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Algebraic K-theory of generalized free products, Part 1

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This paper gives a contribution to the computation of algebraic K-theory in certain cases. The main application is of a geometric nature, a vanishing theorem for Whitehead groups. It was part of the latter result that originally motivated the present work. So it may be appropriate to sketch how one arrives at considering it.

There is a result in 3-dimensional topology which says that any boundary preserving homotopy equivalence between compact 3-manifolds of a certain kind is induced by a homeomorphism [25]; for example the result applies to manifolds obtained from the 3-sphere by removing an open tubular neighborhood of a tame knot. There is no mention of Whitehead torsion in this result. But one should certainly expect Whitehead torsion to enter rather crucially, in view of experience with high-dimensional h-cobordisms on the one hand, and 3-dimensional lens spaces on the other hand. The dilemma suggests:

Conjecture. The Whitehead group of a classical knot group is trivial.

The aforementioned geometric result involves two main steps:

Firstly, by decomposing at a 2-manifold of a certain kind, and repeating this procedure sufficiently often (a finite number of times), certain 3-manifolds can be reduced to 'nothing,' that is, to simply connected pieces. This is a deep result [8].

On the level of fundamental groups, such a decomposition corresponds to a *generalized free product structure* (that is, either a free product with amalgamation, or its companion construction, HNN extension), and the 'reduction to nothing' corresponds to the fact that conversely the fundamental group can be built up out of nothing (the trivial group) by iterated generalized free product. This is how attention is drawn to such groups.

Secondly, the homotopy equivalence under consideration can be *split* at

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the decomposing 2-manifold in the image space, meaning that the restriction of the homotopy equivalence to the pre-image of the 2-manifold, is also a homotopy equivalence.

The analogue of the latter should be the following, from which one may hope to deduce the conjecture above:

Conjecture. For certain amalgamated free products $G = G_1 *_{G_0} G_2$ there is an exact sequence

$$\mathrm{Wh}_1(G_0) \longrightarrow \mathrm{Wh}_1(G_1) \oplus \mathrm{Wh}_1(G_2) \longrightarrow \mathrm{Wh}_1(G) \\ \longrightarrow \mathrm{Wh}_0(G_0) \longrightarrow \mathrm{Wh}_0(G_1) \oplus \mathrm{Wh}_0(G_2) \longrightarrow \mathrm{Wh}_0(G)$$

where Wh_1 denotes the Whitehead group, and $\mathrm{Wh}_0(G) = \tilde{K}_0(ZG)$, the reduced projective class group of the integral group algebra; similarly for certain HNN extensions.

In a sense, the purpose of this paper is to prove these conjectures in their proper setting.

Firstly, the latter conjecture should be reformulated in terms of algebraic K-theory; that is,

$$K_1(R_0) \to K_1(R_1) \oplus K_1(R_2) \to K_1(R) \to K_0(R_0) \to K_0(R_1) \oplus K_0(R_2) \to K_0(R)$$

is exact for group algebras of certain free products with amalgamation.

Secondly, the property that the rings involved are group algebras should be dispensed with. Thus one should use:

(i) a notion of *generalized free product* of rings which in particular captures the amalgamated free product of groups on the level of group algebras,

(ii) a notion of *generalized Laurent extension* of rings which in particular captures the HNN extension of groups on the level of group algebras.

Thirdly, the exact sequence envisaged should be the exact sequence in low degrees of the long exact sequence of a fibration. Thus one should consider the functor from rings to spaces, due to Quillen, $R \mapsto K(R)$, whose homotopy groups give the K-groups of R, $\pi_i K(R) = K_i(R)$, $i = 0, 1, \cdots$.

It turns out that all this can be done.

Going into more detail now, we suppose *rings* always have an identity element which is to be respected by maps. An embedding of rings $\alpha: C \to A$ will be called *pure* if there exists a splitting of C-bimodules, $A = \alpha(C) \oplus A'$. The actual splitting is not part of the data, just its existence. It is convenient though to refer to a fixed complement A' of $\alpha(C)$ in A. We will always have to assume that A' is free as a left C-module (actually, by a little trick the results below can be extended to the case when A' is left projective only, but we will not need this).

We say R is a *generalized free product* if it is the pushout in the category of rings, of pure embeddings $\alpha: C \to A$, $\beta: C \to B$, cf. [5]. Let

$$(K(\alpha), -K(\beta)): K(C) \longrightarrow K(A) \times K(B)$$

be the map in the homotopy category whose second component is the composition of the induced map $K(\beta)$ with a homotopy inverse on the (homotopy everything) H-space $K(B)$. We assume the complements A' and B' of $\alpha(C)$ and $\beta(C)$, respectively, are free from the left.

THEOREM 1. *In this situation there exists a space $\tilde{K}\mathfrak{Nil}(C; A', B')$ whose homotopy type depends only on the ring C and the C-bimodules A' and B'. The loop space $\Omega K(R)$ is the direct product, up to homotopy, of this space $\tilde{K}\mathfrak{Nil}(C; A', B')$ and of the homotopy theoretic fibre of the map $(K(\alpha), -K(\beta))$.*

Alternatively if $\alpha, \beta: C \to A$ are pure embeddings, the *generalized Laurent extension of A with respect to (α, β)* is a ring R which contains A as a subring, and contains an invertible element t so that

$$\alpha(c)t = t\beta(c)\,, \qquad c \in C\,,$$

and which is universal with respect to these properties. We denote

$$(K(\alpha) - K(\beta)): K(C) \longrightarrow K(A)$$

the map in the homotopy category which is the sum (with respect to the H-space structure of $K(A)$) of the maps $K(\alpha)$ and $-K(\beta)$. We assume the complements A' and A'' of $\alpha(C)$ and $\beta(C)$, respectively, are free from the left.

THEOREM 2. *In this situation there exists a space $\tilde{K}\mathfrak{Nil}(C; A', A'', {}_{\alpha}A_{\beta}, {}_{\beta}A_{\alpha})$ whose homotopy type only depends on the ring C and the C-bimodules indicated. The loop space $\Omega K(R)$ is the direct product, up to homotopy, of this space $\tilde{K}\mathfrak{Nil}(C; A', A'', {}_{\alpha}A_{\beta}, {}_{\beta}A_{\alpha})$ and of the homotopy theoretic fibre of the map $(K(\alpha) - K(\beta))$.*

It turns out that a third case, generalizing polynomial extensions, can be included at no extra cost. Let S be a C-bimodule. We assume S is left free. In addition we have to assume that S is finitely generated projective from the right. Let R be the tensor algebra of S.

THEOREM 3. *In this situation there exists a space $\tilde{K}\mathfrak{Nil}(C; S)$. The loop space $\Omega K(R)$ is the direct product, up to homotopy, of $\tilde{K}\mathfrak{Nil}(C; S)$ and the loop space $\Omega K(C)$.*

It is for simplicity of exposition only that the theorems have been formulated to apply to the loop space of $K(R)$. A more unpleasant feature is the appearance of the exotic term $\tilde{K}\mathfrak{Nil}$. Fortunately there is a vanishing theorem. It involves a condition on the ring C.

We say the ring C is *coherent* if its finitely presented right modules form an abelian category (or equivalently, if any finitely generated submodule of a free module is finitely presented). We say C is *regular coherent* if it is coherent and if in addition any finitely presented right module has finite projective dimension. Equivalently, C is regular coherent if any finitely presented right C-module has a finite resolution by finitely generated projective modules.

THEOREM 4. *In any of the situations of Theorems* 1, 2, 3, *a sufficient condition for the space* $\tilde{K}\mathfrak{Nil}(C; \cdots)$ *to be contractible, is that the ring* C *be regular coherent.*

From Theorems 1, 2, and 4, a rather striking computation of the K-theory of certain group algebras can be obtained. It is remarkable that to derive and formulate it properly, one is almost forced to re-introduce Whitehead groups—as a computational tool. Some machinery is required. The space $K(R)$ is in a natural way the *underlying space* of a Γ*-space* in the sense of Segal [22]. It is therefore the 'coefficients' of a homology theory as described by Anderson [2]. For our purposes this means that there is a functor from spaces to spaces, $X \mapsto K(X; R)$ so that $K(\text{pt.}; R) \overset{\simeq}{\to} K(R)$, and $X \mapsto \pi_* K(X; R)$ is a (generalized) homology theory, meaning that it satisfies the Eilenberg-Steenrod axioms except for the dimension axiom.

There is a natural transformation

$$K(BG; R) \longrightarrow K(RG)$$

when BG is the classifying space of a group G, and RG its group algebra over R.

There is a functor from pairs (R, G) to spaces

$$(R, G) \longmapsto \mathrm{Wh}^R(G)$$

which we refer to as the *Whitehead space of* G, *relative to* R, and there is a sequence of the homotopy type of a fibration

$$K(BG; R) \longrightarrow K(RG) \longrightarrow \mathrm{Wh}^R(G)$$

which is natural in (R, G). Thus one could say that the Whitehead space measures to what extent $K(RG)$ differs from a homology theory when R is fixed and G varies (actually, the existence of this fibration is merely a rephrasing of the definition of $\mathrm{Wh}^R(G)$).

We define the *Whitehead groups* of G to be the groups

$$\mathrm{Wh}_i(G) = \pi_i \mathrm{Wh}^Z(G)$$

where Z is the ring of integers. This definition is justified by the fact that it leads to the usual Whitehead groups when the latter are defined (the cases

$i = 0, 1, 2$). Further justification is that all of these groups are related to certain phenomena in geometric topology [27].

THEOREM 5. *There is a class* Cl *of groups which contains: free groups, free abelian groups, torsion free one-relator groups, fundamental groups* $\pi_1 M$ *where* M *is any submanifold of the 3-sphere. For any group* G *in this class, and any regular noetherian ring* R, *the space* $\mathrm{Wh}^R(G)$ *is contractible.*

Formulated in terms of K-theory, Theorem 5 says that $K(BG; R) \to K(RG)$ is a homotopy equivalence if $G \in \mathrm{Cl}$ and R is regular noetherian. This can be applied in two ways. First, in view of the spectral sequence of a generalized homology theory, if a homomorphism of groups $g: G_1 \to G_2$ induces an isomorphism on integral homology then it induces an isomorphism on any homology theory. For example if G_1 is a classical knot group, and G_2 the free cyclic group, the abelianization homomorphism $G_1 \to G_2$ is an integral homology equivalence; as $G_1, G_2 \in \mathrm{Cl}$, their K-groups are thus canonically isomorphic.

Secondly, one may work out the spectral sequence when it nearly collapses. This way one obtains the following which in particular applies to all the groups listed in Theorem 5 except for the free abelian groups. Let H_* denote ordinary homology and $\tilde{K}_*$ reduced K-theory; that is, $\tilde{K}_i(RG)$ is the summand in the canonical splitting $K_i(RG) = K_i(R) \oplus \tilde{K}_i(RG)$.

COROLLARY. *Let* $G \in \mathrm{Cl}$ *and suppose that* $H_i(BG, A) = 0$ *for all* $i \geqq 3$ *and all abelian groups* A. *Let* R *be regular noetherian. Then*

$$\tilde{K}_0(RG) = 0\ , \quad \tilde{K}_1(RG) = H_1(BG, K_0(R))$$

and for $i \geqq 2$ *there is a short exact sequence*

$$0 \longrightarrow H_1(BG, K_{i-1}(R)) \longrightarrow \tilde{K}_i(RG) \longrightarrow H_2(BG, K_{i-2}(R)) \longrightarrow 0\ .$$

From the close connection between 'splitting theorems' for Whitehead groups on the one hand and Siebenmann's treatment of 'infinite simple homotopy types' [23] on the other hand, it is clear, more or less, that the latter theory has an analogue for algebraic K-theory which could be treated with the methods of this paper. There is a corresponding generalization of Karoubi's theory of 'exact sequences of categories' [11] as this theory is a close relative, on the K_0 and K_1 levels, to the analogue of 'infinite simple homotopy theory.'

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I. Structure theory in generalized free product situations

1. *Free products.* As mentioned in the introduction, we say an inclusion of rings $\alpha: C \to A$ is *pure* if there exists a splitting of C-bimodules

$$A = \alpha(C) \oplus A' .$$

It is convenient to fix such a splitting once and for all.

Let $\alpha: C \to A$ and $\beta: C \to B$ both be pure. Then the *free product of A and B, amalgamated at C* (*with respect to* α, β, to be precise), exists; by definition, it is the colimit in the category of rings (with 1, as always) in the diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ B & \longrightarrow & R . \end{array}$$

Our choice of complements A' and B' of $\alpha(C) \subset A$ and $\beta(C) \subset B$, respectively, determines a decomposition of R as C-bimodule which on identification of C, A, B with subrings of R, and other abuse of language, can be described thus, cf. [4],

$$R = C \oplus A' \oplus B' \oplus A' \otimes_C B' \oplus B' \otimes_C A' \oplus A' \otimes_C B' \otimes_C A' \oplus \cdots .$$

Denoting A'_n the term in this decomposition which involves n factors and *has A' on the left* (and B'_n similarly) and putting $A'' = \bigoplus A'_n$, $B'' = \bigoplus B'_n$, we have $R = C \oplus A'' \oplus B''$. Collecting differently, we obtain in obvious fashion

$$R = A \oplus A \otimes_C B'' \text{ and } R = B \oplus B \otimes_C A'' .$$

The point of these latter decompositions is that they are compatible with the left A- (respectively B-) structure on R. It is in facts like these that we use the multiplicative structure of R.

From now on we assume A' and B' are free as left C-modules, and we choose bases. The basis of A' is denoted $\langle A' \rangle$. Our choice determines bases for the other terms in the decomposition of R, e.g., with a convenient abuse of notation we can write $\langle A' \otimes_C B' \rangle = \langle A' \rangle \cdot \langle B' \rangle$. Naturally, as the element of $\langle C \rangle$ we choose the identity.

There are still more bases around. The canonical A-isomorphism $R = A \oplus A \otimes_C B''$ determines a left A-basis of R that we denote T_A. Considered as a subset of R, T_A is the same as $\langle C \rangle \cup \langle B'' \rangle$. We may define T_B similarly.

We will have to work a lot with all these bases. This work is facilitated (and indeed made possible) by the fact that we can put on more structure. We claim *there is a tree T whose set of vertices, T^0, is the disjoint union*

$$T^0 = T_A \cup T_B$$

and whose set of segments is

$$T^1 = \langle R \rangle .$$

To justify the claim, we have to define incidence and do some checking.

Now bases were defined in such a way that we have an isomorphism $\langle A \rangle \times T_A \to \langle R \rangle$ which we abbreviate as $\langle A \rangle \cdot T_A = \langle R \rangle$, and similarly $\langle B \rangle \cdot T_B = \langle R \rangle$. So we declare if $x \in T_A$, say, then $T^1(x)$ (the set of segments incident to x) shall be given by (the values of) $\langle A \rangle \cdot x$. Incidence has thus been defined. We choose $1 \in T_A$ to be the basepoint of T. The following rule puts orientations on the segments: For every vertex x, except the basepoint, there is precisely one segment $s_0(x)$ whose terminal vertex is x, namely $1 \cdot x$. One should think of $s_0(x)$ as the unique segment incident to x that is contained in the shortest path from the basepoint to x. This interpretation of $s_0(x)$ gives at once a length function on T^0 (distance from the basepoint), and by induction on length one sees that T is indeed a tree (one uses that for every $r \in R$, $r = 1$, if $r = a \cdot x = b \cdot y$ where $a \in A$, $b \in B$, $x \in T_A$, $y \in T_B$, then precisely one of a and b is 1).

After these preparatory remarks about the structure of R, we come to the definition of the diagram categories, which we will use constantly.

Definition. A *splitting diagram* consists of right modules

$$M_A, M_B, M_C$$

over the rings, A, B, C, respectively, and a map over R

$$M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R$$

satisfying

$$\kappa(M_A) \subset M_C \otimes_C A \quad \text{and} \quad \kappa(M_B) \subset M_C \otimes_C B \,.$$

It is often convenient to write κ as the difference of canonical components, $\kappa = \kappa_\alpha - \kappa_\beta$, $\kappa_\alpha(M_B) = 0$, $\kappa_\beta(M_A) = 0$. A *map* of splitting diagrams is a triple of maps, over A, B, C, respectively, satisfying the obvious condition. The resulting category is abelian since $(?) \otimes_C R$, etc., are exact functors. An exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \,,$$

part of which is a splitting diagram, is called a *completed splitting diagram*; the map

$$\kappa_\alpha \circ \iota = \kappa_\beta \circ \iota \colon M \longrightarrow M_C \otimes_C R$$

will be referred to as the *cross-term*. A *map* of completed splitting diagrams is a certain quadruple of maps. A completed splitting diagram is called a *Mayer Vietoris presentation* (*of the R-module M*) if the sequence

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

is exact. The category of Mayer Vietoris presentations is not itself abelian. However, if we relax 'exact' to 'order two sequence,' we do get an abelian category. In the latter category, the category of Mayer Vietoris presentations sits as a full subcategory which, by the 3×3 lemma, is closed under extensions.

A *split module* is a splitting diagram

$$M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R$$

where the map κ happens to be an R-isomorphism. The category of split modules is, by definition, a full subcategory of the category of splitting diagrams, and it is itself an abelian category. Eventually it will be useful to consider split modules as Mayer Vietoris presentations of the zero R-module.

Our first result asserts, among other things, that the category of Mayer Vietoris presentations has enough maps.

PROPOSITION 1.1. *Let N be the free right R-module on the basis element n. Let Δ be a finite subtree of T, containing the basepoint. There exists a canonical Mayer Vietoris presentation $\langle N, n, \Delta \rangle$ of N. Also given $m \in M$, there exists a map of $\langle N, n, \Delta \rangle$ into the completed splitting diagram*

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R$$

inducing $n \to m$, *if and only if* Δ *contains a certain finite tree* $\Delta(m)$. *The entire map is uniquely determined by* m.

Proof. Writing $\iota(m)$ in terms of its components, say $\iota(m) = \iota_\alpha(m) + \iota_\beta(m)$, we can express $\iota_\alpha(m)$ in terms of the left A-basis of R,

$$\iota_\alpha(m) = \sum_{x \in T_A} m_x \cdot x ,$$

where $m_x \in M_A$, the m_x are uniquely determined by $\iota_\alpha(m)$, and only finitely many are non-zero. Now κ_α is a right R-map, so

$$\kappa_\alpha(\iota_\alpha(m)) = \sum_{x \in T_A} \kappa_\alpha(m_x) \cdot x$$

and

$$\kappa_\alpha(m_x) = \sum_{a \in \langle A \rangle} m_{x,a} \cdot a$$

by assumption about κ. Therefore

$$\kappa_\alpha(\iota_\alpha(m)) = \sum_{x \in T_A} \sum_{a \in \langle A \rangle} m_{x,a} \cdot a \cdot x .$$

Similarly

$$\kappa_\beta(\iota_\beta(m)) = \sum_{y \in T_B} \sum_{b \in \langle B \rangle} m_{y,b} \cdot b \cdot y .$$

On the other hand, $\kappa_\alpha(\iota_\alpha(m)) = \kappa_\beta(\iota_\beta(m))$ can be expressed in terms of the left C-basis of R, say $\kappa_\alpha(\iota_\alpha(m)) = \sum_{s \in T^1} m_s \cdot s$, and there are for each $s \in T^1$ precisely one product $a \cdot x$, and precisely one product $b \cdot y$, that on evaluation yield s (note that x and y are just the vertices incident to s). Therefore we must have

$$m_{x,a} = m_s = m_{y,b}$$

for those particular indices.

We define $\Delta(m)$ to be the smallest subtree of T which contains all those $x \in T_A$ and $y \in T_B$ for which m_x, respectively m_y, is non-zero.

Let Δ^0 and Δ^1 denote, respectively, the sets of vertices and segments of the given based finite tree Δ. We let

$$\Delta^0_A = \Delta^0 \cap T_A , \quad \Delta^0_B = \Delta^0 \cap T_B .$$

$\langle N, n, \Delta \rangle$ is defined thus. N_A is the free right A-module on the basis elements

$$n_x, \; x \in \Delta^0_A ,$$

and N_B similarly. N_C is the free C-module on the basis

$$n_s, \; s \in \Delta^1 .$$

In order to define $\kappa_\alpha + \kappa_\beta$ (notice 'plus' instead of 'minus'), it is enough to describe its components, a component being the restriction to a summand in the source, projected to a summand in the target. Let $s \in \Delta^1$. Then one of its endpoints, say x, is in Δ_A^0, and the other one, y, is in Δ_B^0. Also there are unique elements $a \in \langle A \rangle$ and $b \in \langle B \rangle$ so that $a \cdot x = s = b \cdot y$ as elements of R (notice if y, say, is the terminal vertex of s, then necessarily $b = 1$). By definition, there are precisely two non-zero components of $\kappa_\alpha + \kappa_\beta$ going into $n_s \cdot C \otimes_C R$, and they are given by

$$n_x \longmapsto n_s \cdot a \quad \text{and} \quad n_y \longmapsto n_s \cdot b \ .$$

The map ι in the sequence

$$n \cdot R \approx N \xrightarrow{\ \iota\ } N_A \otimes_A R \oplus N_B \otimes_B R \xrightarrow{\ \kappa\ } N_C \otimes_C R$$

is defined by

$$\iota(n) = \sum\nolimits_{x \in \Delta_A^0} n_x \cdot x + \sum\nolimits_{y \in \Delta_B^0} n_y \cdot y \ .$$

The cross-term becomes

$$\kappa_\alpha\big(\iota(n)\big) = \sum\nolimits_{s \in \Delta^1} n_s \cdot s \ .$$

If Δ' is another tree containing the base point, and $\Delta' \subset \Delta$, there is a *canonical* projection $\langle N, n, \Delta \rangle \to \langle N, n, \Delta' \rangle$, characterized by $n \to n$. Applying this remark inductively in the situation where Δ'^0 is Δ^0 minus an extreme vertex, and checking what the kernel is, one verifies that indeed $\langle N, n, \Delta \rangle$ is a Mayer Vietoris presentation. The proof of the proposition is now easily completed by comparing the definition of $\langle N, n, \Delta \rangle$ to the analysis of $\kappa_\alpha\big(\iota(n)\big)$ given before.

Our next result asserts that there exist many maps from split modules to Mayer Vietoris presentations. To state it, we need some more language.

An *augmented tree* in T, denoted by ${}^+\Delta$ or some similar symbol, shall consist of subsets ${}^+\Delta^0 \subset T^0$ and ${}^+\Delta^1 \subset T^1$ so that ${}^+\Delta^0$ and all but one, say s, of the elements of ${}^+\Delta^1$ form a tree, and the extra element s, called the *augmentation segment*, is incident to some element of ${}^+\Delta^0$. Or, what is the same, an augmented tree is a subtree of T together with an extra segment stuck on. It is convenient to admit the empty set as an augmented tree. A non-empty augmented tree is called *based* if the augmentation segment is given by $1 \in \langle R \rangle$. We use the notation ${}_A\Delta$ for a based augmented tree if the vertex incident to the augmentation segment is in T_A.

PROPOSITION 1.2. *To any finite based augmented tree ${}^+\Delta$, there is canonically associated a split module $\langle {}^+\Delta \rangle$. And if*

$$M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\ \kappa\ } M_C \otimes_C R$$

is any splitting diagram, and

$$m' \in M_C \cap \operatorname{Im}(\kappa)$$

then there exist finite based augmented trees ${}_A\Delta$ *and* ${}_B\Delta$ *and a map from* $\langle {}_A\Delta\rangle \oplus \langle {}_B\Delta\rangle$ *whose image contains* m'.

Proof. As $m' \in \operatorname{Im}(\kappa)$, there are finite sums so that

$$m' = \kappa_\alpha\left(\sum\nolimits_{x \in T_A} m_x \cdot x\right) - \kappa_\beta\left(\sum\nolimits_{y \in T_B} m_y \cdot y\right).$$

Putting $\kappa_\alpha(m_x) = \sum_{a \in \langle A\rangle} m_{x,a} \cdot a$, where $m_{x,a} \in M_C$, and similarly for $\kappa_\beta(m_y)$, we have

$$m' = \sum\nolimits_{x \in T_A} \sum\nolimits_{a \in \langle A\rangle} m_{x,a} \cdot a \cdot x - \sum\nolimits_{y \in T_B} \sum\nolimits_{b \in \langle B\rangle} m_{y,b} \cdot b \cdot y .$$

On evaluating the double sums, and adding, we express m' in terms of the left C-basis of R,

$$m' = \sum\nolimits_{s \in T^1} m_s \cdot s$$

where $m_s = m_{x,a} - m_{y,b}$ for the unique terms $a \cdot x$ and $b \cdot y$ that on evaluation yield s. Now $m' \in M_C$, i.e., $m_s = 0$ unless $s = 1$. Therefore

$$m_{x,a} - m_{y,b} = m' \quad \text{if} \quad a \cdot x = b \cdot y = 1 ,$$

and $m_{x,a} = m_{y,b}$ otherwise, i.e., if $a \cdot x = b \cdot y \neq 1$.

We now define the split module $\langle {}^+\Delta\rangle$ where ${}^+\Delta$ is any finite based augmented tree. The construction is closely related to the construction of those 'standard' Mayer Vietoris presentations above. We let ${}^+\Delta_A = {}^+\Delta^0 \cap T_A$, and ${}^+\Delta_B = {}^+\Delta^0 \cap T_B$. Then N_A will be the free right A-module on the basis elements

$$n_x,\ x \in {}^+\Delta_A ,$$

and N_B similarly. N_C is the free C-module on the basis

$$n_s,\ s \in {}^+\Delta^1 .$$

The components of the map $\kappa_\alpha + \kappa_\beta$ ('plus' instead of 'minus') are these. If $s \in {}^+\Delta^1$ is different from the basic segment, the summand $n_s \cdot C \otimes_C R$ receives two components of the map. Also if x and y are the vertices incident to s, and, say, $a \cdot x = s = b \cdot y$ as elements of R (where $a \in \langle A\rangle$, $b \in \langle B\rangle$, and one of a and b must be 1) then these components are given by

$$n_x \longmapsto n_s \cdot a \quad \text{and} \quad n_y \longmapsto n_s \cdot b .$$

If s is the basic segment, there is just one component. We have

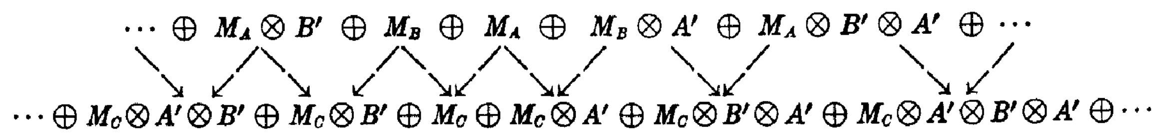
$$\begin{aligned} \kappa\left(\sum\nolimits_{x \in {}^+\Delta_A} n_x \cdot x + \sum\nolimits_{y \in {}^+\Delta_B} n_y \cdot y\right) &= n_1,\ 1 \in \langle R\rangle, \ \text{if } {}^+\Delta \text{ is of type } {}_A\Delta \text{ and} \\ &= -n_1, \ \text{if } {}^+\Delta \text{ is of type } {}_B\Delta. \end{aligned}$$

On comparison, it is now clear that the required map can be defined on $\langle {}_A\Delta\rangle \oplus \langle {}_B\Delta\rangle$ as soon as all the vertices x and y, for which m_x and m_y above are non-zero, are contained in the union of ${}_A\Delta$ and ${}_B\Delta$. (Obviously, there can*not* be a uniqueness assertion in the proposition.)

The remainder of the section is devoted to an analysis of split modules. If

$$M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R$$

is a split module, we can display some of its structure in the diagram



In this diagram, all maps and tensor products are over C. The lower row is the decomposition of $M_C \otimes_C R$ induced from the C-bimodule decomposition of R. Likewise, the upper row assembles to $M_A \otimes_A R \oplus M_B \otimes_B R$; the proof of this uses the fact that $R = A \otimes_C (C \oplus B' \oplus B' \otimes A' \oplus \cdots)$ as left A-module, etc. The tensor products appearing to the left hand side of M_C all have B' as their right-most factor, likewise those to the right have A' as their right-most factor. The point of the diagram is that it depicts κ in terms of component maps, and that the arrows show all component maps that can possibly be non-zero.

Let M^{ur} be the sum of the terms in the *u*pper row that are to the *r*ight hand side of M_C,

$$M^{ur} = M_A \oplus M_B \otimes A' \oplus M_A \otimes B' \otimes A' \oplus \cdots$$

and M^{lr}, M^{ul}, M^{ll} similarly. With

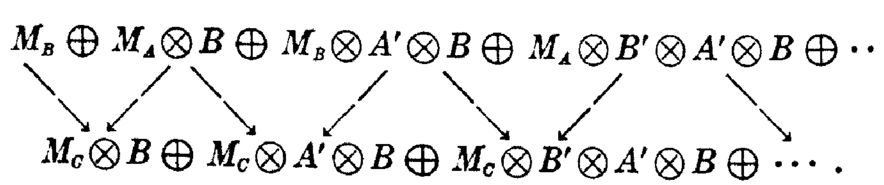
$$\begin{aligned} P &= M_C \cap \operatorname{Im}(M^{ur} \longrightarrow M_C \oplus M^{lr}) \\ &\approx \ker(M^{ur} \longrightarrow M^{lr}) \end{aligned}$$

and Q similarly, we must have

$$M_C = P \oplus Q ,$$

and in particular $\kappa^{-1}(P) \subset M^{ur}$ and $\kappa^{-1}(Q) \subset M^{ul}$.

In the above diagram, any two terms that are situated symmetrically with respect to M_A, add up to an A-module. Hence there are two folded versions of the diagram. The *B-folded diagram* is obtained from the above by folding at the place of M_B and adding up corresponding terms. The resulting diagram of B-modules is



The point of folding is that the terms to the right hand side of M_B are precisely the same as before, except that B has been tensored on from the right. The same is true of the maps, except again the first one. In more condensed notation, the B-folded diagram thus becomes

$$\begin{array}{ccccc} M_B & \oplus & M^{ur}\otimes B & & \\ & \searrow \quad \swarrow & & \searrow & \\ M_C\otimes B & \oplus & & & M^{lr}\otimes B \end{array}$$

and we see that κ induces an isomorphism $\ker(M^{ur} \to M^{lr}) \otimes B \to P \otimes B$. Consequently, the restriction $\kappa \mid M_B$ must be the sum of an isomorphism

$$j_B\colon M_B \xrightarrow{\approx} Q \otimes B$$

and some map

$$k_B\colon M_B \longrightarrow P \otimes B\ .$$

Similarly, $\kappa^{-1} \mid Q \otimes B$ is the sum of j_B^{-1} and some map

$$l_B\colon Q \otimes B \longrightarrow M^{ur} \otimes B\ ,$$

and these decompositions are related by the fact that

$$\kappa \circ \kappa^{-1} \mid Q \otimes B = j_B \circ j_B^{-1} + k_B \circ j_B^{-1} + \kappa \circ l_B$$

and hence

$$k_B \circ j_B^{-1} = (-1)\kappa \circ l_B\ .$$

The point of considering the decomposition of $\kappa^{-1} \mid Q \otimes B$ is that its restriction to Q can be located in the original diagram. From this one sees that $l_B(Q)$ is contained already in the part $M^{ur} \otimes B'$ of M^{ul}. But the map $M^{ur} \otimes B \to P \otimes B$ is of type $(?) \otimes B$, therefore $\kappa(l_B(Q)) \subset P \otimes B'$. Consequently we have proved that the composition

$$Q \longrightarrow Q \otimes B \xrightarrow{j_B^{-1}} M_B \xrightarrow{k_B} P \otimes B$$

has image in $P \otimes B'$. Factoring off the inclusion $P \otimes B' \to P \otimes B$, we thus have a map which we denote

$$q\colon Q \longrightarrow P \otimes B'\ .$$

Likewise we have a map

$$p\colon P \longrightarrow Q \otimes A'\ .$$

Identifying M_B to $Q \otimes B$ by means of j_B, and M_A to $P \otimes A$ by means of j_A, we have proved

PROPOSITION 1.3. *There exists an exact equivalence of the category of split modules with a full subcategory of the category whose objects are the quadruples (P, Q, p, q) where P and Q are C-modules, and p and q are C-maps*

$$p\colon P \longrightarrow Q \otimes A'\ , \quad q\colon Q \longrightarrow P \otimes B'\ .$$

Given such an object as in the proposition, we define a pair of filtrations

$$0 = P_0 \subset P_1 \subset \cdots \subset P ,$$
$$0 = Q_0 \subset Q_1 \subset \cdots \subset Q$$

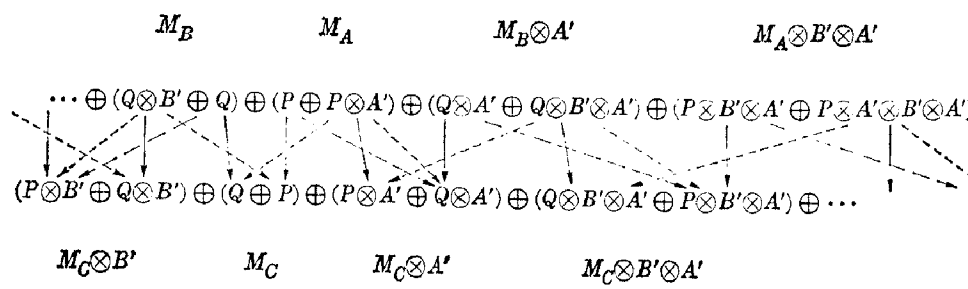
inductively by the rule

$$P_{n+1} = p^{-1}(Q_n \otimes A') , \quad Q_{n+1} = q^{-1}(P_n \otimes B') ,$$

and we call the object (P, Q, p, q) *nilpotent* if these filtrations converge to P and Q, respectively. We let $\mathfrak{Nil}(C; A', B')$ denote the full subcategory of nilpotent objects.

PROPOSITION 1.4. *In Proposition* 1.3, *the subcategory in question is* $\mathfrak{Nil}(C; A', B')$.

In order to verify the assertion, we consider once more the diagram displaying the components on κ. Squeezing in the information we gathered in the meantime, we can rewrite the diagram thus:



The meaning of the different sorts of arrows is this. The solid arrows denote identity maps. The broken ones are p and q, and maps obtained from these by tensoring on an identity. About the remaining maps we cannot say very much (the ring structures of A and B enter) *except* that we know where they go; they are the dotted arrows; e.g., no dotted arrow starts from P.

Conversely, if we have an object (P, Q, p, q) in the above sense, we can construct such a diagram. Let us look at it and figure out the properties of κ.

The basic observation is this. If $m \in M_A \otimes_A R \oplus M_B \otimes_B R$ is such that its image $\kappa(m)$ happens to be in $P \oplus Q$, then the decomposition of m cannot involve an element of the summand $P \otimes A'$ or of any other summand from which a dotted arrow starts. Indeed, if there were a contribution from $P \otimes A'$, say, there had to be a contribution from $Q \otimes B' \otimes A'$, and so on, and we could never stop.

So $\kappa^{-1}(P \oplus Q)$ and $\mathrm{Im}(\kappa) \cap (P \oplus Q)$ are unchanged if we discard all those summands in the source from which a dotted arrow starts. So κ is automatically injective. Also κ is surjective if and only if $P \oplus Q \subset \mathrm{Im}(\kappa)$

which visibly is the case if and only if (P, Q, p, q) is nilpotent.

One aspect of the category $\mathfrak{Nil}(C; A', B')$ is that it visibly depends only on the bimodules A' and B', and so it has *a priori* a much better functorial behavior than the original category of split modules. We are interested here in another aspect, a kind of devissage that becomes indeed very easy once the translation into nilpotent objects has been made.

The rule $(P, Q) \mapsto (P, Q, 0, 0)$, $(P, Q, p, q) \mapsto (P, Q)$ defines maps

$$\mathrm{Mod}_C \times \mathrm{Mod}_C \xrightarrow{i} \mathfrak{Nil}(C; A', B') \xrightarrow{f} \mathrm{Mod}_C \times \mathrm{Mod}_C$$

whose composition is the identity. We call (P, Q, p, q) *finitely generated* if (P, Q) is.

LEMMA 1.5. *If (P, Q, p, q) is finitely generated, it has a finite filtration by finitely generated subobjects, whose quotients are in* $\mathrm{Im}(i)$.

The proof is by more definitions. Call a pair of filtrations

$$0 = P_0 \subset P_1 \subset \cdots \subset P\,,$$
$$0 = Q_0 \subset Q_1 \subset \cdots \subset Q$$

an *assailable filtration* on (P, Q, p, q) if firstly these filtrations converge, and secondly

$$p(P_{n+1}) \subset Q_n \otimes A' \quad\text{and}\quad q(Q_{n+1}) \subset Q_n \otimes B'\,.$$

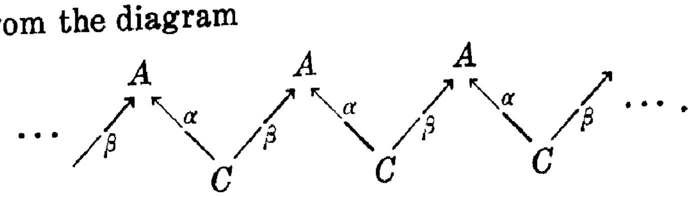
The existence of an assailable filtration is equivalent to nilpotence. Call an assailable filtration *finitely assailable* if firstly the submodules involved are finitely generated and secondly, the filtrations are of essentially finite length. The existence of a finitely assailable filtration is equivalent to the assertion of the lemma. But if (P, Q) is finitely generated, one can construct by downward induction a finitely assailable filtration subordinate to a given assailable filtration.

2. *Laurent extensions.* Let $\alpha, \beta: C \to A$ be inclusions of rings both of which are pure. The *Laurent extension with respect to α and β* is a ring R that contains A, and an invertible element t, and satisfies

$$\alpha(c)t = t\beta(c) \quad\text{if}\quad c \in C\,,$$

and is universal with respect to these properties.

The existence of R can be seen from arguing with free products. Let A^* be the direct limit of the free products with amalgamation constructed inductively from the diagram



A^* has an obvious automorphism (shifting), and we let R be the usual twisted Laurent extension with respect to this automorphism. Then R satisfies the above conditions. However, this is not yet the description that we want.

We fix, once and for all, splittings of C-bimodules

$$A = \alpha(C) \oplus A' , \quad A = \beta(C) \oplus A'' .$$

There are four C-bimodules of interest to us,

$${}_{\alpha}A'_{\alpha},\ {}_{\beta}A_{\alpha},\ {}_{\beta}A''_{\beta},\ {}_{\alpha}A_{\beta} ,$$

e.g., ${}_{\alpha}A'_{\alpha}$ is just A', and ${}_{\beta}A_{\alpha}$ is A with the left and right C-structure induced, respectively, from β and α. We define $\tilde{R}$ to be the direct sum of C and all finite tensor products of these C-bimodules in which the following successions of factors are allowed:

$$\begin{Bmatrix} {}_{\alpha}A'_{\alpha} \\ {}_{\beta}A_{\alpha} \end{Bmatrix} \otimes_C \begin{Bmatrix} {}_{\beta}A''_{\beta} \\ {}_{\beta}A_{\alpha} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} {}_{\beta}A''_{\beta} \\ {}_{\alpha}A_{\beta} \end{Bmatrix} \otimes_C \begin{Bmatrix} {}_{\alpha}A'_{\alpha} \\ {}_{\alpha}A_{\beta} \end{Bmatrix} ,$$

i.e., adjacent indices must be different. In the cases not listed, e.g., ${}_{\alpha}A'_{\alpha} \otimes_C {}_{\alpha}A'_{\alpha}$ is one of them, the cancellation involved in forming the tensor product, is compatible with the multiplication in A. Therefore there is a map $\tilde{R} \otimes_C \tilde{R} \rightarrow \tilde{R}$ which can be seen to induce an associative multiplication on $\tilde{R}$.

Fixing an embedding $A \subset R$, and sending C to $\alpha(C) \subset A$, we obtain a map of C-bimodules $\tilde{R} \rightarrow R$ by the rule

$${}_{\alpha}A'_{\alpha} \longrightarrow A',\ {}_{\beta}A_{\alpha} \longrightarrow tA,\ {}_{\beta}A''_{\beta} \longrightarrow tA''t^{-1},\ {}_{\alpha}A_{\beta} \longrightarrow At^{-1} .$$

Inspection shows this map is multiplicative. So by the universal property of R, it is an isomorphism.

This description of R was suggested by S. Cappell. Still following Cappell, we find it convenient to collect the summands of R, other than C, into four families. These are defined inductively in two ways. The equivalence of the two definitions can be seen by a straightforward inductive argument.

DEFINITION AND LEMMA 2.0.

$$V_1 = {}_{\alpha}A'_{\alpha},\ W_1 = {}_{\beta}A_{\alpha},\ X_1 = {}_{\beta}A''_{\beta},\ Y_1 = {}_{\alpha}A_{\beta} .$$

$$\begin{aligned} V_{n+1} &= V_1 \otimes W_n \oplus Y_1 \otimes V_n = V_n \otimes W_1 \oplus Y_n \otimes V_1 \\ W_{n+1} &= W_1 \otimes W_n \oplus X_1 \otimes V_n = W_n \otimes W_1 \oplus X_n \otimes V_1 \\ X_{n+1} &= X_1 \otimes Y_n \oplus W_1 \otimes X_n = X_n \otimes Y_1 \oplus W_n \otimes X_1 \\ Y_{n+1} &= Y_1 \otimes Y_n \oplus V_1 \otimes X_n = Y_n \otimes Y_1 \oplus V_n \otimes X_1 . \end{aligned}$$

It is convenient to let $V_0 = 0 = X_0$ and $W_0 = C = Y_0$.

From now on we assume A' and A'' are free as left C-modules, and we choose left bases. This choice determines bases $\langle C\rangle$, $\langle V_n\rangle$, etc., of the left C-modules C, V_n, W_n, X_n, Y_n, R. We will construct a tree T whose set of segments is $T^1 = \langle R\rangle$. Its set of vertices, T^0, will be given by a left A-basis of R that we can construct in more or less canonical fashion. Namely, from the first variant of the definition above, we have isomorphisms of C-bimodules, valid for $n \geqq 0$,

$$V_{n+1} \oplus W_n = {}_\alpha A_\alpha \otimes W_n \oplus {}_\alpha A_\beta \otimes V_n ,$$
$$Y_{n+1} \oplus X_n = {}_\alpha A_\alpha \otimes X_n \oplus {}_\alpha A_\beta \otimes Y_n ,$$

and hence a left A-basis of R, isomorphic to the disjoint union

$$\textstyle\bigcup_n(\langle W_n\rangle \cup \langle V_n\rangle \cup \langle X_n\rangle \cup \langle Y_n\rangle) .$$

We let the element in $\langle W_0\rangle$ be the basepoint in T^0.

The definition of incidence in T is facilitated by orienting the segments in such a way that each vertex x, except for the basepoint, is the terminal vertex of precisely one segment, $s_0(x)$. Incidence is then fully described by giving the functions $s_0\colon (T^0 - *) \to T^1$ and $v_0\colon T^1 \to T^0$ where the latter associates to any segment its initial vertex.

We let s_0 be the 'identity' in obvious fashion. The function v_0 is given by the first variant of Definition 2.0, interpreted to be valid for $n \geqq 0$. For example, the image of $\langle V_{n+1}\rangle \subset T^1$ is $\langle W_n\rangle \cup \langle V_n\rangle \subset T^0$, again in obvious fashion.

As in the free product case, the functions s_0 and v_0 combine to give a length function on T, and by induction on length, one can check that T is indeed a tree.

Up to now, our treatment of bases has been 'additive' in the sense that we decomposed R as a module, and used the bases of the pieces. We need information about the multiplicative behavior also. To obtain it, we first have to embed the pieces in R and check what happens to the bases when we identify A to a subring of R, and C to the subring $\alpha(C)$ of A.

By the definition of $\tilde{R}$, and the isomorphism $\tilde{R} \to R$, any C-basis element corresponds to a unique element of R. It is convenient not to distinguish between the two interpretations of such an element.

On the other hand, the left A-structure on R (or rather its pieces) came from the isomorphism

$$R = {}_\alpha A_\alpha \otimes S_1 \oplus {}_\alpha A_\beta \otimes S_2$$

where

$$S_1 = \bigoplus W_n \oplus \bigoplus X_n , \quad S_2 = \bigoplus V_n \oplus \bigoplus Y_n .$$

Denoting T^0_α and T^0_β the A-bases of $A_\alpha \otimes S_1$ and $A_\beta \otimes S_2$, respectively, we had identified T^0_α to $\langle S_1 \rangle$, and T^0_β to $\langle S_2 \rangle$. Now ${}_\alpha A_\alpha$ goes to A in R, and ${}_\alpha A_\beta$ goes to At^{-1}. Therefore under the identification ${}_\alpha A_\alpha \otimes S_1 \oplus {}_\alpha A_\beta \otimes S_2 \to R$, the image of an A-basis element is given by

$$j(x) = x \text{ if } x \in T^0_\alpha \text{ and } j(x) = t^{-1}x \text{ if } x \in T^0_\beta .$$

It is now straightforward to verify that the multiplication in R induces isomorphisms

$$\langle A \rangle \times j(T^0) \longrightarrow T^1 \longleftarrow \langle tA \rangle \times j(T^0) ,$$

and on checking the definition of incidence in T, one finds if $x \in T^0$ then $T^1(x)$, the set of segments incident to x, is given by

$$T^1(x) = \big(\langle A \rangle \cup \langle tA \rangle\big) \cdot j(x) .$$

Definition. A *splitting diagram* consists of right modules

$$M_A,\ M_C$$

over the rings A and C, respectively, and a map over R

$$M_A \otimes_A R \xrightarrow{\kappa} M_C \otimes_C R$$

satisfying $\kappa(M_A) \subset M_C \otimes_C (A \oplus tA)$. We will write $\kappa = \kappa_\alpha - \kappa_\beta$ with

$$\kappa_\alpha(M_A) \subset M_C \otimes_C A\ , \quad \kappa_\beta(M_A) \subset M_C \otimes_C tA\ .$$

A *map* of splitting diagrams is a pair of maps over A and C, respectively, satisfying the obvious condition. The resulting category is abelian since $(?) \otimes_A R$ and $(?) \otimes_C R$ are exact functors.

From the notion of 'splitting diagram' one has the analogous derived notions as in the free product case: *completed splitting diagram, Mayer Vietoris presentation,* and *split module.*

Also the results, when formulated in terms of the tree T, are almost verbatim the same as in the free product case.

PROPOSITION 2.1. *Let N be the free right R-module on the basis element n. Let Δ be a finite subtree of T, containing the basepoint. There exists a canonical Mayer Vietoris presentation $\langle N, n, \Delta \rangle$ of N. Also given $m \in M$, there exists a map of $\langle N, n, \Delta \rangle$ into the completed splitting diagram*

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \xrightarrow{\kappa} M_C \otimes_C R$$

inducing $n \to m$, if and only if Δ contains a certain finite tree $\Delta(m)$. The entire map is uniquely determined by m.

Proof. In terms of the left A-basis T^0 of R, we have a unique expression

$$\iota(m) = \sum_{x \in T^0} m_x \cdot j(x)$$

where $m_x \in M_A$, and only finitely many m_x are non-zero. Then

$$\kappa_\alpha(\iota(m)) = \sum_{x \in T^0} \kappa_\alpha(m_x) \cdot j(x) = \sum_{x \in T^0} \sum_{a \in \langle A \rangle} m_{x,a} \cdot a \cdot j(x)$$

with $m_{x,a} \in M_C$, because $\kappa_\alpha(M_A) \subset M_C \otimes_C A$. Similarly

$$\kappa_\beta(\iota(m)) = \sum_{y \in T^0} \kappa_\beta(m_y) \cdot j(y) = \sum_{y \in T^0} \sum_{b \in \langle tA \rangle} m_{y,b} \cdot b \cdot j(y) .$$

On the other hand, in terms of the left C-basis of R, we have

$$\kappa_\alpha(\iota(m)) = \kappa_\beta(\iota(m)) = \sum_{s \in T^1} m_s \cdot s ,$$

and there is for each s precisely one $(a, x) \in \langle A \rangle \times T^0$, and precisely one $(b, y) \in \langle tA \rangle \times T^0$, so that, on evaluation, $a \cdot j(x) = s = b \cdot j(y)$. Consequently

$$m_{x,a} = m_s = m_{y,b}$$

for these particular indices.

We define $\Delta(m)$ to be the smallest subtree of T which contains all those $x \in T^0$ for which m_x is non-zero.

On inspection of the analogous argument in Proposition 1.1, it is clear that $\langle N, n, \Delta \rangle$ has the required properties if it is defined as follows.

N_A is the free right A-module on the basis n_x, $x \in \Delta^0$; N_C is the free C-module on the basis n_s, $s \in \Delta^1$; and

$$\iota(n) = \sum_{x \in \Delta^0} n_x \cdot j(x) .$$

In order to define $\kappa_\alpha + \kappa_\beta$ ('plus' instead of 'minus') we use the fact that for each $s \in \Delta^1$, and incident $x \in \Delta^0$, there is a unique $a \in \langle A \rangle \cup \langle tA \rangle$ so that

$$s = a \cdot j(x) ;$$

the corresponding component of $\kappa_\alpha + \kappa_\beta$ is then given by

$$n_x \longmapsto n_s \cdot a ,$$

and it follows that

$$\kappa_\alpha(\iota(n)) = \kappa_\beta(\iota(n)) = \sum_{s \in \Delta^1} n_s \cdot s .$$

There is a notion of *based augmented tree* in T, similar to that in Section 1, and by a straightforward variation on the argument, one obtains

PROPOSITION 2.2. *To any finite based augmented tree* ${}^+\Delta$, *there is canonically associated a split module* $\langle {}^+\Delta \rangle$. *And if*

$$M_A \otimes_A R \xrightarrow{\kappa} M_C \otimes_C R$$

is any splitting diagram, and

$$m' \in M_C \cap \mathrm{Im}(\kappa)$$

then there exist finite based augmented trees ${}_\alpha\Delta$ *and* ${}_\beta\Delta$ *and a map from* $\langle {}_\alpha\Delta \rangle \oplus \langle {}_\beta\Delta \rangle$ *whose image contains* m'.

The remainder of this section is devoted to an analysis of split modules. If M_A is an A-module, we can consider it as a C-module in two ways which we indicate by the notation M_α and M_β, respectively.

In terms of the C-bimodules introduced in Definition 2.0, the following is a C-bimodule decomposition of R:

$$\cdots \oplus (W_2 \oplus V_2) \oplus (W_1 \oplus V_1) \oplus C \oplus (Y_1 \oplus X_1) \oplus (Y_2 \oplus X_2) \oplus \cdots .$$

On re-bracketing, we obtain a decomposition with a left A-structure, namely,

$$\begin{array}{ccccccc} \cdots \oplus & (V_2 \oplus W_1) & \oplus (V_1 \oplus C) \oplus Y_1 \oplus & (X_1 \oplus Y_2) & \oplus & (X_2 \oplus Y_3) & \oplus \cdots \\ \cdots \oplus & (A_\alpha \otimes W_1 \oplus A_\beta \otimes V_1) \oplus & A_\alpha \quad \oplus A_\beta \oplus & (A_\alpha \otimes X_1 \oplus A_\beta \otimes Y_1) & \oplus & (A_\alpha \otimes X_2 \oplus A_\beta \otimes Y_2) & \oplus \cdots . \end{array}$$

The point of these particular decompositions is that if we decompose the split module

$$M_A \otimes_A R \xrightarrow{\ \kappa\ } M_C \otimes_C R$$

accordingly, then all the components of κ that can possibly be non-zero are as depicted in the diagram

$$\begin{array}{l} \cdots \oplus (W_\alpha \otimes W_1 \oplus M_\beta \otimes V_1) \ \oplus \ M_\alpha \ \oplus \ M_\beta \ \oplus \ (M_\alpha \otimes X_1 \oplus M_\beta \otimes Y_1) \oplus (M_\alpha \otimes X_2 \oplus M_\beta \otimes Y_2) \\ \quad \searrow \quad \swarrow \qquad \searrow \quad \swarrow \ \searrow \swarrow \ \searrow \quad \swarrow \qquad \searrow \quad \swarrow \quad \searrow \\ M_C \otimes (W_2 \oplus V_2) \oplus M_C \otimes (W_1 \oplus V_1) \oplus M_C \oplus M_C \otimes (X_1 \oplus Y_1) \oplus M_C \otimes (X_2 \oplus Y_2) \quad \oplus \quad \cdots . \end{array}$$

One checks this from Definition 2.0 (first variant) and its relation to multiplication in R. We note a fact which really is a by-product of this checking, namely that the map

$$(M_\alpha \otimes W_1 \oplus M_\beta \otimes V_1) \longrightarrow M_C \otimes (W_1 \oplus V_1)$$

is, in obvious notation, the same as

$$(M_\alpha \longrightarrow M_C) \otimes W_1 \oplus (M_\beta \longrightarrow M_C) \otimes V_1 .$$

Because of the identities (second variant of Definition 2.0)

$$\begin{aligned} V_{n+1} \oplus Y_n &= V_n \otimes_\beta A_\alpha \oplus Y_n \otimes_\alpha A_\alpha , \\ W_{n+1} \oplus X_n &= W_n \otimes_\beta A_\alpha \oplus X_n \otimes_\alpha A_\alpha \end{aligned}$$

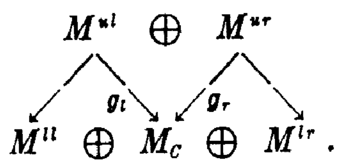
we can fold the diagram at the place of M_α, and add up corresponding terms to obtain the diagram of A-modules

$$\begin{array}{l} M_A \oplus (M_\alpha \otimes_\beta A \oplus M_\beta \otimes_\alpha A) \oplus (M_\alpha \otimes (W_1 \otimes_\beta A \oplus X_1 \otimes_\alpha A) \oplus M_\beta \otimes (V_1 \otimes_\beta A \oplus Y_1 \otimes_\alpha A)) \oplus \cdots \\ \quad \searrow \quad \swarrow \qquad \searrow \qquad\qquad\qquad\qquad\qquad \swarrow \qquad\qquad \searrow \\ M_C \otimes ({}_\beta A \oplus_\alpha A) \otimes M_C \otimes (W_1 \otimes_\beta A \oplus X_1 \otimes_\alpha A \oplus V_1 \otimes_\beta A \oplus Y_1 \otimes_\alpha A) \oplus \cdots . \end{array}$$

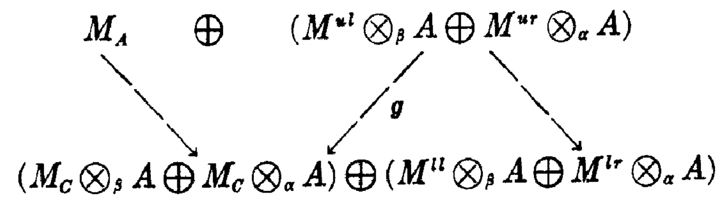
Abbreviating

$$M^{ur} = M_\beta \oplus (M_\alpha \otimes X_1 \oplus M_\beta \otimes Y_1) \cdots \text{ etc.},$$

we can write the first diagram as



Comparing the two diagrams, and using again that κ is an R-map, one sees the folded diagram is of the form



and it is a consequence of the fact noted above that the map g satisfies

$$g = g_l \otimes_\beta A \oplus g_r \otimes_\alpha A .$$

Letting

$$P = \mathrm{Im}(g_l) \approx \ker(M^{*l} \longrightarrow M^{ll}) ,$$
$$Q = \mathrm{Im}(g_r) \approx \ker(M^{*r} \longrightarrow M^{lr}) ,$$

we have $M_C = P \oplus Q$, and we can conclude that $\kappa | M_A$ is the sum of an isomorphism

$$j : M_A \xrightarrow{\approx} P \otimes_\alpha A \oplus Q \otimes_\beta A$$

and some map

$$k : M_A \longrightarrow Q \otimes_\alpha A \oplus P \otimes_\beta A .$$

Similarly, $\kappa^{-1} | P \otimes_\alpha A \oplus Q \otimes_\beta A$ is the sum of j^{-1} and some map

$$l : P \otimes_\alpha A \oplus Q \otimes_\beta A \longrightarrow M^{*l} \otimes_\alpha A \oplus M^{*r} \otimes_\beta A$$

satisfying the relation $k \circ j^{-1} + \kappa \circ l = 0$. We would like to assert that the map l is more special than it appears. We use the fact (from the second variant of Definition 2.0) that

$$M^{*l} \approx M^{*l} \otimes_\beta A_\alpha \oplus M^{*r} \otimes_\alpha A'_\alpha \oplus M_\alpha .$$

Now the restriction $l | P$ can be located in the unfolded diagram, and by definition of the sum decomposition involving l, it has its target in the component $M^{*l} \otimes_\beta A_\alpha \oplus M^{*r} \otimes_\alpha A'_\alpha$ of M^{*l}. By our control on the map g above, we can thus assert that

$$k(j^{-1}(P)) = \kappa(l(P)) \subset P \otimes_\beta A_\alpha \oplus Q \otimes_\alpha A'_\alpha .$$

But the roles of α and β in the definition of splitting diagrams can be interchanged by what is essentially conjugation by t. Therefore by symmetry we can also assert that

$$k(j^{-1}(Q)) \subset Q \otimes_\alpha A_\beta \oplus P \otimes_\beta A''_\beta .$$

Identifying M_A to $P \otimes_\alpha A \oplus Q \otimes_\beta A$ by means of the isomorphism j, we have thus proved

PROPOSITION 2.3. *There exists an exact equivalence of the category of split modules with a full subcategory of the category whose objects are the quadruples (P, Q, p, q) where P and Q are C-modules, and p and q are C-maps*

$$p\colon P \longrightarrow Q \otimes_\alpha A'_\alpha \oplus P \otimes_\beta A_\alpha ,$$
$$q\colon Q \longrightarrow P \otimes_\beta A''_\beta \oplus Q \otimes_\alpha A_\beta .$$

Given such an object as in the proposition, we define a pair of filtrations by induction from $P_0 = 0$ and $Q_0 = 0$,

$$P_{n+1} = p^{-1}(Q_n \otimes_\alpha A'_\alpha \oplus P_n \otimes_\beta A_\alpha) ,$$
$$Q_{n+1} = q^{-1}(P_n \otimes_\beta A''_\beta \oplus Q_n \otimes_\alpha A_\beta) ,$$

and we call the object (P, Q, p, q) *nilpotent* if these filtrations converge to P and Q, respectively. We let $\mathfrak{Nil}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta; {}_\beta A_\alpha, {}_\alpha A_\beta)$ denote the full subcategory of nilpotent objects.

PROPOSITION 2.4. *In Proposition* 2.3, *the subcategory in question is*

$$\mathfrak{Nil}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta; {}_\beta A_\alpha, {}_\alpha A_\beta) .$$

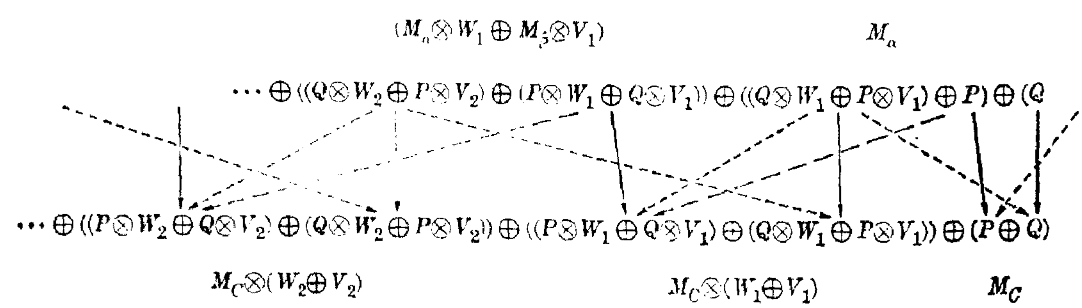
To start the proof, we must do some rewriting. By virtue of Definition 2.0, the isomorphism

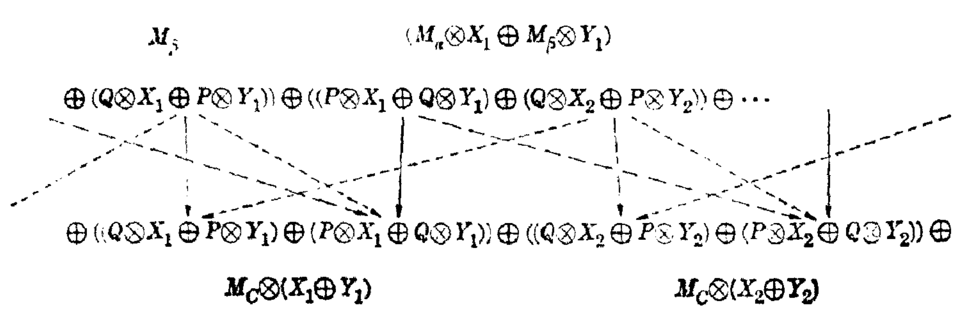
$$j\colon M_A \longrightarrow P \otimes_\alpha A \oplus Q \otimes_\beta A$$

induces isomorphisms

$$M_\alpha \otimes X_n \oplus M_\beta \otimes Y_n \longrightarrow P \otimes X_n \oplus Q \otimes Y_n \oplus P \otimes Y_{n+1} \oplus Q \otimes X_{n+1} ,$$
$$M_\alpha \otimes W_n \oplus M_\beta \otimes V_n \longrightarrow P \otimes W_n \oplus Q \otimes V_n \oplus P \otimes V_{n+1} \oplus Q \otimes W_{n+1} .$$

If we substitute accordingly, the diagram displaying the components of κ becomes





The solid arrows in the diagram are identities. The broken ones are p and q and their induced maps. In the remaining component maps, denoted by dotted arrows, the multiplicative structure of A enters. The point is we need not know anything about the dotted arrows. Even the fact that there is no dotted arrow from $Q\otimes X_1\oplus P\otimes Y_1$ to Q, and its generalizations, is somewhat redundant information. All that matters for us, is the fact that either

$$(P\otimes X_n\oplus Q\otimes Y_n) \text{ or } (P\otimes W_n\oplus Q\otimes V_n)$$

is the source of only one non-identity map, and that this map is the broken arrow with target

$$(P\otimes X_{n+1}\oplus Q\otimes Y_{n+1}) \text{ or } (P\otimes W_{n+1}\oplus Q\otimes V_{n+1})$$

respectively, as claimed in the diagram.

The argument proceeds now as in the free product case. Namely, the diagram as depicted can already be constructed from an object (P, Q, p, q) in the above sense, and in the corresponding map κ one checks it is automatically injective, and it is surjective if and only if $P\oplus Q\subset \mathrm{Im}(\kappa)$ which is the case if and only if (P, Q, p, q) is nilpotent.

By studying the functorial behavior, one sees from the maps $0\rightleftarrows {}_\beta A_\alpha$ and $0\rightleftarrows {}_\alpha A_\beta$ that $\mathfrak{Nil}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta; {}_\beta A_\alpha, {}_\alpha A_\beta)$ has as a retract the category $\mathfrak{Nil}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta)$ of the previous section. And the maps $0\rightleftarrows {}_\alpha A'_\alpha$ and $0\rightleftarrows {}_\beta A''_\beta$ show that another retract is the product of the categories $\mathfrak{Nil}(C; {}_\beta A_\alpha)$ and $\mathfrak{Nil}(C; {}_\alpha A_\beta)$ considered in the next section. In the case when $\alpha, \beta: C\to A$ are both isomorphisms, $\mathfrak{Nil}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta; {}_\beta A_\alpha, {}_\alpha A_\beta)$ actually reduces to that product.

The rule $(P, Q)\mapsto(P, Q, 0, 0)$, $(P, Q, p, q)\mapsto(P, Q)$ defines maps

$$\mathrm{Mod}_C\times \mathrm{Mod}_C \xrightarrow{i} \mathfrak{Nil}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta; {}_\beta A_\alpha, {}_\alpha A_\beta) \xrightarrow{f} \mathrm{Mod}_C\times \mathrm{Mod}_C$$

and as in the previous section we have

LEMMA 2.5. *If (P, Q, p, q) is finitely generated, it has a finite filtration by finitely generated subobjects whose quotients are in* $\mathrm{Im}(i)$.

3. *Polynomial extensions.* Let S be a bimodule on the ring C, and R the tensor algebra of S, so as a C-bimodule

$$R = C\oplus S\oplus S\otimes_C S\oplus S\otimes_C S\otimes_C S\oplus\cdots.$$

We assume S is free as a left C-module, and we fix a left basis that we denote $\langle S\rangle$. This induces the left C-basis

$$\langle R\rangle = \langle C\rangle\cup\langle S\rangle\cup\langle S\otimes S\rangle\cup\cdots.$$

We define an augmented tree ${}^+T$ as follows. Both ${}^+T^0$ and ${}^+T^1$ are isomorphic to the set $\langle R\rangle$. For every vertex $x \in {}^+T^0$, the set ${}^+T^1(x)$ of segments incident to x, is given by evaluating

$$b \cdot x,\ b \in \langle C\rangle \cup \langle S\rangle \ .$$

We orient the segments in such a way that each vertex x is the terminal vertex of precisely one segment $s_0(x)$, and by definition, the function s_0 is given by the 'identity.' So if $b \in \langle S\rangle$ then x is the initial vertex of the segment $b \cdot x$.

From ${}^+T$ we obtain T by omitting the extra segment $1 \in \langle C\rangle$. It is clear that T is indeed a tree. The vertex $1 \in \langle C\rangle$ is the basepoint in T.

Definition. A *splitting diagram* consists of right C-modules

$$M_C,\ M'_C$$

and an R-map

$$M_C \otimes_C R \xrightarrow{\ \kappa\ } M'_C \otimes_C R$$

satisfying

$$\kappa(M_C) \subset M'_C \otimes_C (C \oplus S) \ .$$

There is a canonical way of writing κ as a difference $\kappa = \kappa_0 - \kappa_1$ with

$$\kappa_0(M_C) \subset M'_C \quad \text{and} \quad \kappa_1(M_C) \subset M'_C \otimes_C S \ .$$

A *map* of splitting diagrams is a certain pair of maps over C: the resulting category is abelian since $(?) \otimes_C R$ is an exact functor.

We have the analogous derived notions as in the preceding cases: *completed splitting diagram, Mayer Vietoris presentation,* and *split module.*

PROPOSITION 3.1. *Let N be the free right R-module on the basis element n. Let Δ be a finite subtree of T, containing the basepoint. There exists a canonical Mayer Vietoris presentation $\langle N, n, \Delta\rangle$ of N. And given $m \in M$, there exists a map of $\langle N, n, \Delta\rangle$ into the completed splitting diagram*

$$0 \longrightarrow M \xrightarrow{\ \iota\ } M_C \otimes_C R \xrightarrow{\ \kappa\ } M'_C \otimes_C R$$

inducing $n \to m$, if and only if Δ contains a certain finite tree $\Delta(m)$. The entire map is uniquely determined by m.

Proof. In terms of the left C-basis T^0 of R, we have a unique expression

$$\iota(m) = \textstyle\sum_{x \in T^0} m_x \cdot x$$

where $m_x \in M_C$. Then

$$\kappa_1(\iota(m)) = \textstyle\sum_{x \in T^0} \kappa_1(m_x) \cdot x = \sum_{x \in T^0} \sum_{b \in \langle S\rangle} m_{x,b} \cdot b \cdot x$$

with $m_{x,b} \in M'_C$, and similarly

$$\kappa_0(\iota(m)) = \sum_{y \in T^0} \kappa_0(m_y) \cdot y = \sum_{y \in T^0} m'_y \cdot y$$

with $m'_y \in M'_C$. On the other hand we have, in terms of the C-basis T^1 of R,

$$\kappa_1(\iota(m)) = \kappa_0(\iota(m)) = \sum_{s \in T^1} m_s \cdot s$$

with $m_s \in M'_C$, and the multiplication in R induces an isomorphism

$$\langle S \rangle \times T^0 \longrightarrow T^1 .$$

Therefore

$$m_{x,b} = m_s = m'_y$$

for those (x, b), y, and s such that $b \cdot x = y = s$, as elements of R, and in particular

$$m'_1 = 0 .$$

We let $\Delta(m)$ be the smallest subtree of T which contains those x with $m_x \neq 0$.

The definition of $\langle N, n, \Delta \rangle$ follows the same pattern as in the other cases. N_C is the free C-module on the basis n_x, $x \in \Delta^0$, and N'_C is the free C-module on the basis n_s, $s \in \Delta^1$. In order to define $\kappa_0 + \kappa_1$ ('plus' instead of 'minus') we use that for each $s \in \Delta^1$, and incident $x \in \Delta^0$, there is a unique $b \in \langle C \rangle \cup \langle S \rangle$ so that $s = b \cdot x$ as elements of R. The corresponding component of $\kappa_0 + \kappa_1$ is then given by $n_x \to n_s \cdot b$. By definition,

$$\iota(n) = \sum_{x \in \Delta^0} n_x \cdot x$$

and it follows that

$$\kappa_0(\iota(n)) = \kappa_1(\iota(n)) = \sum_{s \in \Delta^1} n_s \cdot s .$$

This completes the proof.

By a *based augmented tree* in ${}^+T$, we will mean a based subtree of T together with the extra segment in ${}^+T$. As in the other cases, one obtains

PROPOSITION 3.2. *To any finite based augmented tree* ${}^+\Delta$, *there is canonically associated a split module* $\langle {}^+\Delta \rangle$. *And if*

$$M_C \otimes_C R \xrightarrow{\kappa} M'_C \otimes_C R$$

is any splitting diagram, and

$$m' \in M'_C \cap \operatorname{Im}(\kappa)$$

then there exists a finite based augmented tree ${}^+\Delta$ *and a map from* $\langle {}^+\Delta \rangle$ *whose image contains* m'.

The analysis of split modules in this section reduces to almost a triviality. Yet we keep formulating the results in the same way in order to stress the inherent similarity in the different cases. Let

$$M_C \otimes_C R \xrightarrow{\kappa} M'_C \otimes_C R$$

be a split module. As the ring R is graded on the non-negative integers, the isomorphism κ must induce an isomorphism of the degree zero parts of these modules. We identify M_C and M_C' by means of this isomorphism. In terms of the canonical decomposition of R, the map κ can then be displayed in the following diagram which shows all components that can possibly be non-zero:

$$\begin{array}{ccccccccc} & M_C & \oplus & M_C \otimes S & \oplus & M_C \otimes S \otimes S \cdots & & & \\ \swarrow & & \overset{p}{\searrow} & \swarrow & \searrow & \swarrow & \searrow & & \\ M_C & \oplus & M_C \otimes S & \oplus & M_C \otimes S \otimes S & \oplus & \cdots . & & \end{array}$$

The solid arrows are identities, and the broken ones are p, $p \oplus S$, $p \oplus S \oplus S$, etc. We have thus proved

PROPOSITION 3.3. *There exists an exact equivalence of the category of split modules with a full subcategory of the category whose objects are the pairs* (P, p) *where* P *is a* C*-module, and* p *is a* C*-map*

$$p: P \longrightarrow P \otimes S .$$

We define a filtration $0 = P_0 \subset P_1 \subset \cdots \subset P$ inductively by the rule

$$P_{n+1} = p^{-1}(P_n \otimes S)$$

and call the object *nilpotent* if this filtration converges to P. We let $\mathfrak{Nil}(C; S)$ denote the full subcategory of nilpotent objects.

PROPOSITION 3.4. *In Proposition* 3.3, *the subcategory in question is* $\mathfrak{Nil}(C; S)$.

Indeed, given an object (P, p), we can set up a diagram as above. Then $\kappa \mid M_C$ is trivially injective, hence κ is injective. Furthermore κ is surjective if and only if $M_C' \subset \operatorname{Im}(\kappa)$ which visibly is the case if and only if (P, p) is nilpotent.

Finally, the rule $P \mapsto (P, 0)$, $(P, p) \mapsto P$ defines maps

$$\mathrm{Mod}_C \xrightarrow{\ i\ } \mathfrak{Nil}(C; S) \xrightarrow{\ f\ } \mathrm{Mod}_C$$

and as in the preceding sections we have

LEMMA 3.5. *If* (P, p) *is finitely generated, it has a finite filtration by finitely generated subobjects, whose quotients are in* $\operatorname{Im}(i)$.

4. *Dimension, coherence, regularity.* Let M be a right R-module. M is called *coherent* if it has a resolution by finitely generated projective R-modules, and *regular coherent* if this resolution can be taken as finite dimensional.

The ring R is called *coherent*, respectively *regular coherent*, if all its finitely presented right modules are, e.g., R is regular coherent if it is coherent and has finite right global dimension. One can verify by a little diagram chasing that R is coherent if and only if its finitely presented modules form an abelian category.

PROPOSITION 4.1. *Let the ring R be either*

(1) *the free product in the situation $\alpha: C \to A$, $\beta: C \to B$ or*

(2) *the Laurent extension with respect to $\alpha, \beta: C \to A$ or*

(3) *the tensor algebra of the C-bimodule S.*

Assume that the conditions of the preceding sections hold, i.e., *α and β are pure embeddings, and their complements are free from the left; likewise, S is free from the left. Let M be an R-module.*

Then there exist a C-module M_C, an A-module M_A, etc., *and a short exact sequence of R-modules*

(1) $0 \to M_C \otimes_C R \to M_A \otimes_A R \oplus M_B \otimes_B R \to M \to 0$ *or*

(2) $0 \to M_C \otimes_C R \to M_A \otimes_A R \to M \to 0$ *or*

(3) $0 \to M_C \otimes_C R \to M'_C \otimes_C R \to M \to 0$

respectively.

If furthermore C is noetherian, and M finitely presented, and if in case (3) *S is finitely presented from the right, then all the other modules can be taken as finitely presented as well.*

COROLLARY 4.2. *Under the hypotheses of the proposition*

$$\text{r.gl.dim.}(R) \leq \max\big(\text{r.gl.dim.}(A), \text{r.gl.dim.}(B), \text{r.gl.dim.}(C) + 1\big)$$

(*ignore B in case* (2), *and A, B in case* (3)).

If in addition A, B are coherent and C noetherian, then R is coherent.

And if A, B are regular coherent and C regular noetherian, then R is regular coherent.

Indeed, the corollary requires us to construct a certain resolution of an R-module. But the proposition tells us that such a resolution can be constructed by splicing, i.e., by taking the mapping cone of a certain map of resolutions.

The proposition is a rather formal consequence of the results in the preceding sections. As the differences in the three cases are almost in notation only, we describe the argument in just one of the cases. We treat the Laurent extension case.

Let $M^1 \xrightarrow{d_1} M^0 \to M \to 0$ be a free presentation of the given R-module M. There exists a map of Mayer Vietoris presentations

$$\begin{array}{ccccc} M^1 & \xrightarrow{\iota} & M^1_A \otimes_A R & \xrightarrow{\kappa} & M^1_C \otimes_C R \\ \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1_A \otimes R} & & \\ M^0 & \xrightarrow{\iota} & M^0_A \otimes_A R\,. & & \end{array}$$

Indeed, we can choose an isomorphism $M^0 \to M^0_A \otimes_A R$ to begin with, and then construct the rest of the diagram by applying Proposition 2.1 to the generators of M^1. The construction gives M_A^0, M_A^1, M_C^1 free, and finitely generated if M^1 is.

By Proposition 2.2, there exists a split module $M_A^2 \otimes_A R \xrightarrow{\kappa} M_C^2 \otimes_C R$ and a map into the completed splitting diagram

$$0 \longrightarrow \ker(d^1) \longrightarrow \ker(d_A^1) \otimes_A R \longrightarrow M_C^1 \otimes_C R$$

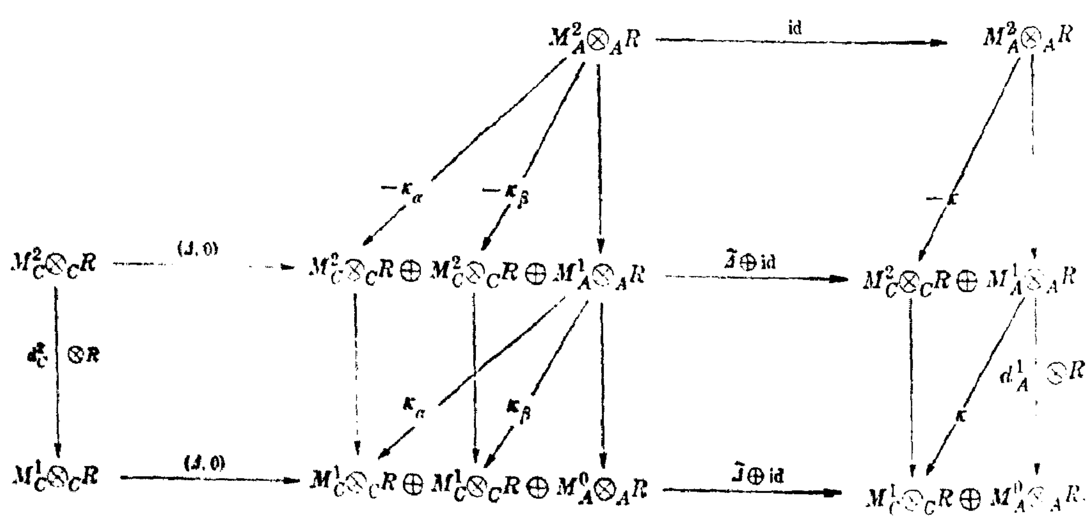
so that $\kappa(\ker(d_A^1)\otimes_A R) \cap M_C^1 \subset \operatorname{Im}(M_C^2)$. As the reverse inclusion holds automatically, we will in fact have the equality

$$\kappa(\ker(d_A^1)\otimes_A R) \cap M_C^1 = \operatorname{Im}(M_C^2)\,.$$

M_A^2 and M_C^2 are given free by the construction; if C is noetherian, and M_C^1 finitely generated, we can assume they are finitely generated. Our data can be collected in a short exact sequence of chain complexes

$$\begin{array}{ccccc} & & M_A^2\otimes_A R & \xrightarrow{\kappa} & M_C^2\otimes_C R \\ & & {\scriptstyle d_A^2}\big\downarrow{\scriptstyle \otimes R} & & {\scriptstyle d_C^2}\big\downarrow{\scriptstyle \otimes R} \\ M^1 & \longrightarrow & M_A^1\otimes_A R & \xrightarrow{\kappa} & M_C^1\otimes_C R \\ \big\downarrow{\scriptstyle d^1} & & {\scriptstyle d_A^1}\big\downarrow{\scriptstyle \otimes R} & & \\ M^0 & \longrightarrow & M_A^0\otimes_A R & & \end{array}$$

and we proceed to deduce the result from this diagram. $M = \operatorname{coker}(d^1)$ is isomorphic to $H_0(X'')$ where X'' is (up to a dimension shift) the mapping cone of the other two complexes, and X'' in turn fits into a short exact sequence $X' \rightarrowtail X \twoheadrightarrow X''$, namely



Here 'id' denotes identity maps, Δ is the diagonal, $\tilde{\Delta}$ the skew codiagonal, and κ_α and κ_β are the summands in the canonical decomposition $\kappa = \kappa_\alpha - \kappa_\beta$.

$H_0(X)$ has an obvious reduction to A, so we will have proved the proposition once we prove that $H_0(X') \to H_0(X)$ is a monomorphism. Inspection of the diagram shows $H_0(X') \to H_0(X)$ is a monomorphism if and only if for any $m \in \ker(d_A^1) \otimes_A R$, $\kappa(m) \in \operatorname{Im}(d_C^2) \otimes_C R$ implies $\kappa_\alpha(m) \in \operatorname{Im}(d_C^2) \otimes_C R$; i.e., if we define

$$\tilde{M} = \{m \in \ker(d_A^1) \otimes_A R \mid \kappa_\alpha(m) - \kappa_\beta(m) \in \operatorname{Im}(d_C^2) \otimes_C R\}$$

then

$$(*) \qquad \kappa_\alpha(\tilde{M}) \subset \operatorname{Im}(d_C^2) \otimes_C R .$$

We prove first

LEMMA 4.3. *Let* $0 \to N \xrightarrow{\iota} N_A \otimes_A R \xrightarrow{\kappa} N_C \otimes_C R$ *be any completed splitting diagram. Then*

$$\kappa_\alpha(\iota(N)) \subset (\operatorname{Im}(\kappa) \cap N_C) \otimes_C R .$$

Proof. In the special case of a Mayer Vietoris presentation, $N_C \subset \operatorname{Im}(\kappa)$, so the lemma is trivial. In general, if $n^* = \kappa_\alpha(\iota(n))$, say, Proposition 2.1 says there exists a 'standard' Mayer Vietoris presentation $\langle N', n', \Delta\rangle$ and a map with $\operatorname{Im}(n') = n$. So the general case follows from the special case.

Consider the diagram

$$\begin{array}{ccccc}
 & & M_A^2 \otimes_A R & \xrightarrow{\approx} & M_C^2 \otimes_C R \\
 & & \Big\downarrow{\scriptstyle \tilde{d}_A^2 \otimes R} & & \Big\downarrow{\scriptstyle d_C^2 \otimes R} \\
\ker(d^1) & \xrightarrow{\iota} & \ker(d_A^1) \otimes_A R & \longrightarrow & M_C^1 \otimes_C R \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle q} & & \Big\downarrow \\
\ker(d^1) & \longrightarrow & \operatorname{coker}(\tilde{d}_A^2) \otimes_A R & \xrightarrow{\tilde{\kappa}} & \operatorname{coker}(d_C^2) \otimes_C R
\end{array}$$

where $\tilde{d}_A^2$ is the same as d_A^2 except for the restriction of the target, and q is the quotient map. The module $\tilde{M}$ can be identified to the kernel of the map from the middle entry to the lower right one, hence $q(\tilde{M}) \subset \ker(\tilde{\kappa})$. But the lower row is exact at the middle, hence from the lemma, and the definition of d_C^2, we obtain $\tilde{\kappa}_\alpha(q(\tilde{M})) = 0$, which is equivalent to $(*)$.

(In [26] I described a short cut to the proof of the proposition; it is a bit too short as it relies on the erroneous statement given as the second part of the lemma in Section 3 of [26].)

II. General theory

5. *Notions of homotopy theory.* Let Δ be the category of ordered sets

$$[n] = (0 < 1 < \cdots < n), \quad n = 0, 1, \cdots$$

and weakly order preserving maps, and Δ^{op} its opposite category. A *simplicial object* in a category $\mathcal{C}$ is a functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$; if $\mathcal{C}$ is the category of *things* we refer to X as a *simplicial thing* and to $X_n = X[n]$ as the *thing in degree n.*

If X is a simplicial set we denote BX its geometric realization. To be precise, we refer to that version of geometric realization which is formed in the category of compactly generated spaces, and which does use the degeneracies. So the functor B commutes with finite products as well as with colimits. A map $f: X \to X'$ of simplicial sets is called a *weak homotopy equivalence* if $Bf: BX \to BX'$ is a homotopy equivalence.

A *bisimplicial object* in $\mathcal{C}$ is a functor $X: \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{C}$; via the diagonal map $\Delta \to \Delta \times \Delta$ one can associate to it a *diagonal simplicial object* with $(\mathrm{diag}\,(X..))_n = X_{nn}$; similarly, a *trisimplicial object* can be diagonalized (in several ways) to a bisimplicial object, etc. There is another obvious way of associating a simplicial set to a bisimplicial set, *condensation* [1]; it gives the same result as diagonalization (to be precise, there is a natural map $\mathrm{cond}(X..) \to \mathrm{diag}(X..)$ and one checks that this map is an isomorphism). Since $\mathrm{cond}(X..)$ is a colimit by definition, since geometric realization commutes with colimits, and since furthermore $B(\Delta^m \times \Delta^n) \to B\Delta^m \times B\Delta^n$ is a homeomorphism, where Δ^n denotes the simplicial set 'standard n-simplex,' this means that $B\mathrm{diag}(X..)$ can also be constructed in the following way. Consider $X..$ as a simplicial object in the category of simplicial sets, apply the geometric realization functor to obtain a simplicial space, and apply geometric realization again to obtain a space. By the above remarks this space will be naturally homeomorphic to $B\,\mathrm{diag}(X..)$. We say $X.. \to Y..$ is a *weak homotopy equivalence* whenever $\mathrm{diag}(X..) \to \mathrm{diag}(Y..)$ is.

Multi-simplicial sets will arise naturally in our work. It will be important that we can work with them directly, without diagonalizing away all the structure. Such work depends on a few basic lemmas which we now collect. It is sufficient to formulate these lemmas for bisimplicial sets as the corresponding lemmas for multi-simplicial sets are immediate consequences, by taking suitable diagonals.

LEMMA 5.1. *Let $X.. \to Y..$ be a map of bisimplicial sets. Suppose that for every n, the map $X._n \to Y._n$ is a weak homotopy equivalence. Then $X.. \to Y..$ is a weak homotopy equivalence.*

Proofs of this lemma are given in [22] and [28]. This lemma will often be used without further comment. A special case is that a simplicial object in the category of contractible simplicial sets is itself contractible (here *contractible* means (weak) homotopy type of a point).

We say a map is *constant* if it factors through a terminal object. A sequence of maps of topological spaces $A \to B \to C$ is called a *fibration up to* *homotopy* if the composed map $A \to C$ is constant, and the resulting map from A to the homotopy theoretic fibre of $B \to C$ is a homotopy equivalence. A sequence of maps of (multi-) simplicial sets will be called a fibration up to homotopy if the sequence of geometric realizations is.

LEMMA 5.2. *Let $X_{..} \to Y_{..} \to Z_{..}$ be a sequence of bisimplicial sets so that $X_{..} \to Z_{..}$ is constant. Suppose that $X_{.n} \to Y_{.n} \to Z_{.n}$ is a fibration up to homotopy, for every n. Suppose further that $Z_{.n}$ is connected, for every n. Then $X_{..} \to Y_{..} \to Z_{..}$ is a fibration up to homotopy.*

This lemma appears to be well known. The following argument goes back to a one line proof, modulo technicalities, by D. Puppe in the case when the $Y_{.n}$ are contractible ("$Z \simeq B(\Omega Z)$; geometric realizations commute among themselves").

Proof. We consider first a special case. Suppose the sequence of the lemma arises in the following way: We are given a simplicial object which in degree n is a pair $(X_{.n}, G_{.n})$ consisting of a simplicial set $X_{.n}$ and a simplicial group $G_{.n}$ acting on $X_{.n}$ from the right. To such a pair is canonically associated a simplicial fibre bundle (or 'twisted cartesian product' [15])

$$X_{.n} \longrightarrow X_{.n\,t}\times NG_n \longrightarrow NG_n$$

where by definition NG_n is the diagonal simplicial set of $(G_{.n})^{\cdot}$. If we omit diagonalizing the bisimplicial sets involved, and assemble for varying n, we obtain a sequence of trisimplicial sets which in tridegree (m, n, k) is

$$X_{m,n} \longrightarrow X_{m,n} \times (G_{m,n})^k \longrightarrow (G_{m,n})^k \ .$$

Clearly then the assertion of the lemma for the sequence above, amounts to the claim that when we diagonalize the latter sequence to a sequence of simplicial sets, and we diagonalize in two steps, then it does not matter which way we do this, which is certainly true.

The general case will be reduced to this special case. Let G be the loop group functor of Kan which to a connected pointed simplicial set L associates a free simplicial group $G(L)$; notice that $G(L)$ is well defined even if L is not reduced [12]. There is a twisted cartesian product $G(L) \to L \times_t G(L) \to L$ so that $L \times_t G(L)$ is contractible; this is also functorial for connected pointed L [12].

In the case at hand, we abbreviate $G_n = G(Z_{.n})$, and $NG_n = \operatorname{diag}(G(Z_{.n})^{\cdot})$. Using the right action of G_n on itself we form the double twisted cartesian product $Z_{.n} \times_t G_{n\,t}\times NG_n$ from which we obtain $Y_{.n} \times_t G_{n\,t}\times NG_n$ by pullback. Since $X_{..} \to Z_{..}$ is a constant map, and G_n and NG_n are naturally pointed, we have a map $X_{.n} \to Y_{.n} \times_t G_{n\,t}\times NG_n$ and a commutative diagram

$$\begin{array}{ccccc}
X_{\cdot n} & \xleftarrow{=} & X_{\cdot n} & \longrightarrow & Y_{\cdot n} \times_t G_n \\
\downarrow & & \downarrow & & \downarrow \\
Y_{\cdot n} & \xleftarrow{\simeq} & Y_{\cdot n} \times_t G_{n\,t} \times NG_n & \xrightarrow{=} & (Y_{\cdot n} \times_t G_n)_t \times NG_n \\
\downarrow & & \downarrow & & \downarrow \\
Z_{\cdot n} & \xleftarrow{\simeq} & Z_{\cdot n} \times_t G_{n\,t} \times NG_n & \xrightarrow{\simeq} & NG_n
\end{array}$$

and the maps denoted '$\simeq$' are weak homotopy equivalences since they are bundle projections with contractible fibre. It follows that $X_{\cdot n} \to Y_{\cdot n} \times_t G_n$ must also be a weak homotopy equivalence.

In view of the naturality of this diagram we can now apply Lemma 5.1 twice to replace the original simplicial object (given by the left column) by a new one (given by the right column). But this is of the special type considered before, and the proof of the lemma is complete.

Nerves. Associated to a small category $\mathcal{C}$ is a simplicial set $N\mathcal{C}$, its *nerve*, where $(N\mathcal{C})_n$ is the set of functors $[n] \to \mathcal{C}$; in other words, an element of $(N\mathcal{C})_n$ is a sequence of n composable morphisms in $\mathcal{C}$. Adhering to the principle of giving up a structure only when we are forced to do so, we will refer to a map of categories $F: \mathcal{C} \to \mathcal{C}'$ as a *homotopy equivalence* whenever the induced map $NF: N\mathcal{C} \to N\mathcal{C}'$ is a weak homotopy equivalence.

A fact to recall is that a natural transformation of a functor $F: \mathcal{C} \to \mathcal{C}'$ induces a simplicial homotopy of the simplicial map NF. For a natural transformation of F is just a functor $\mathcal{C} \times [1] \to \mathcal{C}'$, so it induces $N\mathcal{C} \times \Delta^1 \to N\mathcal{C}'$, using that $N[1] = \Delta^1$. In particular, if F is an equivalence of categories, or if it admits an adjoint, it is a homotopy equivalence. These remarks, due to Segal, and the following two theorems due to Quillen, are basic for doing homotopy theory with categories. We will have to use them again and again, so we will often do so without explicit reference.

Let $F: \mathcal{C} \to \mathcal{C}'$ be a map of small categories, and $X' \in \mathrm{Ob}(\mathcal{C}')$. The *left fibre of F over X'*, denoted F/X', is the category whose objects are the pairs (X, x) where $X \in \mathrm{Ob}(\mathcal{C})$ and $x: F(X) \to X'$ is a morphism in $\mathcal{C}'$, and where a morphism from (X, x) to (Y, y) is a map $f: X \to Y$ in $\mathcal{C}$ so that $x = y \circ F(f)$ ([7], [20]). A morphism $m: X' \to Y'$ in $\mathcal{C}'$ induces an obvious functor $F/m: F/X' \to F/Y'$. Dually, the *right fibre of F over X'* is the category X'/F whose objects are the pairs (X, x), $X \in \mathrm{Ob}(\mathcal{C})$, and $x: X' \to F(X)$.

THEOREM A [20]. *Let $F: \mathcal{C} \to \mathcal{C}'$ be a map of small categories. Suppose that for every $X' \in \mathrm{Ob}(\mathcal{C}')$ the category F/X' is contractible. Then F is a homotopy equivalence.*

THEOREM B [20]. *Let $F: \mathcal{C} \to \mathcal{C}'$ be a map of small categories. Suppose* *that for every morphism $m: X' \to Y'$ in $\mathcal{C}'$, the map $F/m: F/X' \to F/Y'$ is a homotopy equivalence. Then for every $X' \in \mathrm{Ob}(\mathcal{C}')$, the square*

$$\begin{array}{ccc} F/X' & \longrightarrow & \mathcal{C} \\ \big\downarrow & & \big\downarrow F \\ \mathrm{Id}_{\mathcal{C}'}/X' & \longrightarrow & \mathcal{C}' \end{array}$$

is homotopy cartesian.

Dually, one can replace left fibres by right fibres in these theorems.

That a commutative diagram of topological spaces

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow{\sigma} & D \end{array}$$

is *homotopy cartesian* means that the map from A to the homotopy theoretic fibre product $C \times_D D^I \times_D B$ is a homotopy equivalence, where D^I denotes the space of maps from the unit interval to D. If the spaces involved are reasonable (e.g., geometric realizations of simplicial sets) this is equivalent to the property that for every $c \in C$, the map of homotopy theoretic fibres $A \times_C C^I \times_C c \to B \times_D D^I \times_D \sigma(c)$ is a homotopy equivalence; one sees this easily from the fact that a map such as $A \to C$ is a homotopy equivalence if and only if $A \times_C C^I \times_C c$ is contractible for any $c \in C$.

Since the category $\mathrm{Id}_{\mathcal{C}'}/X'$ above has a terminal object it is contractible and there is a canonical nullhomotopy of the map $F/X' \to \mathcal{C}'$. If $\mathcal{C}'$ is connected, Theorem B thus says that the resulting map from $B(F/X')$ to the homotopy theoretic fibre of the map $B\mathcal{C} \to B\mathcal{C}'$ is a homotopy equivalence.

Bicategories. Any small category can be reconstructed from its nerve, or put otherwise, a small category can be considered as a simplicial set of a special kind. The preceding material shows it is useful to be aware of such special structure. Some of the bisimplicial sets we have to work with will also be of a special kind, and it will be useful to recognize the way in which they are special. The relevant notion here is that of a *small bicategory*. A *bicategory* ('catégorie double' [5]) is a structure which is a category in two compatible ways, that is, there are two partially defined composition laws which in particular satisfy the interchange law $(a \cdot b)+(c \cdot d)=(a+c) \cdot (b+d)$. A few examples will now be given, partly for illustration and partly for later reference. These examples will also clarify the way in which we refer to the data involved in a bicategory as *objects*, *horizontal morphisms*, *vertical morphisms*, and *bimorphisms*, respectively. A bicategory will be called *small* if the bimorphisms form a set.

Examples 5.3. (1) Associated to a category $\mathcal{C}$ is a bicategory $\mathrm{bi}(\mathcal{C})$ with

$$\mathrm{Ob}(\mathrm{bi}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C}),\ \mathrm{horMor}(\mathrm{bi}(\mathcal{C})) \xrightarrow{\approx} \mathrm{Mor}(\mathcal{C}),\ \mathrm{vertMor}(\mathrm{bi}(\mathcal{C})) \xrightarrow{\approx} \mathrm{Mor}(\mathcal{C})$$

and $\mathrm{Bimor}(\mathrm{bi}(\mathcal{C}))$ is the class of commutative squares in $\mathcal{C}$,

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \uparrow & & \uparrow \\ \cdot & \longrightarrow & \cdot \end{array} \quad .$$

One can identify $\mathrm{horMor}(\mathrm{bi}(\mathcal{C}))$ to a subclass of $\mathrm{Bimor}(\mathrm{bi}(\mathcal{C}))$, namely those squares in which the vertical arrows are identities. This is analogous to identifying the objects in a category to the identity morphisms. The two composition laws are given by horizontal, or vertical, juxtaposition of squares, respectively.

(2) The bicategory $\mathcal{C}^{\mathrm{Is}}$ is the subbicategory of $\mathrm{bi}(\mathcal{C})$ above with

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{Is}}) = \mathrm{Ob}(\mathcal{C}),\ \mathrm{horMor}(\mathcal{C}^{\mathrm{Is}}) \xrightarrow{\approx} \mathrm{Mor}(\mathcal{C}),$$
$$\mathrm{vertMor}(\mathcal{C}^{\mathrm{Is}}) = \text{isomorphisms in } \mathcal{C},$$

and $\mathrm{Bimor}(\mathcal{C}^{\mathrm{Is}})$ is the class of commutative squares in $\mathcal{C}$,

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \wr\uparrow & & \uparrow\wr \\ \cdot & \longrightarrow & \cdot \end{array}$$

in which the vertical arrows are isomorphisms.

(3) A category $\mathcal{C}$ can be considered as a bicategory in a trivial way,

$$\mathrm{vertMor}(\mathcal{C}) \xrightarrow{\approx} \mathrm{Ob}(\mathcal{C}),\ \mathrm{Bimor}(\mathcal{C}) \xrightarrow{\approx} \mathrm{horMor}(\mathcal{C}) \xrightarrow{\approx} \mathrm{Mor}(\mathcal{C}).$$

This is the subbicategory of $\mathcal{C}^{\mathrm{Is}}$ such that in the squares representing the bimorphisms, the vertical arrows are identities.

(4) To a pair of categories $\mathcal{C}_1, \mathcal{C}_2$, there is associated a bicategory $\mathcal{C}_1 \otimes \mathcal{C}_2$ with

$$\mathrm{Ob}(\mathcal{C}_1 \otimes \mathcal{C}_2) = \mathrm{Ob}(\mathcal{C}_1) \times \mathrm{Ob}(\mathcal{C}_2),\quad \mathrm{horMor}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{Mor}(\mathcal{C}_1) \times \mathrm{Ob}(\mathcal{C}_2),$$
$$\mathrm{verMor}(\mathcal{C}_1 \otimes \mathcal{C}_2) = \mathrm{Ob}(\mathcal{C}_1) \times \mathrm{Mor}(\mathcal{C}_2),\quad \mathrm{Bimor}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{Mor}(\mathcal{C}_1) \times \mathrm{Mor}(\mathcal{C}_2).$$

(5) To an exact category in the sense of Quillen (cf. Section 7), one can associate a bicategory $q\mathcal{A}$ with $\mathrm{Ob}(q\mathcal{A}) = \mathrm{Ob}(\mathcal{A})$,

$$\mathrm{horMor}(q\mathcal{A}) = \text{class of admissible monomorphisms in } \mathcal{A},$$
$$\mathrm{vertMor}(q\mathcal{A}) = \text{class of admissible epimorphisms in } \mathcal{A},$$

and $\mathrm{Bimor}(q\mathcal{A})$ is the class of bicartesian squares in $\mathcal{A}$

$$\begin{array}{ccc} \cdot & \rightarrowtail & \cdot \\ \downarrow & & \downarrow \\ \cdot & \rightarrowtail & \cdot \end{array}$$

in which the horizontal (resp., vertical) arrows are admissible monomorphisms (resp., admissible epimorphisms).

(6) The functor from *categories* to *bicategories* described in example (1) above, when restricted to *small categories*, has a left adjoint, defined on *small bicategories*, which may be called *diagonalization*. When one applies diagonalization to $\mathcal{C}_1 \otimes \mathcal{C}_2$ of example (4) one obtains the usual product of $\mathcal{C}_1$ and $\mathcal{C}_2$. When one applies it to $q\mathcal{A}$ of the preceding example one obtains Quillen's category $Q\mathcal{A}$. The latter fact is implicit in the characterization of $Q\mathcal{A}$ by a universal property [20, p. 18]. Indeed an adaption of this universal property describes the diagonalization functor. It appears though that the diagonalization functor is not, in general, suitable for doing homotopy theory with bicategories. We will not use it at all.

Nerves of bicategories. These are best discussed in a more general framework. One can think of a *category* as a kind of algebraic structure. Specifically, if $\mathcal{B}$ is a category with finite inverse limits, then a *category object* in $\mathcal{B}$ will consist of objects C_0, C_1 in $\mathcal{B}$ ('objects,' 'morphisms') and structure maps s, $t: C_1 \to C_0$ ('source,' 'target'), $i: C_0 \to C_1$ ('identity morphism'), and $c: C_1 \times_{C_0} C_1 \to C_0$ ('composition') where the fibre product is constructed from the diagram

$$C_1 \xrightarrow{t} C_0 \xleftarrow{s} C_1$$

and where the structure maps must satisfy the usual conditions.

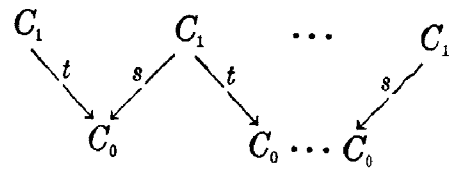
For example, a category object in the category of sets is just a small category. A simplicial category is, by definition, a simplicial object in the category of categories, but a small simplicial category can also be considered as a category object in the category of simplicial sets.

There are two ways of considering a bicategory as a category object in the category of categories. The associated *vertical category object* of a bicategory $\mathcal{D}$ has

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}_0) &= \mathrm{Ob}(\mathcal{D})\,, & \mathrm{Mor}(\mathcal{C}_0) &= \mathrm{horMor}(\mathcal{D}) \\ \mathrm{Ob}(\mathcal{C}_1) &= \mathrm{vertMor}(\mathcal{D})\,, & \mathrm{Mor}(\mathcal{C}_1) &= \mathrm{Bimor}(\mathcal{D}) \end{aligned}$$

and the composition law $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$ is given by the vertical composition law in $\mathcal{D}$.

A category object in $\mathcal{B}$ determines a simplicial object in $\mathcal{B}$, its *nerve*, which is well defined up to unique isomorphism. The object in degree n of this simplicial object is given by an iterated fibre product, the inverse limit of the diagram



with n entries in the upper row (and we agree that in degrees 1 and 0, we obtain C_1 and C_0, respectively).

In particular, from a small bicategory $\mathcal{D}$ we obtain a small simplicial category $N_v\mathcal{D}$, its *vertical nerve*, by first passing to the associated vertical category object and then taking the nerve as indicated. Similarly we can construct $N_h\mathcal{D}$, its *horizontal nerve*. The two bisimplicial sets $N(N_v\mathcal{D})$ and $N(N_h\mathcal{D})$ are canonically isomorphic since the only difference involved is to compute an iterate inverse limit in two different ways.

For later use we record

LEMMA. *Let the category $\mathcal{C}$ be considered as a bicategory in a trivial way, as in example* (3). *Then $N_v\mathcal{C}$ is $\mathcal{C}$, considered as a simplicial category in a trivial way (all face and degeneracy maps are identities).*

LEMMA. *Let $\mathcal{C}^{\mathrm{Is}}$ be the bicategory associated to the category $\mathcal{C}$, as described in example* (2). *Then $N_v\mathcal{C}$ is the simplicial category which in degree n is $\mathcal{C}_n$, the category equivalent to $\mathcal{C}$ in which an object is a sequence of n composable isomorphisms in $\mathcal{C}$.*

LEMMA. *Let $\mathcal{C}_1 \otimes \mathcal{C}_2$ be as described in example* (4). *Then $N(N_v(\mathcal{C}_1 \otimes \mathcal{C}_2))$ is canonically isomorphic to the product of $N\mathcal{C}_1$ and $N\mathcal{C}_2$, provided we consider this product as a bisimplicial set, rather than a simplicial set.*

The proofs of these lemmas are trivial.

6. *Γ-categories and Γ-spaces.* The reference for these is [22], also [1] and [2]. To avoid repetition we speak of *Γ-objects* in a category $\mathcal{C}$. Certain properties are required of $\mathcal{C}$ which we will not spell out. The examples to be kept in mind are the categories of sets, categories (eventually small), simplicial sets, and topological spaces, respectively.

Notation. $\mathcal{S}_*$ is the category of finite pointed sets. The basepoint of any object is denoted $*$.

Definition. A Γ-object in the category $\mathcal{C}$ is a (covariant) functor $F\colon \mathcal{S}_* \to \mathcal{C}$ satisfying

(i) $F\{*\}$ is a terminal object of $\mathcal{C}$;

(ii) For any two pointed sets $(X, *)$, $(Y, *)$, the natural map

$$F\big((X, *) \vee (Y, *)\big) \longrightarrow F(X, *) \times F(Y, *)$$

is a weak equivalence, where the dictionary for *weak equivalence* is

$\mathcal{C}$ = (sets),	isomorphism
(categories),	equivalence of categories
(simplicial sets),	weak homotopy equivalence
(topological spaces),	homotopy equivalence.

Sometimes it may be appropriate to replace (i) by

(i′) $F\{*\}$ is weakly equivalent to a terminal object of $\mathcal{C}$.

Denoting $\{1 \cup *\}$ the object of $\mathcal{S}_*$ with one non-basepoint element, 1, we refer to $F\{1 \cup *\}$ as the *underlying object* of the Γ-object F.

Example. New Γ-objects can be obtained from old ones by composition with a functor that preserves products and the notion of weak equivalence. Examples of such functors are

(1) Q-construction: (small exact categories) $\to$ (small categories),

(2) nerve: (small categories) $\to$ (simplicial sets),

(3) geometric realization: (simplicial sets) $\to$ (topological spaces).

The main reason for considering Γ-spaces is that they provide a convenient way of dealing with 'homotopy everything' H-spaces. In fact [22], giving a Γ-space is equivalent to giving a homotopy everything H-space structure on the underlying space, at least when one considers both notions modulo a suitable notion of equivalence.

As Segal points out in introducing Γ-objects, a Γ-set is just an abelian monoid structure on the underlying set, described very wastefully. Following the same recipe for an action of an abelian monoid on a set, we arrive at what should be thought of as an action of the underlying object of a Γ-object, on some object of $\mathcal{C}$. We codify this as follows.

Notation. $\mathcal{S}_{*0}$ is the category whose objects are the pairs $(X \subset Y)$ in $\mathcal{S}_*$ where X contains at most one non-basepoint element, and where a map from $(X \subset Y)$ to $(X' \subset Y')$ must satisfy that $X \to X'$ is surjective.

It may be convenient to think of an object of $\mathcal{S}_{*0}$ as an object of $\mathcal{S}_*$ together with a distinguished element, possibly absent. Denoting the distinguished element by 0, an object of $\mathcal{S}_{*0}$ can then be described by listing the elements of Y, and the nature of X can be inferred from the occurrence, respectively non-occurrence, of 0 among the elements of Y.

There are maps $p, q\colon \mathcal{S}_{*0} \to \mathcal{S}_*$, $p(X \subset Y) = Y$, $q(X \subset Y) = Y/X$; and q has a unique '$\vee$'-preserving section $s\colon \mathcal{S}_* \to \mathcal{S}_{*0}$.

Definition. A Γ_0-object in $\mathcal{C}$ is a functor $G: \mathcal{S}_{*0} \to \mathcal{C}$ satisfying

(i) $G\{*\}$ is a terminal object;

(ii) $G((X \subset Y) \vee (X' \subset Y')) \to G(X \subset Y) \times G(X' \subset Y')$ is a weak equivalence whenever the left hand term is defined.

We refer to the underlying object $(G \circ s)\{1 \cup *\}$ of the Γ-object $G \circ s$ as the *object that acts*, and to $G\{0 \cup *\}$ as the *object that is being acted on.*

Construction of Γ-categories and Γ_0-categories; cf. [22]. Let $\mathcal{A}$ be a category with 'associative and commutative composition law.' For convenience we assume the composition law is induced from a coproduct on an ambient category $\mathcal{B}$; this covers all the cases we need. In detail, the assumptions are these. $\mathcal{B}$ is a category with coproduct. $\mathcal{A}$ is a subcategory of $\mathcal{B}$. We assume that the embedding is full on isomorphisms, and closed under the coproduct in the sense that with any two morphisms $a_1: A_1 \to A_1'$ and $a_2: A_2 \to A_2'$, $\mathcal{A}$ also contains a representative of $a_1 \amalg a_2: A_1 \amalg A_2 \to A_1' \amalg A_2'$. In addition we assume that $\mathcal{A}$ contains an initial object of $\mathcal{B}$, in fact we will usually ask that $\mathcal{A}$ be pointed by such an object. In this situation we have a Γ-category that we denote $\Gamma_{\mathcal{A}}$, dropping mention of $\mathcal{B}$ and the other data. as follows.

If $(X, *)$ is a finite pointed set, we denote $\mathcal{S}(X, *)$ the category whose objects are the subsets of X *not* containing the basepoint and whose morphisms are the inclusions. A map $(X, *) \to (Y, *)$ is equivalent, via the inverse image, to a functor $\mathcal{S}(Y, *) \to \mathcal{S}(X, *)$ that preserves disjoint unions.

Letting $\mathcal{A}$ and $\mathcal{B}$ be as above, we define $\Gamma_{\mathcal{A}}(X, *)$ as the category whose objects are the functors $\mathcal{S}(X, *) \to \mathcal{B}$ which send

(a) disjoint unions into coproducts,

(b) the empty subset of X to the chosen initial object,

(c) objects into $\mathcal{A}$.

The morphisms in $\Gamma_{\mathcal{A}}(X, *)$ are the natural transformations of functors satisfying that all the (extra) maps involved in the natural transformation are maps in $\mathcal{A}$. Clearly $\Gamma_{\mathcal{A}}$ is indeed a Γ-category. Its underlying object $\Gamma_{\mathcal{A}}\{1 \cup *\}$ is naturally isomorphic to $\mathcal{A}$.

In case $\mathcal{A}$ is not equipped with a distinguished initial object of $\mathcal{B}$, condition (b) must be dropped. In that case, the defining property (i) of a Γ-category must be replaced by the weaker property (i'), and the natural map $\Gamma_{\mathcal{A}}\{1 \cup *\} \to \mathcal{A}$ is only an equivalence of categories, in general.

Γ_0-categories arise similarly. In the above situation, let $\mathcal{D}$ be a subcategory of $\mathcal{B}$ (not pointed) such that the embedding is full on isomorphisms, and closed under coproduct with $\mathcal{A}$ in the following sense. If $d: D \to D'$ is a morphism of $\mathcal{D}$, and $a: A \to A'$ a morphism of $\mathcal{A}$, then $\mathcal{D}$ must contain a representative of $d \amalg a: D \amalg A \to D' \amalg A'$. From these data we obtain a Γ_0-category, $\Gamma_{(\mathcal{D},\mathcal{A})}$, describing an action of $\mathcal{A}$ on $\mathcal{D}$ by a recipe entirely analogous to the above; that is, letting $\mathcal{S}(Y, *)$ have the same meaning as before, we define $\Gamma_{(\mathcal{D},\mathcal{A})}(X \subset Y)$ to be the category of functors $\mathcal{S}(Y, *) \to \mathcal{B}$ which send disjoint unions to coproducts, and the empty set to the chosen initial object in $\mathcal{A}$, and so that the value taken on a particular subset Z of $Y - *$ is an object of $\mathcal{A}$ if $Z \cap X = \emptyset$, respectively of $\mathcal{D}$ if $Z \cap X \neq \emptyset$. As before, we define a morphism to be a natural transformation of functors satisfying the fact that any of the (extra) maps involved in the natural transformation is a map of $\mathcal{A}$, respectively $\mathcal{D}$.

Somewhat more generally, if $\mathcal{B}'$ and $\mathcal{A}'$ are as $\mathcal{B}$ and $\mathcal{A}$ above, and $\mathcal{B}' \to \mathcal{B}$ is a coproduct preserving map inducing a pointed map from $\mathcal{A}'$ to $\mathcal{A}$, we can obtain a Γ_0-category $\Gamma_{(\mathcal{D},\mathcal{A}')}$ by pullback from the diagrams of categories

$$\begin{array}{ccc} \Gamma_{(\mathcal{D},\mathcal{A}')}(X \subset Y) & \dashrightarrow & \Gamma_{\mathcal{A}'}(Y/X) \\ \downarrow & & \downarrow \\ \Gamma_{(\mathcal{D},\mathcal{A})}(X \subset Y) & \longrightarrow & \Gamma_{\mathcal{A}}(Y/X) . \end{array}$$

Generally speaking, the notion of Γ-category allows one to deal with a composition law without using an actual composition map. Though when a composition map is needed, one can be obtained by choosing an adjoint to the equivalence

$$\Gamma_{\mathcal{A}}\{* \cup 1 \cup 2\} \longrightarrow \Gamma_{\mathcal{A}}\{* \cup 1\} \times \Gamma_{\mathcal{A}}\{* \cup 2\}$$

and composing with the map $\Gamma_{\mathcal{A}}\{* \cup 1 \cup 2\} \to \Gamma_{\mathcal{A}}\{* \cup 1\}$ induced from $\{1 \cup 2\} \to \{1\}$. The resulting map $\perp: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is only well defined up to isomorphism, and in general it is neither associative nor commutative. Still it has a certain naturality property which we record for later use.

LEMMA. 6.1. *The map* $\perp: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *extends to a map* $\Gamma_{\mathcal{A}} \times \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{A}}$.

Proof. Following the earlier notation, $\perp: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ can be induced from some coproduct preserving map $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, so we have an induced map of Γ-categories $\Gamma_{(\mathcal{A}\times\mathcal{A})} \to \Gamma_{\mathcal{A}}$. But $\Gamma_{(\mathcal{A}\times\mathcal{A})}$ is canonically isomorphic to $\Gamma_{\mathcal{A}} \times \Gamma_{\mathcal{A}}$.

As a somewhat untypical example we record a particular model of a small category equivalent to the category of finitely generated projective (right) modules over a ring R (with unit, as always) which we will find useful later on.

Definition 6.2. $\mathcal{P}_R$ is the category whose objects are the pairs (m, p)

where p is a projection operator on the free R-module generated by the elements of the standard ordered set with m elements, $0 \leq m < \infty$. A morphism from (m, p) to (m', p') is a map from $\mathrm{Im}(p)$ to $\mathrm{Im}(p')$.

Obvious properties of $\mathcal{P}_R$ are

(i) $R \mapsto \mathcal{P}_R$ is a functor.

(ii) The direct sum on $\mathcal{P}_R$ is represented by a map $\mathcal{P}_R \times \mathcal{P}_R \to \mathcal{P}_R$ which is associative and has a unit, namely the distinguished zero object of $\mathcal{P}_R$.

Simplicial objects associated to Γ*-objects and* Γ_0*-objects.* There is a functor $\varphi\colon \Delta^{\mathrm{op}} \to \mathcal{S}_{*0}$ which to the ordered set $[n] = \{0 < 1 < \cdots < n\}$ associates the basepointed set, with distinguished element 0,

$$\{0, (0 < 1), (1 < 2), \cdots, (n-1 < n), *\}$$

and if $f\colon [m] \to [n]$ is a non-decreasing map, $\varphi(f)$ is defined as follows

$$\varphi(f)(j-1 < j) = \begin{cases} 0 & \text{if} \quad f(0) > j-1 \\ (i-1 < i) & \text{if} \quad f(i-1) \leq j-1 < j \leq f(i) \\ * & \text{if} \quad j > f(m)\,. \end{cases}$$

By composition with the functor $q\colon \mathcal{S}_{*0} \to \mathcal{S}_*$, $(X \subset Y) \mapsto (Y/X)$, we obtain a functor $\psi\colon \Delta^{\mathrm{op}} \to \mathcal{S}_*$. Note that the pointed simplicial set given by ψ, is the standard simplicial circle, $\mathrm{coker}(\Delta^0 \rightrightarrows \Delta^1)$.

Notation. If F is a Γ-object in $\mathcal{C}$ with underlying object $V = F\{1 \cup *\}$, the simplicial object $F \circ \psi$ will be denoted $N_\Gamma V$. Similarly, if G is a Γ_0-object, the simplicial object $G \circ \varphi$ will be denoted $N_\Gamma(W, V)$ when V is the underlying object of the Γ-object $G \circ s$ (recall $s\colon \mathcal{S}_* \to \mathcal{S}_{*0}$ is a certain section of $q\colon \mathcal{S}_{*0} \to \mathcal{S}_*$), and $W = G\{0 \cup *\}$ is the object that is being acted on.

Let a Γ_0-object be given, describing an action of V on W. The natural transformation in $\mathcal{S}_{*0}$ from the identity to $s \circ q$, given by $0 \mapsto *$, induces a map of simplicial objects, $N_\Gamma(W, V) \to N_\Gamma(V)$. Also there is a natural map $W \to N_\Gamma(W, V)$ when W is considered as a simplicial object in a trivial way, and the composition of these two maps is a constant map.

Proposition 6.3. *Let the sequence* $W \to N_\Gamma(W, V) \to N_\Gamma(V)$ *arise from a* Γ_0*-(multi-)simplicial set, in the way described. Suppose that* V *is connected. Then this sequence is a fibration up to homotopy.*

Addendum. If in particular this Γ_0-object is obtained from a Γ-object by means of $p\colon \mathcal{S}_{*0} \to \mathcal{S}_*$ ('forget 0 is distinguished'), the action of V on W is equivalent to (or better, *is*) the 'translation action' of V on itself, and in this case $N_\Gamma(V, V)$ is contractible; so the sequence of the proposition is de-looping of V.

Proof. The special case of the addendum is essentially Proposition 1.4 of [22]. The general case is a straightforward generalization of this special case. (As a technical point, note that a different kind of geometric realization is used in [22], but as pointed out in the appendix to [22], the two notions give the same result, up to homotopy, when they are applied to simplicial spaces that are partial geometric realizations of multi-simplicial sets, as is here the case.) Here is a review of the argument.

When we compose with the geometric realization functor, weak equivalences become honest homotopy equivalences, so, as noted before in another context, we can choose a composition map $BV \times BV \to BV$, and similarly an action map $BW \times BV \to BW$. The H-space BV, being connected by assumption, has a homotopy inverse. So any action of it is invertible. In the case at hand this means that the diagram

$$\begin{array}{ccc} BW \times BV & \longrightarrow & BV \\ \downarrow & & \downarrow \\ BW & \longrightarrow & \text{pt.} \end{array}$$

is homotopy cartesian, where the non-trivial maps are the action, and the projection, respectively. It is this fact that is used in the proof.

The space in degree n of the simplicial space $N_\Gamma(BW, BV)$ is homotopy equivalent to $BW \times (BV)^n$. In terms of this homotopy equivalence, the j^{th} face map d_j: $N_\Gamma(BW, BV)_n \to N_\Gamma(BW, BV)_{n-1}$ is homotopic, for $1 \leqq j \leqq n-1$, to the map induced from the composition $BV \times BV \to BV$ of the j^{th} and $(j+1)^{\text{th}}$ factor BV. The face maps numbered 0 and n correspond, respectively, to the action $BW \times BV \to BW$, and to the projection away from the n^{th} factor BV. The map $N_\Gamma(BW, BV)_n \to N_\Gamma(BV)_n$ corresponds to the projection away from BW, and $BW \to N_\Gamma(BW, BV)_n$ corresponds to the inclusion $BW \to BW \times (BV)^n$ (note that BV is naturally pointed). So, in view of the fact noted above, for any face map d_j, the diagram

$$\begin{array}{ccccccc} BW \times (BV)^n & \xleftarrow{\approx} & N_\Gamma(BW, BV)_n & \longrightarrow & N_\Gamma(BV)_n & \xrightarrow{\approx} & (BV)^n \\ & & \downarrow{\scriptstyle d_j} & & \downarrow{\scriptstyle d_j} & & \\ BW \times (BV)^{n-1} & \xleftarrow{\approx} & N_\Gamma(BW, BV)_{n-1} & \longrightarrow & N_\Gamma(BV)_{n-1} & \xrightarrow{\approx} & (BV)^{n-1} \end{array}$$

is homotopy cartesian, the interesting case being $j = 0$. The assertion of the proposition now follows from Proposition 1.6 of [22]; alternatively, it follows from Lemma 5.2. As to the addendum, our $N_\Gamma(BV, BV)$ corresponds to the PA (a 'simplicial path space') for which a simplicial nullhomotopy is described in the proof of Proposition 1.5 of [22].

Remarks. It may be useful to give an alternative description, in a special case, of the simplicial category $N_\Gamma(\mathcal{A})$ associated to a Γ-category F with underlying category $\mathcal{A} = F\{1 \cup *\}$. That is, suppose the composition law on $\mathcal{A}$ is represented by a map $\perp : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is associative and has a unit 0, for example the category $\mathcal{P}_R$ of 6.2. Then we can define a *monoid category* $\mathcal{A}^\perp$. This is a special case of a bicategory, and $\mathcal{A}^\perp$ is given by

$$\operatorname{Ob}(\mathcal{A}^\perp) = \operatorname{horMor}(\mathcal{A}^\perp) = \{0\}\,,$$
$$\operatorname{vertMor}(\mathcal{A}^\perp) = \operatorname{Ob}(\mathcal{A}),\ \operatorname{Bimor}(\mathcal{A}^\perp) = \operatorname{Mor}(\mathcal{A})\,,$$

the vertical composition law being $\perp$. This bicategory has the same homotopy type as $N_\Gamma(\mathcal{A})$. In fact, taking the vertical nerve, we have a map of simplicial categories

$$N_v(\mathcal{A}^\perp) \longrightarrow N_\Gamma(\mathcal{A})$$

whose degree n part is the splitting given by $\perp$ of the equivalence

$$N_\Gamma(\mathcal{A})_n = F\{* \cup (0 < 1) \cup \cdots \cup (n-1 < n)\} \longrightarrow (\mathcal{A})^n\,.$$

This homotopy equivalence illustrates the fact that the isomorphism commutativity of the composition law is not used at all in the definition of N_Γ.

Though it is true in general that a category with a coherently isomorphism-associative composition law with unit, can be replaced by an equivalent one of the special type, this does not mean that we can dispose of Γ-categories altogether. There are two applications in which we do use the isomorphism commutativity of the composition law. One is in Section 14, the other one is in the following remark.

The construction N_Γ can be iterated. To see this one notes [22] that from a Γ-category one can obtain a $(\Gamma \times \Gamma)$-category, i.e., a functor $\mathcal{S}_* \times \mathcal{S}_* \to$ (categories) with certain properties, simply by composing with the map $\mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$ given by the smash product of pointed sets,

$$(X, *) \wedge (Y, *) = X \times Y \,/\, (X, *) \vee (Y, *)\,.$$

Similarly, let a Γ_0-category giving an action of $\mathcal{A}$ on $\mathcal{B}$ arise in the way described before. Suppose in particular that it arises from a coproduct preserving map inducing $\mathcal{A} \to \mathcal{B}$, and assume in addition that $\mathcal{B}$ itself is also closed under the coproduct. Then inspection of the construction shows that we may as well define a $(\Gamma \times \Gamma_0)$-category, that is a functor $\mathcal{S}_* \times \mathcal{S}_{*0} \to$ (categories) with certain properties. So in this situation we can not only form $N_\Gamma(\mathcal{B}, \mathcal{A})$ but also its 'de-loop' $N_\Gamma(N_\Gamma(\mathcal{B}, \mathcal{A}))$. This ends the remarks.

Below we give a version of Quillen's Theorem A for Γ_0-categories. This requires some preparation.

Any category $\mathcal{C}$ can be considered as a Γ_0-category in a trivial way. Namely, letting $\langle * \rangle$ denote the category with one object and one morphism, we can define a Γ_0-category $G_{\mathcal{C}}$ by

$$G_{\mathcal{C}}(X \subset Y) = \begin{cases} \mathcal{C} & \text{if } X = \{0 \cup *\} \\ \langle * \rangle & \text{if } X = \{*\} . \end{cases}$$

Let G be a Γ_0-category, with associated Γ-category $F = G \circ s$, and $\mathcal{B} = G\{0 \cup *\}$ the category that is being acted on. With $G_{\mathcal{C}}$ as above, let $f: G \to G_{\mathcal{C}}$ be a map of Γ_0-categories, inducing $f_0: \mathcal{B} \to \mathcal{C}$, and let M be an object of $\mathcal{C}$.

LEMMA 6.4. *In this situation there is a canonical* Γ_0*-category* f/M, *with associated* Γ*-category* F, *and* $(f/M)\{0 \cup *\} = f_0/M$.

Proof. We define

$$(f/M)(X \subset Y) = \begin{cases} (f \mid G(X \subset Y))/M & \text{if } X = \{0 \cup *\} \\ G(X \subset Y) & \text{if } X = \{*\} . \end{cases}$$

If $X = \{0 \cup *\}$, an object of $(f \mid G(X \subset Y))/M$ is a pair (A, a) where A is an object of $G(X \subset Y)$ and $a: f(A) \to M$ is a morphism in $\mathcal{C}$. If $z: (X \subset Y) \to (X' \subset Y')$ is a map in $\mathcal{S}_{*0}$, the induced map $(f/M)(z)$ takes (A, a) to $(G(z)(A), a)$ if $X' = \{0 \cup *\}$, respectively to $G(z)(A)$ if $X' = \{*\}$.

It is trivial that F is the associated Γ-category, as asserted, and in particular that the defining property (i) of a Γ_0-category is satisfied, and that (ii) is satisfied in some of the cases. Property (ii) says that whenever we express $(X \subset Y)$ as a coproduct $(X_1 \subset Y_1) \vee (X_2 \subset Y_2)$, and z_1, z_2 are the two obvious retractions, then

$$(f/M)(z_1) \times (f/M)(z_2): (f/M)(X \subset Y) \longrightarrow (f/M)(X_1 \subset Y_1) \times (f/M)(X_2 \subset Y_2)$$

is an equivalence of categories or, what is the same, this map is full and faithful, and surjective on isomorphism classes. We are left to verify (ii) in the case when, say, $X_1 = \{0 \cup *\}$ and hence $X_2 = \{*\}$.

Let (A, a) and (A', a') be objects of $(f/M)(X \subset Y)$. Since G is a Γ_0-category there is a one-one correspondence of morphisms $\alpha: A \to A'$ in $G(X \subset Y)$ and morphisms

$$(\alpha_1, \alpha_2): (G(z_1)(A), G(z_2)(A)) \longrightarrow (G(z_1)(A'), G(z_2)(A'))$$

in $G(X_1 \subset Y_1) \times G(X_2 \subset Y_2)$, given by $(\alpha_1, \alpha_2) = (G(z_1)(\alpha), G(z_2)(\alpha))$. But $f(G(z_1)(\alpha)) = f(\alpha)$ by assumption, hence $a = a'f(\alpha)$ if and only if $a = a'f(G(z_1)(\alpha))$. Thus $(f/M)(z_1) \times (f/M)(z_2)$ is full and faithful.

Similarly if $((B, b), B')$ is an object of $(f/M)(X_1 \subset Y_1) \times (f/M)(X_2 \subset Y_2)$

there is, by assumption about G, an object A of $G(X \subset Y)$ so that $(G(z_1)(A), G(z_2)(A))$ is isomorphic to (B, B') by an isomorphism (α_1, α_2), say. Then (A, a), $a = bf(\alpha_1)$, is an object of $(f/M)(X \subset Y)$, and its image under $(f/M)(z_1) \times (f/M)(z_2)$ is isomorphic to $((B, b), B')$ by the isomorphism (α_1, α_2). This completes the proof of the lemma.

Let $\mathcal{A}$ be the underlying category of the Γ-category $F = G \circ s$. We follow earlier notation in denoting $N_\Gamma(\mathcal{B}, \mathcal{A})$ the simplicial category associated to the Γ_0-category G. Notice that $N_\Gamma(\mathcal{C}, \langle * \rangle)$, the simplicial category associated to the Γ_0-category $G_\mathcal{C}$, is just $\mathcal{C}$ considered as a simplicial category in a trivial way. Assume all the categories involved are small.

Proposition 6.5. *Suppose that for every object M of $\mathcal{C}$, the simplicial category $N_\Gamma(f_0/M, \mathcal{A})$ is contractible. Then $N_\Gamma(\mathcal{B}, \mathcal{A}) \to \mathcal{C}$ is a homotopy equivalence.*

Proof. This follows the argument used to prove Theorem A in [20]. Suppose we wanted to write out this argument for the map $f_0\colon \mathcal{B} \to \mathcal{C}$. Then (with a technical variation to suit the present purposes) we should construct the simplicial category $S(f_0)$ which in degree n is the category

$$S(f_0)_n = \coprod_{M_0 \to \cdots \to M_n} f_0/M_0$$

where the coproduct is taken over all sequences of n morphisms in $\mathcal{C}$, that is, the set $(N\mathcal{C})_n$. There are two natural forgetful maps on the simplicial category $S(f_0)$, one to the category $\mathcal{B}$ (always a homotopy equivalence), and one to the simplicial set $N\mathcal{C}$ (a homotopy equivalence if the hypothesis of Theorem A is satisfied).

In the case at hand, the category $\mathcal{A}$ acts on any of the categories f_0/M (as codified in the preceding lemma). Hence $\mathcal{A}$ also acts on the categories $S(f_0)_n$, for any n; we may codify this by explicitly writing out a Γ_0-category $S(f)_n$. Let $(X \subset Y) \in \mathcal{S}_{*0}$. Then with the usual distinction between the two cases $X = \{* \cup 0\}$, resp. $= \{*\}$, we have

$$S(f)_n(X \subset Y) = \coprod_{M_0 \to \cdots \to M_n} (f \,|\, G(X \subset Y))/M_0$$

if $X = \{* \cup 0\}$, resp. $S(f)_n(X \subset Y) = G(X \subset Y)$ if $X = \{*\}$.

Let the simplicial category associated to the Γ_0-category $S(f)_n$ be denoted

$$N_\Gamma(S(f_0)_n, \mathcal{A})\,;$$

this maps to the set $(N\mathcal{C})_n$, and the pre-image of $(M_0 \to \cdots \to M_n) \in (N\mathcal{C})_n$ is

$$N_\Gamma(f_0/M_0, \mathcal{A})$$

which is contractible by hypothesis.

The simplicial categories $N_\Gamma(S(f_0)_n, \mathcal{A})$ assemble to a bisimplicial category $N_\Gamma(S(f_0), \mathcal{A})$, and the maps $N_\Gamma(S(f_0)_n, \mathcal{A}) \to (N\mathcal{C})_n$ assemble to a map

$$N_\Gamma(S(f_0), \mathcal{A}) \longrightarrow N\mathcal{C} .$$

By what was pointed out just before, this map satisfies the hypothesis of Lemma 5.1, hence it is a homotopy equivalence.

Consider the other natural map $p_n: N_\Gamma(S(f_0)_n, \mathcal{A}) \to N_\Gamma(\mathcal{B}, \mathcal{A})$ which forgets the data relating to $(N\mathcal{C})_n$. Take its nerve. A bisimplex of bidegree (k, l) in the bisimplicial set associated to $N_\Gamma(\mathcal{B}, \mathcal{A})$ is a sequence of morphisms

$$N_0 \longrightarrow \cdots \longrightarrow N_k$$

in the category $G\{0 \cup (0 < 1) \cup \cdots \cup (1 - 1 < 1) \cup *\}$, and the pre-image of $(N_0 \to \cdots \to N_k)$ under the map nerve(p_n) is the set of sequences

$$f(N_k) \longrightarrow M_0 \longrightarrow \cdots \longrightarrow M_n$$

in $\mathcal{C}$. Considering $N_\Gamma(\mathcal{B}, \mathcal{A})$ as a simplicial object in a trivial way, we may assemble the maps p_n to a map

$$p: N_\Gamma(S(f_0), \mathcal{A}) \longrightarrow N_\Gamma(\mathcal{B}, \mathcal{A}) .$$

The pre-image of $(N_0 \to \cdots \to N_k)$ under the map nerve(p) now turns out to be the nerve of the category $f(N_k)/\mathrm{Id}_{\mathcal{C}}$; which is contractible. By Lemma 5.1 therefore, p is a homotopy equivalence.

Let g denote the identity map on the Γ_0-category $G_{\mathcal{C}}$. Putting together the above data for f and for g, we obtain a commutative diagram

$$\begin{array}{ccccc} N_\Gamma(\mathcal{B}, \mathcal{A}) & \longleftarrow & N_\Gamma(S(f_0), \mathcal{A}) & \longrightarrow & N\mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \| \\ N_\Gamma(\mathcal{C}, \langle * \rangle) & \longleftarrow & N_\Gamma(S(g_0), \langle * \rangle) & \longrightarrow & N\mathcal{C} \end{array}$$

in which all the horizontal maps are homotopy equivalences. Consequently the left vertical map is a homotopy equivalence, as asserted.

7. *Exact categories.* According to Quillen [20] a suitable framework for doing algebraic K-theory is the notion of *exact category.* This is an additive category $\mathcal{A}$ equipped with a family of 'exact sequences'

$$(*) \qquad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

satisfying the fact that, roughly speaking, the usual calculus applies. A morphism in $\mathcal{A}$ is called an *admissible monomorphism* if it occurs as the arrow $M' \to M$ in some exact sequence $(*)$; the notation $M' \rightarrowtail M$ will be used for admissible monomorphisms, and for these only. Similarly we speak of *admissible epimorphisms,* notation $M \twoheadrightarrow M''$. An *exact functor* is a functor between exact categories which is additive and which takes each exact sequence to an exact sequence.

We will have to extend this framework slightly to *simplicial exact categories* (simplicial categories in which the face and degeneracy maps are exact functors). Such will in particular arise from the consideration of certain kinds of diagrams in exact categories. Naturally we will want to know that the exact categories constructed in this context are indeed exact. Though this is essentially obvious in all the particular cases considered in this paper, it may be appropriate to say a few words about the general case which is not so obvious. It depends on the fact that the category of exact categories is closed under the formation of 'fibre products' in the following sense.

If $F: \mathcal{A} \to \mathcal{C}$ and $G: \mathcal{B} \to \mathcal{C}$ are any functors with common range, their *fibre product* $\Pi(F, G)$ is defined as the category of triples

$$(A, B; c),\ A \in \mathcal{A},\ B \in \mathcal{B},\ c: F(A) \xrightarrow{\approx} G(B)\ .$$

This is equivalent to the pullback category in special cases, for example if one of F and G is a retraction, but not in general. When the two notions disagree, the fibre product is the correct notion. The assertion is that *if F and G are exact functors then $\Pi(F, G)$ is an exact category in a natural way, and the projections to $\mathcal{A}$ and $\mathcal{B}$ are exact functors.*

Indeed, firstly if F and G are actually exact functors of abelian categories then $\Pi(F, G)$ is an abelian category and the assertion is certainly true.

Secondly, that $\mathcal{A}$ is an exact category means [20] that there is an equivalence of $\mathcal{A}$ with a full subcategory $\mathcal{A}'$ of some abelian category $\mathcal{A}''$, where $\mathcal{A}'$ contains 0 and is closed under extensions in $\mathcal{A}''$, and where furthermore the notion of exact sequence in $\mathcal{A}$ is precisely the one induced from the equivalence $\mathcal{A} \to \mathcal{A}'$. In the situation at hand suppose (for convenience) that each of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ itself is so embedded in an abelian category $\mathcal{A}''$, $\mathcal{B}''$, $\mathcal{C}''$, respectively, and *assume that F and G extend to exact functors $F'': A'' \to C''$ and $G'': B'' \to C''$, respectively.* Then $\Pi(F, G)$ comes equipped with an embedding in $\Pi(F'', G'')$ and is hence an exact category. Furthermore the notion of exact sequence in $\Pi(F, G)$ has an obvious intrinsic meaning which depends only on $F: \mathcal{A} \to \mathcal{C}$ and $G: \mathcal{B} \to \mathcal{C}$, not on the embeddings.

It is this special situation which can easily be seen to hold in any of our concrete applications. As an example which is almost typical for the embeddings that can be used, let $\mathcal{A}$ be the category of short exact sequences in an abelian category $\mathcal{D}$. Then $\mathcal{A}$ embeds in the abelian category $\mathcal{A}''$ given by the chain complexes in $\mathcal{D}$, and the embedding is extension closed in view of the 3×3 lemma.

Finally, to extend the argument to the general case, one must clearly get rid of the random nature of the ambient abelian categories employed. One way to do this is as follows. Quillen has given an intrinsic characterization of exact categories, in terms of axioms on the family of exact sequences, and has pointed out [20] that the axioms imply a particular map on the exact category is a full extension closed embedding into an abelian category, namely the Yoneda embedding which takes each object to the left exact functor it represents. But the Yoneda embedding can be functorial [6], taking an exact functor of exact categories to an exact functor of abelian categories. Thus the above argument may be extended.

We will now introduce certain diagram categories, and fix some notation. Let $\mathcal{B}$ be an exact category. We define $M_n\mathcal{B}$ to be the additive category in which an object is a sequence of n admissible monomorphisms in $\mathcal{B}$,

$$B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n .$$

Suppose $\mathcal{B}$ is pointed by a zero object 0. Then we can define the category $F_n\mathcal{B}$ equivalent to $M_n\mathcal{B}$ in which an object consists of an object of $M_n\mathcal{B}$ together with the choice of an object B_j/B_i in the isomorphism class of $\operatorname{coker}(B_i \to B_j)$ in $\mathcal{B}$, for each pair $i < j$, and $B_j/B_i = 0$, the basepoint, if $i = j$. There is a *subquotient map*

$$q_j\colon F_n\mathcal{B} \longrightarrow \mathcal{B} ;$$

on the object above, q_j takes the value B_j/B_{j-1} if $j > 0$, and B_0 if $j = 0$. The additive category $F_n\mathcal{B}$ is an exact category in an evident way: the notion of exactness is such that the q_j are exact functors. For example, $F_1\mathcal{B}$ is exactly equivalent to the exact category of short sequences in $\mathcal{B}$. By means of the equivalence $F_n\mathcal{B} \to M_n\mathcal{B}$, the latter category can now also be considered as an exact category.

More generally, if $\mathcal{A}$ is a full exact subcategory of $\mathcal{B}$ which contains zero and is closed under extension in $\mathcal{B}$, we can define $M_n(\mathcal{B}, \mathcal{A})$ (respectively, $F_n(\mathcal{B}, \mathcal{A})$) to be the full exact subcategory of $M_n\mathcal{B}$ (respectively, $F_n\mathcal{B}$) whose objects satisfy the condition that for every pair $i \leqq j$, the object B_j/B_i is isomorphic to (respectively, is) an object of $\mathcal{A}$.

It will be convenient to describe the categories $F_n\mathcal{B}$ in a novel way, following Segal (unpublished). Letting $\langle n \rangle$ denote the partially ordered set of pairs (i, j), $0 \leqq i \leqq j \leqq n$, we define $S_n\mathcal{B}$ to be the exact category of functors $B\colon \langle n \rangle \to \mathcal{B}$ satisfying that $B_{(i,j)}$ is the distinguished zero object 0 if $i = j$, and that for any triple $i \leqq j \leqq k$, the sequence

$$B_{(i,j)} \longrightarrow B_{(i,k)} \longrightarrow B_{(j,k)}$$

is short exact. There is an exact isomorphism $S_n\mathcal{B} \to F_{n-1}\mathcal{B}$. More interestingly, there is an exact functor $F_n\mathcal{B} \to S_n\mathcal{B}$ which to the object

$$B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n \ ; \ \{B_j/B_i\}_{i \le j}$$

associates the functor $B: \langle n \rangle \to \mathcal{B}$, $B_{(i,j)} = B_j/B_i$. And finally there is an evident inclusion $B \to F_n\mathcal{B}$ whose composition with $F_n\mathcal{B} \to S_n\mathcal{B}$ is the constant map with value 0.

There is a simplicial exact category $F.\mathcal{B}$ which in degree n is $F_n\mathcal{B}$; the i^{th} face map will just drop B_i. Similarly there is a simplicial exact category $S.\mathcal{B}$ which in degree n is $S_n\mathcal{B}$, and where the i^{th} face map is induced by dropping the number i. The maps $F_n\mathcal{B} \to S_n\mathcal{B}$ above assemble to a map of simplicial exact categories $F.\mathcal{B} \to S.\mathcal{B}$.

Let $f: \mathcal{A} \to \mathcal{B}$ be an exact functor of small exact categories. We assume both $\mathcal{A}$ and $\mathcal{B}$ are pointed by a zero object, and the point is respected by f. Generalizing the definition of $F_n(\mathcal{B}, \mathcal{A})$ above, we define $F_n(f)$ to be the pullback in the diagram of categories

$$\begin{array}{ccc} F_n(f) & \dashrightarrow & S_n\mathcal{A} \\ \downarrow & & \downarrow \\ F_n\mathcal{B} & \longrightarrow & S_n\mathcal{B} \, . \end{array}$$

$F_n(f)$ is again an exact category. Indeed one may take as the definition of an exact sequence in $F_n(f)$ that the associated sequence in $\mathcal{B} \times (\mathcal{A})^n$, associated via the subquotient maps, should be exact.

There is a canonical embedding $\mathcal{B} \to F_n(f)$; its composition with $F_n(f) \to S_n\mathcal{A}$ is the constant map. Assembling for varying n, we obtain a sequence of simplicial exact categories, with constant composition,

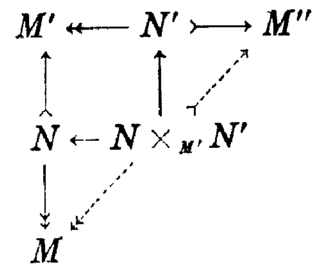
$$\mathcal{B} \longrightarrow F.(f) \longrightarrow S.\mathcal{A}$$

when we consider $\mathcal{B}$ as a simplicial category in a trivial way.

The Q-construction. To an exact category $\mathcal{A}$, Quillen [20] has associated a category $Q\mathcal{A}$, with the same objects as $\mathcal{A}$, in which a morphism from M to M' is an isomorphism class of diagrams in $\mathcal{A}$,

$$M \overset{p}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M'$$

where, as the notation implies, $p: N \twoheadrightarrow M$ is an admissible epimorphism, and $i: M \rightarrowtail M'$ an admissible monomorphism; and where the composite of morphisms $M \twoheadleftarrow N \rightarrowtail M'$ and $M' \twoheadleftarrow N' \rightarrowtail M''$ is represented by the diagonal in the diagram



in which the square is cartesian (bicartesian, in fact).

$\mathcal{A} \mapsto Q\mathcal{A}$ is a functor from *exact categories* to *categories*. It takes an exact equivalence to an equivalence, and it commutes with products and filtering direct limits, up to equivalence. Also it preserves small categories.

If $\mathcal{A}$ is small, its (*exact-sequence-*) K-theory is, by definition, the loop space of the geometric realization of $Q\mathcal{A}$ (this is well defined up to homotopy since $BQ\mathcal{A}$ is 'pointed' by the contractible subspace which arises at the geometric realization of the category formed by the zero objects in $\mathcal{A}$), and the K-groups of $\mathcal{A}$ are the groups $\pi_i \Omega BQ\mathcal{A} = \pi_{i+1} BQ\mathcal{A}$. In practice it is usually preferable though to work with the category $Q\mathcal{A}$ directly.

One of the basic tools in handling Q-construction is the following *additivity theorem* due to Quillen. Let $\mathcal{A}$ be a small exact category, and $\mathcal{A}_1$, $\mathcal{A}_2$ exact subcategories of $\mathcal{A}$. We denote

$$\mathcal{E}(\mathcal{A}_1; \mathcal{A}; \mathcal{A}_2)$$

or $\mathcal{E}$ for short, the exact category whose objects are the exact sequences in $\mathcal{A}$,

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0\,, \quad A_1 \in \mathcal{A}_1\,, \quad A_2 \in \mathcal{A}_2\,.$$

ADDITIVITY THEOREM. *The natural map induced by 'subobject' and 'quotient object,' respectively,*

$$Q\mathcal{E} \longrightarrow Q\mathcal{A}_1 \times Q\mathcal{A}_2\,,$$

is a homotopy equivalence.

The theorem is formulated in [20] only for the special case $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, but the proof carries over without change to the general case.

Here are two immediate applications of the additivity theorem. For the first, cf. [20], suppose the direct sum in $\mathcal{A}$ is represented by a map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Then the split exact sequences in $\mathcal{A}$ give a section of the map $Q\mathcal{E} \to Q\mathcal{A}_1 \times Q\mathcal{A}_2$. Therefore the induced map $BQ\mathcal{E} \to BQ\mathcal{A}_1 \times BQ\mathcal{A}_2$ is actually the retraction part of a deformation retraction. A consequence is that the two maps $BQ\mathcal{E} \to BQ\mathcal{A}$ given by

$$(0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0) \longmapsto A\,, \quad \text{resp.} \quad A_1 \oplus A_2\,,$$

are homotopic, a homotopy being induced by the deformation retraction (and if, e.g., we are working with a distinguished zero object contained in both $\mathcal{A}_1$ and $\mathcal{A}_2$ then the homotopy will preserve the basepoint).

The second application concerns categories which can be identified to categories of the type of $\mathcal{E}$ above. For example if $(k+1)+(l+1)=(n+1)$ there is an equivalence

$$F_n\mathcal{B} \longrightarrow \mathcal{E}(F_k\mathcal{B}; F_n\mathcal{B}; F_l\mathcal{B})$$

which takes $(B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n)$ to the short exact sequence with subobject

$$(B_0 \rightarrowtail \cdots \rightarrowtail B_{k-1} \rightarrowtail B_k \xrightarrow{=} B_k \xrightarrow{=} \cdots \xrightarrow{=} B_k)$$

and quotient object

$$(0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow B_{k+1}/B_k \rightarrowtail \cdots \rightarrowtail B_n/B_k)\ .$$

Hence

$$QF_n\mathcal{B} \xrightarrow{\sim} QF_k\mathcal{B} \times QF_l\mathcal{B}$$

by the additivity theorem. Using this inductively, it follows that the subquotient map induces a homotopy equivalence

$$QF_n\mathcal{B} \xrightarrow{\sim} (Q\mathcal{B})^{n+1}\,,$$
$$(B_0 \rightarrowtail \cdots \rightarrowtail B_n) \longmapsto (B_0, B_1/B_0, \cdots, B_n/B_{n-1})\ .$$

Similarly, the exact category $F_n(f)$ associated to an exact functor $f\colon \mathcal{A} \rightarrow \mathcal{B}$ is equivalent to

$$\mathcal{E}(\mathcal{B}; F_n(f); S_n\mathcal{A})$$

and hence there are homotopy equivalences

$$QF_n(f) \xrightarrow{\sim} Q\mathcal{B} \times QS_n\mathcal{A} \xrightarrow{\sim} Q\mathcal{B} \times (Q\mathcal{A})^n\ .$$

This kind of observation will be put to use in the following material.

Relative versions of the Q-construction. As the Q-construction is functorial it extends to a functor from *simplicial exact categories* to *simplicial categories*. In particular we may apply it to the simplicial exact categories $F.(f)$ and $S.\mathcal{A}$ defined above.

PROPOSITION 7.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be small exact categories, both pointed by a zero object. Let $f\colon \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor preserving the point. Then the sequence of simplicial categories*

$$Q\mathcal{B} \longrightarrow QF.(f) \longrightarrow QS.\mathcal{A}$$

is a fibration up to homotopy. If f is an equivalence, $QF.(f)$ is contractible.

Proof. By the preceding remarks there is for each n a commutative diagram

$$\begin{array}{ccccc} Q\mathcal{B} & \longrightarrow & QF_*(f) & \longrightarrow & QS_*\mathcal{A} \\ \downarrow \scriptstyle{\|} & & \downarrow & & \downarrow \\ Q\mathcal{B} & \longrightarrow & Q\mathcal{B} \times (Q\mathcal{A})^* & \longrightarrow & (Q\mathcal{A})^* \end{array}$$

in which the vertical maps are homotopy equivalences and where the lower row is (trivially) a fibration up to homotopy. As $Q\mathcal{A}$, and hence $(Q\mathcal{A})^*$, is connected, we may apply Lemma 5.2 to conclude that $Q\mathcal{B} \to QF.(f) \to QS.\mathcal{A}$ is a fibration up to homotopy, as asserted.

For the sake of relating related things it is instructive to present a variation on the argument. The map $f: \mathcal{A} \to \mathcal{B}$, being additive, gives rise to a Γ_0-category as described in the preceding section. Applying the Q-construction we obtain another Γ_0-category from which by Proposition 6.3 we obtain a sequence of simplicial categories

$$Q\mathcal{B} \longrightarrow N_\Gamma(Q\mathcal{B}, Q\mathcal{A}) \longrightarrow N_\Gamma(Q\mathcal{A})$$

which is a fibration up to homotopy. There is a commutative diagram of simplicial categories

$$\begin{array}{ccccc} Q\mathcal{B} & \longrightarrow & N_\Gamma(Q\mathcal{B}, Q\mathcal{A}) & \longrightarrow & N_\Gamma(Q\mathcal{A}) \\ \downarrow \scriptstyle{\|} & & \downarrow & & \downarrow \\ Q\mathcal{B} & \longrightarrow & QF.(f) & \longrightarrow & QS.\mathcal{A} \end{array}$$

in which the vertical maps may be interpreted as the forgetful map which takes a split exact sequence (or more generally, split filtration) to an exact sequence (resp., filtration) by forgetting the splitting. By the additivity theorem, the vertical maps are homotopy equivalences in each degree (cf. the first proof above), and therefore homotopy equivalences by Lemma 5.1. Thus the diagram establishes a homotopy equivalence between two fibrations up to homotopy.

To obtain the second assertion of the proposition one may, e.g., appeal to the addendum to Proposition 6.3.

COROLLARY 7.2. *Let $f: \mathcal{A} \to \mathcal{B}$, $g: \mathcal{B} \to \mathcal{C}$ be exact functors of small exact categories, all pointed, and the points preserved by the maps. Then the commutative square of simplicial categories*

$$\begin{array}{ccc} Q\mathcal{B} & \longrightarrow & QF.(f) \\ \downarrow & & \downarrow \\ Q\mathcal{C} & \longrightarrow & QF.(g \circ f) \end{array}$$

is homotopy cartesian.

Proof. In the diagram

$$\begin{array}{ccccc} Q\mathcal{B} & \longrightarrow & QF.(f) & \longrightarrow & QS.\mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \| \\ Q\mathcal{C} & \longrightarrow & QF.(g\circ f) & \longrightarrow & QS.\mathcal{A} \end{array}$$

both rows are fibrations up to homotopy, by the proposition, and the right vertical map is a homotopy equivalence. Therefore the left hand square must be homotopy cartesian.

Two special cases of interest occur when either the map f or the map $g\circ f$ in the corollary is an identity (or an equivalence). The first case can be considered as a more precise version of a 'fibration up to homotopy' in which the composed map is not constant. The second case says if $g: \mathcal{B} \to \mathcal{C}$ is a retraction with section f then $Q\mathcal{B}$ is homotopy equivalent to $Q\mathcal{C} \times QF.(f)$ in an explicit way.

Let $\mathcal{B}$ be a small exact category, and $\mathcal{A}$ a full subcategory of $\mathcal{B}$ that contains zero and is closed under extensions in $\mathcal{B}$, up to isomorphism. Let $M_1(\mathcal{B}, \mathcal{A})$ be the exact category, defined in the preceding section, whose objects are the admissible monomorphisms in $\mathcal{B}$ with cokernel isomorphic to an object of $\mathcal{A}$.

A morphism in $QM_1(\mathcal{B}, \mathcal{A})$ is an isomorphism class of diagrams in $\mathcal{B}$,

$$\begin{array}{ccccc} M_1 & \twoheadleftarrow & N_1 & \rightarrowtail & M_1' \\ \uparrow & & \uparrow & & \uparrow \\ M_0 & \twoheadleftarrow & N_0 & \rightarrowtail & M_0' \end{array}$$

satisfying certain conditions. But there are two composition laws on these diagrams: horizontally the composition law of the Q-construction, and vertically the composition of admissible monomorphisms. The two composition laws are compatible, so these diagrams (or rather their isomorphism classes) are the bimorphisms in a bicategory that we denote $Q(\mathcal{B}, \mathcal{A})$. The category of horizontal morphisms in $Q(\mathcal{B}, \mathcal{A})$ is the category $Q\mathcal{B}$, and the category of vertical morphisms is the category of admissible monomorphisms in $\mathcal{B}$ whose cokernel is isomorphic to an object of $\mathcal{A}$.

Example. (1) Let 0 be the exact category with one object and one morphism. Then $Q(\mathcal{B}, 0)$ is the same as the bicategory $(Q\mathcal{B})^{\mathrm{is}}$ of example 5.3.2. Taking the nerve in the vertical direction we obtain the simplicial category $Q\mathcal{B}_*$ where $\mathcal{B}_*$ is the category equivalent to $\mathcal{B}$ whose objects are the sequences of isomorphisms of length n in $\mathcal{B}$. The face and degeneracy maps are equivalences of categories, so $Q(\mathcal{B}, 0)$ has the homotopy type of $Q\mathcal{B}$.

(2) Considering $Q\mathcal{B}$ as a bicategory in a trivial way, we have a natural

inclusion $Q\mathcal{B} \to Q(\mathcal{B}, 0)$. This is a homotopy equivalence. In fact, passing to vertical nerves, we have in each degree an equivalence of categories.

(3) The bicategory $Q(\mathcal{B}, \mathcal{B})$ is contractible. In fact when we take the nerve horizontally, we obtain a simplicial category which in each degree has an initial element.

The definition of the bicategory $Q(\mathcal{B}, \mathcal{A})$ is not symmetric with respect to admissible monomorphisms and admissible epimorphisms. We may emphasize this by the more explicit notation $Q^{\mathrm{mon}}(\mathcal{B}, \mathcal{A})$. Dually, there is a bicategory $Q^{\mathrm{ep}}(\mathcal{B}, \mathcal{A})$.

PROPOSITION 7.3. *The commutative squares of bicategories*

$$\begin{array}{ccc} Q\mathcal{A} & \longrightarrow & Q(\mathcal{A}, \mathcal{A}) \\ \downarrow & & \downarrow \\ Q\mathcal{B} & \longrightarrow & Q(\mathcal{B}, \mathcal{A}) \end{array}, \qquad \begin{array}{ccc} Q\mathcal{A} & \longrightarrow & Q^{\mathrm{ep}}(\mathcal{A}, \mathcal{A}) \\ \downarrow & & \downarrow \\ Q\mathcal{B} & \longrightarrow & Q^{\mathrm{ep}}(\mathcal{B}, \mathcal{A}) \end{array}$$

are homotopy cartesian.

Proof. The assertion about Q^{ep} reduces to the assertion about Q^{mon} by passing to the dual categories, so it suffices to treat the latter. Taking the nerve in the vertical direction of $Q(\mathcal{B}, \mathcal{A})$ we obtain the simplicial category $QM.(\mathcal{B}, \mathcal{A})$ of the preceding section. Choosing a zero object in $\mathcal{A}$ we can define both $QF.(\mathcal{A}, \mathcal{A})$ and $QF.(\mathcal{B}, \mathcal{A})$ and we obtain a commutative diagram of simplicial categories

$$\begin{array}{ccccc} Q\mathcal{A} & \longrightarrow & QF.(\mathcal{A}, \mathcal{A}) & \longrightarrow & QM.(\mathcal{A}, \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathcal{B} & \longrightarrow & QF.(\mathcal{B}, \mathcal{A}) & \longrightarrow & QM.(\mathcal{B}, \mathcal{A}) \end{array}.$$

The left hand square is homotopy cartesian by Corollary 7.2, and in the right hand square each horizontal map is a homotopy equivalence, being an equivalence of categories in each degree. Hence the big square is homotopy cartesian, as asserted.

Cofinal subcategories. Let as usual $K_0(\mathcal{A})$ denote the class group of the small exact category $\mathcal{A}$, the abelian group with generators $[A]$, $A \in \mathcal{A}$, and relations $[A] = [A'] + [A'']$ for each short exact sequence $A' \to A \to A''$ in $\mathcal{A}$; or what is the same [20], $K_0(\mathcal{A}) = \pi_1 BQ\mathcal{A}$. Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$ which contains zero and is closed under extensions in $\mathcal{A}$. Denote G the quotient group $G = \operatorname{coker}(K_0(\mathcal{B}) \to K_0(\mathcal{A}))$, and $\mathcal{G}$ the associated category with one object $*$, whose set of morphisms is G.

There is a map $f: Q\mathcal{A} \to \mathcal{G}$; by definition, f sends the morphism $M \overset{p}{\twoheadleftarrow} N \rightarrowtail M'$ in $Q\mathcal{A}$ to the element of G represented by $\ker(p)$.

Call $\mathcal{B}$ *cofinal* in A if given $A \in \mathcal{A}$ there exists A' so that $A \oplus A' \in \mathcal{B}$; call it *strongly cofinal** if given any finite number $A_1, \cdots, A_n$ of objects of $\mathcal{A}$, satisfying $f(A_1) = \cdots = f(A_n)$, there exists A' so that $A_i \oplus A' \in \mathcal{B}$ for any i, $i = 1, \cdots, n$.

PROPOSITION 7.4. *Suppose $\mathcal{B}$ is strongly cofinal in $\mathcal{A}$. Let $\mathcal{G}$ be the category associated to the group $G = \operatorname{coker}(K_0(\mathcal{B}) \to K_0(\mathcal{A}))$, and $f: Q\mathcal{A} \to \mathcal{G}$ as described. Then the sequence $Q\mathcal{B} \to Q\mathcal{A} \to \mathcal{G}$ is a fibration up to homotopy.*

Proof. There is but one object, $*$, in $\mathcal{G}$, so there is but one left fibre, $f/*$. The transition maps $f/* \to f/*$ induced from morphisms of $\mathcal{G}$ are isomorphisms since those morphisms are. By Quillen's Theorem B, the sequence $f/* \to Q\mathcal{A} \to \mathcal{G}$ therefore is a fibration up to homotopy, and we are left to show that the natural map $Q\mathcal{B} \to f/*$ is a homotopy equivalence, its composition with $f/* \to Q\mathcal{A}$ being the inclusion $Q\mathcal{B} \to Q\mathcal{A}$.

An object of $f/*$ is a pair (M, g) where $M \in \operatorname{Ob}(\mathcal{A})$ and $g \in G$, and a morphism from (M, g) to (M', g') is a morphism $M \underset{p}{\twoheadleftarrow} N \rightarrowtail M'$ in $Q\mathcal{A}$ subject to the condition that $\ker(p)$ represents $g - g'$ in G. We denote $\mathcal{C}$ the subcategory of $f/*$ whose objects are the $(M, 0)$ and whose morphisms satisfy the fact that $\ker(p)$ is in $\mathcal{B}$. The map $Q\mathcal{B} \to f/*$ is the composition of the two inclusion $k: Q\mathcal{B} \to \mathcal{C}$ and $j: \mathcal{C} \to f/*$, and we will show that both k and j are homotopy equivalences.

As to k, it is sufficient to show that for any $(M, 0) \in \mathcal{C}$, the category $k/(M, 0)$ is contractible, in view of Quillen's Theorem A. An object of $k/(M, 0)$ is equivalent to a morphism $M' \underset{p}{\twoheadleftarrow} N \rightarrowtail M$ in $Q\mathcal{A}$ subject to the condition that M' and $\ker(p)$, and hence also N, are objects of $\mathcal{B}$. Associating to this object just the injection part, $N \rightarrowtail M$ (and, to be precise, choosing an object N in its isomorphism class) gives a natural transformation into a subcategory which is contractible since it has the initial object $0 \rightarrowtail M$. This takes care of k.

To prove j is a homotopy equivalence, we show that for any $(M, g) \in \operatorname{Ob}(f/*)$, the category $(M, g)/j$ is contractible. An object of $(M, g)/j$ is equivalent to a morphism $M \underset{p}{\twoheadleftarrow} N \rightarrowtail M'$ in $Q\mathcal{A}$ satisfying that $\ker(p)$ repre-

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* In an earlier version of this paper, the following proposition had been formulated with 'cofinal' instead of 'strongly cofinal.' Reproducing the proof [Comm. of Alg. **2** (1974)], Gersten pointed out that the argument actually presupposes the stronger condition. The two notions coincide if exact sequences always split in $\mathcal{A}$, but in general the situation is unclear. Note while the proposition says that $BQ\mathcal{B} \to BQ\mathcal{A}$ is homotopy equivalent to a covering map (a representative is $B(f/*) \to BQ\mathcal{A}$, cf. the proof), $BQ\mathcal{B} \to BQ\mathcal{A}$ is not a covering map itself (as for instance it is injective).

Added in proof. An argument of D. Grayson (Localization for flat modules in algebraic K-theory, preprint) shows the distinction between *cofinal* and *strongly cofinal* is unnecessary.

sents g in G. Associating to this object just the projection part, $M \underset{p}{\twoheadleftarrow} N$, gives a deformation retraction into a subcategory whose opposite category we will denote $\mathcal{D}(M, g)$. A morphism in $\mathcal{D}(M, g)$, from $M \underset{p}{\twoheadleftarrow} N$ to $M \underset{p'}{\twoheadleftarrow} N'$, is an admissible epimorphism $N \overset{n}{\twoheadrightarrow} N'$ so that the resulting triangle commutes, and $\ker(n) \in \mathcal{B}$.

To prove $\mathcal{D}(M, g)$ is contractible, it suffices to show two things: (i) $\mathcal{D}(M, g)$ is not empty, (ii) any finite diagram in $\mathcal{D}(M, g)$ is contractible in $\mathcal{D}(M, g)$ (alternatively, any subcategory with finitely many objects is). As to (i), there exists $P \in \mathcal{A}$ representing g in G since $\mathcal{B}$ is cofinal in $\mathcal{A}$; then $M \twoheadleftarrow M \oplus P$ is an object of $\mathcal{D}(M, g)$. As to (ii), let $p_i\colon N_i \to M$ be the objects in a finite diagram in $\mathcal{D}(M, g)$. Let P represent g in G. By hypothesis there exists P' so that $P \oplus P' \in \mathcal{B}$ and $\ker(p_i) \oplus P' \in \mathcal{B}$, all at the same time. Direct sum with $P \oplus P' \to 0$ defines an endofunctor of $\mathcal{D}(M, g)$. This endofunctor admits a natural transformation to the identity functor because $P \oplus P'$ is an object of $\mathcal{B}$. Similarly, its restriction to the finite diagram in question (or the subcategory of $\mathcal{D}(M, g)$ it generates) admits a natural transformation to the constant functor with value $M \oplus P \to M$. The two natural transformations combine to give the required nullhomotopy of the diagram. This takes care of j, and the proof of the proposition is complete.

8. *A splitting lemma.* The purpose of this section is to describe a version of the additivity theorem (Lemma 8.1 below) which applies to the Γ-construction rather than to the Q-construction. For the lemma to be valid, the exact sequences involved must be splittable. The proof of the lemma is related to the proof of the $+ = Q$ theorem, in fact a case of the lemma (Corollary 8.5) amounts to about half the latter, a $\Gamma = Q$ theorem (cf. the next section).

Let $\mathcal{A}$ be a small exact category, and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2$ full subcategories of $\mathcal{A}$ which contain zero and are closed under direct sum. We assume each of $\mathcal{C}_1$ and $\mathcal{C}_2$ is closed under extensions in $\mathcal{A}$. By contrast, $\mathcal{B}_i$ is not assumed closed under extensions, nor need it contain $\mathcal{C}_i$. We let

$$\mathcal{E}_i = \mathcal{E}(\mathcal{B}_i, \mathcal{C}_i)\ , \quad i = 1, 2\ ,$$

be the category whose objects are those of $\mathcal{B}_i$ and whose morphisms are the admissible epimorphisms in $\mathcal{A}$ with kernel in $\mathcal{C}_i$. And we let

$$\mathcal{E} = \mathcal{E}(\mathcal{B}_1, \mathcal{B}_2; \mathcal{C}_1, \mathcal{C}_2)$$

be the category whose objects are the *splittable* short exact sequences in $\mathcal{A}$,

$$0 \longrightarrow B_1 \longrightarrow A \rightleftarrows B_2 \longrightarrow 0\ , \quad B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\ ,$$

and in which a morphism is an admissible epimorphism of short exact sequences

$$\begin{array}{ccccc} B_1 & \rightarrowtail & A & \twoheadrightarrow & B_2 \\ \downarrow & & \downarrow & & \downarrow \\ B_1' & \rightarrowtail & A' & \twoheadrightarrow & B_2' \end{array}$$

subject to the condition that

$$\ker(B_i \longrightarrow B_i') \in \mathcal{C}_i \, .$$

As a technical point, notice that one could include a splitting in the data of an object, and ignore the splittings in the definition of morphisms. This would merely replace $\mathcal{S}$ by an equivalent category.

In view of the direct sum in $\mathcal{A}$, each of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the underlying category of a Γ-category; cf. Section 6. Similarly, so is $\mathcal{S}$.

LEMMA 8.1. *The map $N_\Gamma(\mathcal{S}) \to N_\Gamma(\mathcal{S}_1 \times \mathcal{S}_2)$ is a homotopy equivalence.*

The proof will be given after two lemmas. Choosing zero objects in $\mathcal{S}_1$ and $\mathcal{S}_2$, we may assume all our categories pointed; this will not affect any homotopy types. The projection $\mathcal{S} \to \mathcal{S}_2$ has a section which can be induced by a coproduct preserving map of ambient categories, hence (cf. Section 6) there is a Γ-category describing an action of $\mathcal{S}_2$ on $\mathcal{S}$, and a simplicial category $N_\Gamma(\mathcal{S}, \mathcal{S}_2)$ is defined.

For any $X \in \mathcal{S}_1$ let $\mathcal{S}_{(X)}$ denote the pre-image of X under the projection $\mathcal{S} \to \mathcal{S}_1$. Then $\mathcal{S}_2$ also acts on $\mathcal{S}_{(X)}$, and $N_\Gamma(\mathcal{S}_{(X)}, \mathcal{S}_2)$ is defined.

LEMMA 8.2. *$N_\Gamma(\mathcal{S}_{(X)}, \mathcal{S}_2)$ is contractible.*

Proof. We denote $p\colon \mathcal{S}_{(X)} \to \mathcal{S}_2$ the restriction of the projection $\mathcal{S} \to \mathcal{S}_2$. It has a section $s\colon \mathcal{S}_2 \to \mathcal{S}_{(X)}$ given by sum with X. The category $\mathcal{S}_{(X)}$ has a composition law (we assume it is given by an actual map $\perp$) which is induced from a coproduct,

$$(X \rightarrowtail A) \perp (X \rightarrowtail A') \approx (X \rightarrowtail A \cup_X A')$$

and both p and s are induced by coproduct preserving maps. Hence p and s extend, respectively, to maps of simplicial categories

$$p'\colon N_\Gamma(\mathcal{S}_{(X)}, \mathcal{S}_2) \longrightarrow N_\Gamma(\mathcal{S}_2, \mathcal{S}_2) \, , \quad s'\colon N_\Gamma(\mathcal{S}_2, \mathcal{S}_2) \longrightarrow N_\Gamma(\mathcal{S}_{(X)}, \mathcal{S}_2) \, .$$

It will suffice to show these maps are homotopy equivalences as $N_\Gamma(\mathcal{S}_2, \mathcal{S}_2)$ is contractible (the addendum to Proposition 6.3). s' is a section of p', so we are left to show that $s'p'$ is homotopic to the identity map.

The functor $sp\colon \mathcal{S}_{(X)} \to \mathcal{S}_{(X)}$ cannot be directly related to the identity functor but it can be so related indirectly by a trick of Quillen. The trick is a natural transformation of functors (an isomorphism, in fact)

$$\mathrm{Id} \perp \mathrm{Id} \longrightarrow \mathrm{Id} \perp sp$$

which takes $(X \rightarrowtail A)$ to the map

$$(X \rightarrowtail A) \perp (X \rightarrowtail A) \longrightarrow (X \rightarrowtail A) \perp (X \rightarrowtail X \oplus A/X)$$

obtained by adding two maps, namely

(i) the folding map $(X \rightarrowtail A) \perp (X \rightarrowtail A) \rightarrow (X \rightarrowtail A)$ and

(ii) the composition $(X \rightarrowtail A) \perp (X \rightarrowtail A) \rightarrow 0 \oplus A/X \rightarrow (X \rightarrowtail X \oplus A/X)$.

This natural transformation extends to a simplicial natural transformation (i.e., a simplicial object of natural transformations) on $N_\Gamma(\mathcal{E}_{(X)}, \mathcal{E}_2)$,

$$\mathrm{Id} \perp \mathrm{Id} \longrightarrow \mathrm{Id} \perp s'p'$$

as follows. Using the fact that $\mathcal{E}_{(0)} \xrightarrow{\sim} \mathcal{E}_2$ we may take the above rule to define a natural transformation

$$\mathrm{Id} \perp \mathrm{Id} \longrightarrow \mathrm{Id} \perp \mathrm{Id}$$

on $\mathcal{E}_2$. Now an object in degree n on $N_\Gamma(\mathcal{E}_{(X)}, \mathcal{E}_2)$ is just a sum diagram whose primitive entries are in either $\mathcal{E}_{(X)}$ or $\mathcal{E}_2$ (namely one in the former, and n in the latter), so the two rules together uniquely define a natural transformation on the category in this particular degree n. But for varying n, the natural transformations are mapped to each other by the face and degeneracy functors, assembling as asserted.

On passage to geometric realization, the simplicial natural transformation becomes a homotopy

$$B(\mathrm{Id} \perp \mathrm{Id}) \simeq B(\mathrm{Id} \perp s'p')$$

of maps on $BN_\Gamma(\mathcal{E}_{(X)}, \mathcal{E}_2)$, a homotopy associative (in fact, homotopy everything) H-space with multiplication given by $B(\perp)$. This is a connected space because any object of $\mathcal{E}_{(X)}$ is isomorphic to an object of the form $s(Y)$, $Y \in \mathcal{E}_2$ (it is here that the splittability hypothesis is used). Hence it has a homotopy inverse, and the homotopy $B(\mathrm{Id} \perp \mathrm{Id}) \simeq B(\mathrm{Id} \perp s'p')$ implies a homotopy $B(\mathrm{Id}) \simeq B(s'p')$. This completes the proof of the lemma.

LEMMA 8.3. *The map $N_\Gamma(\mathcal{E}, \mathcal{E}_2) \rightarrow \mathcal{E}_1$ is a homotopy equivalence.*

Proof. According to Proposition 6.5, it is sufficient to show that for any $X \in \mathcal{E}_1$, $N_\Gamma(pr_1/X, \mathcal{E}_2)$ is contractible, where $pr_1 : \mathcal{E} \rightarrow \mathcal{E}_1$ is the projection. $N_\Gamma(pr_1/X, \mathcal{E}_2)$ contains $N_\Gamma(\mathcal{E}_{(X)}, \mathcal{E}_2)$ which is contractible by the preceding lemma. So it suffices to show that the inclusion of $N_\Gamma(\mathcal{E}_{(X)}, \mathcal{E}_2)$ into $N_\Gamma(pr_1/X, \mathcal{E}_2)$ is a homotopy equivalence. By Lemma 5.1, it is enough to show this degree by degree. But in each degree a deformation retraction is given by the natural transformation from the identity functor to the functor which on an object of pr_1/X is given by pushout with its structure map.

Proof of Lemma 8.1. By the preceding lemma, $N_\Gamma(\mathcal{E}, \mathcal{E}_2) \to N_\Gamma(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{E}_2)$ is a homotopy equivalence since $N_\Gamma(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{E}_2)$ is isomorphic to $\mathcal{E}_1 \times N_\Gamma(\mathcal{E}_2, \mathcal{E}_2)$ when we consider the latter $\mathcal{E}_1$ as a simplicial category in a trivial way, and since $N_\Gamma(\mathcal{E}_2, \mathcal{E}_2)$ is contractible. These simplicial categories are naturally underlying objects of Γ-objects, and the resulting map

$$N_\Gamma(N_\Gamma(\mathcal{E}, \mathcal{E}_2)) \longrightarrow N_\Gamma(N_\Gamma(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{E}_2))$$

is also a homotopy equivalence, in view of Lemma 5.1. The bisimplicial category $N_\Gamma(N_\Gamma(\mathcal{E}, \mathcal{E}_2))$ is naturally isomorphic to a bisimplicial category $N_\Gamma(N_\Gamma(\mathcal{E}), N_\Gamma(\mathcal{E}_2))$, and similarly with the other term. In the diagram

$$\begin{array}{ccccc} N_\Gamma(\mathcal{E}) & \longrightarrow & N_\Gamma(N_\Gamma(\mathcal{E}), N_\Gamma(\mathcal{E}_2)) & \longrightarrow & N_\Gamma(N_\Gamma(\mathcal{E}_2)) \\ \downarrow & & \downarrow & & \downarrow \\ N_\Gamma(\mathcal{E}_1 \times \mathcal{E}_2) & \longrightarrow & N_\Gamma(N_\Gamma(\mathcal{E}_1 \times \mathcal{E}_2), N_\Gamma(\mathcal{E}_2)) & \longrightarrow & N_\Gamma(N_\Gamma(\mathcal{E}_2)) \end{array},$$

the rows are fibrations up to homotopy, by Proposition 6.3, since $N_\Gamma(\mathcal{E}_2)$ is connected. The middle vertical map is a homotopy equivalence, as established before, and the right vertical map is an identity map. Consequently the left vertical map must be a homotopy equivalence, as asserted.

As with the additivity theorem, Lemma 8.1 admits an immediate generalization to filtered objects, that is, to splittably filtered objects in the case at hand. Let $\mathcal{A}$ be a small exact category, let $\mathcal{B}_i$, $\mathcal{C}_i$, $1 \leq i \leq n$, be full subcategories which contain zero and are closed under direct sum and where each of the $\mathcal{C}_i$ is closed under extensions in $\mathcal{A}$. Let as before $\mathcal{E}_i = \mathcal{E}(\mathcal{B}_i, \mathcal{C}_i)$ be the category whose objects are those of $\mathcal{B}_i$ and whose morphisms are the admissible epimorphisms with kernel in $\mathcal{C}_i$. We define

$$\mathcal{E} = \mathcal{E}(\mathcal{B}_1, \cdots, B_n; \mathcal{C}_1, \cdots, \mathcal{C}_n)$$

to be the category whose objects are the splittable filtrations (sequences of contractions)

$$A_1 \rightleftarrows A_2 \rightleftarrows \cdots \rightleftarrows A_n$$

in $\mathcal{A}$ where A_i/A_{i-1} is equipped with an isomorphism to an object of $\mathcal{B}_i$, and whose morphisms are the admissible epimorphisms of filtered objects

$$\begin{array}{ccccccc} A_1 & \rightarrowtail & A_2 & \rightarrowtail & \cdots & \rightarrowtail & A_n \\ \downarrow & & \downarrow & & & & \downarrow \\ A'_1 & \rightarrowtail & A'_2 & \rightarrowtail & \cdots & \rightarrowtail & A'_n \end{array}$$

satisfying that for each i the induced map of i^{th} subquotients,

$$A_i/A_{i-1} \twoheadrightarrow A'_i/A'_{i-1}$$

is a map in $\mathcal{E}_i$, that is, has kernel in $\mathcal{C}_i$.

LEMMA 8.4. *The map* $N_\Gamma(\mathcal{E}) \to N_\Gamma(\mathcal{E}_1 \times \cdots \times \mathcal{E}_n)$ *is a homotopy equivalence.*

Proof. This follows by inductive application of Lemma 8.1. For example, one can consider an object of $\mathcal{E}$ as a filtered object of length 2,

$$(A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1} \xrightarrow{=} A_{n-1}) \rightarrowtail (A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1} \rightarrowtail A_n) .$$

In the special case $\mathcal{B}_1 = \cdots = \mathcal{B}_n = \mathcal{A}$ and $\mathcal{C}_1 = \cdots = \mathcal{C}_n = 0$, the category $\mathcal{E}$ is equivalent to a full subcategory of $\mathrm{Is}(F_{n-1}\mathcal{A})$, namely the category of isomorphisms of those filtered objects in $\mathcal{A}$ which are splittable.

COROLLARY 8.5. *Suppose all exact sequences in* $\mathcal{A}$ *are splittable. Then the subquotient map induces a homotopy equivalence*

$$N_\Gamma(\mathrm{Is}(F_{n-1}\mathcal{A})) \xrightarrow{\sim} (N_\Gamma(\mathrm{Is}(\mathcal{A})))^n .$$

9. *Miscellaneous.* In this section we collect some material relating to products, and to chasing them through the $+ = Q$ theorem. This material will be needed for two (somewhat marginal) purposes only: An addendum (involving products) to the 'fundamental theorem,' and a comparison of the 'Whitehead groups' of this paper to the usual Whitehead groups when the latter are defined (this comparison involves products in various settings of K-theory).

The treatment of products in the framework of the Q-construction presupposes some additional machinery. To see this, suppose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are exact categories (equipped with a suitable pairing) and we want to define a bilinear map

$$K_i\mathcal{A} \times K_j\mathcal{B} \longrightarrow K_{i+j}\mathcal{C} .$$

From the point of view of algebraic topology there is a standard way in which such pairings arise, namely the smash product of pointed spaces which induces

$$[S^i, X]_* \times [S^j, Y]_* \longrightarrow [S^i \wedge S^j, X \wedge Y]_* ,$$

that is,

$$\pi_i X \times \pi_j Y \longrightarrow \pi_{i+j}(X \wedge Y) .$$

In the case at hand this means we should seek for a map

$$BQ\mathcal{A} \wedge BQ\mathcal{B} \longrightarrow C$$

where C represents the K-theory of $\mathcal{C}$. But from the point of view of K-theory, $BQ\mathcal{A}$ is off by one dimension ($K_i\mathcal{A} = \pi_{i+1}BQ\mathcal{A}$). Similarly, $BQ\mathcal{B}$ is off by one dimension. Therefore for the program to work, C must be off by *two* dimensions, that is, it must be a de-loop of $BQ\mathcal{C}$. One de-loop of

$BQ\mathcal{C}$ is provided by Proposition 7.1. Another one (actually homotopy equivalent to the former) will be described below. The latter has the advantage that the formula for the product can involve the Q-construction explicitly and still be very simple.

9.1. *De-looping of* $Q\mathcal{A}$ (*the double* Q *construction*). Let $\mathcal{A}$ be a small exact category. Define D to be a subset of the set of commutative diagrams in $\mathcal{A}$ of the type

$$(*) \qquad \begin{array}{ccccc} \cdot & \twoheadleftarrow & \cdot & \rightarrowtail & \cdot \\ \uparrow & & \uparrow & & \uparrow \\ \cdot & \twoheadleftarrow & \cdot & \rightarrowtail & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \twoheadleftarrow & \cdot & \rightarrowtail & \cdot \end{array},$$

namely those diagrams which satisfy the condition that each of the four little squares is *admissible* in the sense that it can be embedded in a 3×3 diagram with short exact rows and columns (the condition is vacuous for the two 'mixed' squares, but non-vacuous for the 'pure' squares).

Here is a possibly more familiar form of the condition. Up to questions of choices (which for the matter at hand are irrelevant) a morphism

$$M \overset{p}{\twoheadleftarrow} M' \rightarrowtail N$$

in $Q\mathcal{A}$ may be identified to a filtered object in $\mathcal{A}$, namely

$$\ker(p) \rightarrowtail M' \rightarrowtail N .$$

Now a diagram $(*)$ satisfies the condition above if and only if it can be similarly identified to a *bifiltered object* (a 2-dimensional lattice of admissible monomorphisms in which the monos themselves form a lattice, that is, the squares are admissible).

The recipe for the composition law in the category $Q\mathcal{A}$ carries over to give a horizontal composition law on the set D: two diagrams $(*)$ can be composed horizontally if the last column of the first diagram coincides with the first column of the second diagram.

There are two technical points involved here however. Firstly it has to be checked that the admissibility condition on the diagrams $(*)$ is preserved under composition, indeed that a diagram $(*)$ is produced at all. These will be clear from an alternative description below.

Secondly the composition law is not quite well defined, due to the choices involved. To make the composition law well defined we proceed by analogy with the definition of morphisms in the category $Q\mathcal{A}$. Namely we pass from D to a quotient set D^* by means of the following equivalence relation: two diagrams $(*)$ are equivalent if and only if they are isomorphic by an isomorphism which restricts to the identity on each of the four objects at the corners.

A vertical composition law can be similarly defined on the set D^*, and the two composition laws are compatible. Hence D^* is the set of bimorphisms in a bicategory that will be denoted

$$QQ\mathcal{A} .$$

To relate $QQ\mathcal{A}$ to $Q\mathcal{A}$, we identify the set of morphisms of $Q\mathcal{A}$ to a set of (equivalence classes of) filtered objects. These are the objects in an exact category $F'_2\mathcal{A}$ equivalent to $F_2\mathcal{A}$. More generally, the set of composable sequences of morphisms of length n in $Q\mathcal{A}$ can be identified to the set of objects in an exact category $F'_{2n}\mathcal{A}$ equivalent to $F_{2n}\mathcal{A}$.

The category $QF'_2\mathcal{A}$ can now be identified to the category formed by the set D^* under the horizontal composition law. Note this explains the admissibility condition above and why it is preserved under composition: everything is due to the exact structure of $F_2\mathcal{A}$. More generally when we take the nerve for the vertical direction of the bicategory $QQ\mathcal{A}$ we obtain a simplicial category $QQ.\mathcal{A}$ of which the category in degree n can be identified to $QF'_{2n}\mathcal{A}$.

Define a category $L\mathcal{A}$. Its objects are the admissible monomorphisms in $\mathcal{A}$, and a morphism in $L\mathcal{A}$ from x to y is a commutative diagram of admissible monomorphisms in $\mathcal{A}$,

$$\begin{array}{ccccc} & X & \overset{x}{\rightarrowtail} & X' & \\ \nearrow & & & & \searrow \\ Y & & \xrightarrow{\quad y \quad} & & Y' . \end{array}$$

We can define a functor

$$L\mathcal{A} \longrightarrow Q\mathcal{A} , \quad (x\colon X \rightarrowtail X') \longmapsto \operatorname{coker}(x) .$$

The definition requires us to choose objects in their isomorphism classes. Supposing $\mathcal{A}$ is equipped with a distinguished zero object 0 we can arrange these choices so that for every $A \in \mathcal{A}$,

$$\operatorname{coker}(\mathrm{Id}_A) = 0 \quad \text{and} \quad \operatorname{coker}(0 \rightarrowtail A) = A .$$

Considering the set $\mathrm{Ob}(\mathcal{A})$ as a category in a trivial way, we also have a functor

$$\mathrm{Ob}(\mathcal{A}) \longrightarrow L\mathcal{A} , \quad A \longmapsto \mathrm{Id}_A$$

and the composition $\mathrm{Ob}(\mathcal{A}) \to L\mathcal{A} \to Q\mathcal{A}$ is the constant functor with value 0.

By a process entirely analogous to the above, we can manufacture a bicategory $QL\mathcal{A}$, and the sequence just given extends to a sequence of bicategories

$$Q\mathcal{A} \longrightarrow QL\mathcal{A} \longrightarrow QQ\mathcal{A}$$

where the category $Q\mathcal{A}$ is considered as a bicategory trivial in the vertical direction, and the composed map in the sequence is a constant map.

PROPOSITION 9.1. *The sequence $Q\mathcal{A} \to QL\mathcal{A} \to QQ\mathcal{A}$ is fibration up to homotopy, and $QL\mathcal{A}$ is contractible.*

Proof. Taking vertical nerves we obtain a sequence of simplicial categories which in degree n is

$$Q\mathcal{A} \longrightarrow QL_n\mathcal{A} \longrightarrow QQ_n\mathcal{A} .$$

As pointed out before, $QQ_n\mathcal{A}$ is equivalent to $QF_{2n}\mathcal{A}$. Similarly $QL_n\mathcal{A}$ is equivalent to $QF_{2n+1}\mathcal{A}$, and the map $QL_n\mathcal{A} \to QQ_n\mathcal{A}$ is equivalent to the map

$$QF_{2n+1}\mathcal{A} \longrightarrow QF_{2n}\mathcal{A} ,$$
$$(A_0 \rightarrowtail \cdots \rightarrowtail A_{2n+1}) \longmapsto (A/A_0 \rightarrowtail \cdots \rightarrowtail A_{2n+1}/A_0) .$$

In view of the additivity theorem the sequence in degree n is therefore homotopy equivalent to the (product) fibration

$$Q\mathcal{A} \longrightarrow Q\mathcal{A} \times (Q\mathcal{A})^{2n+1} \longrightarrow (Q\mathcal{A})^{2n+1} .$$

As $(Q\mathcal{A})^{2n+1}$ is connected, it results from Lemma 5.2 that $Q\mathcal{A} \to QL.\mathcal{A} \to QQ.\mathcal{A}$ is a fibration up to homotopy, as asserted.

To see that $QL\mathcal{A}$ is contractible note first that $L\mathcal{A}$ is contractible, a nullhomotopy being given by the pair of natural transformations in $L\mathcal{A}$,

$$(X \rightarrowtail X') \longmapsto \big((X \rightarrowtail X') \longrightarrow (0 \rightarrowtail X')\big) ,$$
$$(X \rightarrowtail X') \longmapsto \big((0 \rightarrowtail 0) \longrightarrow (0 \rightarrowtail X')\big) .$$

The same formula also works for $Q_nL\mathcal{A}$, the degree n part of the simplicial category $Q.L\mathcal{A}$. Hence $Q_nL\mathcal{A}$ is contractible for any n, and therefore $Q.L\mathcal{A}$ itself is also contractible.

Addendum. The nullhomotopies of the $Q_nL\mathcal{A}$ just described are compatible with the face and degeneracy maps, so they assemble to an explicit nullhomotopy on $Q.L\mathcal{A}$ which in turn induces an explicit nullhomotopy on $BQL\mathcal{A}$ (the geometric realization of the bisimplicial set associated to $QL\mathcal{A}$). Evaluating this nullhomotopy on $BQ\mathcal{A}$ and projecting it to $BQQ\mathcal{A}$ gives a map

$$BQ\mathcal{A} \longrightarrow \Omega BQQ\mathcal{A}$$

which is the homotopy equivalence implied by the proposition. In view of the adjointness between *loop space* and (reduced) *suspension* this map is equivalent to a map (not a homotopy equivalence)

$$\Sigma BQ\mathcal{A} \longrightarrow BQQ\mathcal{A} .$$

The latter map has the advantage that it can be described very directly on the level of nerves. Namely, let us represent suspension by smash product with a simplicial circle which has two 1-simplices, oppositely oriented

 .

Then $\Sigma BQ\mathcal{A} \to BQQ\mathcal{A}$ may simply be described as the map which takes the morphism $M \twoheadleftarrow N \rightarrowtail M'$ in $Q\mathcal{A}$ to the pair of bimorphisms in $QQ\mathcal{A}$,

$$\begin{array}{ccccc} M & \twoheadleftarrow & N & \rightarrowtail & M' \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0 , \end{array} \qquad \begin{array}{ccccc} M & \twoheadleftarrow & N & \rightarrowtail & M' \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0 . \end{array}$$

9.2. *Pairings.* Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be small exact categories. We want to pair the K-theories of the former two into the K-theory of the latter. The appropriate assumption to make is a pairing

$$f\colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$$

which is a *bi-exact* functor in the sense that for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$ the partial functors

$$f(A,\ )\colon \mathcal{B} \longrightarrow \mathcal{C} , \quad f(\ ,B)\colon \mathcal{A} \longrightarrow \mathcal{C}$$

are exact. We will think of f as a tensor product. For technical reasons we assume that each of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ is equipped with a distinguished zero object 0 and that $f(A, 0) = 0 = f(0, B)$ always.

Let $Q\mathcal{A} \otimes Q\mathcal{B}$ denote the bicategory related to the product of $Q\mathcal{A}$ and $Q\mathcal{B}$; cf. 5.3.4.

PROPOSITION 9.2. *The bi-exact functor f induces a map of bicategories*

$$Q\mathcal{A} \otimes Q\mathcal{B} \longrightarrow QQ\mathcal{C}$$

and a map of topological spaces, $BQ\mathcal{A} \wedge BQ\mathcal{B} \to BQQ\mathcal{C}$.

Proof. The map $Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ is defined simply by associating to a pair of morphisms, one from $Q\mathcal{A}$ and one from $Q\mathcal{B}$, their 'tensor product'. It must be checked that the diagram produced is of the type (*) above, i.e., that the arrows are admissible monomorphisms and epimorphisms as claimed, and further that the admissibility conditions are satisfied. But the bi-exactness hypothesis implies that an exact sequence in $\mathcal{A}$ and an exact sequence in $\mathcal{B}$ are paired to a 3×3 diagram in $\mathcal{C}$ whose rows and columns are all short exact. The required properties follow from this.

The geometric realization of $Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ is a map $BQ\mathcal{A} \times BQ\mathcal{B} \to BQQ\mathcal{C}$ which takes $BQ\mathcal{A} \vee BQ\mathcal{B}$ into the basepoint of $BQQ\mathcal{C}$ because of the technical assumption we made. Hence it factors through the smash product as required.

We will now record a (very) few of the naturality properties of the pairing $BQ\mathcal{A} \wedge BQ\mathcal{B} \to BQQ\mathcal{C}$ as these will be needed later. They relate the pairing to two other pairings which are even easier to define. In the notations of 5.3.2 and 5.3.4 these are maps of bicategories

$$Q\mathcal{A} \otimes \mathrm{Is}(\mathcal{B}) \longrightarrow Q\mathcal{C}^{\mathrm{Is}}\,, \quad \mathrm{Is}(\mathcal{A}) \otimes \mathrm{Is}(\mathcal{B}) \longrightarrow \mathrm{Is}(\mathcal{C})^{\mathrm{Is}}$$

inducing maps of topological spaces

$$BQ\mathcal{A} \wedge B\,\mathrm{Is}(\mathcal{B}) \longrightarrow BQ\mathcal{C}^{\mathrm{Is}}\,, \quad B\,\mathrm{Is}(\mathcal{A}) \wedge B\,\mathrm{Is}(\mathcal{B}) \longrightarrow B\,\mathrm{Is}(\mathcal{C})^{\mathrm{Is}}\,.$$

We use a certain embedding

$$\Sigma B\,\mathrm{Is}(\mathcal{A}) \longrightarrow BQ\mathcal{A}\,.$$

It is characterized by the fact that it takes an isomorphism $A \xrightarrow{\approx} A'$ to the pair of commutative triangles in $Q\mathcal{A}$,

$$\begin{array}{ccc} A & \xrightarrow{\approx} & A' \\ & \searrow \quad \swarrow & \\ & 0 & \end{array} \quad , \qquad \begin{array}{ccc} B & \xrightarrow{\approx} & B' \\ & \nwarrow \quad \nearrow & \\ & 0 & \end{array}$$

(its adjoint $B\,\mathrm{Is}(\mathcal{A}) \to \Omega BQ\mathcal{A}$ may be identified to the familiar map). We also use analogous embeddings

$$\Sigma B\,\mathrm{Is}(\mathcal{A})^{\mathrm{Is}} \longrightarrow BQ\mathcal{A}^{\mathrm{Is}}\,, \quad \Sigma BQ\mathcal{A}^{\mathrm{Is}} \longrightarrow BQQ\mathcal{A}$$

(mimic the preceding construction strictly within the horizontal, resp. vertical, direction). By definition of these embeddings we have

Fact 9.2.1. The following diagrams are commutative

$$\begin{array}{ccc} \Sigma B\,\mathrm{Is}(\mathcal{A}) \wedge B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & \Sigma B\,\mathrm{Is}(\mathcal{C})^{\mathrm{Is}} \\ \downarrow & & \downarrow \\ BQ\mathcal{A} \wedge B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & BQ\mathcal{C}^{\mathrm{Is}} \end{array}\,, \qquad \begin{array}{ccc} BQ\mathcal{A} \wedge \Sigma B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & \Sigma BQ\mathcal{C}^{\mathrm{Is}} \\ \downarrow & & \downarrow \\ BQ\mathcal{A} \wedge BQ\mathcal{B} & \longrightarrow & BQQ\mathcal{C} \end{array}\,.$$

A special case of the pairing may be thought of as a multiplicative action of one exact category on another, the case

$$f\colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A}\,.$$

Let us suppose this action has a unit, that is, there is $B_0 \in \mathcal{B}$ so that the partial functor $f(\ ,B_0)$ is the identity on $\mathcal{A}$.

There is a map $S^1 = \Sigma S^0 \to Q\mathcal{B}$ which is given by the pair of morphisms $0 \twoheadleftarrow B_0$, $0 \rightarrowtail B_0$ in $Q\mathcal{B}$. Hence, in view of the preceding,

Fact 9.2.2. The following diagrams are commutative:

$$\begin{array}{ccc} B\,\mathrm{Is}(\mathcal{A}) \wedge \Sigma B(\mathrm{Id}_{B_0} \cup 0) & \xrightarrow{\approx} & \Sigma B\,\mathrm{Is}(\mathcal{A}) \\ \downarrow & & \downarrow \\ B\,\mathrm{Is}(\mathcal{A}) \wedge \Sigma B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & \Sigma B\,\mathrm{Is}(\mathcal{A})^{\mathrm{Is}} \\ \downarrow & & \downarrow \\ B\,\mathrm{Is}(\mathcal{A}) \wedge BQ\mathcal{B} & \longrightarrow & B\,\mathrm{Is}(\mathcal{A})^{Q}, \end{array} \qquad \begin{array}{ccc} BQ\mathcal{A} \wedge \Sigma B(\mathrm{Id}_{B_0} \cup 0) & \xrightarrow{\approx} & \Sigma BQ\mathcal{A} \\ \downarrow & & \downarrow \\ BQ\mathcal{A} \wedge \Sigma B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & \Sigma BQ\mathcal{A}^{\mathrm{Is}} \\ \downarrow & & \downarrow \\ BQ\mathcal{A} \wedge BQ\mathcal{B} & \longrightarrow & BQQ\mathcal{A}\,. \end{array}$$

Here $\mathrm{Is}(\mathcal{A})^{Q}$ denotes the bicategory $Q\mathcal{A}^{\mathrm{Is}}$ with its horizontal and vertical directions interchanged. Note the composed map $\Sigma B\,\mathrm{Is}(\mathcal{A}) \to B\,\mathrm{Is}(\mathcal{A})^{Q}$ may be characterized by the fact that to $A \xrightarrow{\approx} A'$ in $\mathrm{Is}(\mathcal{A})$ it associates the pair of bimorphisms in $\mathrm{Is}(\mathcal{A})^{Q}$,

$$\begin{array}{ccc} A & \xrightarrow{\approx} & A' \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ 0 & \longrightarrow & 0\,, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\approx} & A' \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0\,. \end{array}$$

Similarly, the composed map $\Sigma BQ\mathcal{A} \to BQQ\mathcal{A}$ in the diagram on the right satisfies the fact that it takes a morphism $M \twoheadleftarrow N \rightarrowtail M'$ in $Q\mathcal{A}$ to the pair of bimorphisms in $QQ\mathcal{A}$,

$$\begin{array}{ccccc} M & \twoheadleftarrow & N & \rightarrowtail & M' \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0\,, \end{array} \qquad \begin{array}{ccccc} M & \twoheadleftarrow & N & \rightarrowtail & M' \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0\,. \end{array}$$

Hence the latter map coincides with the map in the addendum to Proposition 9.1, the adjoint of the homotopy equivalence $BQ\mathcal{A} \to \Omega BQQ\mathcal{A}$.

LEMMA 9.2.3. *The inclusion $BQ\mathcal{A} \to BQ\mathcal{A}^{\mathrm{Is}}$ has a canonical left inverse.*

Proof. One defines this map on the level of simplicial sets as a map

$$\mathrm{diag}\, N(N_v Q\mathcal{A}^{\mathrm{Is}}) \longrightarrow NQ\mathcal{A}\,.$$

The map is characterized as follows. One considers a bimorphism in $Q\mathcal{A}^{\mathrm{Is}}$ as a commutative square in $Q\mathcal{A}$, and to this commutative square one associates the composed map from the lower left to the upper right. The map so obtained is a homotopy equivalence since its section is.

Passing to the adjoint situation, with loop spaces instead of suspensions, we may reformulate 9.2.2 thus, in view of 9.2.3,

LEMMA 9.2.4. *In the situation of* 9.2.2, *there are canonical commutative diagrams*

$$\begin{array}{ccc} B\,\mathrm{Is}(\mathcal{A}) \wedge B(\mathrm{Id}_{B_0} \cup 0) & \xrightarrow{\approx} & B\,\mathrm{Is}(\mathcal{A}) \\ \downarrow & & \vdots \\ B\,\mathrm{Is}(\mathcal{A}) \wedge B\,\mathrm{Is}(\mathcal{B}) & & \vdots \\ \downarrow & & \downarrow \\ B\,\mathrm{Is}(\mathcal{A}) \wedge \Omega BQ\mathcal{B} & \longrightarrow & \Omega BQ\mathcal{A}\,, \end{array} \qquad \begin{array}{ccc} BQ\mathcal{A} \wedge B(\mathrm{Id}_{B_0} \cup 0) & \xrightarrow{\approx} & BQ\mathcal{A} \\ \downarrow & & \vdots \\ BQ\mathcal{A} \wedge B\,\mathrm{Is}(\mathcal{B}) & & \vdots \\ \downarrow & & \downarrow \\ BQ\mathcal{A} \wedge \Omega BQ\mathcal{B} & \longrightarrow & \Omega BQQ\mathcal{A}\,. \end{array}$$

Similarly, using 9.2.3 and an analogous map $B\,\mathrm{Is}(\mathcal{C})^{\mathrm{Is}} \to B\,\mathrm{Is}(\mathcal{C})$,

LEMMA 9.2.5. *In the situation of* 9.2.1, *there are canonical diagrams*

$$\begin{array}{ccc} B\,\mathrm{Is}(\mathcal{A}) \wedge B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & B\,\mathrm{Is}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \Omega BQ\mathcal{A} \wedge B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & \Omega BQ\mathcal{C}\,, \end{array} \qquad \begin{array}{ccc} BQ\mathcal{A} \wedge B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & BQ\mathcal{C} \\ \downarrow & & \downarrow \\ BQ\mathcal{A} \wedge \Omega BQ\mathcal{B} & \longrightarrow & \Omega BQQ\mathcal{C} \end{array}$$

of which the first one is commutative, and the second one commutative up to basepoint preserving homotopy.

Putting the two diagrams of 9.2.5 together, we obtain

LEMMA 9.2.6. *In the situation of* 9.2.1, *the following diagram commutes up to basepoint preserving homotopy*

$$\begin{array}{ccc} B\,\mathrm{Is}(\mathcal{A}) \wedge B\,\mathrm{Is}(\mathcal{B}) & \longrightarrow & B\,\mathrm{Is}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \Omega BQ\mathcal{A} \wedge \Omega BQ\mathcal{B} & & \Omega BQ\mathcal{C} \\ \downarrow & & \downarrow \\ \Omega\Omega(BQ\mathcal{A} \wedge BQ\mathcal{B}) & \longrightarrow & \Omega\Omega BQQ\mathcal{C}\,. \end{array}$$

9.3. *Comparison of K-theories.* In order to chase a certain map, we have to go through a variant of Quillen's theorem that $\Omega BQ\mathcal{P}_R$ and $K_0(R) \times BGL^+(R)$ have the same homotopy type.

Let $\mathcal{A}$ be a small exact category. We assume $\mathcal{A}$ is pointed by a zero object 0. As in example 5.3.2, we form bicategories $Q\mathcal{A}^{\mathrm{Is}}$ and $L\mathcal{A}^{\mathrm{Is}}$ where the category $L\mathcal{A}$ is as in 9.1. Considering $\mathrm{Is}(\mathcal{A})$ as a bicategory in a trivial way, degenerating to its category of vertical morphisms, and thinking of the 'Is-directions' of $L\mathcal{A}^{\mathrm{Is}}$ and $Q\mathcal{A}^{\mathrm{Is}}$ as vertical, we can extend the sequence $\mathrm{Ob}(\mathcal{A}) \to L\mathcal{A} \to Q\mathcal{A}$ of 9.1, to a sequence of bicategories

$$\mathrm{Is}(\mathcal{A}) \longrightarrow L\mathcal{A}^{\mathrm{Is}} \longrightarrow Q\mathcal{A}^{\mathrm{Is}}$$

with constant composition. Taking horizontal nerves we obtain a sequence of simplical categories that we denote

$$\mathrm{Is}(\mathcal{A}) \longrightarrow \mathrm{Is}\,L.\mathcal{A} \longrightarrow \mathrm{Is}\,Q.\mathcal{A}\,.$$

For example, the objects of the category $\mathrm{Is}\,Q_1\mathcal{A}$ are the morphisms of $Q\mathcal{A}$, i.e., equivalence classes of certain diagrams in $\mathcal{A}$, and its morphisms are the isomorphisms between such equivalence classes of diagrams.

By the procedure described in Section 6, the direct sum in $\mathcal{A}$ makes each of these simplicial categories naturally the underlying object of a Γ-object, for which we will not introduce extra notation. Passing to the associated Γ_0-objects, and applying Proposition 6.3, we obtain a commutative diagram of bisimplicial categories

$$\begin{array}{ccccc}
\operatorname{Is}(\mathcal{A}) & \longrightarrow & \operatorname{Is} L.\mathcal{A} & \longrightarrow & \operatorname{Is} Q.\mathcal{A} \\
\downarrow & & \downarrow & & \downarrow \\
N_\Gamma(\operatorname{Is}(\mathcal{A}), \operatorname{Is}(\mathcal{A})) & \longrightarrow & N_\Gamma(\operatorname{Is} L.\mathcal{A}, \operatorname{Is} L.\mathcal{A}) & \longrightarrow & N_\Gamma(\operatorname{Is} Q.\mathcal{A}, \operatorname{Is} Q.\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow \\
N_\Gamma(\operatorname{Is}(\mathcal{A})) & \longrightarrow & N_\Gamma(\operatorname{Is} L.\mathcal{A}) & \longrightarrow & N_\Gamma(\operatorname{Is} Q.\mathcal{A})
\end{array}$$

in which the middle and right columns are fibrations up to homotopy since $\operatorname{Is} L.\mathcal{A}$ and $\operatorname{Is} Q.\mathcal{A}$ are connected. By the addendum to Proposition 6.3, the terms in the middle row are contractible. Since $\operatorname{Is} L.\mathcal{A}$ is contractible, all terms in the middle column are contractible as well.

Proposition 9.3.1. *Suppose that exact sequences in* $\mathcal{A}$ *split (non-naturally). Then the bottom row in this diagram, is a fibration up to homotopy.*

Proof. By Lemma 5.2 (or at the cost of some extra considerations, cf. the proof of 6.3, by Proposition 1.6 of [22]) it is sufficient to prove that

$$N_\Gamma(\operatorname{Is}(\mathcal{A})) \longrightarrow N_\Gamma(\operatorname{Is} L_m\mathcal{A}) \longrightarrow N_\Gamma(\operatorname{Is} Q_m\mathcal{A})$$

is a fibration up to homotopy, for each m, because $N_\Gamma(\operatorname{Is} Q_m\mathcal{A})$ is connected. The sequence of categories $\operatorname{Is}(\mathcal{A}) \to \operatorname{Is} L_m\mathcal{A} \to \operatorname{Is} Q_m\mathcal{A}$ is equivalent to the sequence

$$\operatorname{Is}(\mathcal{A}) \longrightarrow \operatorname{Is}(F_{2m+1}\mathcal{A}) \longrightarrow \operatorname{Is}(F_{2m}\mathcal{A})$$

in which the first map is given by

$$A \longmapsto (A \xrightarrow{=} \cdots \xrightarrow{=} A)\,.$$

Hence the sequence $N_\Gamma(\operatorname{Is}(\mathcal{A})) \to N_\Gamma(\operatorname{Is}(L_m\mathcal{A})) \to N_\Gamma(\operatorname{Is}(Q_m\mathcal{A}))$ is homotopy equivalent to the induced sequence

$$N_\Gamma(\operatorname{Is}(\mathcal{A})) \longrightarrow N_\Gamma(\operatorname{Is}(F_{2m+1}\mathcal{A})) \longrightarrow N_\Gamma(\operatorname{Is}(F_{2m}\mathcal{A}))$$

to which Corollary 8.5 applies in view of the hypothesis that exact sequences are splittable in $\mathcal{A}$. The conclusion is that the latter sequence is homotopy equivalent to the (product) fibration

$$N_\Gamma(\operatorname{Is}(\mathcal{A})) \longrightarrow N_\Gamma(\operatorname{Is}(\mathcal{A})) \times (N_\Gamma(\operatorname{Is}(\mathcal{A})))^{2m+1} \longrightarrow (N_\Gamma(\operatorname{Is}(\mathcal{A})))^{2m+1}$$

and the proposition results.

The isomorphism of $\mathrm{Is}(\mathcal{A})$ to the category in degree 1 of the simplicial category $N_\Gamma(\mathrm{Is}(\mathcal{A}))$ gives an embedding $\Sigma B\,\mathrm{Is}(\mathcal{A}) \to BN_\Gamma(\mathrm{Is}(\mathcal{A}))$. The adjoint map $B\,\mathrm{Is}(\mathcal{A}) \to \Omega BN_\Gamma(\mathrm{Is}(\mathcal{A}))$ can be identified to the inclusion of $B\,\mathrm{Is}(\mathcal{A})$ into the homotopy theoretic fibre of the map $BN_\Gamma(\mathrm{Is}(\mathcal{A}), \mathrm{Is}(\mathcal{A})) \to BN_\Gamma(\mathrm{Is}(\mathcal{A}))$, induced from the left column in the diagram of the preceding proposition.

Similar maps correspond to the right hand column of this diagram, and to its upper and lower row, respectively. Putting these maps together, we obtain a commutative diagram

$$\begin{array}{ccc} B\,\mathrm{Is}(\mathcal{A}) & \longrightarrow & \Omega B\,\mathrm{Is}\,Q.\mathcal{A} \\ \downarrow & & \downarrow \\ \Omega BN_\Gamma(\mathrm{Is}(\mathcal{A})) & \longrightarrow & \Omega\Omega BN_\Gamma(\mathrm{Is}\,Q.\mathcal{A}) \end{array}$$

in which the right vertical map is a homotopy equivalence. The lower horizontal map will be a homotopy equivalence whenever the proposition applies. In view of the homotopy equivalence $B\,\mathrm{Is}\,Q.\mathcal{A} = BQ\mathcal{A}^{\mathrm{Is}} \to BQ\mathcal{A}$ we have therefore

COROLLARY 9.3.2. *Suppose that exact sequences split in A. Then there is a basepoint preserving homotopy equivalence $BN_\Gamma(\mathrm{Is}(\mathcal{A})) \to BQ\mathcal{A}$ so that*

$$\begin{array}{ccc} & B\,\mathrm{Is}(\mathcal{A}) & \\ \swarrow & & \searrow \\ \Omega BN_\Gamma(\mathrm{Is}(\mathcal{A})) & \longrightarrow & \Omega BQ\mathcal{A} \end{array}$$

commutes up to basepoint preserving homotopy.

LEMMA ([19], [22, § 4]). *Let R be a ring. There exists a map*

$$K_0(R) \times BGL(R) \longrightarrow \Omega BN_\Gamma(\mathrm{Is}(\mathcal{P}_R))$$

which induces an isomorphism on homology, and so that

$$\begin{array}{ccc} & B\,\mathrm{Is}(\mathcal{A}) & \\ \swarrow & & \searrow \\ K_0(R) \times BGL(R) & \longrightarrow & \Omega BN_\Gamma(\mathrm{Is}(\mathcal{P}_R)) \end{array}$$

commutes up to basepoint preserving homotopy.

The maps in this diagram are the natural ones, that is, the right hand map is the same as that in the preceding corollary, and the left hand map, restricted to $\mathrm{Is}(P)$ where $P \in \mathcal{P}_R$, has components

$$P \longmapsto [P]\,, \quad \mathrm{Is}(P) \hookrightarrow GL(R)\,,$$

where the latter map is induced from the identification of P with a projection operator on a standard free module and any (say, the standard) stabilization of the latter.

Applying the 'plus'-construction of Quillen gives a factorization of the map of this lemma through a homotopy equivalence (unique up to basepoint preserving homotopy on compacta),

$$K_0(R) \times BGL^+(R) \longrightarrow \Omega BN_\Gamma(\mathrm{Is}(\mathcal{P}_R)) .$$

Combining this homotopy equivalence with the one of Corollary 9.3.2, we obtain a homotopy equivalence $K_0(R) \times BGL^+(R) \to \Omega BQ\mathcal{P}_R$.

COROLLARY 9.3.3. *This homotopy equivalence satisfies the fact that*

$$\begin{array}{ccc} K_0(R) \times BGL(R) & \longrightarrow & B\,\mathrm{Is}(\mathcal{P}_R) \\ \downarrow & & \downarrow \\ K_0(R) \times BGL^+(R) & \longrightarrow & \Omega BQ\mathcal{P}_R \end{array}$$

commutes up to basepoint preserving homotopy.

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