Progress in Mathematics, Vol. 95, © 1991 Birkhäuser Verlag Basel

Recent Advances in the Representation Theory of Finite Dimensional Algebras

CLAUS MICHAEL RINGEL

This is a report on advances in the representation theory of finite dimensional algebras in the years 1984 – 1990. During these years, the German research council (DFG) has sponsered a Forschungsschwerpunkt devoted to the representation theory of finite groups and finite dimensional algebras; it started in 1984 and will be finished by 1991.

The topics we have chosen for this report are those related to investigations carried out in the Forschungsschwerpunkt. However, we will not restrict our attention to these investigations, but try to cover the topics in full generality. The reader will observe that two special classes of algebras reappear throughout the report: the hereditary algebras, and the canonical algebras. The module categories of these algebras are quite well understood, as we will outline below. These algebras serve as an important source of inspiration; dealing with them, one may hope to get an answer even to questions which in general may be impossible to attack. Despite of being rather special, one should keep in mind that these classes comprise some of the most important algebras. Also, quite surprising contacts to other parts of mathematics have been found in the last years involving such algebras.

There are many subjects which we have to omit at all. We will refrain from dealing with degenerations of modules. Also, infinite dimensional modules will be discussed only when they shed light on questions dealing with finite dimensional ones. As the title indicates, we will deal with representations of algebras which are *finite dimensional* over some field. Of course, we know that algebras over higher dimensional commutative rings have attracted a lot of interest in the last years, and they have been discussed in the Forschungsschwerpunkt. But we will be able to mention them only in case there is a direct relationship to finite dimensional algebras. We will try to restrict our attention to those results where a full proof is available, at least as a preprint; all other claims may be considered as mere conjectures. Similarly, we regret that we can cover the Russian literature only so far as translations do exist. Anyway, we had to be selective, and the results presented here are those which are not too technical, and which should be of interest to a wider audience.

The reader is advised that several reports are available dealing with the development of the representation theory before 1984: in particular, there were Riedtmann's Bourbaki talk in 1985, and the lectures by Auslander, and Gabriel at the ICM 1986 in Berkeley, see also the Proceedings of the Durham conference 1985.

In spite of its length, the list of references does not try to be complete; besides

the papers quoted in the text, we have included only a few additional ones dealing with related questions.

The author would like to thank D. Happel for his constant help when preparing this survey.

General conventions

We denote by k a commutative field, it will be the base field for all our algebras. Quite often we will restrict to the case of an algebraically closed base field: many problems of representation theory tend to get more lucid under this assumption. All the rings we will consider are supposed to have sufficiently many idempotents, nearly always they even will have a unit element (for example, in case we deal with a finite dimensional algebra), however a subring B of a ring A is not supposed to have the same unit element.

Modules usually will be left modules; module homomorphisms will be written on the opposite side of the scalars, thus usually on the right, so that in this case, the composition of $f : M_1 \to M_2$, $g : M_2 \to M_3$ has to be denoted by fg. For any ring R, we denote by R–mod the category of finitely generated R–modules, by R–Mod the category of all R–modules. If not otherwise stated, any algebra A and any A–module considered will be finite dimensional (over our base field k), and we denote by $\operatorname{rad} A$ the radical of A. We write $s(A)$ for the number of isomorphism classes of simple A–modules, let $E(1), \ldots, E(s(A))$, be the simple A–modules; they may be indexed by the vertices of the (Gabriel) quiver $Q(A)$ of A, note that there is an arrow $x \to y$ in $Q(A)$ if and only if $\operatorname{Ext}^1(E(x), E(y)) \neq 0$ (actually, we may consider $Q(A)$ as a valued quiver by attaching to any arrow the corresponding dimensions of Ext^1 over the endomorphism rings of the given simple modules). The projective cover of $E(x)$ will be denoted by $P(x)$. We denote by $\Gamma(A)$ the Auslander–Reiten quiver of A, and τ or τ_A is the Auslander–Reiten translation.

1. Tame and wild

1.1. The wild behaviour of what now are called wild algebras was first exhibited by Corner, and Brenner. Donovan and Freislich conjectured that there should be a clear distinction between the tame and the wild algebras. In 1979, Drozd presented his tame-and-wild theorem: any finite dimensional algebra over an algebraically closed field is either tame or wild. We refer to [C1] for a complete proof.

Recall that a finite dimensional algebra A over an algebraically closed field k is said to be *tame* provided for any $d \in \mathbf{N}$, there is a finite number of A–$k[T]$–bimodules $M_1, \ldots, M_n$ which are free of rank d as right $k[T]$–modules, such that almost all indecomposable A–modules of dimension d are of the form $M_i \otimes_{k[T]} k[T]/(T - \lambda)$ for some $1 \leq i \leq n$, and $\lambda \in k$. If A is tame, and $d \in \mathbf{N}$, the smallest possible number n of such bimodules is denoted by $\mu_A(d)$.

1.2. In general, we may consider A–$k[T]$–bimodules M which are free of finite rank as right $k[T]$–modules, and the corresponding functors $F_M = M \otimes_{k[T]} -$ from $k[T]$–mod to A–mod. In case almost all the modules $F_M(k[T]/(T - \lambda))$ are indecomposable, and pairwise non–isomorphic, we may call these modules an *affine one-parameter family* of indecomposable modules. Different affine one–parameter

families may intersect non–trivially. In case there are infinitely many (isomorphism classes of) indecomposable modules which belong to two affine one–parameter families, these families will be said to be equivalent. The union of all modules belonging to the affine one–parameter families in one equivalence class may be called a *complete one–parameter family.*

In order to avoid the difficulties of dealing with families of modules (see, for example, the clumsy definition of a complete one–parameter family of indecomposable modules), Crawley–Boevey [C8] has proposed to consider instead generic modules. A *generic* R–module M over an arbitrary ring R is by definition an indecomposable R–module of infinite length, such that M considered as an $\mathrm{End}(M)$–module, is of finite length (its *endolength*). Of course, the generic modules with endomorphism ring a division ring just, form the vertices of the (Cohn) spectrum of R. Note that given a functor of the form F_M as considered above, we obtain a generic module $F_M(k(T))$, where $k(T)$ is the rational function field in one variable over k. The endomorphism ring of a generic module always is a local ring [C9], the proof of this result uses concepts from model theory: a generic module satisfies the descending chain condition on the socalled *pp*–definable subgroups.

Theorem (Crawley–Boevey). *Let A be a finite dimensional algebra over an algebraically closed field. The algebra A is representation finite if and only if there are no generic modules, and A is tame if and only if for any $d \in \mathbb{N}$, there are only finitely many generic modules of endolength d, if and only if for any generic module M, the algebra $\mathrm{End}(M)/\operatorname{rad}\mathrm{End}(M)$ is isomorphic to $k(T)$. Also, in case A is tame, $\mathrm{End}(M)$ is split over its radical, and any two splittings are conjugate.*

The concept of a generic module seems to be so natural that one wonders why it was not considered earlier. Obviously, generic modules should be of interest for arbitrary rings, not just finite dimensional algebras or artinian rings.

Of particular interest will be the generic modules M without selfextensions. For example, any tame hereditary algebra has precisely one generic module, and this module does not have selfextensions. For the generalized Kronecker algebras $K(r) = \begin{bmatrix} k & k^r \\ 0 & k \end{bmatrix}$ with $r \geq 3$, Happel and Unger [HU2] have constructed infinite dimensional generic modules without selfextensions such that the endomorphism ring is a universal division ring of fractions for a free associative k–algebra in finitely many variables.

1.3 Some remarks concerning methods of proof may be appropriate. The use of bocses as introduced by Klejner and Rojter has turned out to be very essential. Drozd's tame–and–wild theorem and the modifications due to Crawley–Boevey deal with bocses, and no other proof seems to be in sight. Of course, the theory of bocses is now more accessible, see [C1] and [C9]. Some of the usual techniques of the representation theory of finite dimensional algebras have been copied for bocses: in particular, the existence of almost split sequences for bocses has been established by Bautista and Klejner [BK], see also [BB]. Methods similar to the usual bocs reduction may be applied also to the case of algebras over fields which are not necessarily algebraically closed, or even to artinian rings. In particular, Crawley–Boevey has shown in this way that given an indecomposable R–module M over some representation finite artinian ring R, there is a simple R–module S such that the division rings $\mathrm{End}(M)/\operatorname{rad}\mathrm{End}(M)$ and $\mathrm{End}(S)$ are isomorphic [C7].

Let A be a finite dimensional algebra over some algebraically closed field, and assume that A is not representation finite. The existence of a generic module for A is derived from the positive solution of the second Brauer–Thrall conjecture. We recall that a solution had been announced by Nazarova and Rojter in 1973. Our optimistic report [7] does not seem to be appropriate. The usual strategy for attacking the problem is to choose a minimal non–zero ideal I, so that by induction, one may assume that A/I is representation finite. The main difficulties arise in the case when the module ${}_AI$ has selfextensions, and it is this case which has been treated insufficiently by Nazarova and Rojter. In the proceedings of the Ottawa conference 1984, Nazarova and Rojter [**NR1**] have presented another approach to the second Brauer–Thrall conjecture: they claim to provide a new reduction of the representations of an algebra to the representations of a completed poset, and to attach to each completed poset a non–completed one of the same representation type. However, *both* reductions do not work! It is not diificult, to exhibit counter examples to the proposed methods of proof, as well as to the actual stated assertions. In the meanwhile, a corrected version of part of the second reduction has been published [**NR2**].

For fields of characteristic different from 2, the first complete proof for the second Brauer–Thrall conjecture has been given by Bautista [**Bau**]. The assumption on the characteristic of the base field has later been removed by Bongartz [**Bo**], and modifications of the proof have been published by Bretscher–Todorov [**BT**] and Fischbacher [**Fi**]. All these proofs rely on the existence theorem for a multiplicative basis. The problem of finding a rather direct proof of the second Brauer–Thrall conjecture still exists. Also, it would be of interest to have a proof for arbitrary base fields. Of course, the case of a perfect base field k follows from that of an algebraically closed field, so the non–perfect base fields remain to be considered.

1.4 There still is the problem of finding a convenient definition of tameness. The definition used by Drozd, as well as Crawley–Boevey's characterization in terms of generic modules involve infinite dimensional modules. One may use instead concepts from algebraic geometry, namely one may consider the sheets of indecomposable modules. Is there a definition of tameness which only involves finite dimensional modules, and avoids any reference to algebraic geometry?

Our survey [8] tried to present such a definition, but without success: we have asked that for any dimension d, there is a finite number of embedding functors F_i from $k[T]$–mod to A–mod such that all but a finite number of indecomposable A–modules of dimension d are of the form $F_i(L)$ for some i and some indecomposable $k[T]$–module L. However, any wild hereditary algebra A over an algebraically closed field k satisfies this condition: Consider the set $\mathcal{M}(x)$ of isomorphism classes of indecomposable A–modules with dimension vector x. We may assume that x is an imaginary root, thus the cardinality of $\mathcal{M}(x)$ is equal to that of k, let $\phi : k \to \mathcal{M}(x)$ be any bijection. Now, define a functor F from $k[T]$–mod to A–mod by sending the $k[T]$–module $L_\lambda[n] = k[T]/(T-\lambda)^n$ to the direct sum $n\phi(\lambda)$ of n copies of $\phi(\lambda)$. The inclusion and projection maps between the various modules $L_\lambda[n]$ with fixed λ shall be sent under F to the inclusion and projection maps between the corresponding direct sums (identify $(n-1)\phi(\lambda)$ with $(n-1)\phi(\lambda) \oplus 0 \subseteq n\phi(\lambda)$.) Clearly, F is an embedding functor, and, by construction, any indecomposable A–module with dimension vector x is of the form $F(X)$ for some simple $k[T]$–module. Of course, the functor F not at all is well–behaved, since ϕ is just a bijection of sets. Also note that under F all exact sequences go to split exact ones.

Two recent results may lead to intrinsic definitions of tameness. First of all, Crawley–Boevey [C1] has shown that for a tame algebra A over an algebraically closed field, almost all indecomposable A–modules M of fixed dimension satisfy $\tau M \cong M$, thus almost all indecomposable A–modules M of fixed dimension belong to tubes of rank one. Bautista has conjectured that this property may characterize the tame algebras. Even for group algebras, this conjecture was solved only recently by Erdmann [E2]. On the other hand, in the category of finitely generated Cohen–Macaulay modules over an isolated hypersurface singularity, all objects are τ–periodic, even in the wild case, as Eisenbud has shown (see [4]).

Second, one may consider the possible endomorphism rings of indecomposable modules. There is a common feeling that the wild algebras may be characterized by the property that any finite dimensional algebra can be realized as a factor ring of the endomorphism ring of some module modulo some ideal. However, first of all no proof that the wild algebras have this property, has been published yet. Second, there are modules over certain tame algebras, for example string modules, which have rather large endomorphism rings. Fortunately, for many tame algebras, there are only finitely many isomorphism classes of algebras which occur as endomorphism rings of indecomposable modules of fixed dimension. (However, the example of the biserial algebra $A = k\langle X,Y\rangle/(X^2,Y^2)$ shows that the indecomposable modules in a one–parameter family may yield a one-parameter family of endomorphism rings: consider the factor rings $A_\lambda = k\langle X,Y\rangle/(X^2,Y^2,XY-\lambda YX)$ as A–modules: this is a one–parameter family of indecomposable A–modules, and $\operatorname{End}_A(A_\lambda) = A_\lambda$.) But even if there are only countably many isomorphism classes of algebras which occur as endomorphism rings of indecomposable modules, it may be conceivable that there are indecomposable modules M_n such that the algebra $k\langle X,Y\rangle/(X,Y)^n$ is isomorphic to a factor ring of $\operatorname{End}(M_n)$, and then at least all finite dimensional local algebras generated by two elements could be realized as factor rings of endomorphism rings. One typical class of indecomposable modules over tame algebras has been studied in detail by Krause [Kr2], the string modules. As Crawley–Boevey [C5] has shown, the maps between string modules may be described combinatorially. Krause shows that the class of factor rings of endomorphism rings of string modules is very restricted: if $\operatorname{rad} A$ is generated by 2 elements, then A can be realized as a factor ring of $\operatorname{End}(M)$ for some string module M only in case

$$\dim_k A/(\operatorname{rad} A)^n \leq 2n^2 - 2n + 1,$$

and even the algebra $k\langle X,Y\rangle/(X,Y)^4$ cannot be realized as factor ring of the endomorphism ring of a string module. One may ask whether there is a polynomial p such that for any indecomposable module M over a tame algebra A, any factor ring A of $\operatorname{End}(M)$ with $\operatorname{rad} A$ generated by two elements satisfies

$$\dim_k A/(\operatorname{rad} A)^n \leq p(n).$$

1.5 In dealing with affine one–parameter families, say given by a functor $F_M = M \otimes_{k[T]} - : k[T]\text{-mod} \to A\text{-mod}$, it sometimes seems to be convenient to look for factorizations of F_M of the form

$$k[T]\text{-mod} \longrightarrow K\text{-mod} \xrightarrow{G} A\text{-mod}$$

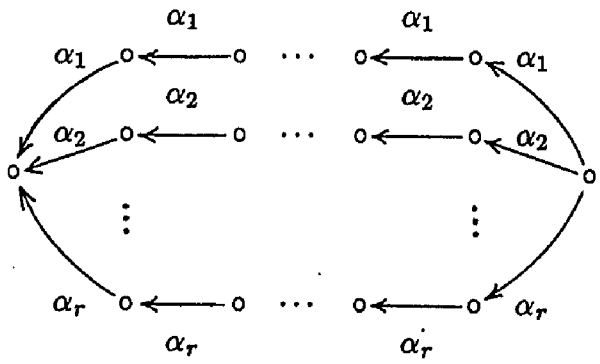
where $K = K(2)$ is the usual Kronecker algebra and G again is exact. First of all, we may ask whether it always will be possible to find such a factorization (for A finite dimensional!). Second, we should remark that G, if it exists, may not be uniquely determined, a typical example are the two embeddings G_1, G_2 of K–mod into A–mod, where A is the hereditary algebra of type $\tilde{A}_{1,2}$

$$G_1(W \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V) = W \overset{\alpha}{\underset{\beta}{\leftleftarrows}} V \quad (V \to 1),$$

$$G_2(W \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V) = W \overset{\alpha}{\underset{\beta}{\leftleftarrows}} V \quad (1 \leftarrow W),$$

where V, W are k–vectorspaces, and $\alpha, \beta : V \to W$ are linear maps. The functors G_1, G_2 coincide on the subcategory of K–mod given by all objects $(V, W, \alpha, 1)$, and this subcategory is equivalent to $k[T]$–mod. We deal here with the situation of a one–parameter family indexed by $\mathbf{P}_1$, where the point $(1 : 0) \in \mathbf{P}_1$ occurs with multiplicity two. In general, it often happens that we deal with a one–parameter family indexed by the projective line $\mathbf{P}_1$, where finitely many points of $\mathbf{P}_1$ occur with a multiplicity greater than 1.

In order to study this phenomenon, consider pairwise different points $\lambda_1, \ldots, \lambda_r$ of $\mathbf{P}_1$, and attach to each λ_i some multiplicity $p_i = p(\lambda_i) \geq 1$ (or, equivalently, take a function $p : \mathbf{P}_1 \to \mathbf{N}_1$ such that $p - 1$ has finite support, thus $p - 1$ is an effective divisor). We can assume that $r \geq 2$, and that $\lambda_1 = \infty, \lambda_2 = 0$, and therefore $\lambda_i \in k \setminus \{0\}$, for all $i \geq 3$. The corresponding canonical algebra $C(p)$ is given by the quiver



with p_i arrows labelled α_i, and the relations

$$\alpha_i^{p_i} = \lambda_i \alpha_1^{p_1} + \alpha_2^{p_2} \quad \text{for} \quad i \geq 3.$$

Let 0 be the sink of the quiver, and ω the source, and let $\Delta(p)$ be the quiver obtained by deleting ω, it is a star with several arms.

The defect of a representation M is by definition $\delta(M) = \dim_k M_\omega - \dim_k M_0$. Denote by $\mathcal{T}$ the representations which are direct sums of indecomposable representations of defect zero. Then we have shown [**R1**] that $\mathcal{T}$ is a standard tubular family of tubular type $(p_1, \ldots, p_n)$, it separates the full subcategory $C(p)$–mod$^-$ of all indecomposable representations of negative defect from the full subcategory $C(p)$–mod$^+$ of all indecomposable representations of positive defect. A more concise proof has been given in [**R3**].

Geigle and Lenzing [**GL1**], see also [**DGL**], have related the category $C(p)$–mod to the category of coherent sheaves $\operatorname{coh}\mathbf{X}(p)$ over what they call the 'weighted projective line' $\mathbf{X}(p)$ of type $p : \mathbf{P}_1 \to \mathbf{N}_1$. In fact, they have shown that the derived categories $D^b(C(p)\text{–mod})$ and $D^b(\operatorname{coh}\mathbf{X}(p))$ are equivalent. The category $\operatorname{coh}\mathbf{X}(p)$ is an abelian category of global dimension 1, thus the structure of $D^b(\operatorname{coh}\mathbf{X}(p))$ is known as soon as we know $\operatorname{coh}\mathbf{X}(p)$. Their direct description of $\operatorname{coh}\mathbf{X}(p)$ therefore yields a completely different, and very illuminating proof for the structure of the category $C(p)$–mod.

Lenzing has stressed the importance of the rank–one modules over a canonical algebra. Here, a module M is said to be a *rank–one module*, provided M is indecomposable, and $\delta(M) = -1$. For example, the radical Q of the injective hull of $E(0)$ is a rank–one module, and will examine the τ–orbit of this module in detail.

First, let M be an arbitrary rank–one module. We claim that τM is a rank–one module, too, and the Auslander–Reiten sequence ending in M is conservative (this means that $\operatorname{proj.dim.} M = 1 = \operatorname{inj.dim.} \tau M$). (Let us outline the proof: In case Z is an indecomposable non–projective module of negative defect, and the Auslander–Reiten sequence ending in Z is conservative, then the dimension vector of τZ can be calculated by applying the Coxeter transformation of A to the dimension vector of Z, and therefore Z and τZ have the same defect. Now, all indecomposable modules of negative defect have projective dimension at most one. Let us show that for any non–projective rank–one module M, we have $\operatorname{Hom}(M, {}_AA) = 0$. Assume we have a non–zero map $\phi : M \to P(a)$, for some indecomposable projective module $P(a)$. Any non-zero submodule U of $P(a)$ has defect $\delta(U) \leq -1$, thus the kernel of ϕ has non–negative defect. But this is possible only in case the kernel is zero, thus M is a submodule of $P(a)$. It is easy to see that the rank–one submodules of any $P(a)$ are projective. This yields a contradiction. As a consequence, the Auslander–Reiten sequence ending in M is conservative, and τM again is a rank–one module.)

As a consequence, we see: In case $\Delta(p)$ is Dynkin, the modules $\tau^{-n}Q$ are rank–one modules, for all $n \in \mathbf{N}$. In case $\Delta(p)$ is wild, the modules $\tau^n Q$ are rank–one modules, for all $n \in \mathbf{N}$. (Consider first the Dynkin case. If $r \leq 2$, then the algebra A is hereditary, thus τ^- respects the defect. Let $r \geq 3$. In this case, $\operatorname{rad} P(\omega)$ is an indecomposable module and a predecessor of Q. It follows that $\operatorname{Hom}(\tau^{-n}Q, P(\omega)) = 0$ for all $n \in \mathbf{N}$. As a consequence, the Auslander–Reiten sequences starting with $\tau^{-n}Q$, for $n \geq 0$, are conservative. For $r \geq 1$, the module $\tau^- P(\omega)$ is of Loewy length 2, its socle is the direct sum of $r-1$ copies of $E(0)$, and its top is multiplicity free (of length r). In the wild case, we have $r \geq 3$, therefore $\tau^- P(\omega)$ is not a rank–one module. Since the set of rank–one modules together with the zero–module is closed under τ, it follows that $\tau^n Q$ cannot be isomorphic to $P(\omega)$, for any $n \in \mathbf{N}$, thus $\tau^n Q$ is a rank–one module, for any $n \in \mathbf{N}$.)

Given an endofunctor $F : \mathcal{U} \to \mathcal{U}$ of some full subcategory $\mathcal{U}$ of A–mod, and a module M in $\mathcal{U}$, Lenzing has introduced the ring

$$\mathcal{A}(F; U) = \bigoplus_{n=0}^{\infty} \operatorname{Hom}(U, F^n U),$$

the product of $f : U \to F^n U$, and $g : U \to F^m U$ is given by the composition $f \cdot F^n(g)$.

Theorem (Geigle–Lenzing) *If $\Delta(p)$ is a Dynkin diagram, then the ring $\mathcal{A}(\tau^-; Q)$ is the simple surface singularity of type $\Delta(p)$.*

Recall that the simple surface singularities are of the form $k[X,Y,Z]/(f)$, where f is given es follows:

(p,q)	$X^{p+q}+YZ$
(2,2,2s)	$X(Y^2+YX^s)+Z^2$
(2,2,2s+1)	$X(Y^2+ZX^s)+Z^2$
(2,3,3)	$Y^3+X^2Z+Z^2$
(2,3,4)	$Y^3+X^3Y+Z^2$
(2,3,5)	$Y^3+X^5+Z^2$.

Theorem (Lenzing) *If $\Delta(p)$ is wild, then the ring $\mathcal{A}(\tau;Q)$ is a ring of automorphic forms.*

For example, for the 14 cases $p=(2,3,7),(2,3,8),\dots,(4,4,4)$, and the base field $k=\mathbb{C}$, one just obtains the 14 rings of automorphic forms with three generators, thus Arnold's 14 exceptional unimodal singularities. Note that we obtain corresponding rings for every base field k, even independent of the characteristic; for the 14 cases, the rings are of the form $k[X,Y,Z]/(f)$, where f is a polynomial which has been calculated explicitly by Hübner **[Hü]**.

Let us add that canonical algebras may be defined more generally for an arbitrary base field. We start with an arbitrary tame hereditary algebra A with $s(A)=2$, thus the A–modules are just the representations of a tame bimodule ${}_FM_G$. Here F,G are division k–algebras, ${}_FM_G$ is an F–G–bimodule on which k operates centrally, and $(\dim_F M)(\dim M_G)=4$; the algebra A being given by $\begin{bmatrix} F & M \\ 0 & G \end{bmatrix}$. Let Ω be the set of isomorphism classes of simple regular A–modules, and take a function $p:\Omega\to\mathbb{N}_1$, such that $p-1$ has finite support. In **[R3]**, we have defined the corresponding canonical algebra $C({}_FM_G,p)$, it has a standard tubular family indexed by Ω, such that the tube with index λ is stable of rank $p(\lambda)$, as we want to have it, and again, this tubular familly is separating. Note that the index set Ω for an arbitrary tame bimodule ${}_FM_G$ has been studied by Crawley–Boevey **[C6]**.

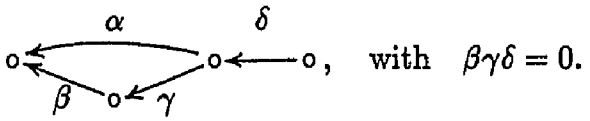
1.6. A tame algebra may be said to be *domestic* with at most n one–parameter families of tubes, provided $\mu_A(d)\le n$ for all $d\in\mathbb{N}$. Typical examples are the tame concealed algebras, these are the endomorphism rings of preprojective (or preinjective) tilting modules over a tame hereditary algebra. They have been classified by Happel and Vossieck. Note that any tilting equivalence class contains precisely one canonical algebra (this is one of the reason for the term 'canonical'). The algebras which are tilting equivalent to tame hereditary algebras have been studied in detail by Assem and Skowroński. The algebras which are tilting equivalent to a hereditary algebra of type $\tilde{A}_n$ can be characterized in terms of quivers and relations: we obtain in this way certain special biserial algebras **[AS1]**. Also, they have shown that any representation infinite algebra which is tilting equivalent to a tame hereditary algebra contains a unique convex subalgebra which is tame concealed, thus it can be obtained from this tame concealed algebra by forming extensions and coextensions (see [AS4]).

We recall that a *cycle* in A–mod is a sequence $X=X_0\to X_1\to\cdots\to X_{n-1}\to X_n=X$ of non–zero and non–invertible maps between indecomposable modules X_i. An indecomposable module M is said to be *directing* provided it does not belong to a cycle, and M is said to be *sincere*, provided every simple A–module appears

as a composition factor of M. Note that an algebra with a sincere directing module M always is a tilted algebra. A tame algebra A with a sincere directing module is domestic with at most two one–parameter families of tubes, as de la Peña [**P3**] has shown, and either A is representation directed, or a finite enlargement of a tame concealed algebra, or a glueing of two tame concealed algebras.

Let $\mathrm{rad}(A\text{–mod})$ be the radical of the category A–mod, and let $\mathrm{rad}^\infty(A\text{–mod})$ denote the intersection of the powers $\mathrm{rad}^n(A\text{–mod})$, with $n \in \mathbb{N}$. Consider any non–zero map $f : X \to Y$, between indecomposable modules X, Y. If X, Y belong to different components of the Auslander–Reiten quiver $\Gamma(A)$, then $f \in \mathrm{rad}^\infty(A\text{–mod})$, whereas in case X, Y belong to one component, and this component is standard (or finite), then $f \notin \mathrm{rad}^\infty(A\text{–mod})$. It follows easily (see [**KSk**]) that $\mathrm{rad}^\infty(A\text{–mod}) = 0$ if and only if A is of finite representation type. Kerner and Skowroński conjecture that $\mathrm{rad}^\infty(A\text{–mod})$ can be nilpotent (or even T-nilpotent) only in case A is domestic, and they prove this for 'standard' selfinjective algebras.

One may call an algebra A *cycle finite,* provided no map in a cycle of A–mod belongs to $\mathrm{rad}^\infty(A\text{–mod})$. Cycle finite algebras have been considered by Assem and Skowroński, they are tame, and maybe they are always of finite growth. But there are even domestic algebras which are not cycle finite, for example



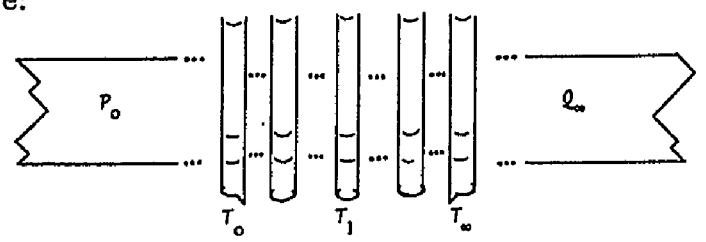
with $\beta\gamma\delta = 0$.

Assem and Skowroński have generalized the notion of a tube to that of a *coil*, these are certain finite enlargements of tubes, where additional projective–injective vertices and nodes are allowed. They call an algebra A a *coil algebra* provided every cycle in A–mod is inside a standard coil, and they show that the minimal representation infinite coil algebras are just the tame concealed ones [**AS5**].

1.7. The difference between finite growth and infinite growth representation type was first observed by Nazarova and Zavadskij when dealing with representations of posets. Note that an algebra A is said to be of *finite growth* provided A is tame and there exists an n such that $\mu_A(d) \leq d^n$ for all $d \in \mathbb{N}$.

Typical examples of algebras of finite growth which are not domestic, are the tubular algebras, in particular the canonical algebras of Euclidean type. The main goal of the lecture notes [**R1**] was to present the structure theory for the module category of a tubular algebra. Some mathematicians have complaint that the book does not contain a definition of tameness: it really only deals with examples of algebras, and the tame ones occurring there are obviously 'tame', so no definition was necessary. One should see that only slowly a general theory of tame algebras is emerging, a theory which seems to show that the examples studied in detail before are the typical building blocks for general tame algebras.

The structure of A–mod, for A a tubular algebra, may be visualized by the following picture:



Here, $\mathcal{P}_0$ is a preprojective component, $\mathcal{Q}_\infty$ is a preinjective component, and all the families $\mathcal{T}_i$ are one-parameter families of tubes, the index set for i being the set of non-negative rational numbers including the symbol ∞. The tubular family $\mathcal{T}_0$ contains projective modules, and $\mathcal{T}_\infty$ contains injective modules, the remaining families $\mathcal{T}_i$ are regular. All the components of a tubular algebra are standard, and the picture indicates the possible maps between components: there are only maps going from left to right. It should be mentioned that this description relies on an understanding of complete one-parameter families, and so it was stimulated by Gabriel's interpretation of the regular modules for the four subspace quiver in terms of tubes, and it was guided by suggestions of Brenner to use tilting modules for getting a complete classification of the indecomposable modules for a tubular algebra. It is sufficient to consider the tubular algebras which are canonical, and the Geigle-Lenzing approach via $\operatorname{coh}\mathbf{X}(p)$ again may be used.

A personal comment: Skowroński has started to rename the tubular algebras. Of course, it is always nice to see the own name in print, especially when it appears parallel to names like Dynkin and Euclid (but they may not even know what their algebras are about). However, it seems that the name 'tubular' is very suggestive, whereas I hope not to look like being made up from tubes. Also, one should keep in mind the guiding intuition of Gabriel, and of Brenner, and the parallel investigations of Zavadskij, so one may speak of Gabriel-Brenner-Ringel-Zavadskij-algebras, but may-be this sounds a little odd? And, as Lenzing has noted, this all relates to Atiyah's classification of vector bundles over elliptic curves, so Atiyah-Gabriel-B-R-Z-algebras!

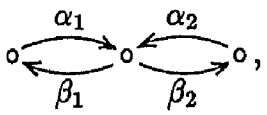
One may construct further examples of algebras of finite growth by making inductively suitable one-point extensions, see **[HR]**, **[PTo]**, in particular, one may consider convex subalgebras of the socalled repetitive algebra $\widehat{A}$ of a tubular algebra A, or also algebras which have $\widehat{A}$ as a Galois covering. In section 5, we will deal with $\widehat{A}$ in more detail. Here we only note that Skowronski (**[Sk2]**, see also **[NS]**) has classified the standard selfinjective algebras B of finite growth: the representation finite ones have been described by Riedtmann, so we can assume that B is representation infinite, and then B has a Galois covering $\widehat{A}$ with A being a tame tubular extension of a tame concealed algebra.

There are corresponding tubular vectorspace categories, see **[R1]**, the representations of those which are posets have also been described by Zavadskij **[Za]**. There are other categories occurring in representation theory which may be described in a similar way by reducing to tubular algebras or tubular vectorspace categories. The most prominent one seems to be the category of lattices for a cyclic group of order 3 over a complete discrete valuation ring where 3 is 4-fold ramified. This problem was solved by Dieterich **[Di]**, he showed that the lattice type is tame, and that we deal with a tubular problem of tubular type $\tilde{D}_4$. Note that in all other cases, the lattice type for finite groups and complete discrete valuation rings had been known before, according to the work of Gudivok and others. The remaining cases are either domestic, or of infinite growth, or wild.

1.8. Let us consider now the tame algebras of infinite growth. The Gelfand-Ponomarev paper on the representations of the Lorentz group has put forward a method which has turned out to be very fruitful, since it can be used for all special biserial algebras. We should remark that Dowbor and Skowroński **[DS]** have clarified the procedure by putting it into the context of covering theory. Of course, what Gelfand and Ponomarev have called the modules of the first kind, the strings, are

just those indecomposable modules which are obtained by using a related covering functor. Dowbor and Skowroński show that the remaining modules, the modules of the second kind, or bands, may be considered as corresponding to modules over the group algebra $k[T,T^{-1}]$ of the infinite cyclic group $\mathbf{Z}$, and that this is really a categorical description: let A be special biserial, then the category obtained from A–mod by factoring out all maps which factor through strings is the categorical sum of copies of $k[T,T^{-1}]$–mod, one copy for each equivalence class of primitive cyclic words.

The main advance concerning tame algebras of infinite growth has been the study of Crawley–Boevey [**C3**] of what he calls 'clans'. In this way, he gave a solution to the Gelfand problem of classifying the indecomposable representations of the quiver



with $\alpha_1\beta_1 = \alpha_2\beta_2$ nilpotent,

posed at the ICM in Nice, 1970. We recall that in 1973, Nazarova and Rojter have shown that this is a tame problem, and they gave a partial solution to the classification problem; however the normal forms which they proposed didn't work. The only assumption needed by Crawley–Boevey is that we deal with a base field k with at least three elements. Starting point of these investigations was his description [**C2**] of the category of finite dimensional A–modules for the algebra

$$A = k\langle X,Y\rangle/(X^2 - X,\, Y^2),$$

of course the A–modules are just given by pairs of square matrices, one being idempotent, the other having square zero.

We have mentioned above that a general theory of tame algebras seems to be emerging. But some crucial questions, even on the level of dealing with examples, are still open. Let us mention at least two: what can be said about the biserial algebras which are not special, for example, only few of the local biserial algebras are special (but all have been conjectured to be tame in [**6**]), and what about the representations of the algebras

$$k\langle X,Y\rangle/(X^2 - (YX)^nY, Y^2 - (XY)^nX),$$

with $n \geq 1$. In characteristic 2, the group algebras of the quaternion 2-groups are, when we disregard socles, of this form, and they are known to be tame by the theorem of Drozd and Bondarenko: the proof uses the realization of a quaternion group as a subgroup of index 2 in a semidihedral group. But for characteristic different from 2, no such trick seems to work.

1.9. Investigations of Kerner on wild hereditary algebras give a completely new interpretation of what may be called the 'wild' behaviour of wild algebras. This seems to be the most spectacular development in representation theory in the last years. In order to pin-point the result, let us recall the main features of wild algebras known before. The intuitive notion of 'wild' algebras was built on investigations of Corner and Brenner who showed that there are finite dimensional k–algebras A,

such that for *any* finite dimensional k–algebra B, there is a full exact embedding of the module category B–mod into A–mod; such an algebra A is said to be *strictly wild*. Clearly, in order to show that A is strictly wild, we only have to find for any $n \in \mathbf{N}$ a full exact embedding of F_n–mod into A–mod, where $F_n = k\langle X_1, \ldots, X_n\rangle$, the free algebra in n (non–commuting) generators X_i, since we may write any finite dimensional k–algebra as a factor algebra of some F_n. For example, the algebra F_2 is strictly wild, a full exact embedding ι_n of F_n–mod into F_2–mod is constructed as follows: the F_n–modules are of the form $(V, \varphi_1, \ldots, \varphi_n)$, where V is a k–space, and φ_i are k–linear endomorphisms of V, and $\iota_n(V, \varphi_1, \ldots, \varphi_n) = (V^{n+2}, \alpha, \beta)$, where

$$\alpha = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ \varphi_1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_n & 1 & 0 \end{bmatrix}.$$

Of course, with F_2 all algebras F_n, where $n \geq 2$, are strictly wild. The mutual embeddings of the module categories of the various strictly wild algebras indicate that these module categories are similarly complicated: a complete classification of the indecomposable A–modules for one strictly wild algebra A would yield a complete classification for any algebra (provided we can control the embeddings effectively, but this often seems to be the case, see the example above): so it is not surprising that no such classification is known (there are some papers who claim to provide one, but they just outline an inductive procedure of what should be done for any fixed dimension). Some features of a strictly wild k–algebra A should be stressed: given any finite dimensional k–algebra B, there is an A–module with endomorphism ring B, in particular, there are indecomposable A–modules which are arbitrarily complicated, and a classification of the A–modules would also yield sort of a classification of all finite dimensional k–algebras.

Assume that A is a representation infinite hereditary finite dimensional k–algebra, where k is an algebraically closed field. We can assume that A is, in addition, basic and connected, thus A is the path algebra of some finite connected quiver Δ without oriented cycles, and Δ is not a Dynkin diagram. Now, A–mod has a preprojective, and a preinjective component. The preprojective modules, as well as the preinjective modules, are easily classified, so it remains to consider the remaining indecomposable modules: they are called the regular modules. One knows that A is tame, if and only if Δ is a Euclidean diagram, and, in this case, the components containing regular modules are stable tubes, and the set of these components is indexed by the projective line $\mathbf{P}_1(k)$ over k. In case Δ is neither a Dynkin nor a Euclidean diagram, it is well–known that A is strictly wild, and the components containing regular modules are of the form $\mathbf{Z}A_\infty$. In both cases, the modules on the boundary of these components are called quasi–simple. Any indecomposable regular module M has a filtration (unique up to isomorphism) $M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$ with all factors M_{i-1}/M_i quasi–simple, and $M_{i-1}/M_i \cong \tau^i(M/M_1)$, the number t is called the quasi–length, the module M/M_1 the quasi–top of M. Any indecomposable regular module is uniquely determined by its quasi–length and its quasi–top, and, conversely, given a quasi–simple module and a positive integer, there is an indecomposable module with this quasi–top and this quasi–length.

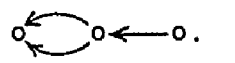
Let us denote the set of regular components of A–mod by $\Omega(A)$. We may use $\Omega(A)$ also as index set for the τ–orbits of the (isomorphism classes of) quasi–simple

A–modules, keeping in mind that the modules belonging to a fixed regular component are determined by the quasi-simple modules in this component.

The main, but rather hopeless problem is to obtain a reasonable description of $\Omega(A)$. Of course, for A tame, $\Omega(A)$ may be identified with $\mathbf{P}_1(k)$ (and for A representation finite, $\Omega(A) = \emptyset$). Assume now that A is wild. It is easy to see ([**Z1**]) that in this case any Auslander–Reiten component contains at most one isomorphism class of indecomposable A–modules with a fixed dimension vector. Thus if we denote by $\mathcal{R}(d)$ the set of isomorphism classes of indecomposable A–modules with dimension vector d, where $d \in \mathbb{N}_0^n$ is an imaginary root, then we may embed $\mathcal{R}(d)$ into $\Omega(A)$, sending the isomorphism class of a module to the component containing it.

The investigations of Kerner center around the problem of describing $\Omega(A)$ in the wild case. What he shows, and this is very amazing, is that the set $\Omega(A)$ may be considered as being independent of A: given two wild hereditary algebras A, A' over the same (algebraically closed) field, there are intrinsic bijections between $\Omega(A)$ and $\Omega(A')$. We stress here the word 'intrinsic': it is rather easy to see (and trivial in case k is an uncountable field) that the set $\Omega(A)$ has the same cardinality as k, thus there are many bijections between $\Omega(A)$ and $\Omega(A')$. The essential steps for constructing Kerner's bijections will be reviewed below. For the algebras A, A', there are at most a countable number of such bijections, indexed by finite sequences of tilting modules. At the moment it is not clear at all, whether bijections with different indices are actually different or not: thus either there is even a unique such bijection, or else, fixing one of these sets $\Omega(A)$, there is a countable group $G(A)$ of automorphisms operating on it, so that at least the orbit space $\Omega(A)/G(A)$ is an intrinsic set, independent of A (note that in case k is uncountable, the set $\Omega(A)/G(A)$ still is uncountable). The reader should keep in mind that the classification problem for wild hereditary algebras deals with many important mathematical problems, such as the n–subspace problems, for $n \geq 5$, or the problem of classifying n-tuples of matrices of the same size with respect to simultaneous multiplication from the left and simultaneous multiplication from the right, for $n \geq 3$, and the assertion is that for all these problems (after deleting the preprojective and the preinjective modules) there are natural bijections!

Let us outline the construction in some detail. Let A be a representation-infinite connected hereditary k–algebra, and ${}_AT$ a tilting module which has no indecomposable preinjective direct summand. As Strauß [**St**] has shown, there is a unique preprojective component in $\mathrm{End}({}_AT)$–mod, and if I denotes its annihilator in $\mathrm{End}({}_AT)$, then $C = \mathrm{End}({}_AT)/I$ is a concealed algebra of the same representation type as A, and we say that A *dominates* C via T. For tame hereditary algebras A, C the domination relation can be read from the Euclidean diagrams, and it coincides with the degeneration relation for the corresponding singularities. Consider the generalized Kronecker algebras $K(r)$, they are wild for $r \geq 3$, and the algebra H defined by the quiver



According to Unger [**U3**], any connected wild hereditary algebra dominates one of the form $K(r)$, with $r \geq 3$, and the algebra H dominates any $K(r)$, with $r \geq 3$; this we will outline below. It follows that the equivalence relation generated by the dominance relation consists of a unique equivalence class. Thus, in order to construct the Kerner bijections, we may restrict to the case of a pair of algebras, one dominating the other.

Let A, C be connected wild hereditary algebras, and assume C is dominated by A via some tilting module ${}_AT$. Let $B = \mathrm{End}({}_AT)$. We are going to construct Kerner's map $\eta_T : \Omega(A) \longrightarrow \Omega(C)$.

Let M be a regular A–module. Then Kerner ([**K2**], [**K4**]) shows that there are integers $s(M), t(M)$ such that $\tau_A^t M$ is generated by ${}_AT$, for all $t \geq t(M)$, and such that $\tau_B^{-s} \mathrm{Hom}_A({}_AT, M)$ is a C–module, for all $s \geq s(M)$.

Theorem (Kerner) *Let M be an indecomposable regular A–module. Let*

$$\eta_T(M) = \tau_B^{-s(\tau_A^{t(M)}M)} \mathrm{Hom}_A({}_AT, \tau_A^{t(M)}M).$$

Then for indecomposable regular modules M, M' in the same component of A–mod, the C–modules $\eta_T(M)$, and $\eta_T(M')$ belong to the same component of C–mod, and the induced map $\eta_T : \Omega(A) \longrightarrow \Omega(C)$ is bijective.

The situation is as follows: altogether we deal with countably many (isomorphism classes of) finite dimensional algebras, the path algebras of finite connected wild quivers without cycles over some fixed algebraically closed field k. For any such algebra A, the set $\Omega(A)$ is defined, and there are at most countably many tilting A–modules, thus there are at most countably many bijections of the form η_T.

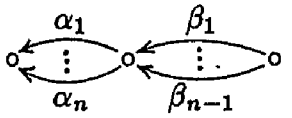
The further aim will be to look for properties of the sets $\Omega(A)$ which are invariant under the maps η_T. If we fix some connected wild hereditary algebra A as the basic example (say $K(3)$, or the 5-subspace quiver) and denote $\Omega = \Omega(A)$, we may consider this set, together with its yet unknown additional structure of properties which are invariant under the Kerner bijections, as a universal index set for handling the representations of wild hereditary algebras.

One may be curious to know what additional wild algebras have Ω as index set for their regular components. Lenzing and de la Peña [**LP**] have shown that given a wild canonical algebra $C(p)$, there are again intrinsic bijections between Ω and the set of all components of $C(p)$–mod$^-$, as well as between Ω and the set of components of $C(p)$–mod$^+$.

It seems worthwhile to indicate part of the proofs: The existence of the number $t(M)$ is an easy consequence of the following lemma [**K2**]: If X, Y are regular modules, then there exists an integer $t(X,Y)$ such that $\mathrm{Hom}_A(\tau^t X, Y) = 0$ for all $t \geq t(X,Y)$. Note that this is the opposite assertion a lemma due to Baer [**B2**]: If X, Y are regular modules, then there exists an integer $s(X,Y)$ such that $\mathrm{Hom}_A(X, \tau^s Y) \neq 0$ for all $s \geq s(X,Y)$. Taking both assertions together, we see that in a regular component, globally, the maps are going in one direction, and this is the opposite direction of the arrows in the Auslander–Reiten quiver. (The two statements are actually interrelated: Kerner uses in his proof Baer's lemma, and one derives from his lemma the existence of the bound $t(M)$ which may be considered as a partial strengthening of Baer's lemma: Let $T = T_p \oplus T_r$, be a tilting module, with T_p preprojective, and T_r regular, let M be a regular A–module, and define $t(M) = t(M, \tau T_r)$. Then, for $t \geq t(M)$, $0 = \mathrm{Hom}_A(\tau^t M, \tau T_r) \cong \mathrm{Ext}^1_A(T_r, \tau^t M)$. Since T_p is preprojective, and $\tau^t M$ is regular, we also have $\mathrm{Ext}^1_A(T_p, \tau^t M) = 0$, thus $\tau^t M$ is generated by ${}_AT$.)

Let us exhibit in detail bijections $\eta_T : \Omega(H) \longrightarrow \Omega(K(n))$, for $n \geq 3$. The indecomposable H–module with dimension vector $(n+1, n, 0)$ will be denoted by $M(n)$. Since $\dim_k \mathrm{Ext}^1_A(E(3), M(n)) = n$, there is a universal extension of the form $0 \to M(n)^n \to N(n) \to E(3) \to 0$, and clearly $N(n)$ is indecomposable and not

preinjective. For $n \geq 1$, we can embed $P(3)$ into $M(n)$, and the cokernel is the direct sum of $n-1$ copies of $M(n+1)$. It follows that $T(n) = M(n) \oplus N(n) \oplus M(n+1)$ is a tilting module, and its endomorphism ring $B(n)$ is given by the quiver



and the relations

$$\alpha_i\beta_i = \alpha_1\beta_1, \quad \alpha_{i+1}\beta_i = \alpha_2\beta_1, \quad \alpha_j\beta_i = 0 \quad \text{for } j \notin \{i, i+1\}.$$

Note that the algebra $B(n)$ is a one-point extension of $K(n)$ by an indecomposable $K(n)$-module $X(n)$ with dimension vector $(2, n-1)$. We have $B(2) = H$. So consider now the cases $n \geq 3$. Under this assumption, all indecomposable $K(n)$-modules with dimension vector $(2, n-1)$ are regular, thus, the preprojective component of $K(n)$-mod is a component of $B(n)$-mod, and therefore H dominates $K(n)$ via $T(n)$. What can we say about components of $B(n)$? By the Kerner lemma, any regular component of $K(n)$-mod has many indecomposable modules M such that both $\mathrm{Hom}_{K(n)}(X(r), M) = 0$ and $\mathrm{Hom}_{K(n)}(X(r), \tau_{K(n)}M) = 0$, and for these modules M we have $\tau_{B(n)}M = \tau_{K(n)}M$; in particular, the $B(n)$-component containing M again will be of the form $\mathbf{Z}A_\infty$, provided it does not contain the module $X(n)$. It is surprising that we obtain in this way all regular components of $B(n)$-mod.

2. Combinatorial Methods I:
The Structure of Auslander–Reiten Components

2.1. Recall that the stable Auslander–Reiten quiver $\Gamma_s(A)$ of an artin algebra A is obtained from the Auslander–Reiten quiver $\Gamma(A)$ of A by deleting all translates of projective or injective vertices (and the corresponding arrows). It is a stable valued translation quiver and the length function yields a subadditive function with values in $\mathbf{N}_1$. By definition, the *regular* components of the Auslander–Reiten quiver of an artin algebra are those components which do not contain projective or injective vertices. Thus, a regular component Γ is always a component of the stable Auslander–Reiten quiver, and the length function is an unbounded additive function on Γ with values in $\mathbf{N}_1$.

A stable valued translation quiver will be called *smooth* provided the valuation is trivial (i.e. $d(\alpha) = d'(\alpha) = 1$, for all arrows α), and any vertex is end point of precisely two arrows. Note that the last condition just means that the corresponding topological realization is a manifold without boundary.

Given any valued quiver $Q = (Q_0, Q_1, s, e, d, d')$, we may follow Riedtmann in order to define a stable valued translation quiver $\mathbf{Z}Q$: the vertex set of $\mathbf{Z}Q$ is given by $\mathbf{Z} \times Q_0$, for any arrow $\alpha : x \to y$ in Q, there are arrows $(z, \alpha) : (z, x) \to (z, y)$ and $\sigma(z, \alpha) : (z-1, y) \to (z, x)$ for any $z \in \mathbf{Z}$, the translation τ is defined by $\tau(z, x) = (z-1, x)$, and $d(z, \alpha) = d'\sigma(z, \alpha) = d(\alpha), d'(z, \alpha) = d\sigma(z, \alpha) = d'(\alpha)$. The structure of connected periodic stable translation quivers with subadditive functions with values in $\mathbf{N}_1$ is known: their universal covering is of the form $\mathbf{Z}Q$, with Q a valued quiver whose underlying graph is either a Dynkin diagram, a Euclidean diagram, or of the

form $A_\infty, B_\infty, C_\infty, D_\infty$, or A_∞^∞. There is the following general structure theorem for non-periodic stable translation quivers with subadditive functions:

Theorem (Zhang) *Let Γ be a connected non-periodic valued stable translation quiver with a non-zero subadditive function f with values in $\mathbb{N}_0$. Then either Γ is smooth and f is additive and bounded, or else $\Gamma = \mathbb{Z}Q$ for some valued quiver Q.*

The proof [Z2] is rather complicated. One has to consider the first homology group of the orbit graph of Γ and some additive function on it measuring the difference between the numbers of forward and of backward arrows in any walk. In order to write Γ in the form $\mathbb{Z}Q$, one has to find a suitable orientation on the orbit graph of Γ. Even in case the orbit graph is countable (and this is the case encountered in representation theory), one needs transfinite induction in order to construct such an orientation.

Corollary *Let Γ be a component of the stable Auslander–Reiten quiver $\Gamma_s(A)$ of an artin algebra A. Then either Γ is periodic, or else $\Gamma = \mathbb{Z}Q$ for some valued quiver Q without oriented cycles.*

On Γ, there is the length function f. Note that it is impossible that f is both additive and unbounded, since f is only additive in case Γ is a regular component, und then f is unbounded, according to Auslander. Thus, in case Γ is non-periodic, we see that $\Gamma = \mathbb{Z}Q$ for some valued quiver Q, and Q cannot have an oriented cycle, since this would yield sectional cyclic paths in $\Gamma(A)$, which is impossible according to Bautista–Smalø.

2.2. It seems to be of interest to know what kind of valued quivers Q actually can occur in $\Gamma = \mathbb{Z}Q$, where Γ is a component of $\Gamma_s(A)$ for some artin algebra A. Of course, Q cannot have oriented cycles. Also, Q has to be symmetrizable.

Let Q be a connected valued quiver without oriented cycles. If Q is a Dynkin or a Euclidean quiver, then any additive function on $\mathbb{Z}Q$ with values in $\mathbb{N}_1$ is bounded, thus $\mathbb{Z}Q$ cannot arise as a regular component of the Auslander–Reiten quiver of an artin algebra.

Consider now the case where Q is neither a Dynkin, nor a Euclidean diagram. In [R2], we have shown that a connected wild hereditary algebra H has a regular tilting module if and only if there are at least three simple H modules. Let H be a connected wild hereditary algebra with at least three simple modules, and let $Q(H)$ be its valued quiver. Let ${}_HT$ be a regular tilting module. Then the connecting component of $B = \mathrm{End}({}_HT)$ is regular and of the form $\mathbb{Z}Q(H)$.

Also, let Q be a connected symmetrizable valued quiver without oriented cycles and assume that after deletion of finitely many vertices and arrows we obtain a disjoint union of quivers of type A_∞ (with trivial valuation). Then there are algebras R with regular components of the form $\mathbb{Z}Q$, see [CR].

2.3. We have asked in [10] whether an Auslander–Reiten component Γ may contain infinitely many isomorphism classes of indecomposable modules of the same dimension. Zhang [Z1] has observed that this is impossible in case we deal with a hereditary algebra. Consider now an arbitrary algebra, and let Γ be a regular component. In case Γ is periodic, it is a regular tube, and therefore for any non-zero additive function f on Γ, the fibres are finite. Thus assume that Γ is non-periodic. By Zhang's theorem, $\Gamma = \mathbb{Z}\Delta$ for some valued quiver Δ. In case Δ is finite, or of the form $A_\infty^\infty, B_\infty, C_\infty$ or D_∞, we have shown in a joint paper with Marmolejo [MR] that

Γ can contain only finitely many isomorphism classes of indecomposable modules of the same dimension. In all other cases, the question seems to be open. Let us remark that it should be difficult to use only combinatorial properties of translation quivers to answer the question. For, let Δ be a finite connected symmetrizable valued quiver with no oriented cycle, but with at least one (non–oriented) cycle. Assume in addition that there are at least three vertices and that Δ is not of type $\tilde{A}_n$. Let Δ' be some infinite covering of Δ. Then, we claim that $\mathbf{Z}\Delta'$ has an additive function f' with values in $\mathbf{N}_1$, such that all fibres of f' are infinite. Namely, as we have mentioned above, there exists a tilted algebra with a regular component of the form $\mathbf{Z}\Delta$. Let f be the length function of this component. Denote by $\pi : \mathbf{Z}\Delta' \to \mathbf{Z}\Delta$ a covering map, and let $f' = f \circ \pi$.

2.4. Let us consider the special case of the group algebra kG of some finite group G over an algebraically closed field k of characteristic p. Let B be a block of kG, and $\delta(B)$ its defect group. According to Webb, any non–periodic component Γ of the stable Auslander–Reiten quiver $\Gamma_s(kG)$ is of the form $\mathbf{Z}\Delta$, where Δ is a Euclidean diagram, or else of the form A_∞, D_∞, or A_∞^∞.

First, let us consider the case when Δ is Euclidean (in particular, Γ is not a regular component). Then the defect group $\delta(B)$ has to be elementary abelian of order 4, and Γ is of the form $\mathbf{Z}\tilde{A}_{1,1}$, or $\mathbf{Z}\tilde{A}_{3,3}$ (see **[Ok]**, **[Bs]**, **[ES]**).

Of course, the defect group $\delta(B)$ is cyclic if and only if the block B is representation finite. Bondarenko and Drozd have shown that the block B is tame, and not representation finite, if and only if $p = 2$, and $\delta(B)$ is dihedral, semidihedral or quaternion. In this case, there is no component of the form $\mathbf{Z}A_\infty$ (if $\delta(B)$ is dihedral, there are also no components $\mathbf{Z}D_\infty$, and for $\delta(B)$ elementary abelian of order 4, or quaternion, all regular components are periodic), as Erdmann **[E1]** has shown.

One may use the knowledge on Auslander–Reiten components in order to obtain information about the block and its representations. This is the main philosophy of Erdmann's treatment of tame blocks **[E1]**. She determines the possible structure of all symmetric algebras with non–singular Cartan matrix which have prescribed components similar to those which are known to exist for tame blocks. In this way, she copies the approach of Riedtmann of classifying the representation finite selfinjective algebras. However, one should observe that there is an intrinsic difference: if A is a representation finite algebra, we know that $\mathrm{rad}^\infty(A\text{–mod}) = 0$, thus we can recover all maps in A–mod from the Auslander–Reiten quiver; in particular, this holds true for maps between indecomposable projective modules. Since the algebra is given by maps between the indecomposable projective modules, it does not seem to be surprising that we are able to recover the algebra. Actually, we know that non–standard representation finite algebras do exist only in characteristic 2. In the representation–infinite case, the situation is different: the maps between indecomposable projective modules usually will belong to $\mathrm{rad}^\infty(A\text{–mod})$, even if we deal with modules in one component, thus there is no way to recover these maps directly from the mesh category. Let us explain one detail of her advance: In order to recover A from $\Gamma(A)$, one needs to know in particular the quiver $Q(A)$. As mentioned above, we may consider the arrows in $Q(A)$ as being special maps $f : P(y) \to P(x)$ between indecomposable projective modules. We also may ask whether they define special meshes in the Auslander–Reiten quiver, and indeed, they do. Consider the cokernel M of f. The middle term of the Auslander–Reiten sequence ending in M is indecomposable **[BR]**, so certain of the meshes with a unique middle term will

correspond to the arrows of $Q(A)$. For example, for A any string algebra, all meshes which have a unique middle term, and which do not belong to homogeneous tubes, arise in this way.

For B a wild block, there are infinitely many components of the form $\mathbf{Z}A_\infty$, [E2]. But also there are always many tubes. In fact, an indecomposable non-projective B-module M is τ-periodic if and only if its complexity is equal to one, thus if and only if the subvariety $X_G(M)$ of the maximal spectrum of the even cohomology ring $H^{ev}(G,k)$ is one-dimensional, and Carleson has shown how to construct indecomposable modules with prescribed irreducible subvarieties, see the book by Benson [Be].

2.5. Directing modules are quite rare. Of course, all the indecomposable modules which belong to a preprojective, or a preinjective component are directing, and also the indecomposable modules in the connecting component of a tilted algebra are directing. Skowroński and Smalø[SS] have shown that an Auslander–Reiten component Γ of a finite dimensional algebra A which consists entirely of directing modules, can have only finitely many τ-orbits. In case Γ is in addition regular, then Γ is the connecting component of some convex subalgebra B of A which is a tilted algebra. It follows that an algebra can have only finitely many components which contain directing modules only.

Recall that a finite dimensional algebra A is said to be *representation directed,* provided it is representation finite, and we can order the indecomposable representations $M_1,\dots,M_n$ in such a way that we have $\mathrm{Hom}(M_i,M_j)=0$ for $i>j$, or, equivalently, provided the Auslander–Reiten quiver is finite and does not have oriented cycles. The position of sincere modules in the Auslander–Reiten quiver of a representation directed algebra has been studied further by de la Peña [P1].

3. Combinatorial Methods II: Quadratic Forms and Roots

Recall that a polynomial $\chi=\chi(X_1,\dots,X_n)$ of the form

$$\chi(X_1,\dots,X_n)=\sum_i X_i^2+\sum_{i<j}\chi_{ij}X_iX_j$$

with integer coefficients χ_{ij} is called an *integral quadratic form* in n variables. The quadratic forms which are of interest in the representation theory of finite dimensional algebras are integral, at least if we deal with an algebraically closed base field. Let χ be an integral quadratic form in n variables. An n-tuple $x=(x_1,\dots,x_n)$ of integers is called a *root* provided $\chi(x_1,\dots,x_n)=1$. Note that the canonical base vectors $e(i)$ with $e(i)_i=1$, and $e(i)_j=0$ for $j\neq i$, are roots. The quadratic form χ defines a symmetric bilinear form $\langle -,-\rangle$ on $\mathbf{Z}^n$, thus any root x defines a reflection σ_x by $\sigma_x(y)=y-\langle y,x\rangle\cdot x$. The group generated by the reflections $\sigma_{e(i)}$ with $1\leq i\leq n$ is called the Weyl group for χ, and the images of the base vectors $e(i)$ under the elements of the Weyl group are called *Weyl roots.* Of course, Weyl roots are roots. In case the coefficients χ_{ij} all are non-positive, so that the quadratic form χ is given by a generalized Cartan matrix, then a corresponding Kac–Moody Lie–algebra is defined, and therefore also the socalled *imaginary roots,* but we stress

that for an imaginary root x, we have $\chi(x) \leq 0$, thus imaginary roots are not roots in the sense defined above.

3.1. Let A be a finite dimensional hereditary algebra, say over an algebraically closed field k, thus A is the path algebra of a quiver $Q(A)$ without oriented cycles. The theorem of Kac asserts that the dimension vectors of the indecomposable A–modules are just the positive roots x of the corresponding quadratic form $\chi_{Q(A)}$. Note that we may distinguish between the real roots (with $\chi_{Q(A)}(x) = 1$) and the imaginary roots (with $\chi_{Q(A)}(x) \leq 0$). If x is a positive real root, there is precisely one indecomposable A–module with dimension vector x. If x is an imaginary root, there is an n–parameter family of indecomposable A–modules with dimension vector x, where $n = 1 - \chi_{Q(A)}(x)$. A positive root x is called a *Schur root* provided there exists a module M with endomorphism ring k and dimension vector x. Observe that x is a real Schur root if and only if there exists an indecomposable module M with dimension vector x such that $\operatorname{Ext}^1(M, M) = 0$.

There is the following inductive procedure for constructing indecomposable modules without selfextensions: Let M_1, M_2 be indecomposable A–modules without selfextensions, assume that

$$\operatorname{Hom}(M_1, M_2) = 0, \quad \operatorname{Hom}(M_2, M_1) = 0, \quad \text{and} \quad \operatorname{Ext}^1(M_2, M_1) = 0,$$

and let $\dim_k \operatorname{Ext}^1(M_1, M_2) = n$. Let (c, d) be a real root for the generalized Cartan matrix $\begin{bmatrix} 2 & -n \\ -n & 2 \end{bmatrix}$. Then there is a unique indecomposable module M with an exact sequence $0 \to dM_2 \to M \to cM_1 \to 0$, and M has no self extensions. The set of modules obtained in this way will be called the *line* determined by M_1, M_2.

Theorem (Schofield) *Let M be an indecomposable A–module M without selfextensions, and let s be the number of isomorphism classes of composition factors of M. Then M belongs to precisely $s - 1$ lines.*

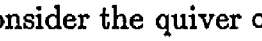
In particular, we see that we obtain all indecomposable modules without selfextensions starting from the simple modules and forming inductively lines.

The main tool for this investigation is the perpendicular category $M^\perp$. Given a set $\mathcal{S}$ of A–modules of projective dimension at most 1, the subcategory

$$\mathcal{S}^\perp = \{M | \quad \operatorname{Hom}(S, M) = 0, \quad \operatorname{Ext}^1(S, M) = 0, \quad \text{for all} \quad S \in \mathcal{S}\}$$

is called the *right perpendicular category* to $\mathcal{S}$, it is an abelian subcategory and the inclusion $\mathcal{S}^\perp \subset A$–mod is exact. This construction is very useful, it has been considered in detail by Geigle–Lenzing [**GL2**]. Note that in case we have in addition $\operatorname{Ext}^1(S, S) = 0$ (so that $\mathcal{S} \subseteq \operatorname{add} S$ for a basic partial tilting module S), the category $\mathcal{S}^\perp$ is equivalent to a module category B–mod, for some finite dimensional algebra B. Happel has pointed out that always $s(A) = s(B) + s(\operatorname{End} S)$. Indeed, take indecomposable A–modules $S_1, \ldots, S_s$ such that $S = \bigoplus S_i$, and let $E_1, \ldots, E_t$ be the Bongartz complement. Let E_i' be the trace of S in E_i. According to [**RS**], this is a proper submodule of E_i, and we let $Q_i = E_i/E_i'$. On the one hand, $\bigoplus Q_i$ is a progenerator for $\mathcal{S}^\perp$ (see [**H7**]), on the other hand, one can show easily that the modules Q_i are indecomposable and pairwise non–isomorphic.

There are several conjectures due to Kac [**Kc**] dealing with representations of quivers. As we have mentioned above, given a real root x, and k an algebraically closed field, there exists a unique indecomposable representation $M(x)$ with dimension

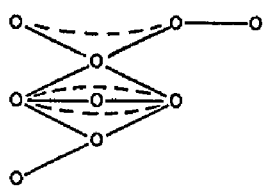
vector x. In case the characteristic of k is non–zero, Kac had shown that $M(x)$ is defined already over the prime field, and Schofield could show that the same is true in characteristic zero, thus solving Conjecture 4 of Kac. On the other hand, Conjecture 9 asserted that given a representation M with endomorphism ring k, there is no non–trivial way of writing the dimension vector of M in the form $y + z$, with $y, z \in \mathbf{N}_0^n$, and $\langle y, z\rangle \geq 0$, $\langle z, y\rangle \geq 0$, however, as Le Brujn [**Le**] has observed, there are obvious counter examples: consider the quiver  , there is a unique indecomposable representation M with dimension vector $(2, 1, 2)$, and its endomorphism ring is k. Let $y = (1, 1, 1)$, and $z = (1, 0, 1)$.

3.2 Let A be a finite dimensional algebra over an algebraically closed field, and assume its quiver has no oriented cycles. Then we may attach to A two integral quadratic forms, namely the Tits form and the Euler form. These forms often will determine the representation type of A, and we even may obtain complete information about the dimension vectors of the indecomposable A–modules in terms of these forms. We refer to the report [**P2**] of de la Peña which surveys the known results: he calls an algebra *good* provided its quiver has no oriented cycles, it satisfies the separation condition, it is Schurian (i.e. $\dim_k \mathrm{Hom}(P, P') \leq 1$, for P, P' indecomposable projective), and there is no full subalgebra which is hereditary of type $\tilde{A}_n$. He conjectures that the good algebras are controlled by the Tits form. Note that the Bongartz criterion asserts that a good algebra is representation finite if and only if it does not contain as a convex subalgebra an algebra from the Happel–Vossieck list.

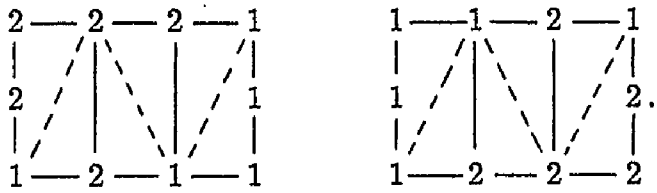
The integral quadratic form χ in n variables is said to be *weakly positive*, provided we have $\chi(x) > 0$ for all positive $x \in \mathbf{Z}^n$ (here, $x \in \mathbf{Z}^n$ is said to be positive, written $x > 0$, provided all coordinates of x are non–negative, and $x \neq 0$). An element $x \in \mathbf{Z}^n$ with all coordinates $x_i \neq 0$ will be said to be *sincere*.

If A is a representation directed algebra, then, as Bongartz has shown, the Tits form is weakly positive, and the dimension vectors of the indecomposable A–modules are just the positive roots. In order to study an indecomposable module, we may assume that its dimension vector is sincere. We recall that A is said to be *sincere* provided there exists a sincere indecomposable A–module.

Bongartz has exhibited a list of 24 series of sincere directed algebras which include all sincere directed algebras A with $s(A) \geq 14$. The list of the remaining sincere directed algebras has been published by Dräxler [**D2**], it should be of interest to obtain a better understanding of these exceptional algebras and their indecomposable modules, or, combinatorially, of the corresponding quadratic forms and their positive roots. For example, Dräxler has observed that any sincere directed algebra has a unique minimal sincere indecomposable module: the coefficients of the corresponding root x_A should be a measure for the zero relations needed to define the algebra (for example, A is given by a fully commutative quiver if and only if all the coefficients of x_A are equal to 1). On the other hand, there usually will be several maximal sincere positive roots, and it is an interesting question to determine the number of such roots in advance. For example, the form



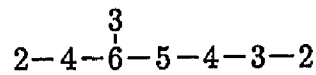
has precisely 10 maximal roots, this is the largest number which can occur. Note that Unger [U5] also has shown that there is only one sincere directed algebra A with indefinite quadratic form χ_A which has more than one maximal sincere positive root, namely



The list of all tame concealed algebras as presented by Happel and Vossieck is a list of integral quadratic forms. This list (as the one given by Dräxler) was produced with the help of a computer. A purely combinatorial approach to this list which abandons the use of a computer, has been given by von Höhne [Hö]. He classifies certain integral quadratic forms which are $\mathbf{Z}$–equivalent to to a quadratic form of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, and singles those out which appear as quadratic forms for tame concealed algebras.

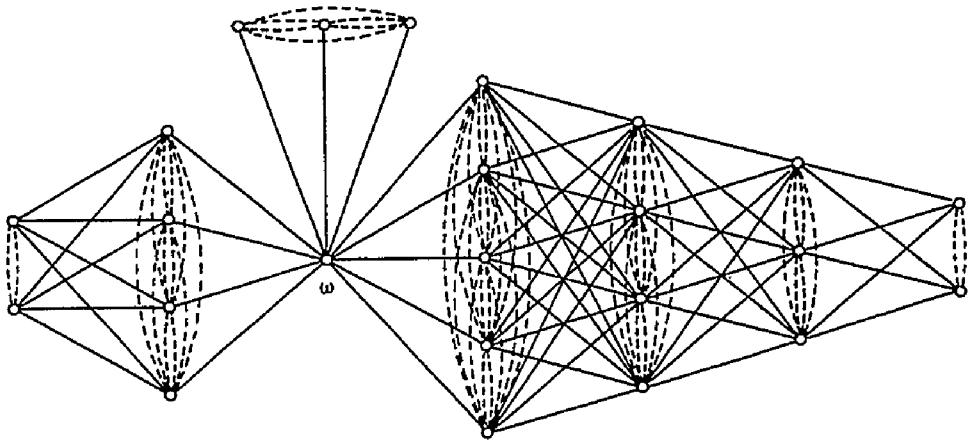
An integral quadratic form χ in n variables is said to be *critical* provided $\chi(x) > 0$ for every non–sincere vector $x \in \mathbf{N}_0^n$, but there is a vector $x \in \mathbf{N}_1^n$ with $\chi(x) \leq 0$. Similarly, χ is said to be *hypercritical* provided $\chi(x) \geq 0$ for every non–sincere vector $x \in \mathbf{N}_0^n$, and there is a vector $x \in \mathbf{N}_1^n$ with $\chi(x) < 0$. The forms which occur in the Happel–Vossieck list are typical critical forms. Unger [U2] has calculated the list of all minimal wild concealed algebras, clearly the corresponding quadratic forms are hypercritical; this list should be of interest for a further study of tame algebras (see [P2]). Note that for every wild concealed algebra A, say of type Δ, with $s(A) \geq 3$, there exists a representation infinite concealed factor algebra B obtained from A by factoring out the twosided ideal generated by some primitive idempotent, such that the type of B is a connected full subquiver of Δ, see [HU1].

3.3. A famous result of Ovsienko asserts that the coordinates of a positive root of a weakly positive integral quadratic form f are bounded by 6. Also, he has shown that if χ is weakly positive, and there exists a positive root for χ with at least one of the coordinates equal to 6, then χ is a form in at least 8 variables. Note that the quadratic form χ_8 for the Lie algebra E_8 has a positive root with coordinates



and is even positive definite.

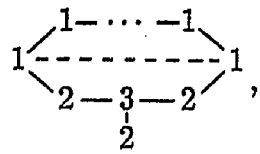
Ostermann and Pott [OP] have shown that a weakly positive quadratic form with a sincere positive root with at least one of the coordinates equal to 6 is a form in at most 24 variables. Also, there is a unique such form χ_{24} with 24 variables:



Any weakly positive quadratic form with a sincere positive root with at least one of the coordinates equal to 6 is a radical extension of the form χ_8 and has χ_{24} as a radical extension. This shows that there is a full control over all such forms.

This result can be applied to representation theory. Assume that A is a representation directed algebra with a sincere indecomposable module X with dimension vector x. Then x is a positive root for the quadratic form χ_A, and χ_A is weakly positive. Ovsienko's theorem asserts that all the coefficients of x, thus all Jordan Hölder multiplicities of X, are bounded by 6. In case one of these multiplicities is equal to 6, we see that χ_A is a radical extension of the quadratic form χ_8, in particular, χ is positive semidefinite. But this implies that χ is either of type E_8 or of type $\tilde{E}_8$, and therefore the number of variables is 8 or 9. Thus A is an algebra with 8 or 9 simple modules. (Actually, the case of 9 simple modules cannot occur).

Let us denote by $\mathcal{F}_n$ the set of weakly positive integral quadratic forms χ which have a sincere positive root with at least one component equal to n, and such that all components of positive roots of χ are bounded by n. Ovsienko's result means that $\mathcal{F}_n$ is empty for $n > 6$, and, as we have seen above, the forms in $\mathcal{F}_6$ are forms in at most 24 variables, and are positive semidefinite. For $n < 6$, It is easy to construct a form in $\mathcal{F}_n$ in $30 - n$ variables (by splitting the central vertex of the form χ_{24} into $7 - n$ vertices). For $n \leq 3$, there are forms in $\mathcal{F}_n$ with arbitrarily many variables: take the forms A_n, D_n, and the following ones (exhibited together with the maximal root)



all these forms are realized by finite dimensional algebras. For $n = 4$ and $n = 5$, we do not know whether $\mathcal{F}_n$ contains finitely or infinitely many forms; but according to Bongartz, only finitely many can be realized by algebras.

3.4. One may use the Hochschild cohomology in order to single out some combinatorial properties of algebras. We assume that the base field is algebraically closed. For a representation directed algebra A, Happel has shown that the Hochschild cohomology groups $H^i(A)$ are zero, for all $i \geq 2$, and, in addition, $H^1(A) = 0$, if and only if the quiver of A is a tree **[H5]**.

For further assertions concerning the interplay of the combinatorics of the quiver (and the relation) defining an algebra on the one hand, and the Hochschild cohomology on the other, we refer to **[H5]**; in **[H10]**,there are several results dealing with the

Auslander algebra of a representation directed algebra, thus with the combinatorial behaviour of the Auslander–Reiten quiver.

3.5. Let A be a connected finite dimensional hereditary algebra, and let C_A be its Cartan matrix. The Coxeter matrix $\Phi_A = -C_A^{-t}C_A$ is an important tool in representation theory, since for any indecomposable non–projective A–module M, with dimension vector $\mathbf{dim}\, M$, we have $\mathbf{dim}\, \tau M = \Phi_A(\mathbf{dim}\, M)$. Let $\rho = \rho_A$ be the spectral radius of Φ_A, it is called the *growth number* for A.

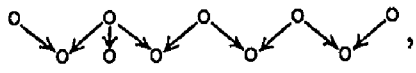
We note that in case A is representation infinite, then ρ is an eigenvalue of Φ_A. Of course, for A tame, we have $\rho = 1$, and we will assume that A is wild.

Let us assume in addition that $\mathrm{rad}^2\, A = 0$, and let (u_{xy}, v_{xy}) be the valuation on the arrow $x \to y$ in $Q(A)$. Then we obtain an $n \times m$ matrix $U = (u_{xy})$, and an $n \times m$ matrix $V = (v_{xy})$, where n is the number of sources, and m the number of sinks of $Q(A)$. In this case, it is very easy to calculate all the eigenvalues of Φ_A, since they are naturally related to the eigenvalues of non–negative matrix UV^t; in particular A'Campo and Subbotin–Stekolshchik have shown that any eigenvector of Φ_A is real or of absolute value 1. This has been used by de la Peña and Takane [**PTa**] and by Zhang [**Z3**] in order to show that there are vectors p, q with positive coefficients such that for any preprojective or regular module X and for any preinjective or regular module Y, we have real numbers $\alpha_p(X) > 0, \alpha_q(X) > 0$, with

$$\lim_{n\to\infty} \frac{1}{\rho^n}\mathbf{dim}\, \tau^{-n}X = \alpha_p(X)p, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\rho^n}\mathbf{dim}\, \tau^{n}Y = \alpha_q(X)q.$$

Also, one can use the vectors p, q in order to obtain a numerical criterion for an indecomposable module M for being preprojective, regular or preinjective: M is preprojective if and only if $\langle \mathbf{dim}\, M, p\rangle > 0$, and preinjective if and only if $\langle q, \mathbf{dim}\, M\rangle > 0$; also, if M is regular, then $\langle p, \mathbf{dim}\, M\rangle > 0$, and $\langle \mathbf{dim}\, M, q\rangle > 0$, see [**PT**]. Here, $\langle -, -\rangle$ is the usual (non–symmetric) bilinear form on $\mathbf{Z}^{s(A)}$ which encodes the homological behaviour of A–modules. It seems to us that all these assertions should be true also in the general case of $\mathrm{rad}^2\, A$ being arbitrary.

Xi [**X4**] has shown that the smallest possible growth number ρ_A for A wild, occurs for the path algebra of the quiver



it is the largest root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, thus approximately 1.176. Note that growth numbers can be used to deal with the structure of Auslander–Reiten components, see [**Z3**].

3.6. These investigations have also other applications. Chains of finite dimensional semisimple algebras (say defined over $\mathbf{Z}$ or $\mathbf{C}$)

$$A_0 \subseteq A_1 \subseteq A_2 \cdots,$$

the inductive limits, and their completions, have been considered in detail in the theory of C^*–algebras, for a survey see [**GHJ**].

In particular, starting with a semisimple algebra A_1 and a semisimple subalgebra A_0, containing the unit element, there is the socalled fundamental construction of Jones of taking as A_2 the endomorphism ring $\mathrm{End}_{A_0}(A_1)$, note that A_1 embeds into

A_2 as the set of right multiplications. We use induction in order to obtain the tower (A_i) corresponding to the pair A_0, A_1, namely let $A_{i+1} = \mathrm{End}_{A_{i-1}}(A_i)$.

Let $B = A_0, C = A_1$, and form the matrix ring $A = \begin{bmatrix} B & C \\ 0 & C \end{bmatrix}$. This is a hereditary algebra with $\mathrm{rad}^2 A = 0$. Note that every hereditary algebra A' with $\mathrm{rad}^2 A' = 0$ is Morita equivalent to one of the form $\begin{bmatrix} B & C \\ 0 & C \end{bmatrix}$. By definition, the Jones index of the tower is just the spectral radius of $U_A V_A^t$, and the Bratelli graph is the underlying quiver of the preprojective component of A. In this way, we see that one may relate questions concerning such towers of algebras to those arising in the representation theory of hereditary algebras. In particular, the values > 4 of the Jones index are related to the growth numbers of wild hereditary algebras: for example, the algebra exhibited by Xi yields a tower with Jones index approximately 4.026, this is the smallest possibe value > 4. See the joint paper [**DR8**] with Dlab.

4. Combinatorial Methods III: Representations of Posets

It was one of the first devices of the new representation theory of finite dimensional algebras, to reduce problems to the consideration of vectorspaces with prescribed subspaces. This is the method of the earlier papers of the Kiev school of Nazarova–Rojter, as well as of Gabriel's work on quivers of finite representation type; but at the same time, we also should mention Corner's and Brenner's investigations on wild behaviour. The school of Nazarova–Rojter showed the importance of the representation theory of posets, in particular, all their attempts to solve the second Brauer–Thrall conjecture use reductions from representations of algebras to representations of posets. The representation theory of posets disposes of rather striking results, there is the Klejner criterion for representation finiteness, Nazarova's criterion for tameness, the Nazarova–Zavadskij criterion for being of finite growth: always, there is given a small number of 'bad' posets which have to be excluded. There are also corresponding lists of the 'good' posets, say the Klejner list of sincere representation finite posets, or Zavadskij's list of the sincere posets of finite growth; they are longer, but still it is possible to overlook them. In contrast, similar results for finite-dimensional algebras cannot be expected, as already the algebras with two simple modules show: there are so many algebras that any list will tend to be useless.

Representations of a poset S were defined by Nazarova–Rojter in terms of partitioned matrices, and Gabriel has introduced the concept of an S–space; the corresponding categories of representations of S and of S–spaces are very similar, the precise relationship was determined by Drozd and Simson.

4.1. Let us assume that the base field k is algebraically closed, and assume we are dealing with a representation finite k–algebra A. The use of S–spaces can best be demonstrated when we add the assumption that A is representation directed. In addition, we may assume that A is basic. Let A–ind be a complete set of indecomposable A–modules, one from each isomorphism class. Let Q be the quiver of A. For any vertex $x \in Q$, let $E(x) \in A$–ind be the corresponding simple A–module, and let $P(x) \in A$-ind be its projective cover. Given an A–module M, and $x \in Q$, we denote by M_x the vectorspace at the vertex x, note that we can identify M_x and

$\mathrm{Hom}_A(P(x), M)$; also, we may identify M and $\bigoplus_{x\in Q_0} M_x$.

For any vertex x of Q, we will define a poset $S(x) = S_A(x)$, and for any A–module M, we will endow the k–vectorspace M_x with subspaces so that it becomes an $S(x)$–space $\tilde{M}_x$. Let A–ind_x be the set of A–modules $M \in A$–ind with $M_x \neq 0$. Then, the $S(x)$–spaces $\tilde{M}_x$, with $M \in A$–ind_x are indecomposable, pairwise non–isomorphic, and, up to isomorphism, any indecomposable $S(x)$–space occurs in this way. We may reformulate this as follows: Consider the direct sum

$$\bigoplus_{M\in A\text{–ind}} M = \bigoplus_{M\in A\text{–ind}} \bigoplus_{x\in Q_0} M_x$$

and identify for any $x \in Q_0$ the direct sum $\bigoplus_{M\in A\text{–ind}} M_x$ with the direct sum of the total spaces of all indecomposable $S(x)$–spaces (one from each isomorphism class).

Let us define $S(x)$, for $x \in Q_0$. Vertices of $S(x)$ are the modules $U \in A$–ind_x, which are different from $P(x)$, such that $(\tau U)_x = 0$. Given two modules U, U' in $S(x)$, let $U \leq U'$ it and only if there is a map $f : U' \to U$ such that $\mathrm{Hom}(P(x), f) \neq 0$. (According to v.Höhne **[Hö]**, it follows from $(\tau U)_x = 0$ that $\dim_k \mathrm{Hom}(P(x), U) \leq 1$, and therefore the relation $\leq$ defined on $S(x)$ is transitive.) Also, any A–module M, the vectorspace M_x may be endowed in a canonical way with subspaces indexed by the elements of $S(x)$: for $U \in S(x)$, take as subspace of $M_x = \mathrm{Hom}(P(x), M)$ the set of maps $P(x) \to M$, which factor through U.

For a proof, we refer to the joint paper **[RV]** with Vossieck, further information, as well as examples, may be found in **[R10]**. The notion behind the result mentioned above is that of a hammock, namely the Auslander–Reiten quiver of the category of $S(x)$–spaces, or, equivalently, the hammock of all modules in A–ind_x, thus of all indecomposable A–modules M with $M_x \neq 0$. Hammocks of this kind have been considered by Brenner **[Br]** in order to give a numerical characterization of finite Auslander–Reiten quivers (the intuition is the following: the modules in A–ind_x strech from $P(x)$ to $Q(x)$, the injective envelope of $E(x)$, these are the pegs to which the hammock is fastened. In between, there are the (Auslander–Reiten) meshes, in this way, the hammock is knitted.) In **[RV]**, we have given a combinatorial definition of *hammocks* as the finite translation quivers Γ with a unique source ω, and no oriented cycle, such that there is an additive function h on Γ (called the *hammock function*) with $h(\omega) = 1$, $h(p) = \sum_{y\to p} h(y)$, for all projective vertices different from ω, and with $h(q) \geq \sum_{q\to y} h(y)$, for all injective vertices. The hammocks turn out to be just the Auslander–Reiten quivers of the categories of S–spaces, where S is a representation finite poset. Given the hammock Γ, we can recover the corresponding poset as follows: its vertices are the projective vertices of Γ different from the unique source, and the partial ordering ist derived from the existence of maps in the mesh category of Γ.

It is not difficult to see that the full translation subquiver of Γ_A given by the vertices $[M]$ with $M \in A$–ind_x form a hammock, with hammock function $\dim_k \mathrm{Hom}(P(x), -)$, the hammock function just counts the Jordan–Hölder multiplicity of the simple module $E(x)$.

In case x is a source of Q, we deal with a one–point extension, say $A = B[M]$, for some algebra B and $M = \mathrm{rad}\, P_A(x)$. Then $S_A(x)$ is given by $\mathrm{Hom}_B(M, -)$. A similar description exists in case x is a sink for Q. The general case where x is an arbitrary vertex of Q can be reduced to the consideration of these two special cases: namely, as Scheuer **[Sr2]** has shown, $S(x)$ has a canonical decomposition into

an ideal $S(x)^-$ and a coideal $S(x)^+$. Let $S(x)^+$ be the subset of $S(x)$ given by all U with $\mathrm{Hom}(U, E(x)) \neq 0$. Then $S(x)^+$ is a coideal, and we denote by $S(x)^-$ its complement. Let A_x^+ be the restriction of A to the subquiver of Q obtained by deleting all proper predecessors of x, similarly, let A_x^- be the restriction of A to the subquiver of Q obtained by deleting all proper successors of x (thus, x is a source for A_x^+, and a sink for A_x^-.) Then $S(x)^+$ can be identified with $S_{A_x^+}(x)$, and $S(x)^-$ with $S_{A_x^-}(x)$.

4.2. It is rather difficult to decide whether a given algebra is representation finite or not. The usual procedure is to use covering theory, and then to invoke the Bongartz criterion. In this way, we have to deal with the Happel–Vossieck list of critical algebras. Of course, it would be easier if we only would have to invoke the Klejner list of critical posets. As we have seen above, the hammock approach allows to attach to each vertex x of a representation directed algebra the poset $S(x)$, but we note that the elements of $S(x)$ already may be rather complicated modules. Dräxler [**D3**] has proposed to consider a smaller poset, namely the full subposet $S'(x)$ of all thin modules, a module M being called *thin* provided $\dim_k M_y \leq 1$, for all vertices y. Actually, for any good algebra A, we may define $S'(x)$ as the set of isomorphism classes of thin indecomposable modules M which are not isomorphic to $P(x)$, such that $M_x \neq 0$, and $(\tau M)_x = 0$. This set becomes a poset by defining $[M] \leq [M']$ provided there is a map $f : M \to M'$ such such that $f_x : M_x \to M'_x$ is non–zero. Dräxler shows that a good algebra is representation finite if and only if all the posets $S'(x)$ are representation finite. Let us stress that this is a very effective criterion: clearly, the thin modules always are easy to write down, thus the sets $S'(x)$ can be computed without difficulty, and then we can use the Klejner list.

4.3. We have considered above representation directed, or, more generally, good algebras. For an arbitrary representation finite algebra A, we can define hammocks $S(x)$ for every vertex x of the quiver of A, by going to the universal covering $\tilde{A}$ of A, or also directly by considering the socalled radical layers in A–mod. If we work with A itself, and not with $\tilde{A}$, and consider the full translation subquiver $\Gamma(x)$ of the Auslander–Reiten quiver $\Gamma(A)$, given by all indecomposable modules M with $M_x \neq 0$, then this will be a hammock if and only if all indecomposable A–modules have k as endomorphism ring [**D4**].

There are other hammocks which appear rather naturally in representation theory, see Scheuer [**Sr1**]; also, we will see in section 6 that for quasi–hereditary algebras, we similarly obtain hammocks such that the hammock function counts the multiplicity of the standard module $\Delta(x)$ in a Δ–good filtration.

For general algebras, one has to replace posets by vectorspace categories. A detailed study of the vectorspace categories $\mathcal{S}(x)$ attached to the vertices of appropriate algebras has been carried out by Xi. His general observations may be found in [**X1**]. For a tame concealed algebra [**X2**], the vectorspace categories $\mathcal{S}(x)$ all contain precisely one critical subset, this subset is convex in $\mathcal{S}(x)$, and leads to a filtration of $\mathcal{S}(x)$ by ideals, which may be described in terms of the defect. Also we should note that one obtains in this way partial tilting modules. Xi also has studied the corresponding problem for tubular algebras [**X3**], and obtains (what one would expect) convex subsets of $\mathcal{S}(x)$ which are tubular, and which may be embedded as convex subsets into the pattern exhibited in [**R1**]. However, what seems to be astonishing, is that these subsets may be rather large, they usually contain several

critical subsets.

4.4. Let us add some remarks concerning the situation when the base field k is not algebraically closed. In this case, we have to take field extensions into account; in particular, we cannot expect that we may deal with posets. A generalization of the notion of a hammock to this setting has not yet been considered, but will be needed. The calculations of Hall polynomials mentioned in section 7 rely on one-point extensions of representation directed algebras over finite fields, so a substitute of the Kleiner list is needed. Fortunately, Klemp and Simson [**KSi**] have exhibited the full list of all sincere directed vectorspace categories of finite type which may be used. For a general report on the interplay between the representation theory of algebras and of vectorspace categories, we refer to Simson [**Si**].

5. Modules with Finite Projective Dimension

5.1. Let A be a finite dimensional k–algebra over some field k. It is not known whether a simple A–module E with finite projective dimension, has to satisfy $\operatorname{Ext}^1(E,E) = 0$. However, if k is algebraically closed and all simple A–modules have finite projective dimension (thus, if A has finite global dimension), then no simple A–module has self–extensions. In terms of quivers, it may be reformulated as follows: the quiver of a finite dimensional algebra of finite global dimension has no loops. Essentially, this is an old result of Lenzing [**L1**]. A recent paper by Igusa [**Ig**] reproves it in the context of algebraic K-theory. Let us remark that Lenzing actually has shown the following stronger theorem: Assume that $\operatorname{Ext}^1(E,E) \neq 0$ for some simple A–module E. Then there exists a module M with infinite projective dimension such that $M/\operatorname{rad} M = E$. The assertion stated in Lenzing's paper is the following: Assume that a is a nilpotent element in some noetherian ring R, such that all left ideals Ra^n have finite projective dimension. Then a belongs to the commutator subgroup $[R,R]$. In order to apply this result, let us assumme that A is a basic finite dimensional k–algebra over some algebraically closed field k. Consider an indecomposable length 2 module X with top and socle isomorphic to $E = A/\operatorname{rad} Ae$, for some primitive idempotent e. Then $[A,A]$ annihilates X. (Note that $1-e$ annihilates X, since A is assumed to be basic. Now, take elements $b,c \in A, x \in X$. Then $[b,c]x = [ebe,ece]x$, since $1-e$ annihilates X. But $[ebe,ece] \in [eAe,eAe] \subseteq \operatorname{rad}^2 eAe \subseteq \operatorname{rad}^2 A$, since for any local algebra B over an algebraically closed field, we have $[B,B] \subseteq \operatorname{rad}^2 B$. Of course, $\operatorname{rad}^2 A$ annihilates X.) On the other hand, there exists an element $a = eae \in \operatorname{rad} A$ with $aX \neq 0$. Since we have seen that a cannot belong to $[A,A]$, it follows that one of the modules Aa^n has infinite projective dimension, therefore also Ae/Aa^n.

Schofield [**Sf1**] has shown that that there is a function f such that the global dimension of the k–algebras of finite global dimension with k–dimension d is bounded by $f(d)$. In general, nothing is known about this bound $f(d)$, however, there are several results dealing with special classes of algebras. There are some classes of algebras A for which the global dimension may be bounded by the number $s(A)$ of simple modules. For example, the quasi–hereditary algebras considered in the next section all have global dimension at most $2s(A) - 2$. On the other hand, there are examples of algebras with $s(A) = 2$ and arbitrarily large global dimension. The first such examples have been exhibited by E. Green, they have been studied in detail

by Happel [H6]. Kirkman and Kuzmanovich [KK] have shown that there are even algebras A with $s(A) = 2$ and $\operatorname{rad}^4 A = 0$ of arbitrarily large finite global dimension. On the other hand, in case $\operatorname{rad}^3 A = 0$, the finitistic dimension is bounded by $s(A)^2$, see [Zi1].

5.2. Given a finite dimensional algebra A, we denote by $\operatorname{fd} A$ the *finitistic dimension* of A, by definition, this is the supremum of proj. dim. M, for all (finite dimensional) A–modules M of finite projective dimension. It has been conjectured a long time ago that the finitistic dimension $\operatorname{fd} A$ is always finite; this conjecture is usually contributed to Bass, to M. Auslander, or to Rosenberg and it has attracted a lot of interest in the last years. The conjecture has been verified for special classes of algebras: by Green, Kirkman and Kuzmanovich [GKK] and by Igusa and Zacharia [IZ] for monomial algebras, by Green and Zimmermann–Huisgen [GZ] for algebras A with $\operatorname{rad}^3 A = 0$.

Note that we also may consider the supremum of proj. dim. M, for arbitrarily, not necessarily finite dimensional A–modules M of finite projective dimension, it is called the *big finististic dimension* and denoted by $\operatorname{Fd} A$. Of course, $\operatorname{fd} A \leq \operatorname{Fd} A$, and a second conjecture asserts that we should have equality. For monomial algebras, Zimmermann–Huisgen [Zi2] has shown that $\operatorname{fd} A$ and $\operatorname{Fd} A$ may differ by at most one. Her proof rests on a structure theorem for modules which occur as second syzygy of some module, thus for kernels of maps between projective modules: they are isomorphic to direct sums of left ideals of A, each of which being generated by a path (note that a monomial algebra A is defined as the factor algebra of the path algebra of a quiver by an ideal I generated by some paths, thus the paths which do not belong to I yield a multiplicative basis of A).

5.3. Following Miyashita [My], an A–module M will be called a tilting module provided proj. dim. M is finite, $\operatorname{Ext}^i(M, M) = 0$ for all $i \geq 1$, and there is an exact sequence $0 \to {}_AA \to M_0 \to \cdots \to M_n \to 0$, where all modules M_i, with $1 \leq i \leq n$ belong to $\operatorname{add} M$. (Actually, Miyashita has considered arbitrary rings and modules, thus he had to add suitable finiteness conditions.) The case of tilting modules of projective dimension at most 1 had been discussed in detail before; these modules have turned out to be very useful.

We recall that Bongartz has shown that any module M with projective dimension at most 1, and $\operatorname{Ext}^1(M, M) = 0$, can be written as a direct summand of a tilting module of projective dimension at most 1. The corresponding assertion for tilting modules of arbitrary projective dimension is no longer true: Rickard and Schofield [RS] have exhibited a module M with finite projective dimension, and $\operatorname{Ext}^i(M, M) = 0$, for all $i \geq 1$, which cannot be written as direct summand of a tilting module: Let K be the Kronecker algebra, let X be an indecomposable A–module of length 2, form the one–point extension $C = K[X]$, let P be the indecomposable projective module corresponding to the extension vertex; now form the one–point coextension $B = [P]C$, and identify the sink and the source of B in order to obtain an algebra A with a node. The algebra A together with the simple module E corresponding to the node is the desired example: E has projective dimension 2, and $\operatorname{Ext}^i(E, E) = 0$, for all $i \geq 1$, and if $\operatorname{Ext}^2(X, E) = 0 = \operatorname{Ext}^2(E, X)$ for some A–module X, then X is in fact a regular K–module, and therefore $\operatorname{Ext}^1(X, X) \neq 0$ in case $X \neq 0$. This shows that E cannot be a direct summand of a tilting module.

Given a module M, let $s(M)$ denote the number of isomorphism classes of indecomposable direct summands of M. There still is the problem whether a module

M with finite projective dimension, with $\mathrm{Ext}^i(M,M) = 0$, for all $i \geq 1$, and with $s(M) = s(A)$ has to be a tilting module.

For example, assume that any indecomposable injective A–module has finite projective dimension, and consider the injective cogenerator $D({}_AA)$, where $D = \mathrm{Hom}_k(-,k)$ denotes the duality with respect to the base field k. Then $D({}_AA)$ has finite projective dimension, $\mathrm{Ext}^i(D({}_AA), D({}_AA)) = 0$, for all $i \geq 1$, and $s(D({}_AA)) = s(A)$. Note that in this case $D({}_AA)$ will be a tilting module if and only ${}_AA$ has also finite injective dimension, but already here, it is not known whether this always has to be so. Algebras A, for which $D({}_AA)$ has finite projective dimension, and ${}_AA$ has finite injective dimension, have been called *Gorenstein algebras*. They form a convenient generalization of algebras of finite global dimension, as well as of self–injective algebras. For an outline of their properties, we refer to the papers by Auslander–Reiten **[AR2]** and Happel **[H9]** in this volume.

Given an A–module M, we denote by $\mathrm{ann}(M)$ its annihilator in A. The modules M which are tilting $A/\mathrm{ann}(M)$–modules, have been considered by D'Este and Happel **[DH]**. They show that these modules yield the representable equivalences which are represented by faithful modules in the sense of Menini and Orsatti **[MO]**.

Let $\mathcal{E}$ be the set of isomorphism classes of basic modules M with projective dimension 1 and without self–extensions, thus the modules M in $\mathcal{E}$ are just the direct summands of tilting modules with projective dimension at most 1. In case we deal with a wild finite quiver without oriented cycles, the elements $\mathcal{E}$ just correspond to the real Schur roots.

Note that $\mathcal{E}$ may be considered as a simplicial complex; here, $[M_1]$ will be a face of $[M_2]$ if and only if M_1 is a direct summand of M_2. Thus, a module M in $\mathcal{E}$ will be a simplex of dimension $s(M) - 1$. We can rephrase the result of Bongartz by saying that all maximal simplices are of dimension $s - 1$, where $s = s(A)$, (this means that $\mathcal{E}$ is a pure simplicial complex).

Consider a module M in $\mathcal{E}$ which is an $(s-2)$–simplex, thus $s(M) = s(A) - 1$. Then there are at most two isomorphism classes of indecomposable modules M', such that $M \oplus M'$ is a tilting module, and there are two such classes if and only if M is faithful, see **[RS]** and **[H7]**. Also, in case M' and M'' are non–isomorphic indecomposable modules such that $M \oplus M'$ and $M \oplus M''$ are tilting modules, then we can assume that $\mathrm{Ext}^1(M'', M') \neq 0$, and then there exists a non–split exact sequence of the form $0 \to M' \to E \to M'' \to 0$, with $E \in \mathrm{add}\, M$. It follws that $\mathcal{E}$ is unramified, and its boundary is given by the non–faithful modules in $\mathcal{E}$. We see that $\mathcal{E}$ is nearly a pseudo–manifold with boundary. In case $\mathcal{E}$ is finite, Riedtmann and Schofield **[RS]** have shown that its geometric realization is just a ball.

In case $\mathcal{E}$ is infinite, Unger **[U6]** shows that $\mathcal{E}$ usually is not locally finite, it may be non–connected, and also inside a connected component, it may be impossible to connect two $(s-1)$–simplices by an alternating sequence of simplices of dimension $s-1$ and $s-2$, such that consecutive simplices are incident. She has considered in detail certain links of simplices. Recall that given a simplex σ of some simplicial complex Σ, its link $\mathrm{lk}(\sigma)$ is the subcomplex of Σ consisting of those simplices τ for which $\sigma \cup \tau$ is a simplex, and $\sigma \cap \tau$ is empty. Thus, given a module M in $\mathcal{E}$, the link $\mathrm{lk}(M)$ is the simplicial complex of all isomorphism classes of modules M' such that $M \oplus M'$ is a basic tilting module of projective dimension at most 1. Assume that M is an $(s-3)$–simplex in $\mathcal{E}$, thus $\mathrm{lk}(M)$ is a graph and any vertex has at most two neighbors. If M is faithful, then $\mathrm{lk}(M)$ is connected, thus it is of the form A_∞^∞ or

$\tilde{A}_r$. If M is not faithful, then $\mathrm{lk}(M)$ is either of the form A_r or the disjoint union of two graphs A_∞.

5.4. Let $\mathcal{X}$ be a subcategory of A–mod. Recall that $\mathcal{X}$ is said to be *resolving* provided it is closed under extensions, under kernels of surjective maps, and contains all projective modules. For example, the classes $\mathcal{P}^i$ of all modules M with $\mathrm{proj.dim.}\, M \leq i$, and the class $\mathcal{P}^\infty$ of all modules with finite projective dimension are resolving. Given an A–module M, a *right $\mathcal{X}$–approximation* of M is a map $g: X \to M$ with $X \in \mathcal{X}$ such that for any map $h: X' \to M$ with $X' \in \mathcal{X}$, there is a map $h': X' \to X$ such that $h = h'g$. In case every A–module has a right $\mathcal{X}$–approximation, $\mathcal{X}$ is said to be *contravariantly finite in* A–mod.

Subcategories of A–mod which are both resolving and contravariantly finite have been studied by Auslander and Reiten [**AR1**], on the basis of previous investigations of Auslander and Buchweitz [**AB**]. On the one hand, there are many important classes of modules which have these properties, on the other hand, the results of Auslander and Reiten give a clear picture of the behaviour of such subcategories.

Assume that $\mathcal{X}$ is a resolving and contravariantly finite subcategory. Let $X_i \to E_i$ be right $\mathcal{X}$–approximations of the simple A–modules $E_1, \ldots, E_n$. Then $\mathcal{X}$ can be recovered from knowing the modules X_i, namely $\mathcal{X} = \mathrm{add}\,\mathcal{F}(X_1, \ldots, X_n)$, where $\mathcal{F}(X_1, \ldots, X_n)$ denotes the class of modules which have a filtration with factors of the form X_j. This result has the following consequence: Assume $\mathcal{P}^\infty$ is contravariantly finite, for some algebra A, and let $X_i \to E_i$ be right $\mathcal{P}^\infty$–approximations. Then $\mathrm{fd}\, A$ is just the maximum of $\mathrm{proj.dim.}\, X_i$, in particular, $\mathrm{fd}\, A$ is finite. We should remark that there do exist examples of algebras such that $\mathcal{P}^\infty$ is not contravariantly finite [**IST**]: Take a one-point coextension $B = [R]K$, where K is the Kronecker algebra, and [R] is indecomposable of length 2, and identify the sink and the source in order to obtain an algebra A with a node. Then $\mathcal{P}^\infty = \mathcal{P}^1$, and these modules are obtained from preprojective and regular B–modules, therefore the simple A–module corresponding to the node cannot have a right $\mathcal{P}^\infty$–approximation.

If $\mathcal{Z}$ is a subcategory of A–mod, let ${}^\circ\mathcal{Z}$ be the full subcategory of all X satisfying $\mathrm{Ext}^i(X, \mathcal{Z}) = 0$, for all $i \geq 1$, and $\mathcal{Z}^\circ$ that of all Y satisfying $\mathrm{Ext}^i(\mathcal{Z}, Y) = 0$, for all $i \geq 1$. If $\mathcal{X}$ is a contravariantly finite and resolving subcategory of A–mod, then $\mathcal{X}^\circ$ has the dual properties: it is a covariantly finite and coresolving subcategory. In this way, we obtain a bijection between the contravariantly finite and resolving subcategories and the covariantly finite and coresolving subcategories.

Auslander and Reiten have related these concepts to tilting theory. If T is a tilting module, then T° is always a covariantly finite and coresolving subcategory. For simplicity, let us now assume that A is of finite global dimension. We obtain a bijection between the isomorphism classes of basic tilting modules, and the covariantly finite and coresolving subcategories, since we can recover the tilting module T from $\mathcal{X} = T^\circ$, due to the fact that $\mathcal{X} \cap \mathcal{X}^\circ = \mathrm{add}\, T$. Also, we obtain $\mathcal{Y} = \mathcal{X}^\circ$ from T as ${}^\circ T$. Alternatively, we can describe the modules in $\mathcal{X}$ as those modules which have a finite T–coresolution, and the modules in $\mathcal{Y}$ as those ones which have a finite T–resolution. Duality shows that the given tilting module T is also a cotilting module, thus for algebras of finite global dimension, the tilting modules and the cotilting modules coincide. An interesting example of the correspondence between tilting modules, contravariantly finite and resolving subcategories and the covariantly finite and coresolving subcategories will be exhibited when we deal with quasi–hereditary algebras.

5.5. Given a finite dimensional k–algebra A, we denote by $\widehat{A}$ the corresponding repetitive algebra, it is constructed as follows: Take copies A_i of A, indexed by the integers, and consider $Q_i = \mathrm{Hom}_k(A, k)$ as an A_{i-1}–A_i–bimodule. Then $\widehat{A}$ is the trivial extension of the ring $\bigoplus_{i\in\mathbf{Z}} A_i$ (with componentwise multiplication) by the bimodule $\bigoplus_{i\in\mathbf{Z}} Q_i$. Note that $\widehat{A}$ is an infinite dimensional algebra (without unit!), the indecomposable projective $\widehat{A}$–modules are just the indecomposable injective $\widehat{A}$–modules, and our main interest lies in the stable category $\widehat{A}$–$\underline{\mathrm{mod}}$ (its objects are the $\widehat{A}$–modules, and the morphisms are the residue classes of module maps modulo maps which factor through projective modules). Note that $\widehat{A}$–$\underline{\mathrm{mod}}$ is a triangulated category. This is a result which essentially is due to Heller, see **[H1]**, **[H4]**.

Theorem (Happel) *For any finite dimensional algebra A, there is an exact embedding (of triangulated categories)*

$$D^b(A\text{-}mod) \longrightarrow \widehat{A}\text{-}\underline{\mathrm{mod}}.$$

The categories $D^b(A\text{-}mod)$ and $\widehat{A}$–$\underline{\mathrm{mod}}$ are equivalent if and only if A has finite global dimension.

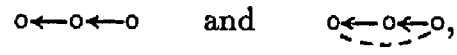
The main difficulty had been the definition of an appropriate embedding functor. Happel's original proof uses induction on the width of a complex. Keller and Vossieck **[KV]**, **[KI]** have proposed a different construction: they define such an embedding as the composition of four rather natural functors. This may be used for a better understanding, but it is still quite complicated. A very natural and straight forward definition has recently been given by Happel **[H8]**: The canonical embedding functor from A–mod to $\widehat{A}$–mod is exact, thus it extends to an exact functor from $D^b(A\text{–mod})$ to $D^b(\widehat{A}\text{–mod})$. Compose this functor with Rickard's functor considered below:

$$D^b(A\text{–mod}) \longrightarrow D^b(\widehat{A}\text{–mod}) \longrightarrow \widehat{A}\text{–}\underline{\mathrm{mod}},$$

in order to obtain the desired embedding. For A of finite global dimension, this embedding functor actually is an equivalence of triangulated categories. In order to decide whether the triangulated categories $D^b(\widehat{A}\text{–mod})$ and $\widehat{A}$–$\underline{\mathrm{mod}}$ may be equivalent also for other algebras, Tachikawa and Wakamatsu **[TW]** have considered Grothendieck groups. Always, the Grothendieck groups of A–mod (modulo exact sequences) and the Grothendieck group of $D^b(\widehat{A}\text{–mod})$ (modulo triangles) are isomorphic, and they show that the Grothendieck groups of A–mod and $\widehat{A}$–$\underline{\mathrm{mod}}$ (modulo triangles) are isomorphic if and only if the determinant of the Cartan matrix of A is equal to ± 1. Since there do exist algebras with infinite global dimension such that the determinant of C_A is ± 1, one needs other methods to decide the question: Happel **[H8]** has shown that for A of infinite global dimension, not all indecomposable objects in the category $D^b(A\text{–mod})$ do have sink or source maps, whereas in $\widehat{A}$–$\underline{\mathrm{mod}}$, all have, thus the categories $D^b(A\text{–mod})$ and $\widehat{A}$–$\underline{\mathrm{mod}}$ cannot be equivalent (even if we forget the triangular structure).

5.6. Algebras A, B will be said to be *tilting equivalent*, provided there is a finite sequence of algebras $A = A_0, A_1, \ldots, A_n = B$, such that for $1 \leq i \leq n$, one of the algebras A_{i-1} and A_i is the endomorphism ring of a tilting module over the other ring. And A and B are said to be *derived equivalent*, provided $D^b(A\text{–Mod})$ and $D^b(B\text{–Mod})$ are equivalent as triangulated categories. One knows that a tilting

module ${}_AT$ with endomorphism ring B yields an equivalence of the derived categories $D^b(A\text{-mod})$ and $D^b(B\text{-mod})$, and also between $D^b(A\text{-Mod})$ and $D^b(B\text{-Mod})$, thus tilting equivalent algebras are derived equivalent. The converse is not true: let A, B be tilting equivalent, such that the trivial extension algebras $T(A), T(B)$ are not Morita equivalent, for example, take the two algebras given as follows:



it is easy to see that the second algebra B is the endomorphism ring of a tilting A–module, and $\dim_k T(A) = 12$, $\dim_k T(B) = 10$, thus $T(A), T(B)$ cannot be isomorphic; they are basic, so they are not Morita equivalent. The only tilting modules for selfinjective algebras are the progenerators, thus they cannot be tilting equivalent. But Rickart [**Rc2**] has shown that with A, B also $T(A), T(B)$ are derived equivalent.

An algebra has been called *piecewise hereditary* and *piecewise tubular* provided it is derived equivalent to a hereditary of a tubular algebra, respectively. These algebras have the advantage that one has full control over the corresponding derived categories. Clearly, if A is hereditary, then the derived category $D^b(A\text{-mod})$ is obtained by taking the disjoint union of countable many copies of A–mod, and adding maps in one direction (using Ext^1), the derived category of any canonical algebra can be calculated by using Happel's theorem, see [**HR**], [**10**], or else using the Geigle–Lenzing equivalence $D^b(C(p)\text{-mod}) \cong D^b(\mathrm{coh}\,\mathbf{X}(p))$, and the fact that $\mathrm{coh}\,\mathbf{X}(p)$ has global dimension 1, so that again we just take countably many copies of $\mathrm{coh}\,\mathbf{X}(p)$ and add maps in one direction. Piecewise hereditary and piecewise tubular algebras are derived equivalent if and and if they are tilting equivalent, see [**HRS**], and [**AS3**]. The algebras which are derived equivalent to a tame hereditary or a tame canonical algebra can be characterized as follows [**AS3**]:

Theorem (Assem–Skowroński) *A finite dimensional algebra A is derived equivalent to a tame hereditary or a tubular algebra if and only if $\widehat{A}$ is locally support finite and cycle finite.*

Here, an infinite dimensional algebra B whose indecomposable projective modules are finite dimensional (like $\widehat{A}$) is said to be *locally support finite* provided for every indecomposable projective B–module P, there are only finitely many simple B–modules which can occur as composition factors of indecomposable B–modules M with $\mathrm{Hom}(P, M) \neq 0$. One may conjecture that $\widehat{A}$ can only be cycle finite, if it is also locally support finite. Assem and Skowroński have shown that in case $\widehat{A}$ is cycle finite, then A itself is either 'simply connected' or else tilting equivalent to a hereditary algebra of type $\tilde{A}_n$.

5.7. Rickard [**Rc1**] has determined precise conditions for derived categories to be equivalent: Let R be an arbitrary rings. Let R-proj be the category of finitely generated projective R–modules. If $\mathcal{R}$ is a full subcategory of R–Mod, let $K^b(\mathcal{R})$ denote the category of bounded complexes over $\mathcal{R}$ modulo homotopy.

Theorem (Rickard). *The rings R, R' are derived equivalent if and only of there is a bounded complex $T^\bullet$ of finitely generated projective left R–modules such that*

(i) *R' is the endomorphism ring of $T^\bullet$ in $K^b(R\text{-proj})$,*
(ii) *$\mathrm{Hom}_{K^b(R\text{-proj})}(T, T[i]) = 0$ for $i \neq 0$,*
(iii) *$\mathrm{add}(T^\bullet)$ generates $K^b(R\text{-proj})$ as a triangulated category.*

A bounded complex $T^\bullet$ with the properties listed in the Theorem may be called a *tilting complex*. König [**Kö**] has given a corresponding characterization for recollements of derived categories of module categories in terms of the existence of suitable partial tilting complexes.

5.8. Rickard [**Rc2**] has shown that two selfinjective algebras A, B which are derived equivalent, also are stable equivalent. Namely, we can obtain the stable category A–$\underline{\text{mod}}$ as a quotient category of the derived category $D^b(A\text{–mod})$ as follows: An object of $D^b(A\text{–mod})$ can be represented by a complex $P^\bullet = (P^i, \delta^i)$ of finitely generated projective modules bounded to the right, with bounded cohomology. Assume that $H^i(P^\bullet) = 0$ for all $i \leq n$, where $n \leq 0$. Let

$$\pi(P^\bullet) = \Sigma^n(\text{Cok}(P^{n-1} \to P^n)),$$

then this defines a functor $\pi : D^b(A\text{–mod}) \to A\text{–}\underline{\text{mod}}$, and this is the quotient functor for the embedding of $K^b(A\text{-proj})$ into $D^b(A\text{–mod})$.

Broué [**Br**] has exhibited examples of selfinjective algebras which are stable equivalent, but not derived equivalent. Alperin and Auslander–Reiten have conjectured that algebras which are stably equivalent, should have the same number of non–projective simple modules. This conjecture has been settled for representation finite algebras by Martinez [**Ma1**], but is open otherwise, even in the case where one of the algebras is local. Martinez [**Ma2**] has shown that it is sufficient to verify the conjecture for selfinjective algebras.

Let us add that algebras which are derived equivalent, always have the same Hochschild cohomology ring [**Ha5**], [**Rc3**].

5.9. In 1979, papers by Bernstein–Gelfand–Gelfand and Beilinson have exhibited algebraic descriptions of the derived category $\text{coh}\, D^b(\mathbf{P}_n)$ of the category of coherent sheaves on the projective complex n–space $\mathbf{P}_n$, namely they construct a finite dimensional algebra Λ_n of finite global dimension and equivalences of categories

$$D^b(\text{coh}\, \mathbf{P}_n) \cong D^b(\Lambda_n\text{–mod}) \quad \text{and} \quad D^b(\text{coh}\, \mathbf{P}_n) \cong \widehat{\Lambda}_n\text{–}\underline{\text{mod}}.$$

The precise relationship between the these equivalences and the one given by Happel has been determined by Dowbor and Meltzer [**DM**]. As an application, one may pass rather freely between the categories $D^b(\text{coh}\, \mathbf{P}_n)$, $D^b(\Lambda_n\text{–mod})$ and $\widehat{\Lambda}_n\text{–}\underline{\text{mod}}$. This may be of interest for explicit calculations of vector bundles over $\mathbf{P}_n$.

Similar descriptions of the categories of coherent sheaves over other algebraic varieties are by now available.

Let X be a nonsingular projective variety. Baer [**B3**] has introduced the concept of a *tilting sheaf* on X, this is a coherent sheaf without selfextensions that generates $C^b(\text{coh}\, X)$ as a triangulated category, and such that its endomorphism ring has finite global dimension. (She even has considered nonsingular 'weighted' projective varieties, in order to cover also the weighted projective lines $\mathbf{X}(p)$, as considered by Geigle and Lenzing.) A tilting sheaf $\mathcal{T}$ induces an equivalence $D^b(\text{coh}\, X) \to D^b(B\text{–mod})$, where B is the endomorphism ring of $\mathcal{T}$. Of course, B is a finite dimensional algebra, thus we see that the existence of a tilting sheaf means that questions concerning coherent sheaves over X may be reduced to questions dealing with modules over some finite dimensional algebra. Meltzer [**Me**] has exhibited tilting sheaves for $X = \mathbf{P}_1 \times \mathbf{P}_1$, and for X the flag variety $\mathbf{F}(1,2)$, and Kapranov [**Ka**] has considered the case of an arbitrary flag variety. Tilting sheaves and related objects have been

studied extensively by Russian mathematicians, we refer to a collection of papers and surveys presented in the Seminaire Rudakov **[Ru]**. Let us note that in all known cases, the quiver $Q(B)$ (and even $Q(\widehat{B})$) may be endowed with an integral value function s such that arrows $x \to y$ only exist in case $s(x) + 1 = s(y)$ (in particular, $Q(B)$ is directed), often these algebras B are 'quadratic', so that we can form $B^!$ and B and $B^!$ tend to be Ext–dual to each other.

6. Quasi–hereditary algebras

Quasi–hereditary algebras have been defined by Cline, Parshall, and Scott ([S], **[CPS2]**, **[PS]**) in order to deal with highest weight categories as they arise in the representation theory of semisimple complex Lie–algebras and algebraic groups. Many algebras which arise rather natural have been shown to be quasi–hereditary: the Schur algebras, the Auslander algebras, and it seems surprising that this class of algebras (which is defined purely in ring theoretical terms) has not been studied before by mathematicians devoted to ring theory. Even when Scott started to propagate quasi–hereditary algebras, it took him some while to find some ring theory resonance.

6.1. The definition of a quasi–hereditary algebra which we will give, follows a suggestion of Soergel **[Soe3]**. We have to start with an algebra A and an ordering of the simple A–modules, thus let $E(i)$, with $i \in \Lambda$, be the set of simple A–modules, where Λ is a (totally) ordered set. For $i \in \Lambda$, let $P(i)$ be the projective cover, and $Q(i)$ the injective envelope of $E(i)$. We denote by $\Delta(i)$ the maximal quotient of $P(i)$ with composition factors of the form $E(j)$, where $j \leq i$, and similarly, let $\nabla(i)$ be the maximal submodule of $Q(i)$ with composition factors of the form $E(j)$, where $j \leq i$. Our notation $\Delta(i), \nabla(i)$ should indicate the shape of the modules: by definition, $\Delta(i)$ has simple top, and $\nabla(i)$ has simple socle.

The algebra A is said to be *quasi–hereditary* with respect to the ordering Λ provided: (a) $\mathrm{End}(\Delta(i))$ is a division ring, for all $i \in \Lambda$, and (b) $\mathrm{Ext}^2(\Delta(i), \nabla(j)) = 0$ for all i, j.

For a quasi–hereditary algebra A, the modules $\Delta(i)$ are called the *standard* modules (in some special cases, they are also called Verma modules, or Weyl modules,) the modules $\nabla(i)$ the *costandard* modules. It turns out that for a quasi–hereditary algebra A, also the dual conditions for the costandard modules are satisfied, since clearly $\mathrm{End}(\Delta(i)) \cong \mathrm{End}(\nabla(i))$. In particular, we see that the opposite of a quasi–hereditary algebra is quasi–hereditary, again.

If $\mathcal{X}$ is a set of modules, we denote by $\mathcal{F}(\mathcal{X})$ the set of modules which have a filtration with factors in $\mathcal{X}$, these modules may be said to be *$\mathcal{X}$–good*. For any $M \in \mathcal{F}(\mathcal{X})$ and $X \in \mathcal{X}$, let $[M : X]$ denote the number of factors isomorphic to X in some $\mathcal{X}$–filtration of M, provided this is well–defined. We will be interested in the Δ–good and the ∇–good modules. Note that in case the condition (a) is satisfied, condition (b) is equivalent to the requirement that ${}_AA$ belongs to $\mathcal{F}(\Delta)$. In this way, we see that for a quasi–hereditary algebra A, the category A–mod together with the set Δ of standard modules becomes a 'highest weight category', with weight set Λ, as defined by Cline, Parshall, Scott, and conversely, any highest weight category with finite weight set is the module category of a quasi–hereditary algebra.

6.2. The usual (and equivalent) definition of quasi–hereditary algebras uses

heredity ideals. A *heredity ideal* of an algebra A is an idempotent ideal I, with $I(\operatorname{rad} A)I = 0$, and such that ${}_AI$ is a projective left module (or, equivalently, I_A is a projective right module). A chain of ideals

$$A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n = 0$$

is called a *heredity chain* provided I_{t-1}/I_t is a heredity ideal of R/I_t, for $1 \leq t \leq n$. A finite dimensional algebra is quasi–hereditary if and only if it has a heredity chain. (Note that a heredity chain may always be refined so that for every t, the indecomposable summands of a fixed module ${}_A(I_{t-1}/I_t)$ are all isomorphic, say isomorphic to some $\Delta(t)$, and then these modules are just the standard modules. Conversely, given an algebra A which satisfies the conditions (a) and (b), then for every t there exists a maximal left ideal I_t of A which belongs to $\mathcal{F}(\{\Delta(t+1), \ldots, \Delta(s(A))\}$, it clearly will be a twosided ideal, and, in this way, we obtain a heredity chain (I_t) for A. Note that the A/I_t–modules are just those A–modules which only have composition factors of the form $E(i)$, with $i \leq t$.

Given a heredity chain $(I_t)_t$ for A, the factor algebras A/I_t again will be quasi–hereditary. An algebra A such that all factor algebras are quasi–hereditary, is necessarily hereditary [**DR1**]. A quasi–hereditary algebra usually will have many ideals, and even heredity ideals, such that the corresponding factor algebras are not quasi–hereditary. Examples of the latter have been given by Agoston [**Ag**] and Wiedemann [**Wi**]. On the other hand, Xi [**X6**] has shown that an algebra A with $A/\operatorname{rad}^n A$ quasi–hereditary for some $n \geq 2$, is quasi–hereditary itself.

An ideal I_t which belongs to a heredity chain is always idempotent, thus we may choose an idempotent e which generates I as an ideal. Then the subalgebra eAe is quasi–hereditary [**DR1**]. On the other hand, for an arbitrary idempotent f in a quasi–hereditary algebra A, the subalgebra fAf may be far away from being quasi–hereditary. In fact, using a construction due to Auslander, a joint paper with Dlab [**DR3**] shows that given an arbitrary finite dimensional algebra B, there exists an idempotent f in some quasi–hereditary algebra A with $B = fAf$. We will see below that there are important classes of quasi–hereditary algebras which are endomorphism rings of faithful modules over selfinjective algebras.

Any ideal I which belongs to a heredity chain of an algebra A gives rise to a recollement: choose an idempotent e which generates I as a twosided ideal:

$$D^b(A/I\text{–mod}) \rightleftarrows D^b(A\text{–mod}) \rightleftarrows D^b(eAe\text{–mod}),$$

thus $D^b(A\text{–mod})$ is built up (via recollments) from the rather trivial derived categories $D^b(D_i\text{–mod})$, where D_i are division rings. This was one of the reasons for Cline, Parshall, Scott to introduce heredity chains, see [**PS**], [**CPS1**].

6.3. Any algebra with directed quiver $Q(A)$ is quasi–hereditary in two ways which are essentially different (provided A is not semisimple): we can take all simple modules as standard modules, then $\mathcal{F}(\Delta) = A\text{–mod}$, or else, we may take the indecomposable projective modules as standard modules, then $\mathcal{F}(\Delta)$ contains only the projective modules.

All algebras A of global dimension at most 2 are quasi–hereditary [**DR1**]: any idempotent ideal I of A with minimal possible Loewy length is a heredity ideal, and the global dimension of A/I again is at most 2. Recall that any representation finite algebra B gives rise to an algebra of global dimension at most 2, take the endomorphism ring of an additive generator of B–mod. For these Auslander–algebras, there

are several choices of standard modules which reflect properties of the B–modules, and are related to the preprojective or preinjective partitions introduced by Auslander and Smalø, or to the Rojter measure on B–mod, see **[DR2]**. On the other hand, Uematsu and Yamagata **[UY]** have exhibited examples of algebras of global dimension 3, which are not quasi–hereditary. It is easy to see that a serial algebra A is quasi–hereditary provided there exists just one heredity ideal, and this happens if and only if there is a simple module with projective dimension 0 or 2 **[UY]**.

There are some classes of quasi–hereditary algebras which may be considered more natural, and which Cline, Parshall, Scott had in mind when they introduced this concept: see the famous 'moose'–notes by Parshall and Scott **[PS]**. Let us direct the attention at least to some of these algebras.

We start with a semisimple finite dimensional complex Lie algebra $\mathfrak{g}$, with a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, and consider the corresponding category $\mathcal{O}$ as defined by Bernstein, Gelfand, Gelfand, it is the category of all finitely generated $\mathfrak{g}$–modules, which are locally $\mathfrak{b}$–finite, and semisimple as $\mathfrak{h}$–modules. The simple objects in $\mathcal{O}$ are indexed by the dual space $\mathfrak{h}^*$, and $\mathcal{O}$ is a highest weight categories with standard modules the Verma modules. Also note that $\mathcal{O}$ is the categorical sum of blocks which are abelian length categories with only finitely many simple modules, and all of them are equivalent to module categories for (finite dimensional) quasi–hereditary algebras. The most interesting case is the principal block $\mathcal{O}_0$ containing the trivial representation of $\mathfrak{g}$. The block $\mathcal{O}_0$ has a unique indecomposable module P which is projective and injective. Its endomorphism ring has been calculated by Soergel **[Soe1]**: Consider the Weyl group action on $\mathfrak{h}^*$, let I be the ideal in the ring R of regular functions on $\mathfrak{h}^*$ generated by the homogeneous invariants of degree at least one; it is known that the factor algebra R/I is a finite dimensional, local, selfinjective $\mathbf{C}$–algebra. Soergel shows that the center of the universal enveloping algebra $U(\mathfrak{g})$ maps surjectively on $\mathrm{End}(P)$, and in this way, there is a canonical identification of R/I with $\mathrm{End}(P)$. Also, he constructs certain R/I–modules $M(w)$, indexed by the elements w of the Weyl group, forms $A = \mathrm{End}(\bigoplus_w M(w))$, and obtains a categorical equivalence between the category $\mathcal{O}_0$ and A–mod. Since A is the endomorphism ring of a faithful module over a self-injective algebra, the dominant dimension of A is at least 2. Soergel shows that A is 'quadratic', so that we can form $A^!$, and A and $A^!$ are Ext–dual to each other (this was conjectured before by Beilinson and Ginsburg **[BG]**). These considerations are extended in **[Soe2]** to other blocks.

Another important class are the Schur algebras. Let us consider the classical case of the Schur algebras of $G = \mathbf{gl}_n(k)$, they are defined as follows: let V be the canonical n–dimensional $\mathbf{gl}_n$–module, and form the r–fold tensor power $V^{\otimes r}$. There is a diagonal action of G on $V^{\otimes r}$, thus an algebra homomorphism

$$T_r : kG \to \mathrm{End}(V^{\otimes r}),$$

where kG is the group algebra of G, and the Schur algebra $S(n,r)$ is just the image of this map. In **[Pa]**, Parshall has shown that the Schur algebras are quasi–hereditary, with standard modules the Weyl modules. Note that for any subgroup H of G, the image of kH under T_r will be a subalgebra of $S(n,r)$, denoted by $S(H)$. Let B^- and B^+ be the Borel subgroups of G of all lower, or upper triangular matrices, respectively. Then $S(B^-)S(B^+) = S(n,r)$, the algebras $S(B^-)$ and (B^+) are quasi-hereditary and isomorphic (since B^- and B^+ are conjugate), see J.A.Green **[Gr]**.

Note that there is the obvious permutation action of the symmetric group Σ_r on $V^{\otimes r}$ on the right; in this way, $V^{\otimes r}$ becomes an $S(n,r)$–Σ_r–bimodule, and it is known that $S(n,r)$ is the full endomorphism ring of $V^{\otimes r}$, considered as a right Σ_r–module. For $n \geq r$, the right Σ_r–module $V^{\otimes r}$ clearly is faithful, and since the group algebra $k\Sigma_r$ is selfinjective, we see that the dominant dimension of the Schur algebra $S(n,r)$ is at least 2.

We remark that Xi **[X6]** has exhibited the structure of the Schur algebras $S(p,p)$, for p a prime, using only the fact that $s(S(p,p)) = s(\Sigma_p) + 1$. In fact, he shows that there are only few quasi–hereditary algebras A with an idempotent e such that eAe is selfinjective and $s(A) = s(eAe) + 1$. Of course, in this way, he also gets a new proof for the structure of $k\Sigma_p$.

6.4. The quasi–hereditary algebras satisfy some rather restrictive conditions. Let A be quasi–hereditary.

First of all, as we have noted above, the global dimension of A is always finite **[PS]**, in fact it is bounded by $2s(A) - 2$, see **[DR1]**. For certain examples, like the Schur algebras, it had been considered a mystery that they are of finite global dimension (for Schur algebras, this had been established by Akin–Buchsbaum and Donkin), before it was realized that this is really a trivial consequence of being quasi–hereditary.

Second, the Loewy length of A is bounded by $2^{s(A)} - 1$. This is easy to see: Let I be a heredity ideal, then $\operatorname{rad} I$ is an A/I–module, thus the Loewy length $LL(A)$ of A is bounded by $1 + 2LL(A/I)$.

Also, we note that given a quasi–hereditary algebra A, we can bound its dimension if we know $A/\operatorname{rad}^2 A$, and the ordering Λ. A quasi–hereditary algebra A is said to be *shallow,* provided for any standard module $\Delta(i)$, its radical is semisimple. And, A is called *deep* provided $\operatorname{rad}\Delta(i)$ is projective when considered as an A/J_{i-1}–module. For any algebra B with $\operatorname{rad}^2 B = 0$, and without loops (this means, $\operatorname{Ext}^1(E,E) = 0$, for any simple module E), and any total ordering Λ of the simple modules, there exists a corresponding shallow algebra $S(B,\Lambda)$ and a deep algebra $D(B,\Lambda)$, such that for any quasi–hereditary algebra A with $B = A/\operatorname{rad}^2 A$, and this ordering Λ, we have **[DR5]**

$$\dim_k S(B,\Lambda) \leq \dim_k A \leq \dim_k D(B,\Lambda).$$

6.5. Important problems of representation theory center around the Jordan–Hölder multiplicities of the standard modules (for example, the Kazhdan–Lusztig conjectures are questions of this kind). For simplicity, let us assume that the base field is algebraically closed. There is the following reciprocity formula

$$[P(j) : \Delta(i)] = [\nabla(i) : E(j)]$$

(it is usually referred to as the Bernstein–Gelfand–Gelfand–reciprocity law), see **[CPS2]**. Recall that the Cartan matrix C of A is by definition the transpose of the matrix $\mathbf{dim}\, P$, thus we can reformulate the reciprocity formula as follows:

$$C^t = (\mathbf{dim}\, \Delta)^t \cdot (\mathbf{dim}\, \nabla).$$

In particular, we see that the determinant of the Cartan matrix is equal to 1. Of particular interest will be the case when there exists a duality $*$ on A–mod with

$E(i)^* = E(i)$, for all i. In this case, we have $\Delta^* \cong \nabla$, thus the reciprocity formula becoms $[P(j) : \Delta(i)] = [\Delta(i) : E(j)]$.

6.6. There is an inductive procedure for obtaining all quasi–hereditary algebras, the 'not–so–trivial–extension' method by Parshall and Scott [**PS**]. Let B be a ring, D a division ring. Recall that the one–point extension of B by a bimodule ${}_BM_D$ is the algebra $\begin{bmatrix} B & M \\ 0 & D \end{bmatrix}$. Similarly, the one–point coextension of B by the bimodule ${}_DN_B$ is the algebra $\begin{bmatrix} B & 0 \\ N & D \end{bmatrix}$. Now assume both bimodules ${}_BM_D, {}_DN_B$ are given, thus we can consider the tensor product ${}_BM \otimes_D N_B$. Let $\tilde{B}$ be a Hochschild extension of B by the bimodule ${}_BM \otimes_D N_B$, thus $\tilde{B}$ is a ring with an ideal J with $J^2 = 0$, and $\tilde{B}/J = B$, so that we may consider J as a B–B–bimodule, and we require in addition that this J is isomorphic to ${}_BM \otimes_D N_B$. The algebra we are looking for is

$$A = \begin{bmatrix} \tilde{B} & M \\ N & D \end{bmatrix}.$$

There is the ideal $I = \begin{bmatrix} J & M \\ N & D \end{bmatrix} \cong \begin{bmatrix} M \otimes N & M \\ N & D \end{bmatrix} \cong \begin{bmatrix} M \\ D \end{bmatrix} \otimes_D [N \;\; D]$, clearly, this is a heredity ideal of A, and $A/I = B$. On the other hand, given a minimal non–zero heredity ideal of a basic finite dimensional algebra A, with a heredity ideal I, choose an idempotent e which generates I as a twosided ideal; note that $D = eAe$ is a division ring, and let $B = A/I, M = \operatorname{rad} Ae, N = \operatorname{rad}(eA)_A$, then clearly we are in the situation depicted above.

Another inductive procedure has been introduced by Mirollo and Vilonen [MV] in the context of categories of perverse sheaves; it avoids the use of Hochschild extensions, and deals instead with tensor products and bimodule maps. Whereas the not–so–trivial–extension method relates A and A/I, where I is a heredity ideal, here we deal with an idempotent e such that the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is bijective, and such that there exists a division subring D which complements $(1-e)AeA(1-e)$ in $(1-e)A(1-e)$. We should remark that for any ideal AeA in a heredity chain, the corresponding multiplication map is bijective, and the second condition will be satisfied in case A/AeA is a division ring, and the base field is perfect. Thus, for algebras over a perfect base field, we obtain in this way an alternative way for constructing all quasi–hereditary algebras, see [**DR4**]. For some time, we had thought that tensor products and bimodule maps are easier to handle than Hochschild extensions, however actual calculations show that sometimes this is not the case. The Hochschild cohomology group $H^2(B, M \otimes N)$ which plays the decisive role in the not–so–trivial–extension approach is just $\operatorname{Ext}^2_B(\operatorname{Hom}_k(N,k), M)$, and its elements are often very easy to handle, see [**DR7**].

6.7. Let us consider the module category of a quasi–hereditary algebra in more detail, following [**R4**] and [**R6**]. The category $\mathcal{F}(\Delta)$ of Δ–good modules has (relative) almost split sequences, thus we can deal with the corresponding (relative) Auslander–Reiten quiver $\Gamma_{\mathcal{F}(\Delta)}$. The relative projective modules in $\mathcal{F}(\Delta)$ are just the projective modules, and for any index $i \in \Lambda$, there is precisely one indecomposable, relative injective module $T(i)$, with $\Delta(i)$ embedded into $T(i)$, and such that $T(i)/\Delta(i)$ has only composition factors of the form $E(j)$, with $j < i$.

There are no loops or sectional cycles in $\Gamma_{\mathcal{F}(\Delta)}$ (however, in contrast to a full

module category, the composition of maps along a sectional path may be zero). It follows that in case the indecomposable Δ–good modules are of bounded length, then there are only finitely many isomorphism classes of indecomposable Δ–good modules: so the analogue of the first Brauer–Thrall conjecture holds. Also, for the stable components for $\Gamma_{\mathcal{F}(\Delta)}$, there are the same restrictions as in the case of a full module category: periodic components will be of the form $\mathbf{Z}\Delta/G$, where Δ is either a Dynkin diagram or else of the form A_∞, and G is a non–trivial group of automorphisms of $\mathbf{Z}\Delta$, and non–periodic components are of the form $\mathbf{Z}\Delta$, where Δ is a connected valued quiver without cyclic paths.

Let us assume that $\mathcal{F}(\Delta)$ is finite, and that the base field is algebraically closed. In this case, we can define for any $i \in \Lambda$ a corresponding hammock (inside the universal covering of $\Gamma_{\mathcal{F}(\Delta)}$), such that the hammock function counts the multiplicities $[M : \Delta(i)]$, for Δ–good modules M. Note that in $\Gamma_{\mathcal{F}(\Delta)}$, we have to take into account multiple arrows: they actually may occur even under our assumption that $\mathcal{F}(\Delta)$ is finite, but fortunately only from an injective vertex to a projective vertex, thus never inside a mesh.

There are corresponding assertions for the category $\mathcal{F}(\nabla)$ of ∇–good modules. Note that we have

$$\mathcal{F}(\Delta)^\circ = \mathcal{F}(\nabla) \quad \text{and} \quad {}^\circ\mathcal{F}(\nabla) = \mathcal{F}(\Delta),$$

and $\mathcal{F}(\Delta)$ is a contravariantly finite and resolving subcategory, whereas $\mathcal{F}(\nabla)$ is a covariantly finite and coresolving subcategory, so we are in the situation studied by Auslander and Reiten, see section 5. The intersection $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is just $\operatorname{add} T$, where $T = \bigoplus T(i)$, thus this is a tilting module, and we can recover from T the structure of A as a quasi–hereditary algebra. This A–module T, and its direct summands seem to be of special interest. In the case of the category $\mathcal{O}$, these modules have been described by Collingwood and Irving [CI]: clearly, the $T(i)$ are just the indecomposable self–dual modules which have a Verma–filtration.

The endomorphism ring A' of T is in a canonical way again a quasi–hereditary algebra, the A'–modules $\Delta'(i) = \operatorname{Hom}(T, \nabla(i))$ being the standard modules, but we have to reverse the ordering of Λ. Then the category of ∇–good A–modules is equivalent (under the functor $\operatorname{Hom}(T, -)$) to the category of Δ'–good A'–modules. Thus, we see that there is no difference between categories which arise as categories of Δ–good or ∇–good modules. Also, the category of Δ–good A–modules is equivalent to the category of ∇'–good A'–modules, thus we see that the two subcategories are really exchanged in A–mod and in A'–mod. If we repeat these considerations, we will return to A in the next step: the endomorphism ring of an additive generator of $\mathcal{F}(\Delta') \cap \mathcal{F}(\nabla')$ is always Morita equivalent to A.

7. Combinatorial Methods IV: Hall Algebras

The free abelian group with basis indexed by the isomorphism classes of finite p–groups may be endowed with a product by counting filtrations of finite p–groups: we obtain what is called the Hall algebra $\mathcal{H}(\mathbf{Z}_p)$ of the ring $\mathbf{Z}_p$ of p–adic integers. It is a commutative and associative ring with identity element and plays an important role in algebra and combinatorics. It first was considered by Steinitz in 1900, and

later, in 1959, Ph. Hall started a detailled investigation. The basic concept may be generalized to fairly arbitrary rings. Under a mild finiteness condition on the ring R, one may define a similar product on the free abelian group with basis indexed by the set $\mathcal{B}$ of isomorphism classes of finite R–modules (where finite means to have finitely many elements, not just finite length), and one obtains an associative ring $\mathcal{H}(R)$ with identity element, the integral Hall algebra of R. In contrast to the case $R = \mathbf{Z}_p$, the Hall algebras in general do not need to be commutative, in fact the main concern are the corresponding Lie–algebras.

7.1. Let R be any ring, and $N_1, \cdots, N_t$, and M finite R–modules. Let $F^M_{N_1 \cdots N_t}$ be the number of filtrations

$$M = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_t = 0$$

of M such that $U_{i-1}/U_i \cong N_i$, for $1 \leq i \leq t$. (Note that in case $N_1, \ldots, N_t$ are in addition simple, we just count the number of composition series with prescribed composition factors).

We call a ring R *finitary,* provided for all finite R–modules M, M' also the extension group $\mathrm{Ext}^1(M, M')$ is finite. Note that all noetherian rings as well as all rings which are finitely generated (over $\mathbf{Z}$) are finitary. Assume that R is a finitary ring. Let $\mathcal{H}(R)$ be the free abelian group with basis $(u_{[M]})_{[M]}$, indexed by the set of isomorphism classes of finite R–modules. We define on $\mathcal{H}(R)$ a multiplication by the following rule

$$u_{[N_1]} u_{[N_2]} = \sum_{[M]} F^M_{N_1 N_2} u_{[M]};$$

note that on the right, we deal with a finite sum, since R is assumed to be finitary. Clearly, $\mathcal{H}(R)$ is an associative ring with 1, the identity element is $u_{[0]}$, and the associativity of this multiplication follows from the fact that the coefficient of $u_{[M]}$ in either $u_{[N_1]}(u_{[N_2]}u_{[N_3]})$ or $(u_{[N_1]}u_{[N_2]})u_{[N_3]}$ is just $F^M_{N_1 N_2 N_3}$. We call $\mathcal{H}(R)$ the *integral Hall algebra* of R. The special case of $R = \mathbf{Z}_p$ is the one considered by Steinitz and Ph. Hall. In contrast to $R = \mathbf{Z}_p$, the Hall algebras in general are not commutative. For example, let $R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, the ring of upper triangular 2×2 matrices over the finite field k. Then there are two non–isomorphic simple R–modules E_1, E_2, and a non–split exact sequence $0 \to E_2 \to P \to E_1 \to 0$, whereas $\mathrm{Ext}^1_R(E_2, E_1) = 0$. It follows that in $\mathcal{H}(R)$, we have

$$u_{[E_1]} u_{[E_2]} = u_{[E_1 \oplus E_2]} + u_{[P]},$$

but

$$u_{[E_2]} u_{[E_1]} = u_{[E_1 \oplus E_2]}.$$

In particular, we can write $u_{[P]}$ as a commutator:

$$u_{[P]} = [u_{[E_1]}, u_{[E_2]}].$$

7.2. Let us assume that A is a representation directed algebra, thus A is representation finite, and we may index the indecomposable A–modules in such a way that $\mathrm{Hom}(X_i, X_j) = 0$ for $i > j$. Also, we assume that A is connected. Then Guo [G2] has shown that the center of $\mathcal{H}(A)$ is trivial (i.e. equal to $\mathbf{Z}$) except in

case all indecomposable A–modules are of length at most 2, thus the center of $\mathcal{H}(A)$ is non–trivial only in case A is serial and $\operatorname{rad}^2 A = 0$. Also, Guo [**G1**] has shown that the Hall algebra $\mathcal{H}(A)$ determines A to a large extend. Namely, if also A' is representation directed, then the rings $\mathcal{H}(A)$, and $\mathcal{H}(A')$ are isomorphic if and only if there is a bijection $E_i \mapsto E_i'$ between the simple A–modules E_i and the simple A'–modules E_i' such that $|\operatorname{Ext}_A^t(E_i, E_j)| = |Ext_{A'}^t(E_i', E_j')|$ for $t = 0, 1, 2$, and all i, j.

7.3. Recall that in the classical case $\mathcal{H}(\mathbf{Z}_p)$, the multiplication coefficients are evaluations of polynomials at p, and these Hall polynomials play a decisive role. The same is true for the Hall algebras $\mathcal{H}(A)$, where A is a representation directed algebra. In the classical case, the Hall polynomials are indexed by a tripel of partitions, since the isomorphism classes of abelian p–groups correspond bijectivly to partitions. Consider now the case of an algebra A with a connected Auslander–Reiten quiver $\Gamma(A)$ (we have in mind the finite dimensional representation finite algebras, and suitable localizations of the path algebras of the cyclic quivers). Let $\mathcal{B}$ be the set of functions $b : \Gamma(A)_0 \to \mathbf{N}_0$ with finite support. The finite dimensional A–modules correspond bijectively to the elements of $\mathcal{B}$: for any vertex $i \in \Gamma(A)_0$, let $M(A, i)$ be a representative of the isomorphism class i, then $b \in \mathcal{B}$ will be attached to the isomorphism class of the module $M(A, b) = \bigoplus b(i)M(A, i)$. Note that the set $\mathcal{B}$ may have different interpretations for certain rings. For example, for A representation directed, we may identify $\Gamma(A)_0$ with the set of positive roots for a corresponding quadratic form, whereas in case we deal with the cyclic quiver with N vertices, we may identify $\mathcal{B}$ with the set of all N–tuples of partitions. In case A is representation directed [**R7**] as well as in case we deal with a cyclic quiver [**R11**], given $a, b, c \in \mathcal{B}$, there exists a monic polynomial $\phi_{ac}^b \in \mathbf{Z}[T]$ such that

$$F_{M(A,a)M(A,c)}^{M(A,b)} = \phi_{ac}^b(q_A)$$

for some fixed number q_A. These polynomials may be called *Hall polynomials,* they depend on the Auslander–Reiten quiver $\Gamma(A)$, but not on A itself.

Some Hall polynomials have been calculated explicitly in [**R8**]. Note that in the classical case, for M indecomposable, we always have $F_{N_1N_2}^M = 0$ or 1. In the case of a representation directed algebra, the situation is much more complicated: even if we assume that all three modules $M(A, a), M(A, b), M(A, c)$ are indecomposable, the polynomial ϕ_{ac}^b may have degree 5. Here is the list of all polynomials ϕ_{ac}^b different from 0 and 1 which occur for a representation directed algebra A which is given by a quiver with relations:

$$T - 2,$$
$$(T-2)^2,$$
$$(T-2)^3,$$
$$T^3 - 5T^2 + 10T - 7,$$
$$(T-2)(T^3 - 4T^2 + 8T - 6),$$
$$T^5 - 6T^4 + 15T^3 - 23T^2 + 25T - 13.$$

There are similar polynomials in case A is given by a species with relations. The polynomials have been calculated using suitable one–point extension algebras, and

counting rational points of certain algebraic varieties. It may be of interest to obtain an interpretation of the various coefficients. Let us remark that for the polynomials exhibited above we have $|\phi^b_{ac}(1)| = 1$, in the general case, we have $|\phi^b_{ac}(1)| \leq 3$.

7.4. In those cases where Hall polynomials do exist, we may introduce the *generic Hall algebra* $\mathcal{H}(A, \mathbf{Z}[T])$ and the *degenerate Hall algebra* $\mathcal{H}(A)_1$ as follows: In order to obtain $\mathcal{H}(A, \mathbf{Z}[T])$, take the free $\mathbf{Z}[T]$–module with basis $(u_b)_{b \in \mathcal{B}}$, and define the multiplication by

$$u_a u_c = \sum_{b \in \mathcal{B}} \phi^b_{ac} u_b.$$

In order to obtain $\mathcal{H}(A)_1$, take the free abelian group with basis $(u_b)_{b \in \mathcal{B}}$, and define the multiplication by

$$u_a u_c = \sum_{b \in \mathcal{B}} \phi^b_{ac}(1) u_b.$$

Of course, both are associative rings with 1.

There is the following interesting property of the numbers $F^M_{N_1 N_2}$, for A–modules N_1, N_2, M, where A is a k–algebra over a finite field k. If N_1, N_2 are indecomposable, and M is decomposable, then $|k| - 1$ divides $F^M_{N_1 N_2} - F^M_{N_2 N_1}$. This has the following consequence: We may consider $\Gamma(A)_0$ as a subset of $\mathcal{B}$, by identifying an element of $\Gamma(A)_0$ with its characteristic function. Then $T - 1$ divides $\phi^b_{ac}(1) - \phi^b_{ca}(1)$ in case a, c belong to $\Gamma(A)_0$, and b not. As a consequence, the subgroup of $\mathcal{H}(A)_1$ with basis $(u_b)_{b \in \Gamma(A)_0}$ is a Lie subring of $\mathcal{H}(A)_1$. By definition, this subgroup is the free abelian group on the set of isomorphism classes of indecomposable A–modules, thus we may consider it as the Grothendieck group $K(A\text{–mod})$ of all finite A–modules modulo split exact sequences. In particular, we see that $K(A\text{–mod})$ is, in a natural way, a Lie ring.

Theorem *Let A be a hereditary algebra of Dynkin type Δ. Let $\mathfrak{g}$ be the semi-simple complex Lie algebra of type Δ, with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Then $K(A\text{–mod})$ is a Chevalley $\mathbf{Z}$–form of $\mathfrak{n}_+$, and $\mathcal{H}(A)_1$ may be identified with the corresponding Kostant $\mathbf{Z}$–form of the universal enveloping algebra $U(\mathfrak{n}_+)$.*

Indeed, the elements u_b with $b \in \Gamma(A)_0$ themselves form a Chevalley $\mathbf{Z}$–basis of $\mathfrak{n}_+$. This is a consequence of the explicit determination of certain Hall polynomials. In particular, it is necessary to know the corresponding values $|\phi^b_{ac}(1)|$, as mentioned above.

Let us add that the rather technical form of the basis elements of the Kostant $\mathbf{Z}$–form of $U(\mathfrak{n}_+)$ gets a very natural interpretation in terms of the Hall algebra. Since A is representation directed, we may assume that we have indexed the indecomposable A–modules $M(1), M(2), \ldots, M(m)$, such that $\mathrm{Hom}(M(i), M(j)) = 0$ for $i > j$. Let $b : \{1, 2, \ldots, m\} \to \mathbf{N}_0$ be an element of $\mathcal{B}$, and consider $u^{b(1)}_{M(1)} u^{b(2)}_{M(2)} \cdots u^{b(m)}_{M(m)}$ in $\mathcal{H}(A)$. We want to write it as a linear combination in our basis $(u_{[M]})_{[M]}$, and we see that the only non–zero coefficient can occur for $M = \bigoplus b(i)M(i) = M(b)$, since any filtration of the type considered has to split. Also, the only remaining coefficient can be calculated without difficulty; in $\mathcal{H}(A)_1$, it is the evaluation of a polynomial ϕ, with $\phi(1) = \prod b(i)!$. It follows that

$$u_1^{b(1)} u_2^{b(2)} \cdots u_m^{b(m)} = \prod b(i)! \cdot u_b,$$

and therefore

$$\frac{u_1^{b(1)}}{b(1)!}\frac{u_2^{b(2)}}{b(2)!}\cdots\frac{u_m^{b(m)}}{b(m)!} = u_b.$$

The theorem brings to an end investigations started by Gabriel: In 1972, he showed that for a hereditary algebra A of type A_n, D_n, E_6, E_7, E_8, the indecomposable modules correspond bijectively to the positive roots of the corresponding simple complex Lie algebra. This result was extended to all hereditary algebras of Dynkin type in joint work with Dlab. Thus, it was known for a long time that we may identify $K(A\text{–mod}) \otimes_{\mathbf{Z}} \mathbf{C}$ with $\mathfrak{n}_+$ as $\mathbf{C}$–spaces, and there was the problem whether it is possible to recover the Lie multiplication of $\mathfrak{n}_+$ in terms of the representation theory of A. We see that the Hall algebras provide a possibility to do so.

7.5. One may ask whether it is possible to use representations of algebras to recover the whole Lie algebra $\mathfrak{g}$ and not only $\mathfrak{n}_+$. It is easy to define an extended Hall algebra $\mathcal{H}'(A)$ in order to obtain $U(\mathfrak{b})$ as the corresponding degenerate extended Hall algebra: We form the twisted polynomial ring over $\mathcal{H}(A)$ by adjoining variables $X_1, \ldots, X_{s(A)}$, such that $[X_i, u_{[M]}] = (\dim M)_i u_{[M]}$. Schofield [**Sc4**] has proposed another way by dealing with the varieties of all composition series of modules with fixed dimension, and he is able to construct in this way the complete Lie algebra $\mathfrak{g}$.

7.6. The Hall algebra approach presented above yields, in fact, a q–analogue of the enveloping algebra of $U(\mathfrak{n}_+)$, and it turns out that this is really the quantization of the universal enveloping algebra as defined by Jimbo and Drinfeld [**Dr**]. For details, we refer to our survey [**R10**]. One may consider in the same way any hereditary algebra A say of type Δ, and one obtains a canonical ring homomorphism from the quantization of the universal enveloping algebra of the Kac–Moody Lie algebra of type Δ, or better from a $\mathbf{Z}$–form, into the Hall algebra of A. The image will be the subalgebra generated by the simple modules, or the semisimple modules, we call this the composition algebra, or the Loewy algebra of A. At least in the case of a Euclidean diagram, we can show that we obtain in this way a realization of the corresponding quantum groups. Let us add that Lusztig [**Lu**] has used this approach in order to define canonical bases for the universal enveloping algebras of all finite dimensional semisimple Lie algebras over $\mathbf{C}$.

If we want to handle the Euclidean quivers, the main difficulties arise already in the special case of an oriented cycle. Of course, the corresponding path algebra is not a finite dimensional algebra, but even if we only are interested in the remaining cases (which give finite dimensional algebras), we have to consider the modules which belong to non–homogeneous tubes, and this is just the case of dealing with the (locally nilpotent) representations of an oriented cycle. The corresponding composition algebra has been exhibited in detail in [**R11**], and we think that the combinatorial methods introduced should be of interest elsewhere. Let us remark that Guo [**G3**] has considered the structure of the complete Hall algebra of an oriented cycle.

It seems surprising that the parameter q used for the quantization of the universal enveloping algebra $U(\mathfrak{b})$ has an interpretation as a variable which stands for the cardinality of a finite field, but this is what we encounter when identifying the quantization with a Hall or Loewy algebra.

REFERENCES

[Ag] Agoston, I.: Quotients of quasi–hereditary algebras. C. R. Math. Rep. Acad. Sci. Canada 11 (1989), 99-103.

[AS1] Assem, I.; Skowronski, A.: Iterated tilted algebras of type $\tilde{A}_n$, Math. Z. 195 (1987), 269-290

[AS2] Assem, I.; Skowronski, A.: On some classes of simply connected algebras. Proc. London Math. Soc. 56 (1988), 417-450.

[AS3] Assem, I.; Skowronski, A.: Algebras with cycle finite derived categories. Math. Ann. 280 (1988), 441-463.

[AS4] Assem, I.; Skowronski, A.: Algèbres pré-inclinées et catégories dérivées. In: Sém. d'Algèbre Dubreil–Malliavin. Springer LNM. 1404 (1989), 1-34.

[AS5] Assem, I.; Skowronski, A.: Minimal representation–infinite coil algebras. Manuscr. Math. 67 (1990), 305-331.

[AB] Auslander, M.; Buchweitz, R.–O.: The homology theory of maximal Cohen–Macaulay approximations. Mém. Soc. Math. France N.S. (1989).

[APT] Auslander, M.; Platzeck, M. I.; Todorov, G.: Homological theory of idempotent ideals. (To appear).

[AR1] Auslander, M.; Reiten, I.: Applications of contravariantly finite subcategories. Preprint 8/1989 Univ. Trondheim.

[AR2] Auslander, M.; Reiten, I.: Cohen–Macaulay and Gorenstein Artin algebras. This volume.

[BS] Bakke, Ø.; Smalø, S.O.: Modules with the same socles and tops as a directing module are isomorphic. Comm. Algebra 15 (1987), 1-10.

[Bau] Bautista, R.: On algebras of strongly unbounded representation type. Comment. Math. Helv. 60 (1985), 392-399.

[BK] Bautista, R.; Kleiner, M.: Almost split sequences for relatively projective modules. J.Algebra 135 (1990), 19-56

[B1] Baer, D.: Homological properties of wild hereditary Artin algebras. In: Representation Theory I. Springer LNM 1177 (1986), 1-12

[B2] Baer, D.: Wild hereditary Artin algebras and linear methods. Manuscr. Math. 55 (1986), 69-82.

[B3] Baer, D.: Tilting sheaves in representation theory of algebras. Manuscr. Math. 60 (1988), 323-347.

[B4] Baer, D.: A note on wild quiver algebras and tilting modules. Comm. Algebra 17 (1989), 751-757

[BGL] Baer, D.; Geigle, W.; Lenzing,H.: The preprojective algebra of a tame hereditary Artin algebra. Comm. Algebra 15 (1987), 425-457.

[BG] Beilinson, A.; Ginsburg, V.: Mixed categories, Ext-duality abd representations (Results and conjectures). (To appear).

[Be] Benson, D.: *Modular Representation Theory: New Trends and Methods.* Springer LNM 1081 (1984).

[Bs] Bessenrodt–Timmerscheidt, Ch.: The Auslander–Reiten quiver of a modular group algebra revisited. Preprint SM-DU-159 Duisburg (1989).

[Bo] Bongartz, K.: Indecomposables are standard. Comment. Math. Helv. 60 (1985), 400-410.

[BS] Bongartz, K.; Smalø, S.O.: Modules determined by their tops and socles. Proc. Amer. Math. Soc. 96 (1986), 34-38.

[Br] Brenner. Sh.: A combinatorial characterization of finite Auslander–Reiten quivers. In: Representation Theory I. Springer LNM 1177 (1986), 13-49.

[BT] Bretscher, O; Todorov, G.: On a theorem of Nazarova and Roiter. In: Representation Theory I. Springer LNM 1177 (1986), 50-54.

[Bé] Broué, M.: News about perfect isometries. (Unpublished).

[BB] Burt, W.L.; Butler, M.C.R.: Almost split sequences for bocses. (To appear).

[BR] Butler, M.C.R.; Ringel, C.M.: Auslander–Reiten sequences with few middle terms,with applications to string algebras. Comm.Alg.15 (1987),145-179.

[CPS1] Cline, E.; Parshall, B.; Scott, L.: Algebraic statification in representation categories. J. Algebra. 117 (1988), 504-521

[CPS2] Cline, E.; Parshall, B.; Scott, L.: Finite dimensional algebras and highest weight categories. J. reine angew. Math. 391 (1988), 85-99

[CPS3] Cline, E.; Parshall, B.; Scott, L.: Duality in highest weight categories. In: Classical Groups and Related Topics. Contemp. Math. 82 (1989), 7-22.

[CI] Collingwood, D.H.; Irving, R.S.: A decomposition theorem for certain self–dual modules in the categorie $\mathcal{O}$. Duke Math. J. 58 (1989), 89–102.

[C1] Crawley–Boevey, W.: On tame algebras and bocses. Proc. London Math. Soc.(3), 56 (1988),451-483.

[C2] Crawley–Boevey, W.: Functorial filtrations and the problem of an idempotent and a square–zero matrix. J. London Math. Soc. (2) 38 (1988), 385-402.

[C3] Crawley–Boevey, W.: Functorial filtrations II: Clans and the Gelfand problem. J. London Math. Soc. (2) 40 (1989), 9-30.

[C4] Crawley–Boevey, W.: Functorial filtrations III: Semi–dihedral algebras. J. London Math. Soc. (2) 40 (1989), 31-39.

[C5] Crawley–Boevey, W.: Maps between representations of zero–relation algebras. J. Algebra 126 (1989), 259-263

[C6] Crawley–Boevey, W.: Tame hereditary algebras, hereditary orders and their curves. (To appear).

[C7] Crawley–Boevey, W.: Matrix reduction for artinian rings, and an application to rings of finite representation type. Preprint 90-020 SFB 343 Bielefeld.

[C8] Crawley–Boevey, W.: Tame algebras and generic modules. Preprint 89-028 SFB 343 Bielefeld.

[C9] Crawley–Boevey, W.: Lectures at the Tsukuba Workshop 1990. (To appear).

[CR] Crawley–Boevey, W.; Ringel, C.M.: Algebras whose Auslander–Reiten components have large regular components. Preprint 90-052 SFB 343 Bielefeld.

[CU] Crawley–Boevey, W.; Unger, L.: Dimensions of Auslander–Reiten translates for representation–finite algebras. Comm. Algebra 17 (1989), 837-842.

[DH] D'Este, G.; Happel, D.: Representable equivalences are represented by tilting modules. Rend. Sem. Mat. Univ. Padova 82 (1990), 77-80.

[Di] Dieterich, E.: *Solution of a non–domestic tame classification problem from intergal representation theory of finite groups* ($\Lambda = RC_3, v(3) = 4$.) Memoirs Amer.Math.Soc. (To appear).

[DR1] Dlab, V.; Ringel, C.M.: Quasi–hereditary algebras. Illinois J. Math. 33 (1989), 280-291

[DR2] Dlab, V.; Ringel, C.M.: Auslander algebras as quasi–hereditary algebras. J. London Math. Soc. (2) 39 (1989), 457-466.

[DR3] Dlab, V.; Ringel, C.M.: Every semiprimary ring is the endomorphism ring of a projective module over a quasi–hereditary ring. Proc. Amer.Math.Soc. 107 (1989), 1-5.

[DR4] Dlab, V.; Ringel, C.M.: A construction for quasi–hereditary algebras. Compositio Math. 70 (1989), 155-175.

[DR5] Dlab, V.; Ringel, C.M.: The dimension of a quasi–hereditary algebra. In: Topics in Algebra. Banach Center Publ. 26. (To appear).

[DR6] Dlab, V.; Ringel, C.M.Filtrations of right ideals related to projectivity of left ideals. In: Sém. d'Algèbre Dubreil–Malliavin. Springer LNM. 1404 (1989), 95–107.

[DR7] Dlab, V.; Ringel, C.M.: The Hochschild cocycle corresponding to a long exact sequence. Tsukuba J. Math (To appear).

[DR8] Dlab, V.; Ringel, C.M.: Towers of semi–simple algebras. J. Funct. Anal. (To appear)

[DGL] Dowbor, P.; Geigle, W.; Lenzing, H.: *Graded Sheaf Theory and Group Quotients, with Applications to Representations of Finite Dimensional Algebras.* (To appear).

[DM] Dowbor, P.; Meltzer, H.: On equivalences of Bernstein–Gelfand–Gelfand, Beilinson and Happel. Preprint 257 Humboldt–Universität Berlin (1990).

[DS] Dowbor, P.; Skowroński, A.: Galois coverings of representation–infinite algebras. Comment. Math. Helv. 62 (1987), 311-337.

[D1] Dräxler, P.: U-Fasersummen in darstellungsendlichen Algebren. J. Algebra 113 (1988), 430-437.

[D2] Dräxler, P.: *Aufrichtige gerichtete Ausnahmealgebren.* Bayreuther Math. Schriften 29 (1989).

[D3] Dräxler, P.: Fasersummen über dünnen s–Startmoduln. Arch.Math. 54 (1990), 252-257.

[D4] Dräxler, P.: On indecomposable modules over directed algebras. (To appear).

[Df] Drinfeld, V.G.: Quantum groups. In: Proc. Intern. Congr. Math. 1986. Amer. Math. Soc. (1987), Vol.1, 798–820.

[E1] Erdmann, K.: *Tame Blocks and Related Algebras.* Springer LNM 1428 (1990).

[E2] Erdmann, K.: On Auslander–Reiten components for wild blocks. This volume.

[ES] Erdmann, K.; Skowroński, A.: On Auslander–Reiten components of blocks and self–injective biserial algebras. (To appear).

[Fi] Fischbacher, U.: Une nouvelle preuve d'un théorème de Nazarova et Roiter. C.R.Acad.Sc. Paris I 300 (1985), 259-262.

[FP] Fischbacher, U.; de la Peña, J.: Algorithms in representation theory of algebras. In: Representation Theory I. Springer LNM 1177 (1986), 94-114.

[GP] Gabriel, P.; de la Peña, J.: Quotients of representation–finite algebras. Comm. Algebra 15 (1987), 279-308.

[GL1] Geigle, W.; Lenzing, H.: A class of weighted projective curves arising in the representation theory of finite dimensional algebras. In: Singularities, Representations of Algebras, and Vector Bundles. Springer LNM 1273 (1987), 265-297.

[GL2] Geigle, W.; Lenzing, H.: Perpendicular categories with applications to representations and sheaves. J. Algebra (To appear).

[Gr] Green, J.A.: On certain subalgebras of the Schur algebra. (To appear).

[GKK] Green, E.L.; Kirkman, E.; Kuzmanovich, J.: Finitistic dimension of finite dimensional monomial algebras. J. Algebra 136 (1991), 37-50.

[GZ] Green, E.L.; Zimmermann–Huisgen, B.: Finitistic dimension of artinian rings with vanishing radical cube zero. Math. Z. (To appear).

[GHJ] Goodman, F.M., de la Harpe, P.; Jones, V.F.R.: *Coxeter Graphs of Algebras.* Math. Sci. Res. Inst. Publ. 14 (1989).

[G1] Guo, J.Y.: The isomorphism of Hall algebras. Preprint 90-060 SFB 343 Bielefeld.

[G2] Guo, J.Y.: The center of a Hall algebra. Preprint 90-061 SFB 343 Bielefeld.

[G3] Guo, J.Y.: The Hall algebra of a cyclic serial algebra. (In preparation)

[H1] Happel, D.: On the derived category of a finite–dimensional algebra. Comment. Math. Helv. 62 (1987), 339-389.

[H2] Happel, D.: Iterated tilted algebras of affine type. Comm. Alg. 15 (1987), 29-46

[H3] Happel, D.: Repetitive categories. In: Singularities, Representations of Algebras, and Vector Bundles. Springer LNM 1273 (1987), 298-317

[H4] Happel, D.: *Triangulated Categories in the Representation Theory of Finite–dimensional Algebras.* London Math. Soc. LNS 119 (1988).

[H5] Happel, D.: Hochschild cohomology of finite–dimensional algebras. In: Sém. d'Algbre. Dubreil–Malliavin. Springer LNM 1404 (1989), 108-126.

[H6] Happel, D.: A family of algebras with two simple modules and Fibonacci numbers. Archiv Math. (To appear)

[H7] Happel, D.: Partial tilting modules and recollement. Preprint 89-016 SFB 343 Bielefeld.

[H8] Happel, D.: Auslander–Reiten triangles in derived categories of finite–dimensional algebras. Proc. Amer.Math.Soc. (To appear).

[H9] Happel, D.: On Gorenstein algebras. This volume.

[H10] Happel, D.: Hochschild cohomology of Auslander algebras. In: Topics in Algebra. Banach Center Publ. 26. (To appear).

[HRS] Happel, D.; Rickard, J.; Schofield, A.: Piecewise hereditary algebras. Bull. London Math.Soc. 20 (1988), 23-28.

[HR] Happel, D.; Ringel, C.M.: The derived category of a tubular algebra. In: Representation Theory I. Springer LNM 1177 (1986). 156-180.

[HU1] Happel, D.; Unger, L.: Factors of concealed algebras. Math. Z. 201 (1989), 477-483.

[HU2] Happel, D.; Unger, L.: A family of infinite–dimensional non–selfextending bricks for wild hereditary algebras. Proc. Tsukuba Conf. 1990 (To appear).

[Hi] Hille, L.: Assoziative gestufte Algebren und Kippfolgen mit $\dim(X)+1$ Stufen auf projektiven glatten algebraischen Mannigfaltigkeiten. Diplomarbeit Humboldt Univ. Berlin (1990).

[HM] Hoshino, M.; Miyachi, J.: Tame triangular matrix algebras over self–injective algebras. Tsukuba J. Math. 11 (1987), 383-391.

[Hö] v. Höhne; H. J.: On weakly positive unit forms. Comm. Math. Helv. 63 (1988), 312-336.

[Hü] Hübner, T.: (In preparation).

[Ig] Igusa, K.: Notes on the loop conjecture. J. Pure and Appl. Algebra. (To appear).

[IST] Igusa, K.; Smalø, S.; Todorov, G.: Finite projectivity and contravariant finiteness. (To appear).

[IT] Igusa, K.; Todorov, G.: A numerical characterization of finite Auslander–Reiten quivers. In: Representation Theory I. Springer LNM 1177 (1986), 181-198.

[IZ] Igusa, K.; Zacharia, D.: Syzygy pairs in a monomial algebra. Proc. Amer. Math. Soc. 108 (1990) 601-604.

[Ir] Irving, R.: BGG algebras and the BGG reciprocity principle. J. Algebra 135 (1990), 363-380.

[Kc] Kac, V. G.: Root systems, representations of quivers and invariant theory. In: Invariant Theory. Springer LNM 996 (1983), 74-108.

[Kp] Kapranov, M.M.: On the derived categories of coherent sheaves on some homogeneous spaces. Inv. Math. 92 (1988), 479-508.

[Kl] Keller, B.: Chain complexes and stable categories. Manuscr. Math. 67 (1990), 379-417.

[KV] Keller, B.; Vossieck, D: Sous les catégories dérivées. C.R.Acad. Sci. Paris I 305 (1987) 225-228.

[K1] Kerner, O.: Preprojective components of wild hereditary algebras. Manuscr. Math. 61 (1988), 429-445.

[K2] Kerner, O.: Tilting wild algebras. J.London Math.Soc.(2) 39 (1989), 29-47.

[K3] Kerner, O.: Universal exact sequences for torsion theories. In: Topics in Algebra. Banach Center Publ. 26. (To appear).

[K4] Kerner, O.: Stable components of wild tilted algebras. J.Algebra (To appear).

[KSk] Kerner, O.; Skowroński, A.: On module categories with nilpotent infinite radical. (To appear).

[KK] Kirkman, E.; Kuzmanovich, J.: Algebras with large homological dimension. (To appear)

[KSi] Klemp,B.; Simson, D.: Schurian sp–representation–finite right peak PI–rings and their indecomposable socle projective modules. J. Algebra 134 (1990), 390-468.

[Kö] König, S.: Tilting complexes, perpendicular categories, and recollement of derived module categories of rings. (To appear)

[Kr1] Krause, H.: Maps between tree and band modules. J.Algebra (To appear).

[Kr2] Krause, H.: Endomorphismem von Worten in einem Köcher. Dissertation. Bielefeld 1991.

[Le] Le Bruyn, L.: Counterexamples to the Kac–conjecture on Schur roots. Bull. Sci. Math. 110 (1986), 437-448.

[L1] Lenzing, H.: Nilpotente Elemente in Ringen von endlicher globaler Dimension. Math. Z. 108 (1969), 313-324.

[L2] Lenzing, H.: Curve singularities arising from the representation theory of tame hereditary Artin algebras. In: Representation Theory I. Springer LNM 1177 (1986), 199-231.

[L3] Lenzing, H.: Canonical algebras and rings of automorphic forms. (In preparation).

[LP] Lenzing, H.; de la Peña, J. A.: The Auslander–Reiten components of a canonical algebra. (In preparation).

[Lu] Lusztig, G.: Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc. 3 (1990), 447-498

[MP] Marmaridis, N.; de la Peña, J.A.: Quadratic forms and preinjective modules. J. Algebra 134 (1990), 326-343.

[MR] Marmolejo, E.; Ringel, C.M.: Modules of bounded length in Auslander–Reiten components. Arch. Math. 50 (1988), 128-133.

[Ma1] Martínez–Villa, R.: The stable equivalence for algebras of finite representation type. Comm. Algebra 13 (1985), 991-1018.

[Ma2] Martínez–Villa, R.: Some remarks on stably equivalent algebras. Comm. Algebra (To appear).

[Me] Meltzer, H.: Tilting bundles, repetitive algebras, and derived categories of coherent sheaves. Preprint 193 Humboldt–Universität Berlin (1988)

[MO] Menini, C.; Orsatti, A.: Representable equivalences between categories of modules and applications. Rend. Sem. Mat. Univ. Padova 82 (1989), 203-231.

[MV] Mirollo,R.; Vilonen,K.: Bernstein–Gelfand–Gelfand reciprocity on perverse sheaves. Ann. Sci. E.N.S. 20 (1987), 311-324.

[My] Miyashita, Y.: Tilting modules of finite projective dimension. Math. Z. 193 (1986), 113-146.

[NR1] Nazarova, L.A.; Rojter, A.V.: Representations of completed partially ordered sets. In: Proc. Fourth Intern. Conf. Representations of Algebras. Vol.1. Carleton–Ottawa LNS 1 (1985).

[NR2] Nazarova, L.A.; Rojter, A.V.: Representations of bipartite completed posets. Comment. Math. Helv. 63 (1988), 498-526.

[NS] Nehring, J.; Skowroński, A.: Polynomial growth trivial extensions of simply connected algebras. Fund. Math. 132 (1989), 117-134.

[Ok] Okuyama, T.: On the Auslander–Reiten quiver of a finite group. J. Algebra 110 (1987), 425-430.

[OP] Ostermann, A.; Pott, A.: Schwach positive ganze quadratische Formen, die eine aufrichtige, positive Wurzel mit einem Koeffizienten 6 besitzen. J. Algebra 126 (1989), 80-118.

[Pa] Parshall, B.: Finite dimensional algebras and algebraic groups. In: Classical Groups and Related Topics. Contemp. Math. 82 (1989).

[PS] Parshall, B; Scott, L.L.: *Derived Categories, Quasi–hereditary Algebras, and Algebraic Groups.* Proc. Ottawa–Moosonee Workshop Algebra. Carleton–Ottawa Math. LNS 3 (1988), 1-105

[Pt] Partharasarathy, R.: t-structures dans la catégorie dérivée associée aux représentations d'un carquois. C.R. Acad. Sci. Paris I 304 (1987), 355-357.

[P1] de la Peña, J.: On omnipresent modules in simply connected algebras. J. London Math. Soc. (2) 36 (1987), 385-392.

[P2] de la Peña, J.: *Quadratic forms and the representation type of an algebra.* Ergänzungsreihe 90-003 SFB 343 Bielefeld.

[P3] de la Peña, J.: Tame algebras with sincere directing modules. (To appear).

[PS] de la Peña, J., Simson, D.: Prinjective modules, reflection functors, quadratic forms and Auslander–Reiten sequences. (To appear).

[PTa] de la Peña, J.; Takane, M.: Spectral properties of Coxeter transformations and applications. Arch. Math. 55 (1990), 120-134.

[PTo] de la Peña, J.; Tomé, B.: Iterated tubular algebras. J. Pure Appl. Algbera (To appear).

[Rc1] Rickard, J.: Morita theory for derived categories. J. London Math.Soc. 39 (1989), 436-456

[Rc2] Rickard, J.: Derived categories and stable equivalence. J. Pure Appl. Algebra 61 (1989), 436-456.

[Rc3] Rickard, J.: Derived equivalences as derived functors. (To appear).

[RS] Rickard, J.; Schofield, A.: Cocovers and tilting modules. Proc. Cambridge Phil.Soc. 106 (1989) 1-5.

[RS] Riedtmann, Ch.; Schofield, A.: On a simplicial complex associated with tilting modules. Prépubl. l'Inst. Fourier. 137 Grenoble (1989)

[R1] Ringel, C.M.: *Tame Algebras and Integral Quadratic Forms.* Springer LNM 1099 (1984).

[R2] Ringel, C.M.: The regular components of the Auslander–Reiten quiver of a tilted algebra. Chinese Ann. Math. B 9 (1988), 1-18.

[R3] Ringel, C.M.: The canonical algebras. (With an appendix by W. Crawley–Boevey). In: Topics in Algebra. Banach Center Publ. 26. (To appear).

[R4] Ringel, C.M.: The category of modules with good filtrations over a quasi–hereditary algebra has almost split sequences. Math.Z. (To appear).

[R5] Ringel, C.M.: On contravariantly finite subcategories. In: Proc. Tsukuba Conf. 1990 (To appear)

[R6] Ringel, C.M.: The category of good modules over a quasi–hereditary algebra. In: Proc. Tsukuba Conf. 1990 (To appear)

[R7] Ringel, C.M.: Hall algebras. In: Topics in Algebra. Banach Centre Publ. 26. Warszawa (To appear).

[R8] Ringel, C.M.: Hall polynomials for the representation–finite hereditary algebras. Adv. Math. 84 (1990), 137-178.

[R9] Ringel, C.M.: From representations of quivers via Hall and Loewy algebras to quantum groups. In: Proc. Novosibirsk Conf. Algebra 1989 (To appear)

[R10] Ringel, C.M.: Hall algebras and quantum groups. Invent. Math. 101 (1990), 583-592

[R11] Ringel, C.M.: The composition algebra of a cyclic quiver. (To appear)

[RV] Ringel, C.M.; Vossieck, D.: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216-246.

[Ru] Rudakov, A. N. et. al.: *Helices and Vector Bundles. Seminaire Rudakov.* London Math. Soc. LNS 148 (1990).

[Sr1] Scheuer, T.: More hammocks. In: Topics in Algebra. Banach Centre Publ. 26. Warszawa (To appear).

[Sr2] Scheuer, T.: The canonical decomposition of the poset of a hammock. Proc. London Math. Soc. (To appear).

[Sw1] Schewe, W.: The set of $\mathbb{Z}$–coverings for a finite translation quiver. Dissertation Bielefeld 1989.

[Sw2] Schewe, W.: Fundamental domains for representation–finite algebras. Rapport 74 Univ. Sherbrooke (1990).

[Sf1] Schofield, A.: Bounding the global dimension in terms of the dimension. Bull. London Math. Soc. 17 (1985), 393-394.

[Sf2] Schofield, A.: The field of definition of a real representation of a quiver Q. (To appear)

[Sf3] Schofield, A.: The internal structure of real Schur representations. (To appear)

[Sf4] Schofield, A.: (To appear).

[Sc] Scott, L.L.: Simulating algebraic geometry with algebra I: The algebraic theory of derived categories. Proc. Symp. Pure Math. 47 (1987), 271-281.

[Si] Simson, D.: Module categories and adjusted modules over traced rings. Diss. Math. 269 (1990).

[Sk1] Skowroński, A.: Group algebras of polynomial growth. Manuscr. Math. 59 (1987), 499-516.

[Sk2] Skowroński, A.: Selfinjective algebras of polynomial growth. Math. Ann. 285 (1989), 177-199.

[Sk3] Skowroński, A.: Algebras of polynomial growth. Topics in Algebra. Banach centre publ. 26. (To appear).

[SS] Skowroński, A.; Smalø, S.O.: Directing modules. Preprint 8/1990 Univ. Trondheim.

[Sm] Smalø, S.O.: Functorial finite subcategories over triangular matrix rings. Preprint 6/1989 Trondheim.

[Soe1] Soergel, W.: Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. 3 (1990), 421-445.

[Soe2] Soergel, W.: Parabolisch–singuläre Dualität für Kategorie $\mathcal{O}$. Preprint MPI Bonn 89-68.

[Soe3] Soergel, W.: Construction of projectives and reciprocity in an abstract setting. (To appear).

[St] Strauß, H.: Tilting modules over wild hereditary algebras. Thesis. Carleton University 1986. Abstract: C.R.Math., Acad.Sci.Canada 9 (1987), 161-166.

[TW] Tachikawa, H., Wakamatsu, T.: Cartan matrices and Grothendieck groups of stable categories. (To appear).

[UY] Uematsu, M.; Yamagata, K.: On serial quasi–hereditary rings. Hokkaido Math. J. 19 (1990), 165-174.

[U1] Unger, L.: Lower bounds for indecomposable, faithful, preinjective modules. Manuscr. Math. 57 (1986), 1-31.

[U2] Unger, L.: The concealed algebras of the minimal wild hereditary algebras. Bayreuther Math. Schriften 31 (1990), 145-154.

[U3] Unger, L.: On wild tilted algebras which are squids. Archiv Mathematik (To appear).

[U4] Unger, L.: Schur modules over wild, finite dimensional path algebras with three non isomorphic simple modules. J. Pure Appl. Algebra. (To appear).

[U5] Unger, L.: On the number of maximal sincere modules over sincere directed algebras. J.Algebra 133 (1990), 211-231.

[U6] Unger, L.: One–dimensional links of the simplicial complex of partial tilting modules. (In preparation).

[W1] Wakamatsu, T.: On modules with trivial selfextensions. J.Algebra 114 (1988), 106-114.

[W2] Wakamatsu, T.: Stable equivalences for selfinjective algebras and a generalization of tilting modules. J. Algebra 134 (1990), 298-325.

[Wi] Wiedemann, A.: Quotients of quasi–hereditary algebras. (To appear).

[X1] Xi, Ch.: Die Vektorraumkategorie zu einem unzerlegbaren projektiven Modul einer Algebra. J. Algebra (To appear).

[X2] Xi, Ch.: Die Vektorraumkategorie zu einem Punkt einer zahmen verkleideten Algebra. J. Algebra (To appear).

[X3] Xi, Ch.: Die Vektorraumkategorie zu einem unzerlegbaren projektiven Modul einer tubularen Algebra. Manuscr. Math. 69 (1990), 223-235.

[X4] Xi, Ch.: On wild hereditary algebras with small growth numbers. Comm. Algebra 18 (1990), 3413-3422.
[X5] Xi, Ch.: Minimal elements of the poset of a hammock. Preprint 90-016 SFB 343 Bielefeld.
[X6] Xi, Ch.: Quasi–heredity of algebras and their factor algebras. Preprint 90-59 SFB 343 Bielefeld.
[X7] Xi, Ch.: Symmetric algebras as endomorphism rings of large projective modules over quasi–hereditary algebras. Preprint 90-59 SFB 343 Bielefeld.
[Za] Zavadskij, A. G.: Classification of representations of posets of finite growth. in: Proc. Fourth Intern. Conf. Representations of Algebras. Vol.2. Carleton–Ottawa LNS 2 (1984), 36.01-36.15
[Z1] Zhang, Y.: The modules in any component of the AR–quivaer of a wild hereditary artin algebra are uniquely determined by their composition factors. Archiv Math. 53 (1989), 250-251.
[Z2] Zhang, Y.: Eigenvalues of Coxeter transformations and the structure of the regular components of the Auslander–Reiten quiver. Comm. Algebra. 17 (1989), 2347-2362.
[Z3] Zhang, Y.: The structure of stable components. Can. J. Math. (To appear).
[Zi1] Zimmermann–Huisgen, B.: Bounds on finitistic and global dimension for artinian rings with vanishing radical cube. (To appear).
[Zi2] Zimmermann–Huisgen, B.: Predicting syzygies over monomial algebras. (To appear).

Older surveys

[1] Auslander, M.: The what, where and why of almost split sequences. Proc. Intern. Congr. Math. 1986. Amer. Math. Soc.(1987), Vol.1, 338-345.
[2] Gabriel, P.: Darstellungen endlichdimensionaler Algebren Proc. Intern. Congr. Math. 1986. Amer. Math. Soc.(1987), Vol.1, 378-388.
[3] Reiten, I.: An introduction to the representation theory of Artin algebras. Bull. London Math. Soc. 17 (1985), 209-233.
[4] Reiten, I.: Finite dimensional algebras and singularities. In: Singularities, Representations of Algebras, and Vector Bundles. Springer LNM 1273 (1987), 35-57.
[5] Riedtmann, Ch.: Algèbre de type de représentation fini. Séminaire Bourbaki. 37e année 1984-1985, n. 650. Astérisque 133-134 (1986), 335-350.
[6] Ringel, C.M.: The representation type of local algebras. In: Representations of Algebras. Springer LNM 488 (1975), 282-305.
[7] Ringel, C.M.: Report on the Brauer–Thrall conjectures. In: Representation Theory I. Springer LNM 831 (1980), 104-136
[8] Ringel, C.M.: Tame algebras. In: Representation Theory I. Springer LNM 831 (1980), 137-287
[9] Ringel, C.M.: Indecomposable representations of finite dimensional algebras. Proceedings Intern. Conf. Math. Warszawa 1983, (1984), 425–436.
[10] Ringel, C.M.: Representation theory of finite dimensional algebras. Durham Lectures 1985. London Math. Soc. Lecture Note Series 116 (1986), 7–79.