Proceedings of Symposia in Pure Mathematics
Volume 32, 1978

ALGEBRAIC K-THEORY OF TOPOLOGICAL SPACES. I

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This paper is concerned with a functor $A(X)$, from spaces to spaces, which is in some ways analogous to the algebraic K-theory functor $K(R)$ which goes from rings to spaces. The two are related by a natural transformation from $A(X)$ to a suitable K-theory. The natural transformation itself is induced by a Hurewicz map.

The functor $A(X)$ is of some interest in itself, for example there are about as many definitions of $A(X)$ as there are definitions of $K(R)$. More significantly however it can be used to obtain information about the Whitehead spaces $\mathrm{Wh}^{\mathrm{PL}}(X)$ and $\mathrm{Wh}^{\mathrm{Diff}}(X)$ whose homotopy groups are the PL, resp. Diff, concordance groups, stabilized with respect to dimension.

The plan of the exposition is as follows.

§1 discusses a K-theory of simplicial rings. This may be regarded as a model for the study of one aspect of $A(X)$.

§2 gives the quick definition of $A(X)$, via the plus construction.

§3 describes the Whitehead spaces and their relation to $A(X)$.

§4 introduces what by analogy may be called a nonadditive exact-sequence-K-theory.

§5 indicates the proof of the main results.

1. K-theory of simplicial rings. Let $R.$ be a simplicial ring (with unit); then $\pi_0 R.$ is a ring, and $\pi_0\colon R. \to \pi_0 R.$ can be considered as a map of simplicial rings. Let M_n denote $(n \times n)$-matrices, and $\mathrm{GL}_n \to M_n$ the inclusion. Define $\widehat{\mathrm{GL}}_n(R.)$ to be the pullback in the diagram

$$\begin{array}{ccc} \widehat{\mathrm{GL}}_n(R.) & \longrightarrow & M_n(R.) \\ \downarrow & & \downarrow \\ \mathrm{GL}_n(\pi_0 R.) & \longrightarrow & M_n(\pi_0 R.) \end{array}$$

AMS (MOS) subject classifications (1970). Primary 55—xx, 57—xx.

and $\widehat{\mathrm{GL}}(R.) = \text{dir lim } \widehat{\mathrm{GL}}_n(R.)$. This is a simplicial monoid, and one can form its classifying space $B\widehat{\mathrm{GL}}(R.)$ (to be interpreted, say, as the geometric realization of a certain bisimplicial set). By definition of $\widehat{\mathrm{GL}}(R.)$, the map

$$\pi_1 B\widehat{\mathrm{GL}}(R.) \to \pi_1 B\mathrm{GL}(\pi_0 R.)$$

is an isomorphism, so $\pi_1 B\widehat{\mathrm{GL}}(R.)$ has perfect commutator group, and one can apply Quillen's plus construction to form $B\widehat{\mathrm{GL}}(R.)^+$, together with a natural transformation $B\widehat{\mathrm{GL}}(R.) \to B\widehat{\mathrm{GL}}(R.)^+$ which abelianizes π_1 and induces an isomorphism on homology. $B\widehat{\mathrm{GL}}(R.)^+$ is a simple space.

REMARK. If $\pi_0 R. = 0$ then $R.$ is contractible (multiply by a path from 1 to 0); hence $B\widehat{\mathrm{GL}}(R.)$ is contractible, and $B\widehat{\mathrm{GL}}(R.)^+$ is not a very interesting space. This shows that our $B\widehat{\mathrm{GL}}(R.)^+$ is very much different, in general, from the K-theory of a simplicial ring defined by Anderson **[2]** (one forms $B\mathrm{GL}(R.)^+$ (no $\hat{}$) by taking the plus degreewise); for example, the Karoubi-Villamayor K-theory, which agrees with Quillen's for regular noetherian rings, is the K-theory, in Anderson's sense, of a connected simplicial ring **[2]**; as a more immediate example, consider the real numbers. There appears to be only one case where the two definitions agree for general reasons, that of a graded ring considered as a differential graded ring in a trivial way and turned into a simplicial ring by means of the Dold-Kan functor. A question in that context is if there is any relation, in general, between the K-theory of a simplicial ring and that of the graded ring of its homotopy groups.

Henceforth we assume that $1 \neq 0$ in $\pi_0 R.$.

DEFINITION. $K(R.) = Z \times B\widehat{\mathrm{GL}}(R.)^+$.

It is a special case of the following proposition that the functor $R. \mapsto K(R.)$ preserves weak homotopy equivalences, that is, if $R. \to R'.$ induces an isomorphism on all homotopy groups then so does the induced map $K(R.) \to K(R'.)$. Thus for example if $R. \xrightarrow{\sim} \pi_0 R.$ then $K(R.)$ gives the Quillen K-theory of $(\pi_0 R.)$-modules (in this paper, K-theory of a ring will mean that of its free, rather than projective, modules).

Following the usual convention, we will call a map $X \to X'$ *k-connected* if, for any basepoint in X, the induced map $\pi_j X \to \pi_j X'$ is an isomorphism for $j < k$ and an epimorphism for $j = k$. Bookkeeping with this convention will be easier to follow if one keeps in mind that a k-connected map has $(k-1)$-connected homotopy fibre(s).

PROPOSITION 1.1. *If $R. \to R'.$ is k-connected, and $k \geqq 1$, then $K(R.) \to K(R'.)$ is $(k+1)$-connected.*

PROOF. There is a commutative diagram of simplicial sets,

$$\begin{array}{ccc} M_n(R._{(0)}) & \longrightarrow & M_n(R'._{(0)}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GL}}_n(R.)_{(e)} & \longrightarrow & \widehat{\mathrm{GL}}_n(R'.)_{(e)} \end{array}$$

where the subscript (0), resp. (e), denotes the zero-, resp. identity-, component, and where the vertical maps are the isomorphisms given by addition of 1. The top map is k-connected by hypothesis, and $k \geqq 1$; hence $B\widehat{\mathrm{GL}}(R.) \to B\widehat{\mathrm{GL}}(R'.)$ is $(k+1)$-

connected. Hence $B\widehat{\mathrm{GL}}(R.)^+ \to B\widehat{\mathrm{GL}}(R'.)^+$, being a map of simple spaces, is also $(k+1)$-connected [**6**].

The proposition admits the following quantitative refinement.

PROPOSITION 1.2. *Let $R. \to R'.$ be k-connected, where $k \geqq 1$. Then*

$$\pi_{k+1}\, \textit{fibre}(K(R.) \to K(R'.)) \xrightarrow{\approx} H_0(\pi_0 R'., \pi_k \textit{fibre}(R. \to R'.)) \quad (\textit{Hochschild homology})$$
$$\approx \pi_k\, \textit{fibre}(R. \to R'.)\,/\,(ar - ra)$$

where $a \in \pi_k\, \textit{fibre}(R. \to R'.)$, and $r \in \pi_0 R'.$

PROOF. By the Hurewicz theorem, an equivalent assertion is that

$$H_{k+1}\ \mathrm{fibre}(K(R.) \to K(R'.)) \xrightarrow{\approx} H_0(\pi_0 R'., \pi_k \mathrm{fibre}(R. \to R'.)).$$

In order to prove this, we will compare the Serre spectral sequences of the vertical maps in the diagram:

$$\begin{array}{ccc} B\widehat{\mathrm{GL}}(R.) & \longrightarrow & B\widehat{\mathrm{GL}}(R.)^+ \\ \downarrow & & \downarrow \\ B\widehat{\mathrm{GL}}(R'.) & \longrightarrow & B\widehat{\mathrm{GL}}(R'.)^+ \end{array}$$

Let A be an abelian group. We are interested in the cases $A = Z$, and $A = \boldsymbol{Q}$, the rationals, in view of a later application. Then the spectral sequence of the left vertical map is

$$E^2_{p,q} \Rightarrow H_{p+q}(B\widehat{\mathrm{GL}}(R.), A)$$

where $E^2_{p,q}$ is given by

$$\mathrm{dir\ lim}_{(n)}\ H_p(B\widehat{\mathrm{GL}}_n(R'.), H_q(\mathrm{fibre}(B\widehat{\mathrm{GL}}_n(R.) \to B\widehat{\mathrm{GL}}_n(R'.)), A)).$$

We have

$$\pi_i\ \mathrm{fibre}(B\widehat{\mathrm{GL}}_n(R.) \to B\widehat{\mathrm{GL}}_n(R'.)) \xrightarrow{\approx} \pi_{i-1}\ \mathrm{fibre}(\widehat{\mathrm{GL}}_n(R.) \to \widehat{\mathrm{GL}}_n(R'.))$$
$$\xrightarrow{\approx} \pi_{i-1}\ \mathrm{fibre}(M_n(R.) \to M_n(R'.)) \xrightarrow{\approx} M_n(\pi_{i-1}\mathrm{fibre}(R. \to R'.))$$

for all i, this being trivially true for small i because of the connectivity assumption. Furthermore the action of $\pi_1 B\widehat{\mathrm{GL}}_n(R'.)$ on the former group corresponds, under the isomorphism, to the conjugation action of $\pi_0\, \widehat{\mathrm{GL}}_n(R'.) \to^{\approx} \mathrm{GL}_n(\pi_0 R'.)$ on the latter. Let j be the smallest number so that $\pi_{j-1}\mathrm{fibre}(R. \to R'.)$ is nonzero. Then by the Hurewicz theorem and universal coefficient theorem,

$$H_j(\mathrm{fibre}(B\widehat{\mathrm{GL}}_n(R.) \to B\widehat{\mathrm{GL}}_n(R'.)), A) \approx M_n(\pi_{j-1}\mathrm{fibre}(R. \to R'.)) \otimes A;$$

hence $E^2_{p,j}$ is given by

$$\mathrm{dir\ lim}_{(n)}\ H_p(B\widehat{\mathrm{GL}}_n(R'.), M_n(\pi_{j-1}\mathrm{fibre}(R. \to R'.) \otimes A))$$

where the action is the conjugation action.

Let the spectral sequence of the right-hand vertical map be denoted

$$E'^2_{p,q} = H_p(B\widehat{\mathrm{GL}}(\mathrm{R}'.)^+, B_q) \Rightarrow H_{p+q}(B\widehat{\mathrm{GL}}(R.)^+, A)$$

where $B_q = H_q(\mathrm{fibre}(B\widehat{\mathrm{GL}}(R.)^+ \to B\widehat{\mathrm{GL}}(R'.)^+), A)$. The action in $H_p(B\widehat{\mathrm{GL}}(R'.)^+, B_q)$ is trivial. By the fundamental property of the plus construction, the map of spectral sequences is an isomorphism on the abutment, $H_*(B\widehat{\mathrm{GL}}(R.), A) \to^{\approx} H_*(B\widehat{\mathrm{GL}}(R.)^+, A)$ and on the base, $H_p(B\widehat{\mathrm{GL}}(R'.), B_q) \to^{\approx} H_p(B\widehat{\mathrm{GL}}(R'.)^+, B_q)$. Since the fibres are connected, we have $A \approx E^2_{0,0} \approx E'^2_{0,0} \approx B_0$; hence $E^2_{p,0} \to^{\approx} E'^2_{p,0}$. Furthermore $E^2_{p,q} \to^{\approx} E'^2_{p,q}$ for $0 < q \leqq k$ since these terms are zero. It follows that we must have

$$E^{\infty}_{0,k+1} \xrightarrow{\approx} E'^{\infty}_{0,k+1}$$

and from this that $E^2_{0,k+1} \to^{\approx} E'^2_{0,k+1}$, i.e., that

$$\operatorname{dir\,lim}_{(n)} H_0(\mathrm{GL}_n(\pi_0 R'_\cdot), M_n(\pi_k \operatorname{fibre}(R_\cdot \to R'_\cdot) \otimes A)) \xrightarrow{\approx} H_0(\mathrm{GL}(\pi_0 R'_\cdot), B_{k+1}) \xleftarrow{\approx} B_{k+1}.$$

But clearly the trace map of the coefficients induces an isomorphism of the former term to $H_0(\pi_0 R'_\cdot, \pi_k \operatorname{fibre}(R_\cdot \to R'_\cdot) \otimes A)$. This completes the proof.

Proposition 1.1 admits a relativization which will be given in the following proposition. To state it properly we need the following notions. Let an (m, n)-*connected square* denote a commutative square in which the horizontal maps are m-connected, and the vertical maps n-connected. Call it *k-homotopy cartesian* if the map of the homotopy fibres of the vertical maps is $(k + 1)$-connected. For example, there is a result in algebraic topology, *homotopy excision*, which says that an (m, n)-connected pushout diagram of cofibrations is k-homotopy cartesian with $k = (m - 1) + (n - 1)$.

PROPOSITION 1.3. *A $(k - 1)$-homotopy cartesian, $(m - 1, n - 1)$-connected square of simplical rings, with $m, n \geqq 2$ and $k \leqq m + n - 2$, is taken by the functor K to a k-homotopy cartesian, (m, n)-connected square.*

PROOF. The functor $B\widehat{\mathrm{GL}}$ produces a k-homotopy cartesian (m, n)-connected square. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be such a square in general. After replacing of $A \to B$ and $C \to D$ by cofibrations, if necessary, their pushout diagram is $(m + n - 2)$-homotopy cartesian, by homotopy excision. From this one sees that k-homotopy cartesianness of the square is equivalent, for $k \leqq m + n - 2$, to the map $B \cup_A C \to D$ being $(k + 2)$-connected. In the case at hand we can have $(B \cup_A C)^+ = B^+ \cup_{A^+} C^+$. This shows firstly that the resulting spaces are nilpotent, so [6] the connectivity survives under the plus construction. Secondly it shows that the above argument can be traced backward, and the assertion follows.

From Propositions 1.1 and 1.3 one has a spectral sequence relating $K(R_\cdot)$ to the $K(\mathrm{Sk}^j R_\cdot)$ where Sk^j denotes the j-coskeleton, $E^2_{p,q} \Rightarrow \pi_{p+q} K(R_\cdot)$, $p \geqq 0$, $q \geqq 1$, with

$$\begin{aligned} E^2_{p,q} &= \pi_{p+q} \operatorname{fibre}(K(\mathrm{Sk}^{q-1} R_\cdot) \to K(\mathrm{Sk}^{q-2} R_\cdot)) \quad \text{if } q > 1, \\ E^2_{p,1} &= \pi_{p+1} K(\pi_0 R_\cdot) \end{aligned}$$

and

$$E^2_{0,q} \approx H_0(\pi_0 R_\cdot, \pi_{q-1} R_\cdot) \quad \text{if } q > 1$$

by Propsition 1.2

Kiyoshi Igusa has discussed a related spectral sequence in a context of what he calls higher Whitehead groups [11]. In this context Igusa points out that by (an analogue of) Proposition 1.3, there is a 'stable range', $p \leqq q - 2$, in which $E^2_{p,q}$ can be identified to a certain 'stable' group. We will not enter this discussion here. Rather we will discuss a somewhat different kind of stabilization which will be needed later on.

Let $G(X)$ denote the *loop group* in the sense of Kan [12]. $X \mapsto G(X)$ is a functor from pointed connected simplicial sets to free simplicial groups, and $G(X)$ is a loop group for X in the sense that there exists a principal simplicial $G(X)$-bundle over X, natural in X, with (weakly) contractible total space $X_t \times G(X)$ (a 'twisted cartesian product').

If R is any ring, one may form the simplicial group algebra $R[G(X)]$, hence $K(R[G(X)])$ is defined. For example, Proposition 1.1 says, if $X \to X'$ is k-connected then so is $K(R[G(X)]) \to K(R[G(X')])$ provided that $k \geqq 2$.

The functor $K(R[G(X)])$ is related to the usual K-theory of a group ring. Indeed, the map $G(X) \to \pi_0 G(X) = \pi_1 X$ induces a natural transformation

$$K(R[G(X)]) \to K(R[\pi_1 X]).$$

This is by no means an equivalence in general. For example,

$$G(X) \to 1 \cdot G(X) \subset R[G(X)]$$

may be considered as a Hurewicz map; and Proposition 1.2 yields

$$\pi_2 \operatorname{fibre}(K(R[G(X)]) \to K(R[\pi_1 X])) \approx (\pi_2 X \otimes R[\pi_1 X])/\pi_1 X.$$

Using a suitable pointed simplicial set as a model, we can decompose the $(n+1)$-sphere, $n \geqq 2$, into its upper and lower hemispheres D^{n+1}_+ and D^{n+1}_-; this gives for any pointed simplicial set X a pushout diagram

$$\begin{array}{ccc} S^n \wedge (X \cup *) & \longrightarrow & D^{n+1}_+ \wedge (X \cup *) \\ \downarrow & & \downarrow \\ D^{n+1}_- \wedge (X \cup *) & \longrightarrow & S^{n+1} \wedge (X \cup *) \end{array}$$

in which all maps are n-connected. By Proposition 1.3, the (n, n)-connected square

$$\begin{array}{ccc} K(R[G(S^n \wedge (X \cup *))]) & \longrightarrow & K(R[G(D^{n+1}_+ \wedge (X \cup *))]) \\ \downarrow & & \downarrow \\ K(R[G(D^{n+1}_- \wedge (X \cup *))]) & \longrightarrow & K(R[G(S^{n+1} \wedge (X \cup *))]) \end{array}$$

is therefore $(2n-2)$-homotopy cartesian, consequently the map

$$\begin{aligned} &\Omega^n \operatorname{fibre}(K(R[G(S^n \wedge (X \cup *))]) \to K(R)) \\ &\longrightarrow \Omega^n \operatorname{fibre}(K(R) \to K(R[G(S^{n+1} \wedge (X \cup *))])) \\ &\simeq \Omega^{n+1} \operatorname{fibre}(K(R[G(S^{n+1} \wedge (X \cup *))]) \to K(R)) \end{aligned}$$

is $(n-1)$-connected.

DEFINITION. $K^S(R[G(X)]) = \operatorname{dir\,lim} \Omega^n \operatorname{fibre}(K(R[G(S^n \wedge (X \cup *))]) \to K(R))$.

REMARK. Having to use basepoints and connectivity is somewhat artificial but unfortunately necessary since we are using $G(X)$. For example one may expect that there is a natural transformation

$$K(R[G(X)]) \to K^S(R[G(X)]).$$

This does indeed exist, but it cannot (or at least not obviously) be obtained with the present definition.

Notice that $K^S(R[G(X)])$ is really a functor of two variables, R and X. For example if $R = R'[G(Y)]$ there does not appear to be any reason why $K^S(R) = K^S(R[G(*)])$ and $K^S(R'[G(Y)])$ should be the same.

LEMMA 1.4. *The functor $X \mapsto K^S(R[G(X)])$ is an (unreduced) homology theory, that is, $X \mapsto \pi_* K^S(R[G(X)])$ satisfies the Eilenberg-Steenrod axioms except for the dimension axiom.*

PROOF. The thing to verify is excision. Suppose X_0, X_1, X_2 are simplicial subsets of X so that $X_1 \cup_{X_0} X_2 \to^{\approx} X$. Then the square

$$\begin{array}{ccc} \Omega^n\tilde{K}(R[G(S^n \wedge (X_0 \cup *))]) & \longrightarrow & \Omega^n\tilde{K}(R[G(S^n \wedge (X_1 \cup *))]) \\ \downarrow & & \downarrow \\ \Omega^n\tilde{K}(R[G(S^n \wedge (X_2 \cup *))]) & \longrightarrow & \Omega^n\tilde{K}(R[G(S^n \wedge (X \cup *))]) \end{array}$$

where $\tilde{K}(R[G(Y)]) = \text{fibre}(K(R[G(Y)]) \to K(R))$, is $(n-2)$-homotopy cartesian, by Proposition 1.3. So in the limit the square becomes homotopy cartesian, which is the assertion of excision.

The only use so far of the curious functor $R \mapsto K^S(R)$ is the following result which will be needed later.

PROPOSITION 1.5.

$$\pi_i K^S(Z) \otimes Q \approx \begin{cases} Q, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

The proof depends on a result of F. T. Farrell and W. C. Hsiang, adapting a technique of Borel [3]. Let $\mathrm{GL}_n(Z)$ act by conjugation on $M_n(Q)$, the rational $(n \times n)$-matrices. The trace $M_n(Q) \to Q$ induces a surjective map

$$\text{dir lim } H_*(\mathrm{GL}_n(Z), M_n(Q)) \to \text{dir lim } H_*(\mathrm{GL}_n(Z), Q).$$

LEMMA (FARRELL AND HSIANG [7]). *This map is an isomorphism.*

PROOF OF PROPOSITION. It suffices to show that, for large n,

$$\begin{aligned} B_q &= H_q(\text{fibre}(B\widehat{\mathrm{GL}}(Z[G(S^n)])^+ \to B\,\mathrm{GL}(Z)^+), Q) \\ &\approx \begin{cases} Q & \text{if } q = 0, n, \\ 0 & \text{if } q \leqq 2n-2,\ q \neq 0, n. \end{cases} \end{aligned}$$

This is proved by the method of Proposition 1.2, the comparison of the Serre spectral sequences for rational homology of the vertical maps in the diagram:

$$\begin{array}{ccc} B\widehat{\mathrm{GL}}(Z[G(S^n)]) & \longrightarrow & B\widehat{\mathrm{GL}}(Z[G(S^n)])^+ \\ \downarrow & & \downarrow \\ B\mathrm{GL}(Z) & \longrightarrow & B\mathrm{GL}(Z)^+ \end{array}$$

In the notation set up in the proof of Proposition 1.2, suppose r is the smallest number so that $E^2_{p,r} \to E'^2_{p,r}$ si not an isomorphism, for some p. Suppose $r \leqq 2n-2$

and $r \neq n$. The assumption now means that $E'^2_{p,r} \neq 0$. It follows that $E'^2_{0,r} \neq 0$ since this is the coefficient group. Since $E^2_{p,q} \to^{\approx} E'^2_{p,q}$ for $q < r$, and $E^2_{0,r} = 0$, it follows that $E'^2_{0,r}, E'^3_{0,r}, \cdots$, cannot be hit by a nonzero differential. Hence $E'^2_{0,r} \to^{\approx} E'^{\infty}_{0,r}$ must contribute to the abutment, a contradiction.

So suppose the first deviation occurs for $r = n$. The above argument still shows that $E^2_{0,n} \to^{\approx} E'^2_{0,n}$, i.e., that

$$\text{dir lim } H_0(\mathrm{GL}_k(Z),\ M_k(Q)) \xrightarrow{\approx} H_0(\mathrm{GL}(Z),\ B_n) \xleftarrow{\approx} B_n.$$

But

$$\text{dir lim } M_k(Q) \longrightarrow \text{dir lim } H_0(\mathrm{GL}_k(Z),\ M_k(Q)) \xrightarrow{\approx} Q$$

is given by the trace, and it is also the coefficient map, i.e., $H_n(\ , Q)$ applied to the map of fibres. Hence by the lemma of Farrell and Hsiang, $E^2_{p,n} \to^{\approx} E'^2_{p,n}$ for all p, and there is no deviation after all. This completes the proof.

2. The functor $A(X)$**.** By way of introduction, a definition will be given which is very close to that of the K-theory of a simplicial ring. It presupposes the notion of 'ring up to homotopy'.

A *ring up to homotopy* consists of an *underlying space*, R, plus a lot of additional structure. This is (i) the additive-group-law-up-to-homotopy, i.e., a homotopy everything H-space (E_∞ space) with underlying space R satisfying that $\pi_0 R$ with the induced monoid structure is in fact a group; (ii) the multiplication-law-up-to-homotopy, i.e., a homotopy associative H-space with unit and higher coherence conditions (A_∞ space) with underlying space R; (iii) homotopy distributivity relating the additive and multiplicative structure, with higher coherence conditions.

The notion of a commutative ring up to homotopy has been successfully codified by May [14]. By dropping some of the structure one may expect to obtain from this one workable notion of a ring up to homotopy.

If R is (the underlying space of) a ring up to homotopy, one can form a new ring up to homotopy, the ring of $(n \times n)$-matrices. The underlying space $M_n(R)$ is simply the product of n^2 copies of R; similarly the additive structure is formed componentwise. However it is the multiplicative structure of $M_n(R)$ in which we are interested.

By restricting to a certain union of components one obtains an A_∞ space with underlying space $\widehat{\mathrm{GL}}_n(R)$, the pullback in the diagram:

$$\begin{array}{ccc} \widehat{\mathrm{GL}}_n(R) & \longrightarrow & M_n(R) \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{GL}_n(\pi_0 R) & \longrightarrow & M_n(\pi_0 R) \end{array}$$

We let $\widehat{\mathrm{GL}}(R) = \text{dir lim } \widehat{\mathrm{GL}}_n(R)$. This is of interest only if $1 \neq 0$ in $\pi_0 R$ which we now assume.

DEFINITION. $K(R) = Z \times B\widehat{\mathrm{GL}}(R)^+$.

There is a 'universal' ring up to homotopy (in the same sense in which the ring of integers is the universal ring with unit). The underlying space of this universal ring up to homotopy is the space

$$\mathrm{Q} = \text{dir lim } \Omega^n S^n.$$

If G is any simplicial group one can form the 'group algebra' $Q[G]$. It has underlying space dir lim $\Omega^n S^n(|G| \cup *)$, the space whose homotopy groups are the stable homotopy groups of $(|G| \cup *)$ (or, what is the same, the framed bordism groups of $|G|$).

If X is a pointed connected simplicial set, let as before $G(X)$ denote the loop group of X.

DEFINITION. $A(X) = K(Q[G(X)]) = Z \times B\widehat{GL}(Q[G(X)])^+$.

The notion of 'ring up to homotopy', and the technical problems it entails, enter in this construction of $A(X)$ in the following way: We need that the multiplication on $\widehat{GL}(Q[G(X)])$ is sufficiently associative for the classifying space to exist, and we want a canonical choice for the latter.

On the other hand it turns out that what is $B\widehat{GL}(Q[G(X)])$ for all practical purposes, can also be constructed very directly. This will be described below. It follows that $A(X)$ can also be constructed very directly; and that practically all of the results of the preceding section carry over to $A(X)$, quite independently of a worked out theory of rings up to homotopy, since the proofs do not involve the actual construction of a classifying space of a $\widehat{GL}$.

In order to obtain a more direct definition of $A(X)$, let G be any simplicial group which is a loop group for X in the sense that there exists a principal simplicial G-bundle over X with (weakly) contractible total space $X_t \times G$, the latter being a free right simplicial G-set. For example, $G = G(X)$, the loop group of Kan.

Define a category $\mathscr{S}(G)$ as follows. The objects are pointed simplicial left G-sets Y which are free in the sense that for all k, $g \in G_k$, $y \in Y_k$, one has $g(y) = y$ if and only if either $g = 1$ or $y = *$. The morphisms in $\mathscr{S}(G)$ are G-maps.

Let $h\mathscr{S}(G)$ be the subcategory of $\mathscr{S}(G)$ of those G-maps which are weak homotopy equivalences of the underlying simplicial sets.

Let $\mathscr{S}(G)^n_k$ be the full subcategory of those objects for which

$$|(X_t \times G) \times^G Y| \simeq_{\mathrm{rel}|X|} |X| \vee \bigvee_{j=1,\cdots,k} S^n_j$$

(homotopy equivalence, relative to the subspace $|X|$, to $|X|$ wedge k spheres of dimension n), and let

$$h\mathscr{S}(G)^n_k = \mathscr{S}(G)^n_k \cap h\mathscr{S}(G).$$

The latter categories are interrelated by a composition law

$$h\mathscr{S}(G)^n_k \times h\mathscr{S}(G)^n_{k'} \to h\mathscr{S}(G)^n_{k+k'}$$

and a suspension map

$$h\mathscr{S}(G)^n_k \to h\mathscr{S}(G)^{n+1}_k.$$

There is a particular object $(G \cup *)$ in $\mathscr{S}(G)^0_1$. By adding the identity on its n-fold suspension, one obtains a stabilization map $h\mathscr{S}^n_k \to h\mathscr{S}(G)^n_{k+1}$.

LEMMA 2.1. *There is a homotopy equivalence*

$$\Omega\, |\mathrm{dir}\lim_{(n)} h\mathscr{S}(G)^n_k| \simeq \widehat{GL}_k(Q[G(X)])$$

under which composition of loops in the former space corresponds to matrix multiplication in the latter.

COROLLARY AND/OR DEFINITION. $A(X) \simeq Z \times |\mathrm{dir}\lim_{(n,k)} h\mathscr{S}(G(X))^n_k|^+$.

There is a natural transformation

$$A(X) \to K(Z[G(X)])$$

which may be described in two ways. Firstly one may 'linearize' the category $h\mathcal{S}(G)^n_k$, i.e., one applies the functor which sends a free pointed simplicial G-set to the free simplicial $Z[G]$-module generated by the nonbasepoint elements; a natural transformation is then given by the Hurewicz map which takes an element to the generator it represents. Secondly, one has the map $B\widehat{\mathrm{GL}}(\mathrm{Q}[G(X)])^+ \to B\widehat{\mathrm{GL}}(Z[G(X)])^+$ induced from the ring map $\pi_0 \colon \mathrm{Q} \to Z$. In view of the linear analogue of Lemma 2.1, the two maps are the same, up to homotopy. As $\mathrm{Q}[G(X)] \to Z[G(X)]$ is a rational homotopy equivalence, and 1-connected, the proof of 1.1 (or 1.2) shows:

PROPOSITION 2.2. *The natural transformation* $A(X) \to K(Z[G(X)])$ *is a rational homotopy equivalence.*

In particular, $A(*) \to K(Z)$ is a rational homotopy equivalence.

Let $\Omega^{\mathrm{fr}}(X)$ (framed bordism) denote the space dir lim $\Omega^n S^n(|X| \cup *)$. The Barratt-Priddy-Quillen-Segal theorem may be stated to say that

$$\Omega^{\mathrm{fr}}(X) \simeq Z \times \left|\text{dir lim } h\mathcal{S}(G(X))^0_n\right|^+;$$

hence one has a natural transformation $\Omega^{\mathrm{fr}}(X) \to A(X)$, and, clearly, the composite $\Omega^{\mathrm{fr}}(*) \to A(*) \to K(Z)$ is the usual map. In particular, all the elements of $K_*(Z)$ that come from $\pi_*\Omega^{\mathrm{fr}}(*) = \pi^S_*$ also come from $\pi_*A(*)$.

The source of the natural transformation $\Omega^{\mathrm{fr}}(X) \to A(X)$ may be enlarged. For simplicity this will be described only in the case where $X = *$. Notice that in this case, $h\mathcal{S}(G(*))^n_k = h\mathcal{S}(*)^n_k$ is simply the category of pointed simplicial sets, of the weak homotopy type of a certain wedge of spheres, and the maps are pointed maps which are weak homotopy equivalences.

Let these spheres be endowed with orientations. Define $h\bar{\mathcal{S}}(*)^n_1$ to be the subcategory of orientation preserving weak homotopy equivalences. Then

$$\Omega\left|h\bar{\mathcal{S}}(*)^n_1\right| \simeq SG_n$$

the space of pointed maps of S^n of degree $+1$, and

$$\left|\text{dir lim } h\bar{\mathcal{S}}(*)^n_1\right| \simeq BSG.$$

More generally, let $h\bar{\mathcal{S}}(*)^n_k$ consist of those maps in $h\mathcal{S}(*)^n_k$ which are given as a wedge of k maps in $h\bar{\mathcal{S}}(*)^n_1$, followed by some permutation. Then by the Barratt-Priddy-Quillen-Segal theorem,

$$\left|\text{dir lim}_{(n,k)}\ h\bar{\mathcal{S}}(*)^n_k\right|^+ \simeq \Omega^{\mathrm{fr}}(BSG)$$

and thus one has a natural transformation $\Omega^{\mathrm{fr}}(BSG) \to A(*)$. If one thinks of $A(*)$ as the K-theory of the ring up to homotopy Q, this natural transformation may be pictured as given by the inclusion of the (positive) monomial matrices in Q.

Let $\tilde{\Omega}^{\mathrm{fr}}(BSG)$ be the homotopy fibre of the (naturally split) map $\Omega^{\mathrm{fr}}(BSG) \to \Omega^{\mathrm{fr}}(*)$. Then we have a diagram of fibrations:

$$\begin{array}{ccccc} \tilde{\Omega}^{\mathrm{fr}}(BSG) & \longrightarrow & \Omega^{\mathrm{fr}}(BSG) & \longrightarrow & \Omega^{\mathrm{fr}}(*) \\ \downarrow & & \downarrow & & \downarrow \\ \text{fibre}(A(*) \to K(Z)) & \longrightarrow & A(*) & \longrightarrow & K(Z) \end{array}$$

Now $\pi_i \tilde{\Omega}^{\text{fr}}(BSG) = 0$ if $i = 0$ or 1, and $\approx Z_2$ if $i = 2$, and the same is true for π_i fibre$(A(*) \to K(Z))$, cf. Corollary 2.7 below. By chasing a representative element through the latter computation, it is not difficult to see that

$$\pi_2 \tilde{\Omega}^{\text{fr}}(BSG) \xrightarrow{\approx} \pi_2 \text{ fibre}(A(*) \to K(Z)).$$

Somewhat surprisingly, it appears that π_2 fibre$(A(*) \to K(Z)) \to \pi_2 A(*)$ is the zero map, or equivalently, that $\pi_3 A(*) \to K_3(Z)$ is not surjective. This comes from an indirect (and involved) argument of Igusa which says that a particular kind of 2-parameter family of cell complexes must exist (cf. the remark after Theorem 3.3 in the next section). An explicit description of this 2-parameter family has not been found so far. It is certain to be complicated, though, since it is closely related to an explicit description of the exotic element of $K_3(Z)$ **[13]**.

Here are the analogues of Propositions 1.1 and 1.3.

PROPOSITION 2.3. *If $X \to X'$ is n-connected, $n \geqq 2$, then so is $A(X) \to A(X')$.*

PROPOSITION 2.4. *The functor A preserves k-homotopy cartesian (m, n)-connected squares, provided that $m, n \geqq 2$, $k \leqq m + n - 2$.*

The analogue of Proposition 1.2 is not quite a quantitative statement since it involves computing framed bordism in a fibration. One can do better in special cases. Specifically, one can get spectral sequences from Postnikov towers. The case of the Postnikov tower of X itself seems to be of least interest here, so we will not deal with it.

A ring up to homotopy, with underlying space R, has a Postnikov tower. The jth term has underlying space $\text{Sk}^j R$, the j-coskeleton of R. We define $K^j(R) = K(\text{Sk}^j R)$. In the case of $R = \text{Q}[G(X)]$, the functor

$$K^j(\text{Q}[G(X)]) = Z \times B\widehat{\text{GL}}(\text{Sk}^j(\text{Q}[G(X)]))^+$$

can again be defined directly, in a way that avoids the general notion of ring up to homotopy. We have, in this case,

$$K^0(\text{Q}[G(X)]) = K(Z[\pi_1 X]).$$

The analogue of Proposition 1.2 is

PROPOSITION 2.5. *Let $j \geqq 1$. Then $K^j(R) \to K^{j-1}(R)$ is $(j + 1)$-connected, and*

$$\pi_{j+1}\text{fibre}(K^j(R) \to K^{j-1}(R)) \approx H_0(\pi_0 R, \pi_j R).$$

In particular if $R = \text{Q}[G(X)]$, this is $\approx H_0(\pi_1 X, \pi_j \Omega^{\text{fr}} G(X))$ which, for $j = 1$, is $H_0(\pi_1 X, (Z_2 \oplus \pi_2 X) \otimes Z[\pi_1 X])$.

The tower of maps $K(R) \cdots \to K^j(R) \to \cdots$, gives rise to a spectral sequence $E^2_{p,q} \Rightarrow \pi_{p+q} K(R)$, $p \geqq 0$, $q \geqq 1$, with

$$E^2_{p,q} = \pi_{p+q}\text{fibre } (K^{q-1}(R) \to K^{q-2}(R)) \quad \text{if } q \geqq 2,$$

where the term $E^2_{0,q}$ is given by the proposition, and

$$E^2_{p,1} = \pi_{p+1} K^0(R) = \pi_{p+1} K(\pi_0 R).$$

Similarly, we may consider the Postnikov tower on the 'coefficient ring' Q, and define

$$A^j(X) = K((\mathrm{Sk}^j Q)[G(X)]).$$

Then $A^0(X) = K(Z[G(X)])$, and the analogue of Proposition 1.2 is

PROPOSITION 2.6. *Let $j \geqq 1$. Then $A^j(X) \to A^{j-1}(X)$ is $(j+1)$-connected, and*

$$\pi_{j+1}\mathrm{fibre}(A^j(X) \to A^{j-1}(X)) \approx H_0(\pi_1 X, \pi_j^S \otimes Z[\pi_1 X])$$
$$\approx \pi_j^S \otimes H_0(\pi_1 X, Z[\pi_1 X]) \quad (\textit{conjugation action}).$$

Again the tower of maps $A(X) \cdots \to A^j(X) \to \cdots$, gives rise to a spectral sequence $E^2_{p,q} \Rightarrow \pi_{p+q}A(X)$, $p \geqq 0$, $q \geqq 1$, with

$$E^2_{p,q} = \pi_{p+q}\mathrm{fibre}(A^{q-1}(X) \to A^{q-2}(X)) \quad \text{if } q \geqq 2,$$
$$E^2_{p,1} = \pi_{p+1}K(Z[G(X)]).$$

In the special case $X = *$, either of the two towers of maps above specializes to

COROLLARY 2.7. *There is a tower of maps $A(*) \cdots \to A^j(*) \to \cdots$ approximating $A(*)$, with*
(i) $A^0(*) = K(Z)$,
(ii) *$A^j(*) \to A^{j-1}(*)$ is $(j+1)$-connected,*
(iii) *π_{j+1} fibre$(A^j(*) \to A^{j-1}(*)) \approx \pi_j^S$.*

Lastly we have to consider the stabilization of the functor $A(X)$. Arguing as in the preceding section we may define it as

$$A^S(X) = \mathrm{dir\ lim}\ \Omega^n\ \mathrm{fibre}(A(S^n \wedge (X \cup *)) \to A(*)).$$

In detail, the nth map in the system is the map

$$\Omega^n\ \mathrm{fibre}(A(S^n \wedge (X \cup *)) \to A(*)) \to \Omega^n\ \mathrm{fibre}(A(*) \to A(S^{n+1} \wedge (X \cup *)))$$

which is the map of fibres in the diagram:

$$\begin{array}{ccc} A(S^n \wedge (X \cup *)) & \longrightarrow & A(D^{n+1}_+ \wedge (X \cup *)) \\ \downarrow & & \downarrow \\ A(D^{n+1}_- \wedge (X \cup *)) & \longrightarrow & A(S^{n+1} \wedge (X \cup *)) \end{array}$$

Anticipating from §5 that A is really a functor on simplicial sets, not necessarily pointed nor connected, we see that $A(S^0 \wedge (X \cup *))$ makes sense, and, clearly,

$$A(X) \xrightarrow{\sim} \mathrm{fibre}(A(S^0 \wedge (X \cup *)) \to A(*)).$$

Therefore there is a natural transformation

$$A(X) \to A^S(X).$$

The analogue of Lemma 1.4 is

LEMMA 2.8. *The functor $X \mapsto A^S(X)$ is a homology theory.*

The 'coefficients' of this homology theory is the space $A^S(*)$. In view of Proposition 2.2, the natural transformation

$$A^S(*) \to K^S(Z)$$

is a rational homotopy equivalence. Hence Proposition 1.5 gives

PROPOSITION 2.9.

$$\pi_i A^S(*) \otimes \mathbf{Q} \approx \begin{cases} \mathbf{Q} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

3. The Whitehead spaces and their relation to $A(X)$**.** If X is a PL manifold, denote $\mathscr{C}_0^n(X)$ the groupoid in which an object is a PL h-cobordism whose lower face is a compact codimension zero submanifold of $X \times I^n$ (where I^n is the n-cube); a morphism in $\mathscr{C}_0^n(X)$ is a PL isomorphism which is the identity on $X \times I^n$, the isomorphism need not preserve the upper face. More generally, let $\mathscr{C}_k^n(X)$ be the groupoid of PL k-parameter families of such h-cobordisms, the parameter domain being the k-simplex. Define $\mathscr{C}^n(X)$ to be the simplicial groupoid which in degree k is $\mathscr{C}_k^n(X)$.

Multiplication with the interval gives a map $\mathscr{C}^n(X) \to \mathscr{C}^{n+1}(X)$, and one defines $C^{\mathrm{PL}}(X) = \operatorname{dir\,lim} |\mathscr{C}^n(X)|$. The functor $X \mapsto C^{\mathrm{PL}}(X)$ extends, canonically up to homotopy, to a functor from spaces to spaces; this kind of argument is well known, it is described in **[9]** in one case.

In view of the composition law 'gluing at$X$', $C^{\mathrm{PL}}(X)$ is the underlying space of a Γ-space in the sense of Segal **[17]**; hence it is canonically an infinite loop space. In particular there is a canonical (connected) deloop $\mathrm{Wh}^{\mathrm{PL}}(X)$, the PL *Whitehead space.*

More or less by definition of this space, $\pi_1 \mathrm{Wh}^{\mathrm{PL}}(X)$ gives a stable classification of h-cobordisms, and $\pi_2 \mathrm{Wh}^{\mathrm{PL}}(X)$ classifies stable concordances. In view of Hatcher's stability theorem **[8]**, $\pi_{i+2} \mathrm{Wh}^{\mathrm{PL}}(X)$ is actually isomorphic to the ith concordance group of X if X is a compact PL manifold whose dimension is sufficiently large (depending on i). Cf. Hatcher's article **[9]** for a summary of known results.

Note that $\mathrm{Wh}^{\mathrm{PL}}(*) \simeq *$ in view of the (stable) h-cobordism theorem and the Alexander trick.

THEOREM 3.1. *There is a map* $A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$, *well defined up to homotopy. Its homotopy fibre, denoted* $h(X, A(*))$, *is a homology theory.*

COROLLARY. *The 'coefficients' of this homology theory is* $A(*)$.

PROOF. $\mathrm{Wh}^{\mathrm{PL}}(*) \simeq *$, so by definition of $h(X, A(*))$ there is a homotopy fibration $h(*, A(*)) \to A(*) \to *$.

Theorem 3.1 is entirely a nonmanifold theorem; the proof starts from Hatcher's 'parametrized h-cobordism theorem' **[8]**, a nonmanifold reformulation of $\mathrm{Wh}^{\mathrm{PL}}(X)$, and from that point on, manifolds just are not used anywhere in the argument—except maybe an occasional simplex. The proof will be indicated in §5.

REMARK. As will be apparent later, $A(*)$ is the underlying space of a Γ-space, hence the coefficients of a homology theory by the recipe of **[1]**. This homology theory coincides with $h(X, A(*))$. For general reasons again, there is a natural transformation

$$h(X, A(*)) \to A(X)$$

which turns out to coincide with the map of Theorem 3.1. By naturality therefore there is a diagram whose rows are homotopy fibrations:

$$\begin{array}{ccccc} h(X, A(*)) & \longrightarrow & A(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{PL}}(X) \\ \downarrow & & \downarrow & & \downarrow \\ h(X, K(Z)) & \longrightarrow & K(Z[G(X)]) & \longrightarrow & \mathrm{Wh}(G(X)) \\ \downarrow & & \downarrow & & \downarrow \\ h(B\pi_1 X, K(Z)) & \longrightarrow & K(Z[\pi_1 X]) & \longrightarrow & \mathrm{Wh}(\pi_1 X) \end{array}$$

Here the upper row is the fibration of Theorem 3.1, and the lower row is a fibration studied in [**18**]. Concerning the middle row, $K(Z[G(X)])$ is as defined in §1, and the map

$$h(B(G(X)),\ K(Z)) \to K(Z[G(X)])$$

is defined similarly as the map $h(B\pi_1 X,\ K(Z)) \to K(Z[\pi_1 X])$ in [**18**]. The term $\mathrm{Wh}(G(X))$ in the middle row can be (and is) defined so that the row is a homotopy fibration. This ends the remark.

While there appears to be no direct way to obtain an analogous result for $\mathrm{Wh}^{\mathrm{Diff}}(X)$, the smooth analogue of $\mathrm{Wh}^{\mathrm{PL}}(X)$, it turns out that one may proceed indirectly, using known results about $\mathrm{Wh}^{\mathrm{Diff}}(X)$ and about its relation to $\mathrm{Wh}^{\mathrm{PL}}(X)$, to obtain a result which is just as good. The argument is as follows.

The stabilization procedure to construct $A^S(X)$ from $A(X)$, and the fact that $A^S(X)$ is a homology theory, presuppose only certain formal properties of $A(X)$ and so carry over to other functors sharing these properties. In particular, we may stabilize the whole fibration of Theorem 3.1, and obtain a diagram of homotopy fibrations

$$\begin{array}{ccccc} h(X, A(*)) & \longrightarrow & A(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{PL}}(X) \\ \downarrow\sim & & \downarrow & & \downarrow \\ h^S(X, A(*)) & \longrightarrow & A^S(X) & \longrightarrow & (\mathrm{Wh}^{\mathrm{PL}})^S(X) \end{array}$$

and the left-hand vertical map is a homotopy equivalence since $h(X,\ A(*))$ is a homology theory already, and therefore unchanged by stabilization. Hence the right-hand square is homotopy cartesian.

It follows from smoothing theory (Burghelea, Lashof and Rothenberg [**5**]) that $F(X)$, the homotopy fibre of $\mathrm{Wh}^{\mathrm{Diff}}(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$, is a homology theory. Hence as before, stabilization gives a diagram of fibrations

$$\begin{array}{ccccc} F(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{Diff}}(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{PL}}(X) \\ \downarrow\sim & & \downarrow & & \downarrow \\ F^S(X) & \longrightarrow & (\mathrm{Wh}^{\mathrm{Diff}})^S(X) & \longrightarrow & (\mathrm{Wh}^{\mathrm{PL}})^S(X) \end{array}$$

in which the left-hand vertical map is a homotopy equivalence. Hence the right-hand square is homotopy cartesian.

Putting these two squares together, we obtain the diagram

$$\begin{array}{ccccc} A(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{PL}}(X) & \longleftarrow & \mathrm{Wh}^{\mathrm{Diff}}(X) \\ \downarrow & & \downarrow & & \downarrow \\ A^S(X) & \longrightarrow & (\mathrm{Wh}^{\mathrm{PL}})^S(X) & \longleftarrow & (\mathrm{Wh}^{\mathrm{Diff}})^S(X) \end{array}$$

in which both squares are homotopy cartesian. Hence the homotopy fibres of the vertical maps are all homotopy equivalent, and are mapped to each other by homotopy equivalence. But $\mathrm{Wh}^{\mathrm{Diff}}(S^n) \to \mathrm{Wh}^{\mathrm{Diff}}(*)$ is a $(2n-2)$-connected map **[5]** (cf. Hatcher **[9]** for a more direct argument, using only Morlet's lemma of disjunction). Hence

$$(\mathrm{Wh}^{\mathrm{Diff}})^S(*) \simeq *$$

and (since $(\mathrm{Wh}^{\mathrm{Diff}})^S$ is a homology theory) $(\mathrm{Wh}^{\mathrm{Diff}})^S(X) \simeq *$. Hence the homotopy fibre of the right-hand vertical map is $\mathrm{Wh}^{\mathrm{Diff}}(X)$, and it follows therefore that there is a homotopy fibration $\mathrm{Wh}^{\mathrm{Diff}}(X) \to A(X) \to A^S(X)$.

This leads to a numerical result. Namely by Proposition 2.2, and thanks to Borel **[3]**,

$$\pi_i A(*) \otimes \boldsymbol{Q} \approx \pi_i K(Z) \otimes \boldsymbol{Q} \approx \begin{cases} \boldsymbol{Q} & \text{if } i = 0, \\ \boldsymbol{Q} & \text{if } i = 5, 9, 13, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

And by Propositions 1.5 and 2.9, and thanks to Farrell and Hsiang **[7]**,

$$\pi_i A^S(*) \otimes \boldsymbol{Q} \approx \pi_i K^S(Z) \otimes \boldsymbol{Q} \approx \begin{cases} \boldsymbol{Q} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Hence

THEOREM 3.2.

$$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*) \otimes \boldsymbol{Q} \approx \begin{cases} \boldsymbol{Q} & \textit{if } i = 5, 9, 13, \cdots,. \\ 0 & \textit{otherwise,} \end{cases}$$

COROLLARY. *The smooth Alexander trick fails rationally.*

One way to visualize the map $A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$ of Theorem 3.1 is to replace $A(X)$ by another functor, not too far removed from it, and then map the latter. This functor is related to the idea of an elementary expansion. We will refer to it as the *combinatorial Whitehead space* of X, denoted $\mathrm{Wh}^{\mathrm{Comb}}(X)$. Its definition, which is rather involved, will now be given.

First we need a very rigid notion of cell complex, in fact we want k-parameter families of such. Working in the framework of topological spaces, a diagram

$$\begin{array}{ccc} Y_0 & \subset & Y \\ & \searrow \quad \swarrow & \\ & \Delta^k & \end{array}$$

where Δ^k is the k-simplex, will be called a *k-parameter family of cell complexes* from Y_0 to Y if it is endowed with the following data:

(i) a finite filtration $Y_0 \subset Y_1 \subset \cdots \subset Y$, over Δ^k,

(ii) for each $j > 0$, an attaching map over Δ^k

$$(S^{n_1} \cup \cdots \cup S^{n_{ij}}) \times \Delta^k \to Y_{j-1}$$

and an isomorphism over Δ^k

$$Y_j \xrightarrow{\approx} Y_{j-1} \cup_{(\cup_i S^{n_i} \times \Delta^k)} \Big(\bigcup_i D^{n_i+1} \times \Delta^k \Big);$$

these data are subject *only* to the equivalence relation of refinement: two cells which are attached simultaneously, may also be attached one after the other (in any order).

Similarly, we define a *k-parameter family of expansions* from Y_0 to Y, by not attaching disks D^{n+1} along spheres S^n, but attaching pairs (D^{n+1}, D^n_+) along pairs (D^n_-, S^{n-1}). Again, the structure is supposed to be very rigid, subject only to the equivalence relation of refinement, as above. In particular, we insist here that the particular pairing of cells *is* part of the data, and not subject to change.

DEFINTION. $\mathscr{E}(X)_k$ is the category in which:

(i) an object is a k-parameter family of cell complexes from $X \times \Delta^k$ to some Y which is 'acyclic', i.e., the inclusion $X \times \Delta^k \to Y$ is a homotopy equivalence;

(ii) a morphism from $(Y, \cdots)$ to $(Y', \cdots)$ is a k-parameter family of expansions from Y' to Y such that the cell structure in $(Y, \cdots)$ coincides with the cell structure induced from the expansion.

We let $\mathscr{E}(X)$ be the simplicial category which in degree k is $\mathscr{E}(X)_k$. Its geometric realization, $E(X) = |\mathscr{E}(X)|$, will be referred to as the *expansion space*.

An interesting question is if the 'two-index-theorem' holds for $E(X)$. That is, if one defines $E^{i,i+1}(X)$ by insisting that all the cells involved have dimension either i or $i+1$, is it true that dir $\lim_{(i)} E^{i,i+1}(X)$ is homotopy equivalent to $E(X)$? This is far from being obviously true, in fact it might well be wrong.

In view of the composition law 'gluing at X', $E(X)$ is the underlying space of a Γ-space, hence canonically an infinite loop space, and $\mathrm{Wh}^{\mathrm{Comb}}(X)$ is defined as the deloop.

THEOREM 3.3. *There is a map* $A(X) \to \mathrm{Wh}^{\mathrm{Comb}}(X)$, *well defined up to homotopy. The sequence*

$$\Omega^{\mathrm{fr}}(X) \to A(X) \to \mathrm{Wh}^{\mathrm{Comb}}(X)$$

is, canonically, the homotopy type of a fibration.

REMARK. Continuing the discussion of what $\pi_2 A(*)$ is (cf. the material just before Proposition 2.3), we have $\pi_i \mathrm{Wh}^{\mathrm{Comb}}(*) = 0$ if $i = 0, 1$, and

$$\pi_2\, A(*) \approx \pi_2\, K(Z) \oplus \pi_2\, \mathrm{Wh}^{\mathrm{Comb}}(*)$$

and the map

$$Z_2 \approx \pi_2\, \tilde{\Omega}^{\mathrm{fr}}(BSG) \xrightarrow{\approx} \pi_2\, \mathrm{fibre}\,(A(*) \to K(Z)) \longrightarrow \pi_2\, \mathrm{Wh}^{\mathrm{Comb}}(*)$$

is surjective. In view of what this map means geometrically, the candidate for a nontrivial element in $\pi_2 \mathrm{Wh}^{\mathrm{Comb}}(*) = \pi_1 E(*)$ is represented by a 'rolling collapse', a circle of cell complexes $S^n \bigcup_{S^n} D^{n+1}$ where the attaching map is homotopic to the identity, and varies through the nontrivial element of π_1^S. The aforementioned argument of Igusa is that this element, and hence $\pi_1 E(*)$, must be zero for the following reason. There must exist a 2-parameter family of cell complexes which over the boundary of the parameter domain consists of an odd number of rolling collapses, plus maybe a few circles of expansions ('nonrolling' collapses); for if such a 2-parameter family would not exist, it would follow by a (tricky) geometric argument that π_3^S should split off $K_3(Z)$, in contradiction to the result of Lee and Szczarba [**13**]. This ends the remark.

Here is how to map $\mathrm{Wh}^{\mathrm{Comb}}(X)$ to $\mathrm{Wh}^{\mathrm{PL}}(X)$: Take an acyclic cell complex, fatten the cells until a handle decomposed h-cobordism (of suitably large dimension) is obtained, and then forget the handle structure. To make this idea work, one redefines the expansion space using handle decomposed h-cobordisms instead of cell complexes. Thus if X is a PL manifold, define $\mathscr{E}_0^n(X)$ to be the category in which an object consists of

(i) an object of the category $\mathscr{C}_0^n(X)$ (cf. the beginning of the section),

(ii) a handle decomposition of the h-cobordism (including all the necessary data to describe it *completely*), up to an equivalence relation of rearrangement of handles (as in the definition of the expansion space),

and where a morphism is a standard handle cancellation, or composition of such (subject to an equivalence relation of rearrangement). More generally define $\mathscr{E}_k^n(X)$ to be the category whose objects are the PL k-parameter families of such handle decomposed h-cobordisms, and whose morphisms are the PL k-parameter families of standard handle cancellations. $\mathscr{E}^n(X)$ is the simplicial category which in degree k is $\mathscr{E}_k^n(X)$.

Forgetting the handle structure gives a map $\mathscr{E}^n(X) \to \mathscr{C}^n(X)$. This uses that a standard handle cancellation involves a canonical isomorphism of the underlying manifolds.

In order that this define a map $\mathrm{Wh}^{\mathrm{Comb}}(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$, one needs that $\mathscr{E}^n(X)$ is sufficiently close to $\mathscr{E}(X)$. The latter is seen as follows. One maps $\mathscr{E}^n(X)$ to $\mathscr{E}(X)$ by squeezing each handle to its core. Then a handle-by-handle argument shows that this map is highly connected (depending on the difference of $\dim(X) + n$ and the index of the handle; each test for homotopy equivalence in the limit involves a finite diagram only, so there is a highest handle index).

The map $\mathrm{Wh}^{\mathrm{Comb}}(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$ so obtained satisfies that

$$\begin{array}{ccc} & A(X) & \\ \swarrow & & \searrow \\ \mathrm{Wh}^{\mathrm{Comb}}(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{PL}}(X) \end{array}$$

commutes up to homotopy.

It appears that a careful wording of the argument actually yields a smooth analogue, a factorization up to homotopy of $\mathrm{Wh}^{\mathrm{Comb}}(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$ through a map $\mathrm{Wh}^{\mathrm{Comb}}(X) \to \mathrm{Wh}^{\mathrm{Diff}}(X)$. This leads to a startling consequence. Namely in view of the resulting homotopy commutative diagram

$$\begin{array}{ccc} & A(X) & \\ \swarrow & & \searrow \\ \mathrm{Wh}^{\mathrm{Diff}}(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{PL}}(X) \end{array}$$

the homotopy fibre of $A(X) \to \mathrm{Wh}^{\mathrm{Diff}}(X)$ can be identified to the homotopy fibre of

$$\mathrm{fibre}(A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)) \longrightarrow \mathrm{fibre}(\mathrm{Wh}^{\mathrm{Diff}}(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)),$$

a map of homology theories. Hence $\mathrm{fibre}(A(X) \to \mathrm{Wh}^{\mathrm{Diff}}(X))$ is also a homology theory, and stabilization gives a diagram of fibrations

$$\begin{array}{ccccc} ? & \longrightarrow & A(X) & \longrightarrow & \mathrm{Wh}^{\mathrm{Diff}}(X) \\ \downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow \\ ? & \longrightarrow & A^S(X) & \longrightarrow & (\mathrm{Wh}^{\mathrm{Diff}})^S(X) \end{array}$$

in which the left-hand vertical map is a homotopy equivalence. Because $(\mathrm{Wh}^{\mathrm{Diff}})^S(X) \simeq *$, it follows that $A(X)$ actually splits,

$$A(X) \simeq \mathrm{Wh}^{\mathrm{Diff}}(X) \times A^S(X).$$

It appears that this splitting is hard to reconcile with the computation of $\pi_3 \mathrm{Wh}^{\mathrm{Diff}}(*)$ described in §3 of **[9]**.

Finally a word about the map $A(X) \to \mathrm{Wh}^{\mathrm{Comb}}(X)$. The very description of this map requires the machinery of the following sections, involving a very particular way of constructing simplicial objects. An indication of the nature of this kind of simplicial structure can be seen from the following remarks. For ease of notation we consider only the case $X = *$.

Let $\mathscr{C}$ be the category whose objects are the finite pointed simplicial sets of the homotopy type of, say, a wedge of two spheres of dimension twenty, and whose morphisms are the weak homotopy equivalences which happen to be monomorphisms. Then $\mathscr{C}$ maps to $A(*)$ and hence, by composition, to $\mathrm{Wh}^{\mathrm{Comb}}(*)$. By wishful thinking (a bit too wishful really, but not too far off either) let us insist on the following:

(i) $\mathrm{Wh}^{\mathrm{Comb}}(*)$ can be described as a simplicial object in such a way that

$$\mathrm{nerve}(\mathscr{C}) \to \mathrm{Wh}^{\mathrm{Comb}}(*)$$

can be a simplicial map,

(ii) on 1-simplices, this map is given by

$$(Y_0 \overset{\sim}{\rightarrowtail} Y_1) \mapsto Y_1/Y_0.$$

What does such wishful thinking imply about the simplicial structure of $\mathrm{Wh}^{\mathrm{Comb}}(*)$?

A 2-simplex in $\mathrm{nerve}(\mathscr{C})$ is a sequence $(Y_0 \rightarrowtail^{\sim} Y_1 \rightarrowtail^{\sim} Y_2)$, and its faces are given, respectively, by $(Y_1 \rightarrowtail^{\sim} Y_2)$, $(Y_0 \rightarrowtail^{\sim} Y_2)$, $(Y_0 \rightarrowtail^{\sim} Y_1)$. We conclude that there must be a 2-simplex in $\mathrm{Wh}^{\mathrm{Comb}}(*)$ whose faces are given by Y_2/Y_1, Y_2/Y_0, Y_1/Y_0, a particular arrangement of the terms in the cofibration sequence $(Y_1/Y_0) \rightarrowtail (Y_2/Y_0) \twoheadrightarrow (Y_2/Y_1)$.

What is the general conclusion for n-simplices?

4. An exact sequence K-theory in nonadditive categories. Call a simplicial set finite if the number of nondegenerate simplices is finite. Suppose we want to define $[Y]$, the reduced Euler characteristic of a pointed finite simplicial set Y. Then we may take $[Y]$ to be an element of the abelian group $\mathfrak{eul}$ with generators the finite pointed simplicial sets Y, and relations

(i) $[Y] = [Y']$ if $Y \rightarrowtail^{\sim} Y'$,

(ii) $[Y] = [Y'] + [Y'']$ if $Y' \rightarrowtail Y \twoheadrightarrow Y''$ is a cofibration sequence.

This is analogous to the definition of the projective class group $K_0(R)$, the abelian group with generators the finitely generated projective R-modules P, and relations

(i′) $[P] = [P']$ if $P \approx P'$,

(ii′) $[P] = [P'] + [P'']$ if $P' \rightarrowtail P \twoheadrightarrow P''$ is short exact.

In the latter case, relation (i′) is redundant since it is implied by (ii′). In the former case we might try to find a single type of relation which is equivalent to (i) and (ii) together. However it is clear that the only thing we may gain in doing so is a loss of simplicity.

Hence if we want to interpret eul as a low-dimensional homotopy group of some space, this space be better not constructed as a simplicial set (e.g., the nerve of a category); rather we should look for a bisimplicial set (e.g., the nerve of a simplicial category). Also it is hard to imagine that eul could be a π_0 in a reasonably direct way since relation (ii) is a typical π_1-relation.

On the other hand, eul will clearly be the π_1 of any simplicial category $\mathscr{E}.$ that in low degrees satisfies

(0) $\mathscr{E}_0$ is the trivial category with one object and one morphism,

(1) $\mathscr{E}_1$ is the category of weak equivalences $Y \to^{\sim} Y'$ of finite pointed simplicial sets,

(2) $\mathscr{E}_2$ is a category whose objects are the cofibration sequences $Y' \rightarrowtail Y \twoheadrightarrow Y''$ of finite pointed simplicial sets; and the faces of $(Y' \rightarrowtail Y \twoheadrightarrow Y'')$ are given by the collection Y', Y, Y'' (in this order, or in reverse order).

This is as far as the structure of $\mathscr{E}.$ can be suggested by the relations (i) and (ii) above. We must now provide our own choice of morphisms for the category $\mathscr{E}_2$. The most natural choice is to let a morphism in $\mathscr{E}_2$ be a weak equivalence of cofibration sequences, that is, a commutative diagram

$$\begin{array}{ccccc} Y' & \rightarrowtail & Y & \twoheadrightarrow & Y'' \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \bar{Y}' & \rightarrowtail & \bar{Y} & \twoheadrightarrow & \bar{Y}'' \end{array}$$

where the rows are cofibration sequences, and the vertical arrows weak equivalences.

A moment's reflection shows that the sequence $\mathscr{E}_0$, $\mathscr{E}_1$, $\mathscr{E}_2$, can be continued in a simple way. Namely we may generally define $\mathscr{E}_k$ to be the category in which an object is a sequence of cofibrations

$$* \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_k$$

and where a morphism is a commutative diagram

$$\begin{array}{ccccccc} * & \rightarrowtail & Y_1 & \rightarrowtail \cdots \rightarrowtail & Y_k \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ * & \rightarrowtail & Y_1' & \rightarrowtail \cdots \rightarrowtail & Y_k' \end{array}$$

in which the vertical arrows are weak equivalences. The new definition of $\mathscr{E}_2$ is equivalent to the old one since by the gluing lemma $Y_2/Y_1 \to Y_2'/Y_1'$ will now also be a weak equivalence.

In order to assemble the $\mathscr{E}_k$ to a simplicial category, we must define face and degeneracy maps. The following rule for face maps from $\mathscr{E}_k$ to $\mathscr{E}_{k-1}$ extends the rule given in (2) above for the case $k = 2$. The ith face map, for $i > 0$, just drops Y_i from the sequence

$$* \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_k.$$

But the 0th face map, as it drops $*$, must force Y_1 to become a new $*$, that is, the 0th face is given by the sequence

$$* \rightarrowtail Y_2/Y_1 \rightarrowtail \cdots \rightarrowtail Y_k/Y_1.$$

With this rule, the simplicial identities for iterated face maps are satisfied—except possibly for a choice problem with the choice of cokernels.

The degneracy maps from $\mathscr{E}_k$ to $\mathscr{E}_{k+1}$ can be defined in the obvious way, by the insertion of an identity map at the appropriate place of the sequence.

The above choice problem with the face maps is not serious. In fact there is a standard trick to avoid such choice problems, to replace the category in question by an equivalent one to incorporate all the necessary choices. In the case at hand, one may proceed as follows.

Let Δ denote the category of ordered sets [0], [1], $\cdots$, $[n] = (0 < 1 < \cdots < n)$, and weakly monotonic maps.

Define $\langle n \rangle$ to be the partially ordered set of pairs (i, j), $0 \leqq i \leqq j \leqq n$, where $(i, j) \leqq (i', j')$ if and only if $i \leqq i'$ and $j \leqq j'$. Considering $\langle n \rangle$ as a category, we may identify it to $\mathfrak{Mor}[n]$, the category whose objects are the morphisms in the ordered set $[n]$ when the latter is considered as a category. The notation emphasizes that $[n] \mapsto \mathfrak{Mor}[n]$ is a covariant functor on the category Δ.

DEFINITION 4.1. $\mathscr{E}'_n$ is the category whose objects are the functors

$$Y\colon \mathfrak{Mor}[n] \to \text{(pointed finite simplicial sets)}, \quad (i, j) \mapsto Y_{(i,j)}$$

satisfying

(i) for any i, $Y_{(i,i)}$ equals the (distinguished) zero object,

(ii) for any triple $i \leqq j \leqq k$, the sequence

$$Y_{(i,j)} \to Y_{(i,k)} \to Y_{(j,k)}$$

is a cofibration sequence,

and whose morphisms are the natural transformations of functors, satisfying that all the maps involved are weak equivalences.

An equivalence from $\mathscr{E}'_n$ to $\mathscr{E}_n$ is given by the forgetful map

$$Y \mapsto (Y_{(0,0)} \rightarrowtail Y_{(0,1)} \rightarrowtail \cdots \rightarrowtail Y_{(0,n)})$$

and it is clear from the definition that the $\mathscr{E}'_n$ assemble to a simplicial category $\mathscr{E}'_\cdot$.

We have thus achieved a fairly natural classification of the Euler characteristics of finite pointed simplicial sets by the elements of some homotopy group, namely $\pi_1|\mathscr{E}'_\cdot|$. But, after all, one expects Euler characteristics to be classified by a π_0. So we should really consider the loop space $\Omega|\mathscr{E}'_\cdot|$.

THEOREM. $\Omega|\mathscr{E}'_\cdot| \simeq A(*)$.

This is a special case of Theorem 5.7 of the next section. To prove it, one has to consider variants of Definition 4.1.

Other variants of Definition 4.1 have to be considered for other purposes. It is therefore desirable to have an abstract version of this definition. The ingredients we need are: a category, and notions of 'cofibration' and 'weak equivalence' in this category. These must satisfy certain conditions, to ensure that Definition 4.1 makes sense. Preferably they should also satisfy other conditions of a general nature which we can expect to hold in cases of interest, and to be useful in proofs.

Thus a *category with cofibrations and weak equivalences* shall mean a category $\mathscr{C}$ together with subcategories $co(\mathscr{C})$ and $w(\mathscr{C})$ satisfying the following three groups of conditions:

(I) $\mathscr{C}$ has a (distinguished) zero object 0;

(II) (1) Isomorphisms in $\mathscr{C}$ are cofibrations (i.e., are morphisms in $co(\mathscr{C})$).

(2) For every object A of $\mathscr{C}$, the morphism $0 \to A$ is a cofibration.

(3) $co(\mathscr{C})$ is closed under cobase change; this means, if in the diagram

$$\begin{array}{ccc} \bullet & & \\ \uparrow & & \\ \bullet & \longrightarrow & \bullet \end{array}$$

the left vertical arrow is a cofibration, then the pushout of the diagram exists in $\mathscr{C}$, and the right vertical arrow will also be a cofibration.

(III) (1) Isomorphisms in $\mathscr{C}$ are weak equivalences (i.e., are morphisms in $w(\mathscr{C})$).

(2) If in the diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \to & C \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ B' & \leftarrowtail & A' & \to & C' \end{array}$$

the left-hand horizontal arrows are cofibrations, and all vertical arrows are weak equivalences, then the map

$$B \cup_A C \to B' \cup_{A'} C'$$

is a weak equivalence.

REMARK. The axioms have been chosen for their simplicity, so that they reproduce easily under chains of constructions. The axioms are sufficiently general to include many uninteresting cases; for example if $\mathscr{C}$ has 0 and colimits, and $co(\mathscr{C}) = \mathscr{C}$; or $w(\mathscr{C}) = \mathscr{C}$. In either of the latter two cases one may expect the homotopy type associated below not to be very interesting either; this is indeed the case.

In practice there will never be any doubt about what the cofibrations are, so the category $co(\mathscr{C})$ will be dropped from the notation in the following definition. By contrast there will, as a rule, be several categories of weak equivalences to choose from, so the category $w(\mathscr{C})$ has to be specified in the notation.

DEFINITION 4.2. $wS_n\mathscr{C}$ is the category whose objects are the functors

$$\begin{aligned} A: \mathfrak{Mor}[n] &\to \mathscr{C} \\ (i, j) &\mapsto A_{(i,j)} \end{aligned}$$

satisfying

(i) $A_{(i,i)} = 0$, the distinguished zero object, for every i;

(ii) for every triple $i \leqq j \leqq k$, the morphism $A_{(i,j)} \to A_{(i,k)}$ is a cofibration, and

$$\begin{array}{ccc} A_{(i,j)} & \longrightarrow & A_{(i,k)} \\ \downarrow & & \downarrow \\ A_{(j,j)} & \longrightarrow & A_{(j,k)} \end{array}$$

is a pushout;

and whose morphisms are the natural transformations of functors satisfying that all the maps involved are weak equivalences.

$wS.\mathscr{C}$ is the simplicial category which in degree n is $wS_n\mathscr{C}$.

COROLLARY. $|wS.\mathscr{C}|$ *is canonically an infinite loop space.*

PROOF. It follows from the axioms that $(A, A') \to A \vee A' = A \cup_0 A'$ defines a composition law on $|wS.\mathscr{C}|$; therefore $|wS.\mathscr{C}|$ is the underlying space of a Γ-space, with respect to this composition law, whence the assertion **[17]**.

The present construction generalizes the Q-construction of Quillen: Let $\mathscr{A}$ be an exact category in the sense of **[16]**, $co(\mathscr{A})$ the category of admissible monomorphisms, and $w(\mathscr{A})$ the category of isomorphisms. Then there is a natural homotopy equivalence $|wS.\mathscr{A}| \simeq |Q\mathscr{A}|$ (Quillen, unpublished).

Because of the added generality one cannot expect many of the general results on $Q\mathscr{A}$ to carry over directly. In fact, only Theorem 2 of **[16]** has a direct analogue here to which we refer as the *additivity theorem.*

The additivity theorem says, roughly, if $\mathscr{C}$ is a category with cofibrations and weak equivalences, then the category $E(\mathscr{C})$ whose objects are the diagrams $(A \rightarrowtail B)$ in $\mathscr{C}$ can be made into a category with cofibrations and weak equivalences in a natural way, and

$$\begin{gathered} wS.E(\mathscr{C}) \to wS.\mathscr{C} \times wS.\mathscr{C} \\ (A \rightarrowtail B) \mapsto (A, B/A) \end{gathered}$$

is a homotopy equivalence.

That is, cofibration sequences can be replaced by sum diagrams. It thus appears that, philosophically speaking, Definition 4.2 is just another version of the bar construction, applicable in another unusual situation.

5. The functor $A(X)$**, revisited.** Let X be a simplicial set. Denote by $\mathscr{C}(X)$ the category of pairs (Y, s) where $s\colon X \to Y$ is an injective map, and where a map from (Y, s) to (Y', s') is a map of simplicial sets, $f\colon Y \to Y'$, such that $fs = s'$.

A map $f\colon (Y, s) \to (Y', s')$ is called a weak homotopy equivalence, or *h-map*, for short, if $|f|\colon |Y| \to |Y'|$ is a homotopy equivalence relative to the subspace $|X|$;

it is called a *simple map* if $|f|$ has contractible point inverses (and in particular is surjective). Simple maps can be defined without recourse to geometric realization, but this will not be done here. Any simple map is an h-map. Let $s\mathscr{C}(X)$ denote the category whose objects are those of $\mathscr{C}(X)$ and whose morphisms are the simple maps.

Denote by $s\mathscr{C}_f(X)$ the full subcategory of $s\mathscr{C}(X)$ of those (Y, s) which are *finite*, i.e., which satisfy that all but finitely many of the nondegenerate simplices of Y are contained in the simplicial subset $s(X)$;

and by $s\mathscr{C}_f^h(X)$ the full subcategory of $s\mathscr{C}_f(X)$ of those (Y, s) which are *acyclic*, i.e., which satisfy that $s\colon X \to (Y, s)$ is an h-map; equivalently, that $|s(X)|$ is a deformation retract of $|Y|$.

Hatcher's main result in **[8]** is that $\Omega\ \mathrm{Wh}^{\mathrm{PL}}(X)$ is homotopy equivalent to $s\mathscr{C}_f^h(X)$. (Actually, Hatcher uses simple PL maps of polyhedra. The translation into simple maps of simplicial sets is nontrivial, but not too surprising either. Also, because of the ambiguity of PL mapping cylinders, the formulation of Hatcher's theorem in terms of simple maps of polyhedra requires additional justification.) It will be indicated below how to prove Theorem 3.1 from the homotopy equivalence between $\Omega\mathrm{Wh}^{\mathrm{PL}}(X)$ and $s\mathscr{C}_f^h(X)$. There is a more direct proof, independent of Hatcher's theorem, but the present proof is simpler at least in the regard that manifolds do not have to be used anymore.

To relate $s\mathscr{C}_f^h(X)$ to $A(X)$, the machinery of the preceding section has to be used. But $\mathscr{C}(X)$ has no cokernels, so we have to modify it.

Let $\mathscr{R}(X)$ be the category in which an object is a triple (Y, r, s) where $r\colon Y \to X$ is a retraction of simplicial sets, and s is a section of r; and where a map $f\colon (Y, r, s) \to (Y', r', s')$ is a map $f\colon Y \to Y'$ such that $fs = s'$ and $r = r'f$.

$\mathscr{R}(X)$ has a distinguished zero object $0 = (X, \mathrm{id}, \mathrm{id})$, and it is a category with cofibrations in a natural way: the cofibrations are the maps $(Y', r', s') \to (Y, r, s)$ with $Y' \to Y$ injective. The axioms put down in the preceding section clearly hold. To any cofibration is associated a 'cofibration sequence'

$$(Y', r', s') \rightarrowtail (Y, r, s) \twoheadrightarrow (Y, r, s) \cup_{(Y', r', s')} 0.$$

There are four notions of weak equivalence in $\mathscr{R}(X)$ that we have to be aware of. In either case, the subcategory of weak equivalences will be denoted by prefixing any of the letters i, s, h, h_X, respectively, whichever applies.

(i) $i\mathscr{R}(X)$ is the category of *isomorphisms* in $\mathscr{R}(X)$;

(s) $s\mathscr{R}(X)$ is the category of *simple maps*, i.e., maps such that $|f|$ has contractible point inverses; or equivalently, where the associated map in $\mathscr{C}(X)$ is simple;

(h) $h\mathscr{R}(X)$ is the category of *weak homotopy equivalences*, or *h-maps*, for short; by definition, a map f in $\mathscr{R}(X)$ is in $h\mathscr{R}(X)$ if and only if the associated map in $\mathscr{C}(X)$ is in $h\mathscr{C}(X)$;

(h_X) $h_X\mathscr{R}(X)$ is the category of *hereditary weak homotopy equivalences*, i.e., maps $(Y, r, s) \to (Y', r', s')$ such that for any $X' \subset X$, the induced map $r^{-1}(X') \to r'^{-1}(X')$ is an h-map in $\mathscr{R}(X')$; these maps are mainly introduced here for the purpose of making it clear that we will *not* use them.

We have $i\mathscr{R}(X) \subset s\mathscr{R}(X) \subset h_X\mathscr{R}(X) \subset h\mathscr{R}(X)$.

Similarly as before, the subscript 'f' added to the notation of $\mathscr{R}(X)$, or any of its subcategories, will refer to the full subcategory of finite objects, i.e., those (Y, r, s) satisfying that all but finitely many of the nondegenerate simplices of Y are contained in $s(X)$;

and the superscript 'h' added to the notation of $\mathscr{R}(X)$ or $\mathscr{R}_f(X)$, or any of their subcategories, will refer to the full subcategory of those (Y, r, s) for which $s\colon X \to (Y, r, s)$ is an h-map.

One of the many categories now defined is $s\mathscr{R}_f^h(X)$. We would like to prove that $s\mathscr{R}_f^h(X) \to s\mathscr{C}_f^h(X)$ is a homotopy equivalence. Unfortunately this is wrong in general, simply for the trivial but still annoying reason that if $s\mathscr{R}_f^h(X)$ is not contractible then the two maps

$$s\mathscr{R}_f^h(X) \to s\mathscr{R}_f^h(X \times \Delta^1)$$

are not homotopic.

The counterexample suggests the remedy, namely we must allow things to 'move'. Fortunately there is a simple way to allow for such moving without loss of functoriality, namely, to replace everything in sight by k-parameter families, with varying k, of the same kind of thing. An organized way of doing so, is to introduce a dummy simplicial direction, in replacing X by the simplicial object of its higher path spaces. To be precise,

DEFINITION. If K, L are simplicial sets, let L^K denote the function space, the simplicial set which in degree n is the set of maps

$$K \times \Delta^n \to L.$$

If F: (simplicial sets) $\to \mathscr{D}$, $X \mapsto F(X)$, is any functor, define a functor from simplicial sets to the simplicial objects in $\mathscr{D}$, or equivalently, a simplicial object of functors $F.$: (simplicial sets) $\to \mathscr{D}$ by letting $F_k(X) = F(X^{\Delta^k})$.

Parenthesis. To understand the meaning of this construction, and why it helps, one should note that a functor F: (simplicial sets) $\to$ (simplicial sets) may not respect 'homotopy' in any sense, for example the functor 0-skeleton, $X \mapsto \mathrm{sk}_0(X)$. However there is a natural transformation of functors from simplicial sets to simplicial sets, $F \to \check{F}$ with the following properties:

(i) $\check{F}$ respects simplicial homotopies,

(ii) if F respects simplicial homotopies, then $F(X) \to^{\sim} \check{F}(X)$.

Indeed, one defines

$$\check{F}(X) = \mathrm{diag}\, F.(X).$$

Exercise. What is $\check{\mathrm{sk}}_0$? This ends the parenthesis.

For the purpose of better readability, the notation $F(X^{\Delta^\cdot})$ will be used instead of the more precise $F.(X)$.

LEMMA 5.1. *$s\mathscr{C}^h_f(X) \to^{\sim} s\mathscr{C}^h_f(X^{\Delta^\cdot})$; if X satisfies the extension condition then also*

$$s\mathscr{R}^h_f(X^{\Delta^\cdot}) \xrightarrow{\sim} s\mathscr{C}^h_f(X^{\Delta^\cdot}).$$

REMARK. h_X-maps are redundant. For if X satisfies the extension condition then

$$s\mathscr{R}^h_f(X^{\Delta^\cdot}) \xrightarrow{\sim} h_X\mathscr{R}^h_f(X^{\Delta^\cdot}).$$

From now on, $\mathscr{R}_f(X)$ and $\mathscr{R}^h_f(X)$ will be considered as categories with cofibrations and weak equivalences (with several choices for the latter) in the sense of the preceding section. Thus by Definition 4.2 we have, for example, for each n the category $sS_n\mathscr{R}^h_f(X)$ and the simplicial category $sS_n\mathscr{R}^h_f(X^{\Delta^\cdot})$, and we have a simplicial category $sS.\mathscr{R}^h_f(X)$ and a bisimplicial category $sS.\mathscr{R}^h_f(X^{\Delta^\cdot})$. Henceforth we assume that X satisfies the extension condition.

LEMMA 5.2. *The 'subquotient' map*

$$sS_n\mathscr{R}^h_f(X^{\Delta^\cdot}) \to (s\mathscr{R}^h_f(X^{\Delta^\cdot}))^n$$
$$(A\colon \mathfrak{Mor}[n] \to \mathscr{R}^h_f(X^{\Delta^k})) \mapsto (A_{(0,1)}, A_{(1,2)}, \cdots, A_{(n-1,n)})$$

is a weak homotopy equivalence.

The idea of proof is, if $X \to (Y', r', s')$ is an h-map then a cofibration sequence $(Y', r', s') \rightarrowtail (Y, r, s) \twoheadrightarrow (Y'', r'', s'')$ can be 'moved' to a split one because (Y', r', s') is acyclic.

The lemma implies that $|sS.\mathscr{R}^h_f(X^{\Delta^\cdot})|$ is homotopy equivalent to the canonical deloop (from the composition law) of $|s\mathscr{R}^h_f(X^{\Delta^\cdot})|$. Because of Hatcher's theorem, Lemma 5.1, and the definition of $\mathrm{Wh}^{\mathrm{PL}}(X)$ as a canonical deloop, we have therefore

PROPOSITION 5.3. $|sS.\mathcal{R}^h_f(X^{\Delta^\cdot})| \simeq \mathrm{Wh}^{\mathrm{PL}}(X)$.

Using this transcription, we may compare $\mathrm{Wh}^{\mathrm{PL}}(X)$ to other functors. The following diagram involves the forgetful maps 'simple maps are h-maps' (the horizontal arrows) and 'acyclic objects are objects' (the vertical arrows):

$$\begin{array}{ccc} sS.\mathcal{R}^h_f(X^{\Delta^\cdot}) & \longrightarrow & hS.\mathcal{R}^h_f(X^{\Delta^\cdot}) \\ \downarrow & & \downarrow \\ sS.\mathcal{R}_f(X^{\Delta^\cdot}) & \longrightarrow & hS.\mathcal{R}_f(X^{\Delta^\cdot}) \end{array}$$

LEMMA 5.4. (i) *This square is homotopy cartesian*; (ii) $hS.\mathcal{R}^h_f(X^{\Delta^\cdot})$ *is contractible.*

PROOF OF (ii). In each bidegree, the category in this bidegree has a terminal object (in fact, a zero object) and is hence contractible. This implies the assertion, in view of a well-known result on the geometric realization of multisimplicial sets.

Part (i) of the lemma is a special case of a general result which is deduced from the additivity theorem in a similar way as Propositions 7.1—7.3 in **[18]**.

In view of the lemma, the left and bottom arrows in the diagram form a homotopy fibration. The next results below identify the other terms in this fibration. This yields Theorem 3.1.

One may thus say that Theorem 3.1 is obtained by comparison of two notions of weak equivalence in $\mathcal{R}_f(X)$, namely 'h-map' and 'simple map'. From this point of view, Theorem 3.3 is obtained similarly, by comparison of 'h-map' and 'isomorphism'.

LEMMA 5.5. *The functor* $X \mapsto sS.\mathcal{R}_f(X^{\Delta^\cdot})$ *is a homology theory.*

To prove the lemma one compares $sS.\mathcal{R}_f(X^{\Delta^\cdot})$ to the functor $X \mapsto sS.\mathcal{R}_f(\mathrm{sk}_0(X^{\Delta^\cdot}))$ which is a homology theory by the recipe of how to associate a homology theory to a Γ-space **[1]**; cf. **[18]** for a more detailed treatment.

One uses that, for any X, the skeleton filtration $\mathrm{sk}_j(X)$ induces a filtration of the identity functor on $\mathcal{R}_f(X)$,

$$\mathrm{sk}^*_j(Y, r, s) = (r^{-1}(sk_j(X)), \cdots).$$

The key fact is that sk^*_j does induce an endomorphism of $sS.\mathcal{R}_f(X)$, because

(i) simple maps are hereditary (as opposed to h-maps),

(ii) the objects are *not* required to be acyclic.

The additivity theorem implies that the identity map on $sS.\mathcal{R}_f(X)$ is homotopic to the map induced from

$$\mathrm{sk}^*_0 \vee \mathrm{sk}^*_1/\mathrm{sk}^*_0 \vee \cdots \vee \mathrm{sk}^*_j/\mathrm{sk}^*_{j-1} \vee \cdots.$$

This means that for the purposes of $sS.\mathcal{R}_f(X)$, one may restrict to those objects in $\mathcal{R}_f(X)$ which are given as a sum of 'objects with small support'.

LEMMA 5.6. $hS.\mathcal{R}_f(X) \xrightarrow{\sim} hS.\mathcal{R}_f(X^{\Delta^\cdot})$.

THEOREM 5.7. $\Omega\, |hS.\, \mathcal{R}_f(X)\,| \simeq A(X)$.

The main steps in the proof of this theorem will be indicated in the remaining material.

DEFINITION. $\mathcal{R}(X)^n_k$ is the full subcategory of those (Y, r, s) such that $|s|$ is homotopy equivalent, relative to $|X|$, to the inclusion of $|X|$ into ($|X|$ wedge k spheres of dimention n).

LEMMA 5.8. $Z \times |\text{dir lim}_{(k,n)}\, h\mathscr{R}(X)_k^n|^+ \simeq A(X)$.

PROOF. The old definition of $A(X)$ was only given for connected pointed X. So we assume X is connected and pointed. Let G be any loop group for X, for example the $G(X)$ of Kan, and $h\mathscr{S}(G)_k^n$ the category considered in §2, in defining $A(X)$. There are functors

$$h\mathscr{R}(X)_k^n \to h\mathscr{S}(G)_k^n, \quad (Y, r, s) \mapsto r^*(X_t \times G)/(X_t \times G)$$

(and the action changed from right to left), and

$$h\mathscr{S}(G)_k^n \to h\mathscr{R}(X)_k^n, \quad Y \mapsto (X_t \times G) \times^G Y$$

and these functors are adjoint. Hence $h\mathscr{R}(X)_k^n \to^\sim h\mathscr{S}(G)_k^n$; hence

$$Z \times |\text{dir lim}_{(k,n)}\, h\mathscr{R}(X)_k^n|^+ \xrightarrow{\sim} Z \times |\text{dir lim}_{(k,n)}\, h\mathscr{S}(G)_k^n|^+$$

and the latter is $A(X)$, by §2.

Let $\mathscr{R}_f(X)_k^n = \mathscr{R}_f(X) \cap \mathscr{R}(X)_k^n$.

LEMMA 5.9. $h\mathscr{R}_f(X)_k^n \to^\sim h\mathscr{R}(X)_k^n$.

The composition law on $\mathscr{R}_f(X)$ induces a composition law on $\mathscr{R}_f(X)^n$, the union of the categories $\mathscr{R}_f(X)_k^n$; hence this is naturally the underlying category of a Γ-category, and so is the subcategory $h\mathscr{R}_f(X)^n$. Hence a simplicial category $N_\Gamma(h\mathscr{R}_f(X)^n)$, the nerve with respect to the composition law, is defined, cf. **[18]** for details. By Segal **[17]** there is a natural homotopy equivalence

$$\Omega\, |N_\Gamma(h\mathscr{R}_f(X)^n)| \xrightarrow{\sim} Z \times |\text{dir lim}_{(k)}\, h\mathscr{R}_f(X)_k^n|^+;$$

hence one is reduced to showing that

$$\text{dir lim}\, |N_\Gamma(h\mathscr{R}_f(X)^n)| \xrightarrow{\sim} |hS.\mathscr{R}_f(X)|.$$

The category $\mathscr{R}_f(X)^n$ is a category with cofibrations (this requires some care). Hence $hS.\mathscr{R}_f(X)^n$ is defined. There is a map of simplicial categories $N_\Gamma(h\mathscr{R}_f(X)^n) \to hS.\mathscr{R}_f(X)^n$ which sends each sum diagram to a cofibration sequence by forgetting part of the data.

LEMMA 5.10. $\text{dir lim}\, N_\Gamma(h\mathscr{R}_f(X)^n) \to^\sim \text{dir lim}\, hS.\mathscr{R}_f(X)^n$.

Define functors *cone*, $C\colon \mathscr{R}(X) \to \mathscr{R}(X)$, $C(Y, r, s) =$ (mapping cylinder of $r, \cdots$), and *suspension*, $S = \text{coker}(\text{id} \to C)$. When the cone and suspension are considered as endomorphisms of $hS.\mathscr{R}_f(X)$, one has by the additivity theorem a homotopy $C \to^\sim \text{id} \vee S$. In particular, the suspension represents a homotopy inverse to the identity on $hS.\mathscr{R}_f(X)$, and the map

$$hS.\mathscr{R}_f(X) \to \text{dir lim}_{(S)}\, hS.\mathscr{R}_f(X)$$

is a weak homotopy equivalence.

LEMMA 5.11. $\text{dir lim}_{(n)}\, hS.\mathscr{R}_f(X)^n \to^\sim \text{dir lim}_{(S)}\, hS.\mathscr{R}_f(X)$.

This ends the indication of proof of Theorem 5.7.

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