_0) \cup \mathcal{Z}(T_1, T_0 - T_2) \cup \mathcal{Z}(T_1, T_2))$$

is the set of non-zero $w \in W$ with indecomposable cokernel. But

$$|\mathcal{O}| = (q-1)(q^2-2).$$

For we may choose $0 \neq w_0$ arbitrarily. If $w_1 = 0$, then we must choose $w_2 \notin \{0, w_0\}$, so there are $q-2$ possibilities. If $w_1 \neq 0$, we may choose w_2 arbitrarily. Thus, for fixed $w_0 \neq 0$, there are $q - 2 + (q-1)q = q^2 - 2$ possibilities.

9.4. *Cases* 13 *and* 15. First, we deal with Case 13. We consider the following realisation of Y

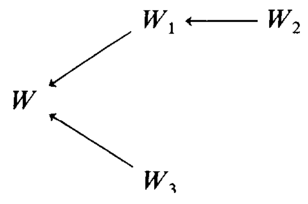
$$\begin{array}{ccc} & & k^2 \\ & \swarrow & \\ K^2 & & \\ & \nwarrow & \\ & & K[1, a] \end{array}$$

where both maps are the inclusion maps. Test modules are the simple injective modules N_1 and N_2, where $\mathrm{End}(N_1)=k$, $\mathrm{End}(N_2)=K$. The simple submodules U of Y are given by the one-dimensional K-subspaces of K^2. The subspace $0K$ gives a submodule U so that Y/U splits off a copy of N_1. The remaining one-dimensional K-subspaces are of the form $K[1, w_1]$ with $w_1 \in K$. So let U be given by $K[1, w_1]$. Then $\mathrm{Hom}(N_1, Y/U) \neq 0$ if and only if $K[1, w_1] \cap k^2 \neq 0$, thus if and only if $w_1 \in k$. Similarly, $\mathrm{Hom}(N_2, Y/U) \neq 0$ if and only if $K[1, w_1] \cap K[1, a] \neq 0$, thus if and only if $w_1 = a$. It follows that the set of simple submodules U with Y/U indecomposable corresponds bijectively to the set of elements of K different from a and not in k; thus $\varphi^{y}_{ze} = q^2 - q - 1$.

In Case 15, an additional one-dimensional k-subspace of k^2 is specified, but the test modules remain the same. So the Hall polynomial φ^{y}_{ze} is the same as above. [The only difference between Cases 13 and 15 is the following: In Case 13, we have $\mathrm{End}(Y) \cong K$, whereas the additional one-dimensional k-subspace in Case 15 yields $\mathrm{End}(Y) = k$. This is not visible when we calculate φ^{y}_{ze}, but can be seen by looking at the full rotational equivalence class: we obtain from Case 15 by rotation the Case 19 with Hall polynomial $(T+1)(T^2-T-1)$.]

9.5. *Case* 16. In order to facilitate the calculations, we assume that the characteristic of k is not 2. Note that for determining the Hall polynomial φ^{y}_{ze}, we only need infinitely many evaluations (or even just five evaluations, since one easily observes that the degree of φ^{y}_{ze} is at most 4), so we may specify the characteristic of k. Under our assumption, we may suppose that $a \in K - k$ is chosen so that $a^2 \in k$.

We choose the following realision of Y. Let $W = K^3$. Note that W may be considered as a six-dimensional k-space with elements $[w_0 + w_1 a, w_2 + w_3 a, w_4 + w_5 a]$, where $[w_0, ..., w_5] \in k^6$, and $\mathscr{R} = k[T_0, ..., T_5]$ denotes the corresponding ring of regular functions. Let W_3 be the K-subspace $0K0$ of W; thus $W_3 = \mathscr{Z}(T_0, T_1, T_4, T_5)$, and we consider also two k-subspaces W_1 and W_2, namely $W_2 = k0k = \mathscr{Z}(T_1, T_2, T_3, T_5)$, and $W_1 = W_2 + K[1, 1, a]$. Since $a^2 \in k$, we see that W_1 is the k-subspace generated by $[1, 0, 0]$, $[0, 0, 1]$, $[0, 1, a]$, and $[a, a, 0]$; thus $W_1 = \mathscr{Z}(T_1 - T_3, T_2 - T_5)$. The module Y is



where all maps are the inclusion maps. As before, we identify the element $w \in W$ with the corresponding map $E \to Y$, where

$$E = \begin{pmatrix} & 0\ 0 \\ K & \\ & 0 \end{pmatrix},$$

which sends $1 \in K$ to $w \in W$.

There are three test modules N_1, N_2, N_3, with dimension vectors

$$\begin{matrix} & 1\ 1 \\ 1 & \\ & 1 \end{matrix}, \quad \begin{matrix} & 1\ 0 \\ 0 & \\ & 0 \end{matrix}, \quad \begin{matrix} & 2\ 2 \\ 1 & \\ & 0 \end{matrix},$$

respectively. Let $0 \neq w \in W$.

We claim that $\mathrm{Hom}(N_1, \mathrm{Cok}\, w) \neq 0$ if and only if $w \in \mathcal{X}(T_0 T_5 - T_1 T_4)$. For a non-zero homomorphism $N_1 \to \mathrm{Cok}\, w$ yields elements $x \in W_2$ and $y \in W_3$, not both zero, such that $x - y \in Kw$ and $x - y = zw$, with $z \in K$. Since $W_2 \cap W_3 = 0$, we have $z \neq 0$; thus $w = z^{-1}x - z^{-1}y$. Let $x = [\lambda, 0, \mu]$, with $\lambda, \mu \in k$. If we write $w = [w_0 + w_1 a, w_2 + w_3 a, w_4 + w_5 a]$, we see that $w_0 + w_1 a = \lambda z^{-1}$, and $w_4 + w_5 a = \mu z^{-1}$; thus $[w_0, w_1]$ and $[w_4, w_5]$ are linearly dependent in k^2, and therefore $w_0 w_5 - w_1 w_4 = 0$. Also conversely, assume $w \in W = k^6$ with $[w_0, w_1]$ and $[w_4, w_5]$ linearly dependent in k^2. Let z' be a non-zero vector in k^2 such that both $[w_0, w_1]$ and $[w_4, w_5]$ are multiples of z'. It follows that $(z')^{-1}w$ belongs to $W_2 + W_3$; thus there are $x \in W_2$, $y \in W_3$, not both zero, with $x - y = z'w \in Kw$, and so we obtain a non-zero homomorphism $N_1 \to \mathrm{Cok}\, w$.

Next, we claim that $\mathrm{Hom}(N_2, \mathrm{Cok}\, w) \neq 0$ if and only if $w \in \mathcal{X}((T_0 - T_2)(T_5 - T_2) - (T_1 - T_3)(T_4 - a^2 T_3))$. For $\mathrm{Hom}(N_2, \mathrm{Cok}\, w) \neq 0$ if and only if $Kw \cap W_1 \neq 0$, thus if and only if w belongs to

$$\{zv \mid v \in W_1, z \in K\} = \{[\lambda y + x, x, \mu y + xa] \mid x, y \in K; \lambda, \mu \in k\}.$$

But $w = [\lambda y + x, x, \mu y + xa]$ means that

$$\begin{aligned} \lambda y = (\lambda y + x) - x &= (w_0 + w_1 a) - (w_2 + w_3) a \\ &= (w_0 - w_2) + (w_1 - w_3) a, \\ \mu y = (\mu y + xa) - xa &= (w_4 + w_5 a) - (w_2 + w_3)\, a^2 \\ &= (w_4 - w_3 a^2) + (w_5 - w_3) a, \end{aligned}$$

and thus that $w \in \mathcal{X}((T_0 - T_2)(T_5 - T_3) - (T_1 - T_3)(T_4 - a^2 T_3))$.

Finally, $\mathrm{Hom}(N_3, \mathrm{Cok}\, w) \neq 0$ if and only if $w \in \mathcal{X}(T_2, T_3)$. For, if $\mathrm{Hom}(N_3, \mathrm{Cok}\, w) \neq 0$, then there exists a one-dimensional K-subspace in $Kw + W_2$ which is different from Kw. The k-dimension of $Kw + W_2$ shows that $Kw + W_2$ must be a K-subspace; therefore $Kw + W_2 = K0K$, in particular, $w \in K0K$. Conversely, let $w \in K0K$. If $w = [\lambda z, 0, \mu z]$, for some $z \in K$ and $\lambda, \mu \in k$ (with $z \neq 0$, and $[\lambda, \mu] \neq [0, 0]$), then Cok w contains an indecomposable submodule with dimension vector

$$\begin{matrix} & 1\ 1 \\ 0 & \\ & 0 \end{matrix},$$

namely generated by the residue class of $[\lambda, 0, \mu] \in W_2$ in Cok w. But an indecomposable module with dimension vector

$$\begin{matrix} & 1\ 1 \\ 0 & \\ & 0 \end{matrix}$$

is generated by N_3; thus $\mathrm{Hom}(N_3, \mathrm{Cok}\, w) \neq 0$. Otherwise, $Kw + W_2 = K0K$, and Cok w has an indecomposable submodule isomorphic to N_3 itself.

It follows that the set of non-zero $w \in W$ with indecomposable cokernel is

$$\mathcal{O} = W - (\mathscr{Z}(T_0T_5 - T_1T_4) \cup \mathscr{Z}((T_0 - T_2)(T_5 - T_3) - (T_1 - T_3)(T_4 - a^2T_3)) \cup \mathscr{Z}(T_2, T_3)).$$

But $|\mathcal{O}| = (q^2 - 1)\, q(q^3 - 2q^2 - q + 3)$, as an easy, but again tedious, calculation shows. Again, one may use instead a computer, calculating $|\mathcal{O}|$ for the fields k with $|k| = 3, 5, 7, 11$, and 13, and a^2 an integer which is not a square modulo $|k|$, and using Lagrange interpolation. Note that we know that $|\mathcal{O}|$ must be divisible by $q^2 - 1$, and $|\mathcal{O}| = (q^2 - 1)\, \varphi^{y}_{ze}$.

9.6. *Case* 21. Here, we deal with a field extension $k \subset K$ of degree 3, and we denote by a, b elements of K such that $1, a, b$ is a k-basis of K. Let $W = K^2$, and we consider the k-subspace $W' = k^2 + k[a, b]$ of W. Thus Y is given by the inclusion map

$$W \leftarrow W'.$$

The elements of W are pairs $w = [w_0, w_1]$. Given a non-zero element $w \in W$, we consider the submodule $U = (Kw \leftarrow 0)$ of Y, and denote the inclusion again by w. If $w_0 = 0$, $w_1 \neq 0$, then Cok w splits off a copy of the simple injective module N (which is our test module), since we may assume $w_1 = 1$. Thus, we may assume $w_0 = 1$. Clearly, Cok w is indecomposable if and only if $K[1, w_1] \cap W' \neq 0$, if and only if there is $0 \neq z \in K$ with $[z, zw_1] \in W'$.

Now $[z, zw_1] \in W'$, if and only if there are $\lambda_1, \lambda_2, \lambda_3 \in k$, with $z = \lambda_1 + \lambda_3 a$, $zw_1 = \lambda_2 + \lambda_3 b$. We see that Cok w is indecomposable if and only if there are $\lambda_1, \lambda_2, \lambda_3 \in k$, with $(\lambda_1+\lambda_3 a) w_1 = \lambda_2 + \lambda_3 b$, and $[\lambda_1, \lambda_3] \neq 0$, thus if and only if $w_1 \notin \{(\lambda_2+\lambda_3 b)/(\lambda_1+\lambda_3 a) \mid \lambda_1, \lambda_2, \lambda_3 \in k, \lambda_1+\lambda_3 a \neq 0\} = V$. We claim that $V \subseteq K$ is the disjoint union of k and $V' = \{(\lambda_2+b)/(\lambda_1+a) \mid \lambda_1, \lambda_2 \in k\}$. Clearly, both k and V' are contained in V. Conversely , the elements of the form $(\lambda_2+\lambda_3 b)/(\lambda_1+\lambda_3 a)$ in V with $\lambda_3 = 0$ belong to k, those with $\lambda_3 \neq 0$ belong to V'. Also, $V' \cap k = \varnothing$, since $(\lambda_2+b)/(\lambda_1+a) = \lambda_4 \in k$ would simply be $\lambda_2 + b = \lambda_4\lambda_1 + \lambda_4 a$, and thus $b \in k + ka$, impossible. Finally, we observe that for $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in k$, $(\lambda_2+b)/(\lambda_1+a) = (\lambda_2'+b)/(\lambda_1'+a)$ implies $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$. For we obtain $(\lambda_2+b)(\lambda_1'+a) = (\lambda_2'+b)(\lambda_1+a)$; thus $\lambda_2\lambda_1' + \lambda_2 a + \lambda_1' b = \lambda_2'\lambda_1 + \lambda_2' a + \lambda_1 b$, and by Assumption 1, a, b is a k-basis of K. It follows that $|V| = |k| + |V'| = q + q^2$; therefore the number of elements $w = [1, w_1]$ with indecomposable cokernel is $q^3 - q^2 - q = q(q^2 - q - 1)$, and thus $\varphi_{ze}^{y} = q(q^2 - q - 1)$. This completes the proof.

10. Hereditary Algebras

It remains to single out the polynomials which actually occur for a hereditary algebra with a given Dynkin diagram as underlying graph.

So let R be a finite-dimensional k-algebra with centre k which is representation-finite, hereditary, and connected. Let Δ be its underlying graph and Φ^+ the corresponding set of positive roots. Let Ω be the orientation on Δ defined by R, and let $(-,-)_\Omega$ be the corresponding bilinear form. Note that for a general representation-directed algebra A, we have used in Section 3 a bilinear form $\langle -,- \rangle$, and for $A = R$, we have $\langle -,- \rangle = (-,\ -)_\Omega$. (The notation $\langle -,- \rangle$ is quite standard in the representation theory of algebras, but it is in conflict with the use of the notation $\langle -,- \rangle$ in some accounts of Lie theory [Hu].)

THEOREM 3. *Let $x, y, z \in \Phi^+$. If $y \neq x + z$, then $\varphi_{zx}^{y} = 0$. Thus, let $y = x + z$, and then either $(x, z)_\Omega < 0$ or $(z, x)_\Omega < 0$. We assume $(z, x)_\Omega < 0$; thus $0 \leqslant (x, z)_\Omega \leqslant 5$. Then $\varphi_{xz}^{y} = 0$, and φ_{zx}^{y} is one of the following polynomials*:

$(x, z)_\Omega$	$\Delta = A_n, D_n, E_m$	$\Delta = B_n, C_n, F_4$	$\Delta = G_2$
0	φ_1	φ_1	φ_1
1	φ_2	$\varphi_8, \zeta_2\varphi_1$	$\zeta_2\varphi_1$
2	φ_3	φ_9, φ_{10}	$\zeta_3\varphi_1$
3	φ_4, φ_5	$\zeta_2\varphi_9$	φ_{12}
4	φ_6	φ_{11}	
5	φ_7		

Proof. We have seen in 1.1 that $\varphi_{zx}^{y} = 0$ for $y \neq x + z$; thus we assume now $y = x + z$. Since the polynomials φ_{xz}^{y} only depend on the Auslander–

Reiten quiver of R, we can assume that k is a finite field with at least three elements; in particular given any indecomposable R-module M, we have $[\operatorname{End} M : k] \leqslant 3$ and, in addition $\operatorname{End}(M) = k$ in case $\Delta = A_n, D_n, E_m$. It follows that for any $x_1, x_2 \in \Phi^+$, we have

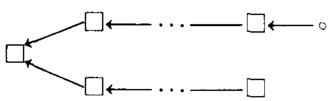
$$(x_1, x_2)_\Omega = \dim_k \operatorname{Hom}(M(x_1), M(x_2)) - \dim_k \operatorname{Ext}^1(M(x_1), M(x_2))$$

(see [R1]). We can assume that there is not path in Γ_R from $M(z)$ to $M(x)$, otherwise interchanging x and z. According to (1.2 and) 1.3, we can assume that $M(x) = E$ is simple projective (note that the reflection functors used preserve the dimensions of $\operatorname{Hom}(M(x_1), M(x_2))$ and $\operatorname{Ext}^1(M(x_1), M(x_2))$ for $\{x_1, x_2\} = \{x, z\}$; thus $(x, z)_\Omega$ and $(z, x)_\Omega$ are preserved). According to Theorems 1 and 2, we see that $\varphi^y_{zx} \neq 0$; thus $\varphi^y_{zx}(|k|) \neq 0$, since $|k| \neq 2$: It follows that there is an exact sequence $0 \to M(x) \to M(y) \to M(z) \to 0$. In particular, $\operatorname{Ext}^1(M(z), M(x)) \neq 0$. Since $\operatorname{Hom}(M(z), M(x)) = 0$ and $\operatorname{Ext}^1(M(x), M(z)) = 0$, it follows that $(z, x)_\Omega < 0$, and $(x, z)_\Omega = \dim_k \operatorname{Hom}(M(x), M(z)) \geqslant 0$. In case $(x, z)_\Omega = 0$, we clearly have $\varphi^y_{zx} = 1$. Thus we assume $(x, z)_\Omega \geqslant 1$.

If $\Delta = A_n, D_n$, or E_m, the number $(x, z)_\Omega$ is the multiplicity of E as composition factor of Z, then after tightening of the support , we see that we are in one of the Cases 2, ..., 9 of Theorem 2; thus we deal with the indicated polynomials. If $\Delta = G_2$, we directly apply Theorem 2 and obtain the exhibited polynomials: in Case 21, the multiplicity of E as composition factor of Z is 1; thus $(x, z)_\Omega = \dim_k \operatorname{End}(E) = 3$. In Cases 23 and 24, this multiplicity is 1 or 2, respectively, and $\dim_k \operatorname{End}(E) = 1$.

It remains to consider $\Delta = B_n$, C_n, and F_4. We have to verify that the tightening of the support does not lead to one of the Cases 2, ..., 9 (in case the endomorphism rings of (Mx), $M(y)$, and $M(z)$ are isomorphic). Note that $\operatorname{Hom}(M(x), M(y)) \neq 0$, $\operatorname{Ext}^1(M(z), M(x)) \neq 0$ shows that we can assume that $M(x) = E$ is the only simple projective R-module, again using 1.2. Also, we may assume that $Z = M(z)$ is faithful. For exhibiting R and Z, we use the conventions as in Theorem 2.

First, we deal with $\Delta = B_n$. Say R is of the form

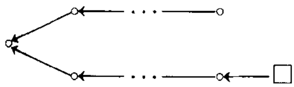


where both arms are of length at least 1, and dim Z is

$$\begin{matrix} & 2 \cdots 2 \ 2 \\ 1 & \\ & 1 \cdots 1 \end{matrix}$$

with at least two 1's. Clearly, tightening the support leads to Case 13 of Theorem 2; thus we obtain $\varphi^y_{zx} = \varphi_9$. On the other hand, $(x, z)_\Omega = 2$.

Similarly, for $\Delta = C_n$, the algebra R is of the form

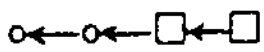
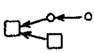
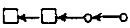


where the upper arm may be of length zero, and dim Z is

$$\begin{matrix} & 1 \cdots 1 & \\ 1 & & \\ & 2 \cdots 2 & 1 \end{matrix}$$

(with no 2 in case the lower arm is of length 1). Tightening of the support yields either Case 12 provided Z only lives on the lower arm, and Case 14 otherwise. So, we obtain either $\zeta_2\varphi_1$ or φ_8, and we note that $(x, z)_\Omega = 1$.

Finally, consider $\Delta = F_4$. The cases to be considered are listed in the following table.

R	dim Z	$(x, z)_\Omega$	Case	φ^y_{zx}
	1 2 1 1	1	12	$\zeta_2\varphi_1$
	1 2 2 1	1	12	$\zeta_2\varphi_1$
	1 3 2 1	1	14	φ_8
	1 1 1 1	1	14	φ_8
	1 2 2 1	2	17	φ_{10}
	2 2 2 1	2	18	φ_9
	2 3 2 1	3	19	$\zeta_2\varphi_9$
	1 1 2 1	2	15	φ_9
	2 2 1 1	2	13	φ_9
	4 2 2 1	4	16	φ_{11}
	1 3 4 2	2	13	φ_9

Nearly all entries are easily checked. We only mention that in the case $\dim Z = 1321$, the module $\tilde{Z}$ is the direct sum of a module with dimension vector 110 and endomorphism ring k, and one with dimension vector 2211 and endomorphism ring K. And that in the case $\dim Z = 1342$, the module $\tilde{Z}$ is the direct sum of two copies of 1111 with endomorphism ring k, and one copy of 1120 with endomorphism ring K.

This completes the proof.

COROLLARY 1. *Let $x, z \in \Phi^+$. If $x + z \notin \Phi^+$, define $N_{zx} = 0$. If $y = x + z \in \Phi^+$, define $N_{zx} = \varphi^y_{zx}(1) - \varphi^y_{xz}(1)$. Then $|N_{zx}| = r_{zx}$.*

Proof. One of the numbers $\varphi^y_{zx}(1)$, $\varphi^y_{xz}(1)$ is zero, and the other has absolute value r_{zx}, according to Theorem 1.

COROLLARY 2. *The Lie algebra $K(R-\text{mod}) \otimes \mathbb{Q}$ is generated by the elements $u_{e_1}, ..., u_{e_n}$, where $e_1, ..., e_n$ are the simple roots in Φ^+.*

Proof. If y is a non-simple root in Φ^+, there is a simple root $e \in \Phi^+$ with $z = y - e \in \Phi^+$. By induction, u_z belongs to the Lie subalgebra generated by $u_{e_1}, ..., u_{e_n}$, and $[u_z, u_e] = N_{ze} u_y$, with $0 \neq N_{ze} \in \mathbb{Z}$.

COROLLARY 3. *The Lie algebras $K(R-\text{mod}) \otimes \mathbb{C}$ and $\mathbf{n}_+$ are isomorphic.*

Proof. As Serre [Se] has shown, $\mathbf{n}_+$ is the complex Lie algebra with generators $u_1, ..., u_n$ and relations $(\text{ad}\, u_i)^{-a_{ji}+1}(u_j) = 0$. Since the elements $u_{e_1}, ..., u_{e_n}$ of $K(R-\text{mod}) \otimes \mathbb{C}$ satisfy the relations $(\text{ad}\, u_{e_i})^{-a_{ij}+1}(u_{e_j}) = 0$, we obtain a Lie algebra homomorphism $\mathbf{n}_+ \to K(R-\text{mod})$. According to Corollary 2, this map is surjective. Since both $\mathbb{C}$-algebras have the same dimension, it is an isomorphism.

11. The Relevance of the Polynomials

As we have shown the twelve polynomials $\varphi_1, ..., \varphi_{12}$ exhibited above are of important for a combinatorial study of modules over a representation-finite hereditary algebra, and, more generally, over a representation-directed algebra. They have been determined using the representation theory of partially ordered sets and vectorspace categories, and conversely, they may be interpreted as counting corresponding short exact sequences in certain exact categories. Similarly, we may use them to count isomorphism classes of distinguished triangles in some triangulated categories, for example the derived category of a representation-finite hereditary algebra. It would be interesting to know whether these polynomials occur also outside representation theory.

The combinatorial relevance of the classical Hall polynomials for a discrete valuation ring is well recognized [M]. Note that the classical Hall polynomials are of interest only for decomposable modules: the only non-zero polynomial occurring for indecomposable modules over a discrete valuation ring is $\varphi_1 = 1$. If we deal with a connected representation-finite hereditary algebra, not all φ_i will occur at the same time, but in the cases different from A_n we have to take into account some non-constant polynomial even when dealing with indecomposable modules. It will be interesting, but presumably not easy, to handle also the Hall polynomials for decomposable modules. For a first attempt to understand this situation, we refer to Section 4 of [R3] where the degenerate Hall algebra is shown to be the Kostant $\mathbb{Z}$-form of the corresponding enveloping algebra. The generic Hall algebra itself will be investigated in detail in a forthcoming publication [R4] where we will consider the relationship to quantum groups [Df].

Given an infinite field k, a representation finite hereditary k-algebra R, and indecomposable R-modules X, Y, Z with $\dim X + \dim Z = \dim Y$, it has been shown in the joint paper [DR3] with Dlab that there is an exact sequence $0 \to X \to Y \to Z \to 0$ or $0 \to Z \to Y \to X \to 0$. (See also [B] for an extension of this result.) The Hall polynomials we have exhibited show that the same assertion is true for k a finite field provided $|k| \neq 2$. Namely, the polynomials we have found have non-zero values for $|k| \neq 2$, since the only linear factors which arise are T, $T+1$, and $T-2$.

The results of our investigation have been reported at the Antwerp conference on Perspectives in Ring Theory and the Ottawa–Moosonee workshop in 1987, and then at the Banach centre in Warsaw in 1988. The Antwerp conference was sponsored by NATO, so we took the opportunity to outline the military relevance of the exhibited polynomials, at least those which occur in the splitting case. The military significance of ring theory in general is well documented in the literature [V1, V2, VL, KPS], and the polynomial rings seem to be very attractive due to the involvement of indeterminates. Some of our polynomials are well known in military science. In particular, the polynomials φ_0 and φ_1 seem to play a decisive role, and of course the use of φ_0 usually outnumbers that of φ_1. Note that

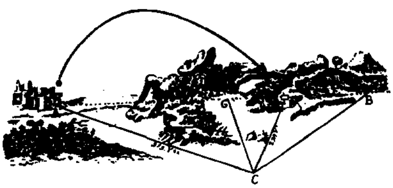


FIGURE 2

the polynomial φ_2 is the classical plundering polynomial. The existence of the polynomial φ_3 was already observed in antiquity. Apparently, it approximates quite well the trajectory of any missile, with the T-coordinate pointing to hell. Figure 2, from an older military handbook (reproduced, by permission of the publisher, from Ref. [A]), will clarify what is going on. The polynomials φ_4, φ_5, φ_6, φ_7 seem to be new, and we are sure that they may help to improve our military power. Maybe, with the use of these polynomials, we may be able to kill all human beings not only five times, but even six times. Such an improvement would be a great achievement of mankind.

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