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An un-delooped version of algebraic K-theory

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Abstract

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Problems working with the Segal operations in algebraic K-theory of spaces—constructed by F. Waldhausen (1982)—arose from the absence of a nice groupcompletion on the category level. H. Grayson and D. Gillet (1987) introduced a combinatorial model $G.$ for K-theory of exact categories. For dealing with K-theory of spaces we need an extension $wG.$ of their result to the context of categories with cofibrations and weak equivalences. Our main result is that in the presence of a suspension functor—as in the case of retractive spaces—the $wG.$ construction on the category of prespectra is an un-delooped version of the K-theory of the original category. In a sequel to this paper we show that Grayson's formula (1988) for Segal operations works as intended.

1. Introduction

In this paper we start an analysis of the Segal operations in algebraic K-theory of spaces $A(X)$. We call the operations Segal operations since they can be regarded as extending the power operations in stable homotopy [9]. These operations, $\Theta = (\Theta_n) : A(*) \to \prod_n A(B\Sigma_n)$, were constructed by Waldhausen in [11] in his proof of the analogue of the Kahn–Pridy theorem for $A(*)$. These

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operations have been difficult to work with since their construction appeals to universal properties of the plus construction and since they are extended from their behaviour on spherical models (cf. [11, remark on p. 395]).

There are many ways to obtain K-theory, each having its advantages. Problems with defining operations arose from the absence of a nice group completion on the category level. Grayson and Gillet introduced a combinatorial model $G.$ for K-theory of exact categories (in the sense of Quillen) in [2]. Their model is a simplified model of the loop space of Waldhausen's $S.$-construction. For dealing with K-theory of spaces we need an extension, $wG.$, of their result to the context of categories, $\mathscr{C}$, with cofibrations and weak equivalences.

In the presence of a suspension functor (as in the case of retractive spaces), we apply the Grayson and Gillet construction to the category of prespectra. Our main result is that this is an un-delooped version of the K-theory of the original category (Remark 2.7). This is a consequence of the fact that the construction gives the desired model for categories which we call pseudo-additive (Theorem 2.6). This situation includes the cases treated in [2].

In general we have to iterate their construction infinitely many times (Theorem 2.8). Our proof is more in the spirit of [12], the key point being the proof of the additivity theorem for $wG.$. Technically the material is centered around a suitable generalization of [12, Proposition 1.5.5] from which the result follows by comparison of fibration sequences. One of the motivations behind this model was that it should be used for constructing operations more directly.

Grayson used this model in [3] to provide a framework for long exact sequences in algebraic K-theory and in [4] to define formulas for λ-operations in K-theory. (These formulas also give formulas for the Segal operations.)

In a sequel to this paper [6] we will show that Grayson's formula for Segal operations, interpreted for categories with cofibrations and weak equivalences, works as intended. The total Segal operation, its applications and structure will be studied using these models.

2. Definitions and statements of theorems

Before we can state our theorem we need some definitions and some facts from [12]. Notations not explicitly recalled are taken from [12].

Definition 2.1. (1) $\mathscr{C}of$ is the category whose objects are the small categories with cofibrations and whose morphisms are the exact functors [12, p. 320].

(2) $w\mathscr{C}of$ is the category whose objects are the small categories with cofibrations and weak equivalences and whose morphisms are the exact functors [12, p. 326].

An object $\mathscr{C} \in \mathscr{C}of$ can be considered as an object in $w\mathscr{C}of$ by letting $w\mathscr{C} := i\mathscr{C}$, the category of isomorphisms in $\mathscr{C}$.

$S.: w\mathscr{C}of \to \Delta^{op} - w\mathscr{C}of$ is the functor defined in [12, p. 328]. Δ^{op} is the index category for simplicial objects.

For $X. \in \Delta^{op} - w\mathscr{C}of$ we let $PX.$ denote the corresponding path object $PX_n = X_{n+1}$. The boundary map $d_0 : X_{n+1} \to X_n$ defines a morphism $d_0 : PX. \to X.$. This path object is simplicially homotopic to the constant simplicial object X_0 by a homotopy that is described in [12, Lemma 1.5.1, p. 341]. Note also the map $X_1 = PX_0 \to PX.$ where PX_0 is considered as a constant simplicial object.

Definition 2.2 [2]. $G.: w\mathscr{C}of \to \Delta^{op} - w\mathscr{C}of$ is defined by the cartesian square

$$\begin{array}{ccc} G.\mathscr{C} & \longrightarrow & PS.\mathscr{C} \\ \downarrow & & \downarrow{\scriptstyle d_0} \\ PS.\mathscr{C} & \xrightarrow{d_0} & S.\mathscr{C} \end{array}$$

in $\Delta^{op} - w\mathscr{C}of$.

The maps $\mathscr{C} \cong (PS.\mathscr{C})_0 \to PS.\mathscr{C}$ and the zero map $\mathscr{C} \to PS.\mathscr{C}$ give an inclusion $\mathscr{C} \to G.\mathscr{C}$ where $\mathscr{C}$ is considered as a constant simplicial object. This map can be viewed as a map of multisimplicial categories and in this way it becomes the first map of a sequence of maps

$$\begin{aligned} &\mathscr{C} \to G.\mathscr{C} \to G.(G.\mathscr{C}) \to \cdots \to (G.)^n\mathscr{C} \\ &\quad \to G.((G.)^n\mathscr{C}) =: (G.)^{n+1}\mathscr{C} \to \cdots \end{aligned}$$

The colimit will be denoted by $G^{\infty}_{\cdot}\mathscr{C}$, which is a multisimplicical object in $w\mathscr{C}of$.

In [2] $G.$ was only defined for exact categories in the sense of Quillen [7], but the definition makes sense in the general case: The weak equivalences in $G_n\mathscr{C}$ are given by the pullback

$$\begin{array}{ccc} wG.\mathscr{C} & \longrightarrow & wPS.\mathscr{C} \\ \downarrow & & \downarrow \\ wPS.\mathscr{C} & \longrightarrow & wS.\mathscr{C} \end{array} \tag{1}$$

Similarly the cofibrations are given by

$$\mathrm{cof}(PS_n\mathscr{C}) \underset{\mathrm{cof}(S_n\mathscr{C})}{\times} \mathrm{cof}(PS_n\mathscr{C}) .$$

That this makes $G.\mathscr{C}$ a simplicial object in $w\mathscr{C}of$ follows from the fact that $G_n\mathscr{C}$ is equivalent to the fiber product $\prod (d_0, d_0)_n$ (cf. [12, Lemma 1.1.6, p. 325]). This in turn follows from the fact that

$$i_1 PS_n\mathscr{C} \to i_1 S_n\mathscr{C} \to i_1 S_n\mathscr{C} \underset{\mathrm{Ob}(S_n\mathscr{C})}{\times} \mathrm{Ob}(PS_n\mathscr{C})$$

has a section. (Here i_1 denotes the set of isomorphisms. The pullback is taken over (source, $\mathrm{Ob}(d_0)$) and the map is induced by $(i_1 d_0$, source).)

It will sometimes be convenient to use the fiber product $\coprod(d_0, d_0)$. instead of $G.\mathscr{C}$. We denote the fiberproduct by $\underline{G}.\mathscr{C}$. Since $|wPS.\mathscr{C}|$ is contractible to $|wS_0\mathscr{C}|$, which is a point, it follows that diagram (1) defines a map

$$|wG.\mathscr{C}| \to \Omega|wS.\mathscr{C}| .$$

We will show that this is a weak equivalence when $\mathscr{C}$ has the following property (P).

Definition 2.3. (i) A category $\mathscr{C} \in w\mathscr{C}of$ is said to be *pseudo-additive* (P) if there is a given a natural sequence P of exact weak equivalences between the functors

$$A \rightarrowtail C \mapsto A \rightarrowtail C \cup_A C \quad \text{resp.} \quad A \rightarrowtail C \cup_A (A \vee C/A)$$

in $\mathscr{C}_A$, the category of cofibrant objects under A for each A in $\mathscr{C}$. This pseudo-additive structure is also required to be naturally exact in the cofibration $A \rightarrowtail C$.

(ii) A *pseudo-additive* functor is an exact functor which respects this extra structure.

(iii) The category of pseudo-additive categories and functors is denoted by $w\mathscr{P}add$. It comes with a forgetful functor to $w\mathscr{C}of$.

Remark 2.4. (1) Finite limits, fiberproducts and categories of filtered objects can be formed in $w\mathscr{P}add$, provided they exist in $w\mathscr{C}of$.

(2) There are natural extensions of the $wG.$- and $wS.$-constructions to the $w\mathscr{P}add$ situation.

A category $\mathscr{C}$ with cofibrations and weak equivalences is pseudo-additive (P) in the following cases:

(3) Cofibration sequences in $\mathscr{C}$ are split.

(4) $\mathscr{C}$ has a product such that the maps

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : C \cup_A C \to C \times C/A \leftarrow C \vee C/A \ (\cong (C \cup_A (A \vee C/A))$$

are exact weak equivalences.

(5) $\Sigma^\infty \mathscr{C}$ is the category of prespectra associated to a category $\mathscr{C} \in w\mathscr{C}of$ with a cylinder functor satisfying Cyl1 and Cyl2 in [12, p. 348] and the cylinder axiom [12, p. 349]. Then $\mathscr{C}$ has a suspension functor Σ and $\Sigma^\infty\mathscr{C}$ is defined as

$$\mathrm{colim}(\mathscr{C} \xrightarrow{\Sigma} \mathscr{C} \xrightarrow{\Sigma} \mathscr{C} \to \cdots) .$$

One can also define Σ^∞ as the homotopy colimit, and hence use the familiar description of prespectra. All what follows is true for both definitions. $\Sigma^\infty\mathscr{C}$ is again a category in $w\mathscr{C}of$. Σ^∞ can be viewed as a functor with values in $w\mathscr{P}add$.

For (5) the following lemma is needed:

Lemma 2.5 [12, Lemma 1.8.2, p. 368]. *To $X \rightarrowtail A$ in $\mathscr{C}_X$ there is naturally associated a chain of exact weak equivalences in $\mathscr{C}_{\Sigma X}$,*

$$(\Sigma X \rightarrowtail \Sigma A \cup_{\Sigma X} \Sigma A) \sim (\Sigma X \rightarrowtail \Sigma A \cup_* \Sigma A/\Sigma X). \qquad \square$$

Theorem 2.6. *The map $|wG.\mathscr{C}| \rightarrow \Omega|wS.\mathscr{C}|$ is a weak equivalence if $\mathscr{C}$ is pseudo-additive* (P).

This will be shown below.

Remark 2.7. For exact categories [7] 2.4(4) is fulfilled, with isomorphisms as weak equivalences, giving the case originally treated by Grayson and Gillet. Theorem 2.6 is a generalisation of Theorem 3.1 in [2], which states that $|g.\mathscr{C}| \rightarrow \Omega|s.\mathscr{C}|$ is a weak equivalence if $\mathscr{C}$ is an exact category [7]. Here $g_n\mathscr{C} = \mathrm{Ob}(G_n\mathscr{C})$ and $s_n\mathscr{C} = \mathrm{Ob}(S_n\mathscr{C})$. That this is a special case of Theorem 2.6 follows from the fact that $|g.\mathscr{C}| \rightarrow |iG.\mathscr{C}|$ is a weak equivalence. This will be discussed later. That $|s.\mathscr{C}| \rightarrow |iS.\mathscr{C}|$ is a weak equivalence is proved in [12, part (2) of the corollary to Lemma 1.4.1, p. 335]. By [12] the $wS.$-construction is insensitive, up to weak equivalence, for forming $\Sigma^\times\mathscr{C}$. Hence $wG.\Sigma^\times$ is a model for $\Omega|wS.|$.

We will later define a map $|wG^\times_.\mathscr{C}| \rightarrow \Omega|wS.\mathscr{C}|$ and show the following:

Theorem 2.8. *The map $|wG^\times_.\mathscr{C}| \rightarrow \Omega|wS.\mathscr{C}|$ is a weak equivalence for all $\mathscr{C} \in w\mathscr{C}of$.*

For $C \in w\mathscr{C}of$ let $E(\mathscr{C})$ denote the category of cofibration sequences $A \rightarrowtail C \twoheadrightarrow B$ in $\mathscr{C}$.

$$(f, g) : E(\mathscr{C}) \rightarrow \mathscr{C} \times \mathscr{C}, \qquad (A \rightarrowtail C \twoheadrightarrow B) \mapsto (A, B).$$

It is shown in [12, p. 325] that $E(\mathscr{C}) \in w\mathscr{C}of$ and that (f, g) is an exact functor.

It will be convenient to collect the properties of $|wG.\mathscr{C}|$ and $|wG^\times_.\mathscr{C}|$ that are needed in the proof of the theorem.

Definition 2.9. Let $\mathscr{M}$ be a class of objects in $w\mathscr{C}of$. We call a functor $F. : w\mathscr{C}of \rightarrow \Delta^{\mathrm{op}}\text{-Sets}_*$ *$\mathscr{M}$-admissible* if the following holds for all $\mathscr{A}$ and $\mathscr{B}$ in $w\mathscr{C}of$ and for all $\mathscr{C}$ in $\mathscr{M}$.

(1) $F.(\mathrm{pt}) \xrightarrow{\sim} \mathrm{pt}$ is a weak equivalence.

(2) $F.(\mathscr{A} \times \mathscr{B}) \xrightarrow{\sim} F.(\mathscr{A}) \times F.(\mathscr{B})$ (given by the projections) is a weak equivalence.

(3) $F.(E(\mathscr{C})) \xrightarrow{\sim} F.(\mathscr{C} \times \mathscr{C})$ (given by (f, g)) is a weak equivalence. (We say that *the additivity theorem holds for F.*.)

(4) $\pi_0(F.(\mathscr{A}))$ is a group under the operation induced by the categorical sum in $\mathscr{A}$.

(5) $F.$ transforms naturally isomorphic exact functors to homotopic maps.

We will only use the cases when $\mathscr{M}$ is $w\mathscr{C}of$ or $w\mathscr{P}add$. In [12] it is shown that $wS.$ is $w\mathscr{C}of$-admissible. We will show that $wG.$ is $w\mathscr{P}add$-admissible:

(2.9(1) and 2.9(2) hold trivially. (2.9(3) is covered by the following:

Theorem 2.10 (Additivity theorem). *For any $\mathscr{C} \in w\mathscr{P}add$ the map*

$$|wG.(f, g)| : |wG.(E(\mathscr{C}))| \to |wG.\mathscr{C}| \times |wG.\mathscr{C}|$$

is a weak equivalence.

This is the main ingredient in the proof of Theorem 2.6. Conversely it is a consequence of that theorem and the additivity theorem for $|wS.\mathscr{C}|$. 2.9(4) for $|wG.\mathscr{C}|$ is Lemma 2.11. We will discuss 2.9(5) in Proposition 2.13.

Lemma 2.11. *$\pi_0(wG.\mathscr{C})$ is a group under the operation induced by the categorical sum.*

Proof. This reduces to showing that the monoid $\pi_0(g.\mathscr{C})$ is a group. For this, using the notation of [2], put

$$-\left[\frac{B_0}{B'_0}\right] := \left[\frac{B'_0}{B_0}\right].$$

To see that this gives an inverse, we have to provide a homotopy

$$\frac{B_0}{B'_0} \vee \frac{B'_0}{B_0} \simeq \frac{*}{*}.$$

Such a homotopy can be defined by a 1-simplex in $g.\mathscr{C}$. This in turn, again using the notation of [2], is described by the following diagram:

$$\begin{array}{ccc}
* & \rightarrowtail & B_0 \vee B'_0 \\
 & & \downarrow \quad \downarrow \\
 & & B_0 \vee B'_0 \\
 & & \nwarrow \times \nearrow \\
 & & B'_0 \vee B_0 \\
 & & \uparrow \quad \uparrow \\
* & \rightarrowtail & B'_0 \vee B_0
\end{array} \qquad \square$$

For the next lemma we need some notation.

$\mathscr{C}^{[n]}$ denotes the category of functors from $[n] := (0 \to 1 \to \cdots \to n)$ to $\mathscr{C}$ and $\mathscr{C}(n, i)$ is the subcategory of $\mathscr{C}^{[n]}$ of those functors which take values in $i\mathscr{C}$. The category $\mathscr{C}(n, i)$ is in $w\mathscr{C}of$ by [12]. $iF.(\mathscr{C})$ is the diagonal of the bisimplicial set $n \mapsto F.(\mathscr{C}(n, i))$.

Lemma 2.12. *For a functor $F. : w\mathscr{C}of \to \Delta^{op}$-Sets$_*$ the following are equivalent:*

(1) *A natural isomorphism η of exact functors $f_0, f_1 : \mathscr{C} \to \mathscr{D}$ in $w\mathscr{C}of$ induces a homotopy of the maps $F.f_0$ and $F.f_1$.*

(2) *$F.(\mathscr{C}) \to iF.(\mathscr{C})$ is a weak equivalence.*

Proof. (1)$\Rightarrow$(2) by the argument in [12, p. 335].

(2)$\Rightarrow$(1) follows, if we can show that $iF.$ satisfies (1).

This is seen as follows. For $\varphi : [n] \to [1]$ define a functor $\mathscr{C}^{[n]} \to \mathscr{D}^{[n]}$ by

$$(C_0 \to \cdots \to C_n) \mapsto f_{\varphi(0)}(C_0) \to \cdots \to f_{\varphi(n)}(C_n),$$

where, if $\varphi(k) = 0$ and $\varphi(k+1) = 1$, we use the composition of

$$\eta_{C_k} : f_0(C_k) \to f_1(C_k) \quad \text{and} \quad f_1(C_k) \to f_1(C_{k+1}).$$

The remaining maps are given by f_0 and f_1 respectively.

The construction is simplicial in the variable n and hence gives a simplicial homotopy between $iF.(f_0)$ and $iF.(f_1)$. □

Proposition 2.13. *A weak equivalence of exact functors $f_0, f_1 : \mathscr{C} \to \mathscr{D}$ induces a homotopy between $|wG.f_0|$ and $wG.f_1|$.*

Proof. This is shown in the same way as the corresponding proposition for $wS.$ in [12, Proposition 1.3.1, p. 330]. □

Lemma 2.14. *The map $|g.\mathscr{C}| \to |iG.\mathscr{C}|$ is a weak equivalence.*

Proof. The argument is similar to the $S.$-case [12, p. 335]. □

3. Reduction

Theorem 2.6 will be a consequence of a variant of Proposition 1.55 in [12]. One main ingredient is a reformulation of the additivity theorem.

Proposition 3.1 [12, Proposition 1.3.2]. *Each of the following assertions implies all the three others for a functor $F. : w\mathscr{C}of \to \Delta^{op}$-Sets$_*$:*

(1) *The following projection is a homotopy equivalence*

$$F.(E(\mathscr{A}, \mathscr{C}, \mathscr{B})) \to F.(\mathscr{A}) \times F.(\mathscr{B}), \qquad A \rightarrowtail C \twoheadrightarrow B \mapsto (A, B).$$

(2) *The following projection is a homotopy equivalence*

$$F.(E(\mathscr{C}))\to F.(\mathscr{C})\times F.(\mathscr{C})\,, \qquad A\rightarrowtail C\twoheadrightarrow B\mapsto (A,B)\,.$$

(3) *The following two maps are homotopic (resp. weakly homotopic)*

$$F.(E(\mathscr{C}))\to F.(\mathscr{C})\,, \qquad A\rightarrowtail C\twoheadrightarrow B\mapsto C\,, \qquad \textit{resp. } A\vee B\,.$$

(4) *If $j'\rightarrowtail j\twoheadrightarrow j''$ is a cofibration sequence of exact functors $\mathscr{C}'\to\mathscr{C}$, then there exists a homotopy*

$$F.(j)\simeq F.(j')\vee F.(j'')\,(=F.(j'\vee j''))\,. \qquad \square$$

Recall the $N.$ construction from [12, p. 367]. $N.: w\mathscr{C}of\to\Delta^{\mathrm{op}}-w\mathscr{C}of$ is defined using the categorical sum. $N_n\mathscr{C}$ is equivalent to $\mathscr{C}^n$, but contains also compatible sum diagrams which make $N.$ simplicial. It even carries a Γ-structure [8].

Consider a cartesian square

$$\begin{array}{ccc} \mathscr{G}f. & \longrightarrow & PS.\mathscr{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle d_0} \\ \mathscr{A}. & \xrightarrow{f_0} & S.\mathscr{C} \end{array}$$

of simplical objects in $w\mathscr{C}of$ (resp. in $w\mathscr{P}add$). That $\mathscr{G}f.\in w\mathscr{C}of$ follows by the same argument we used to show that $G.\mathscr{C}\in w\mathscr{C}of$. Let $\mathscr{C}\to\mathscr{G}f.$ be induced by the zero functor $\mathscr{C}\to\mathscr{A}.$ and the inclusion $\mathscr{C}\xrightarrow{\simeq} PS_0\mathscr{C}\to PS.\mathscr{C}$. The following is the generalisation of [12, Proposition 1.55] we will use.

Proposition 1.55′. *If $F.$ is an $\mathscr{M}$-admissible functor and each $\mathscr{G}f_n$ is in $\mathscr{M}$, then*

$$F.(\mathscr{C})\to F.(\mathscr{G}f.)\to F.(\mathscr{A}.)$$

is a fibration sequence up to homotopy.

We will need Proposition 1.55′ in the cases, where $\mathscr{M}$ is $w\mathscr{C}of$ or $w\mathscr{P}add$.

Proof. First we show that for each n

$$F.(\mathscr{C})\to F.(\mathscr{G}f_n)\to F.(\mathscr{A}_n) \tag{2}$$

is weakly equivalent to a fibration sequence. Adapting the proof in [12, Proposition [1.55] we define an exact functor

$$\mathscr{A}_n\times\mathscr{C}\to\mathscr{G}f_n \quad \text{by} \quad (a,c)\mapsto(a,c\vee s_0f_n(a))\,.$$

Here $\vee$ is a sum in $(PS\mathscr{C})_n$. The map s_0 is the zeroth degeneracy $S_n\mathscr{C} \to S_{n+1}\mathscr{C} = (PS\mathscr{C})_n$ and c is interpreted as an element of $(PS\mathscr{C})_n$ via $\mathscr{C} \xrightarrow{\cong} (PS\mathscr{C})_0 \to (PS\mathscr{C})_n$.

We obtain maps of sequences.

$$\begin{array}{ccccc}
F.(\mathscr{C}) & \longrightarrow & F.(\mathscr{A}_n) \times F.(\mathscr{C}) & \longrightarrow & F.(\mathscr{A}_n) \\
\| & & \uparrow & & \| \\
F.(\mathscr{C}) & \longrightarrow & F.(\mathscr{A}_n \times \mathscr{C}) & \longrightarrow & F.(\mathscr{A}_n) \\
\| & & \downarrow & & \| \\
F.(\mathscr{C}) & \longrightarrow & F.(\mathscr{G}f_n) & \longrightarrow & F.(\mathscr{A}_n)
\end{array}$$

The top sequence is a product fibration, which by the assumptions on $F.$ is weakly equivalent to the sequence in the middle.

The map $F.(\mathscr{A}_n \times \mathscr{C}) \to F.(\mathscr{G}f_n)$ is a weak equivalence by an argument which uses the additivity theorem for $F.$ and parallels exactly the discussion in [12, Proposition 1.55].

We have now proved that (2) is a fibration sequence. It remains to show that the diagonal of (2) is a fibration sequence up to homotopy. For this we choose the following argument.

First we observe that the spaces have natural Γ-structures $n \mapsto F.(N_n\mathscr{C})$. We use this structure to form a classifying space (cf. [8] or [12]). Call this $BF.$, which is still an $\mathscr{M}$-admissible functor and hence gives a fibration sequence up to homotopy

$$BF.(\mathscr{C}) \to BF.(\mathscr{G}f_n) \to BF.(\mathscr{A}_n)$$

by the above argument.

Since the base spaces now are connected, we can apply [11, Lemma 5.2] and conclude that

$$BF.(\mathscr{C}) \to BF.(\mathscr{G}f.) \to BF.(\mathscr{A}.)$$

is a fibration sequence up to homotopy. We note that, since the Γ-structure is natural, B commutes with taking diagonals. We now use the fact [8] that $\Omega BF.(\mathscr{C})$ is a natural group completion and obtain a map of sequences

$$\begin{array}{ccccc}
F.(\mathscr{C}) & \longrightarrow & F.(\mathscr{G}f.) & \longrightarrow & F.(\mathscr{A}.) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega BF.(\mathscr{C}) & \longrightarrow & \Omega BF.(\mathscr{G}f.) & \longrightarrow & \Omega BF.(\mathscr{A}.)
\end{array}$$

Since it follows from the assumptions on $F.$ that the spaces in the top sequence are grouplike we can deduce from [8] that the vertical maps are weak equivalences. □

Proof of Theorem 2.6 from Proposition 1.55′. (a) Let $f = d_0 : \mathcal{A}. = PS.\mathcal{C} \to S.\mathcal{C}$. This gives for a $w\mathcal{P}add$-admissible functor $F.$ a fibration sequence

$$F.\mathcal{C} \to F.G.\mathcal{C} \to F.PS.\mathcal{C}\ .$$

Since $PS.\mathcal{C}$ is simplicially contractible we obtain that

$$F.\mathcal{C} \to F.G.\mathcal{C}$$

is a weak equivalence. With $F. = |wS.|$, this gives a weak equivalence $|wS.\mathcal{C}| \to |wS.G.\mathcal{C}|$. With $F. = |wG.|$, this gives a weak equivalence $|wG.\mathcal{C}| \to |wG.G.\mathcal{C}|$.

(b) Let $f = 1_{S.\mathcal{C}} : S.\mathcal{C} \to S.\mathcal{C}$. This gives a fibration sequence

$$F.(\mathcal{C}) \to F.(PS.\mathcal{C}) \to F.(S.\mathcal{C})$$

with contractible total space. The contraction of $PS.\mathcal{C}$ and a contraction of $F.(\mathrm{pt})$ to a point gives a map $F.(\mathcal{C}) \to \Omega F.(S.\mathcal{C})$ which is a weak equivalence.

(c) Now consider the diagram

$$\begin{array}{ccc} F.(\mathcal{C}) & \xrightarrow{=} & F.(\mathcal{C}) \\ \downarrow & & \downarrow \\ F.(G.(\mathcal{C})) & \longrightarrow & F.(PS.\mathcal{C}) \\ \downarrow & & \downarrow \\ F.(PS.\mathcal{C}) & \longrightarrow & F.(S.\mathcal{C}) \end{array}$$

Since the columns are fibration sequences up to homotopy, it follows that the square is homotopy cartesian. We will use the case $F. = wG.$.

Consider the map of squares

$$\begin{array}{ccc} |wG.\mathcal{C}| & \longrightarrow & |wPS.\mathcal{C}| \\ \downarrow & & \downarrow \\ |wPS.\mathcal{C}| & \longrightarrow & |wS.\mathcal{C}| \end{array} \longrightarrow \begin{array}{ccc} |wG.G.\mathcal{C}| & \longrightarrow & |wG.PS.\mathcal{C}| \\ \downarrow & & \downarrow \\ |wG.PS.\mathcal{C}| & \longrightarrow & |wG.S.\mathcal{C}| \end{array}$$

where the component maps are given by inclusions $\mathcal{D} \to G.\mathcal{D}$, for the appropriate $\mathcal{D}$'s.

Consider first the commutative diagram

$$\begin{array}{ccc} wS.\mathscr{C} & \xrightarrow{\alpha} & |wS.G.\mathscr{C}| \\ & \searrow^{\gamma} & \downarrow{\scriptstyle\beta} \\ & & |wG.S.\mathscr{C}| \end{array}$$

Here α is a weak equivalence by (a). β is an isomorphism. Hence γ is a weak equivalence.

Two of the components of the map of squares are maps of contractible spaces. The remaining map is a weak equivalence by (a).

The right-hand square is homotopy cartesian by (c). Since the map of squares is a weak equivalence, it follows that the left square is homotopy cartesian. The simplicial contraction of $|wPS.\mathscr{C}|$ to a point then gives a weak equivalence $|wG.\mathscr{C}| \to \Omega|wS.\mathscr{C}|$. □

Observation 3.2. The map $|wG.\mathscr{C}| \to \Omega|wS.\mathscr{C}|$ is natural in $\mathscr{C}$. This means that we can replace $\mathscr{C}$ by $N.\mathscr{C}$ and hence we observe that the map is an infinite loop map with respect to the structure given by categorical sum. But this structure can be used to define the infinite loop space structure on $|wS.\mathscr{C}|$. (Actually, by the additivity theorem the map $|wS.N.\mathscr{C}| \to |wS.S.\mathscr{C}|$ is a weak equivalence. This follows by concatenation of [12, Proposition 1.55] and [12, Lemma 1.8.6]).

Remaining proofs

It remains to show the additivity theorem (Theorem 2.10).

Proof. For each A consider the following diagram of bisimplicial sets:

$$\begin{array}{ccccc} f^{-1}(m,0,(\mathrm{id},v_n)^*(A)) & \xrightarrow{(2)} & f^{-1}(m,n,A) & \longrightarrow & w.G.E(\mathscr{C}) \\ \downarrow{\scriptstyle(3)} & & \downarrow{\scriptstyle(4)} & & \downarrow{\scriptstyle(5)} \\ \Delta_1^m \times w.G.\mathscr{C} & \xrightarrow{(1)} & \Delta_1^m \times \Delta_2^n \times w.G.\mathscr{C} & \longrightarrow & w.G.(\mathscr{C}\times\mathscr{C}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle(6)} \\ \Delta_1^m & \xrightarrow{(\mathrm{id},v_n)} & \Delta_1^m \times \Delta_2^n & \xrightarrow{A} & w.G.\mathscr{C} \end{array}$$

$w.G.\mathscr{D}$ denotes the dimensionwise nerve of the simplicial category $wG.\mathscr{D}$. The diagonal of (5) is the map of the theorem. The map (6) is induced by projection to the first factor (i.e. the composition of (6) with (5) picks up the subobject of a cofibration sequence). The subscripts 1 and 2 on the standard simplices denote the corresponding simplicial directions, '1' is the 'w-direction'. v_n is the 'last vertex'-map. All squares in the diagram are pullback squares.

By a weak equivalence of bisimplicial sets we will mean a map whose diagonal is a weak equivalence. Recall that the diagonal of a map $f_{..} : X_{..} \to Y_{..}$ of bisimplicial sets is a weak equivalence if each $f_{.n}$ (or each $f_{n.}$) is a weak equivalence of simplicial sets. This will give us freedom to freeze different simplicial directions when considering different maps. To see that (5) is a weak equivalence we will compare the fibres of the map (6) with the fibres of the composition of (6) with (5).

(1) is a weak equivalence. (2) is a weak equivalence by [12, p. 339]. (For that argument we fix the second direction.) To prove that (5) is a weak equivalence it suffices by the argument in [5, Proposition 15.4] (shown in the Appendix) to see that (4) is a weak equivalence for all A and hence by the above that (3) is a weak equivalence for all A. This is a consequence of the pseudo-additivity. For this step we can, to begin with, fix the second direction.

Actually let

$$j : f^{-1}(m, 0, (\mathrm{id}, v_n)^*(A)) \to f^{-1}(m, 0, (\mathrm{id}, v_n)^*(A))$$

be the functor given by

$$(A \rightarrowtail C \twoheadrightarrow C/A) \mapsto (A \rightarrowtail A \vee C/A \twoheadrightarrow C/A) .$$

We will show that j is homotopic to the identity, from which it follows that (3) is a weak equivalence.

By the pseudo-additivity structure there is a natural sequence of weak equivalences pictured as follows

$$\begin{array}{ccccc} A & \rightarrowtail & C \cup_A (A \cup C/A) & \twoheadrightarrow & C/A \vee C/A \\ \| & & {\scriptstyle (6)}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle (7)}\big\downarrow{\scriptstyle \simeq} \\ A & \rightarrowtail & C \cup_A C & \twoheadrightarrow & C/A \vee C/A \end{array}$$

where (6) is the sequence given by the pseudo-additive structure and (7) is given by computing the cofibers using the base category. This means that there is a sequence of weak equivalences

$$\mathrm{id} \cup_A j \simeq \mathrm{id} \cup_A \mathrm{id} .$$

By Proposition 2.13 the functors $|\mathrm{id} \cup_A j|$ and $|\mathrm{id} \cup_A \mathrm{id}|$ are homotopic. Since by Lemma 3.3 below the space $|f^{-1}(m, 0, (\mathrm{id}, v_n)^*(A))|$ is a grouplike infinite loop space, we can subtract id and obtain that j is homotopic to the identity. □

Lemma 3.3. $|f^{-1}(m, 0, (\mathrm{id}, v_n)^*(A))|$ *is grouplike.*

Proof. Just as in the case of $wG.\mathscr{C}$ we will show that $\pi_0(f^{-1}(m, 0, (\mathrm{id}, v_n)^*(A)))$

is a group by exhibiting an inverse for the operation given by sum along A. It suffices to do this when the w-dimension is fixed.

The inverse is given by

$$-\left[\frac{A}{A'} \rightarrowtail \frac{C}{C'} \twoheadrightarrow \frac{B}{B'}\right] := \left[\frac{A}{A'} \rightarrowtail \frac{A \vee B'}{A' \vee B} \twoheadrightarrow \frac{B'}{B}\right].$$

Actually the zerosimplex

$$\frac{A}{A'} \rightarrowtail \frac{C \vee B'}{C' \vee B} \twoheadrightarrow \frac{B \vee B'}{B' \vee B}$$

is connected to the zerosimplex

$$\frac{A}{A'} \rightarrowtail \frac{A}{A'} \twoheadrightarrow \frac{*}{*}$$

via the 1-simplex

$$\begin{pmatrix} A = A \\ \downarrow \\ * \\ \| \\ * \\ \uparrow \\ A' = A' \end{pmatrix} \begin{matrix} \rightarrowtail \\ \\ \rightarrowtail \\ \\ \rightarrowtail \\ \\ \rightarrowtail \end{matrix} \begin{pmatrix} A \rightarrowtail C \vee B' \\ \downarrow \\ C/A \vee B' \\ \nwarrow\!\!\!\nearrow \\ C'/A' \vee B \\ \uparrow \\ A' \rightarrowtail C' \vee B \end{pmatrix} \begin{matrix} \twoheadrightarrow \\ \\ = \\ \\ = \\ \\ \twoheadrightarrow \end{matrix} \begin{pmatrix} * \rightarrowtail B \vee B' \\ \| \\ B \vee B' \\ \nwarrow\!\!\!\nearrow \\ B' \vee B \\ \| \\ * \rightarrowtail B' \vee B \end{pmatrix}$$

□

Remark 3.4. Let $\mathscr{S}^{\mathrm{f}}_{*}$ be the category of finite pointed simplical sets, $\mathscr{C}$ the corresponding category of finite suspension prespectra and $\mathscr{E}^{\mathrm{f}}_{*}$ the category of finite pointed sets. The natural functors

$$\mathscr{E}^{\mathrm{f}}_{*} \to \mathscr{S}^{\mathrm{f}}_{*} \to \mathscr{C}$$

give the following commutative diagram with weak equivalences an indicated:

$$\begin{array}{ccccc} wG.(\mathscr{E}^{\mathrm{f}}_{*}) & \longrightarrow & wG.(\mathscr{S}^{\mathrm{f}}_{*}) & \longrightarrow & wG.(\mathscr{C}) \\ \simeq\downarrow & & \downarrow & & \downarrow\simeq \\ \Omega wS.(\mathscr{E}^{\mathrm{f}}_{*}) & \longrightarrow & \Omega wS.(\mathscr{S}^{\mathrm{f}}_{*}) & \xrightarrow{\sim} & \Omega wS.(\mathscr{C}) \\ \simeq| & & \| & & \\ QS^0 & \longrightarrow & A(*) & & \end{array}$$

The upper-left vertical map is a weak equivalence since the assertion in the additivity theorem is induced by an equivalence of categories. On the other hand the upper-right vertical map is a weak equivalence by the pseudo-additive structure exhibited in Lemma 2.5.

Note that the diagram gives a description of the natural map from QS^0 to $A(*)$ using $G.$-models.

4. Iteration of the $G.$-construction

Theorem 2.8 is a consequence of the following:

Theorem 4.1. *For any $\mathscr{C} \in w\mathscr{C}of$ there are weak equivalences:*

$$wG^{\times}_{\bullet}\mathscr{C} \xrightarrow{\sim} \Omega wG^{\times}_{\bullet}S.\mathscr{C} \xleftarrow{\sim} \Omega wS.\mathscr{C} . \qquad \square$$

Aside from first explaining the maps this amounts to showing that $wG^{\times}_{\bullet}$ is a $w\mathscr{C}of$-admissible functor. As usual $g^{\times}_{\bullet}$ will be the version of $wG^{\times}_{\bullet}$ suppressing the w-direction.

We enter again the discussion following Proposition 3.1:

The first map is given by the composition of the maps in (a) and (b) of the proof of Theorem 2.6 from Proposition 1.55′ for $F. = wG^{\times}_{\bullet}$.

The second map is by taking loops in the iteration of the cited map (a) composed with the switch of simplical coordinates $wS.G^{\times}_{\bullet}\mathscr{C} \xrightarrow{\cong} wG^{\times}_{\bullet}S.\mathscr{C}$. The second map of the theorem is a weak equivalence by (a) of the cited proof since $wS.$ is a $w\mathscr{C}of$-admissible functor.

The first map of Theorem 4.1 will be a weak equivalence also by (b) of the cited proof once we have shown, that $wG^{\times}_{\bullet}$ too is a $w\mathscr{C}of$-admissible functor.

Properties (1), (2), (4), (5) follow from the corresponding properties for $wG.$ by taking colimits. So it remains to show the following:

Lemma 4.2. *$wG^{\times}_{\bullet}$ satisfies the additivity theorem.*

Proof. By the usual reduction it suffices to show, that $g^{\times}_{\bullet}$ satisfies the additivity theorem on categories with cofibrations. By the argument in [12, p. 332] with $\Omega^{\times} wS^{\times}_{\bullet}$ replaced by $g^{\times}_{\bullet}$ it suffices to show, that the two maps

$$t : \mathrm{ob}[E(\mathscr{C})] \to g.\mathscr{C} , \qquad [A \rightarrowtail C \twoheadrightarrow B] \mapsto C$$

and

$$s \vee q : \mathrm{ob}[E(\mathscr{C})] \to g.\mathscr{C} , \qquad [A \rightarrowtail C \twoheadrightarrow B] \mapsto A \vee B$$

are connected by a sequence of simplicial homotopies. (Whenever $\mathscr{C}$ stands for $g_{\cdot}^{n}\mathscr{S}$, then $s \vee q$ is always computed making choices in $\mathscr{D}$.)

The required homotopies are given by the three one-simplices

$$\begin{array}{ccc} t & \rightarrowtail & t \vee (s \vee q) \\ & & \downarrow \\ & & s \vee q \\ & & \| \\ * & \rightarrowtail & s \vee q \end{array} \tag{3}$$

$$\begin{array}{ccc} s \vee (s \vee q) & \rightarrowtail & t \vee (s \vee q) \\ & & \downarrow \\ & & q \\ & & \uparrow \\ s & \rightarrowtail & s \vee q \end{array} \tag{4}$$

$$\begin{array}{ccc} s \vee q & \rightarrowtail & s \vee (s \vee q) \\ & & \downarrow \\ & & s \\ & & \| \\ * & \rightarrowtail & s \end{array} \tag{5}$$

□

Appendix

Here we give the argument from [5] used in the proof of the additivity theorem. Let $f : E \to B$ be a map of bisimplicial sets. The pullback square

$$\begin{array}{ccc} f^{-1}(x) & \longrightarrow & E \\ \downarrow & & \downarrow \\ \Delta_1^m \times \Delta_2^n & \xrightarrow{x} & B \end{array}$$

defines a functor $f^{-1} : \Delta \times \Delta / B \to \mathscr{B}is$, where $\mathscr{B}is$ is the category of bisimplicial sets and $f^{-1}(x)$ is the fibre over x. We will show that if a map of bisimplicial sets over B induces weak equivalences of fibres over all $x \in \mathrm{Ob}(\Delta \times \Delta / B)$ then it is a weak equivalence.

The maps $f^{-1}(x) \to E$ combine to a map $p_f : \mathrm{hocolim}_{\Delta \times \Delta / B} f^{-1} \to E$. Here $\mathrm{hocolim}_{\Delta \times \Delta / B} f^{-1}$ stands for the trisimplicial set

$$(k,l,m) \longmapsto \coprod_{\substack{\Delta_1^{m_0}\times\Delta_2^{n_0}\to\cdots\to\Delta_1^{m_k}\times\Delta_2^{n_k} \\ \searrow_x \quad \downarrow \\ B}} f^{-1}(x)_{(l,m)}$$

The diagonal of this space is the ho-colim of [1]. The squares in the diagram

$$\begin{array}{ccc}
\operatorname{hocolim}_{\Delta\times\Delta/B} g^{-1} & \xrightarrow{p_g} & E_1 \\
\downarrow & & \downarrow h \\
\operatorname{hocolim}_{\Delta\times\Delta/B} f^{-1} & \xrightarrow{p_f} & E_2 \\
\downarrow & & \downarrow f \\
\operatorname{hocolim}_{\Delta\times\Delta/B} (\mathrm{id}_B)^{-1} & \xrightarrow{p_{\mathrm{id}_B}} & B
\end{array}$$

where $g = f \circ h$ are cartesian. The map $p_{\mathrm{id}_B} : \operatorname{hocolim}_{\Delta\times\Delta/B}(\mathrm{id}_B)^{-1} \to B$ is a map of trisimplicial sets, when considering the bisimplicial set B as a trisimplicial set in a trivial way. Then, freezing the two original simplicial directions, we have a dimensionwise weak equivalence. To see this we note that the inverse image of $b \in B_{l,m}$ is the nerve of the category of bisimplices in B under b, which is contractible since this category has an initial object. Observing that $p_f : \operatorname{hocolim}_{\Delta\times\Delta/B} f^{-1} \to E$ has the same fibres as $p_{\mathrm{id}B}$, it follows that p_f is a weak equivalence. Hence h is a weak equivalence iff $\operatorname{hocolim}_{\Delta\times\Delta/B}(h)$ is a weak equivalence, and this is the case if each of the maps between the fibres $g^{-1}(x) \to f^{-1}(x)$ are weak equivalences for all $x \in \mathrm{Ob}(\Delta \times \Delta/B)$, since ho-colim preserves weak equivalences [1].

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