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THE INTRODUCTION OF TRUTH PREDICATES INTO FIRST-ORDER LANGUAGES*

0. This paper is a shortened and slightly changed version of my 1976 paper. In the present paper I will only deal with the question in which way it might be possible to extend first-order languages to languages with truth predicates.[1] The problems of the Liar paradox and of the introduction of truth predicates have often been treated and different solutions have been proposed. But such proposals are not very useful unless they are developed within a precise theory of language and, what is more important, within a theory of language extensions. It is astonishing that there were for a long time no serious systematic attempts to answer the questions: what type of language extension should the introduction of a truth predicate be regarded as, and under which conditions can such a predicate be introduced in a unique way.[2] Only if one tries to answer these questions is it possible to find a complete and adequate solution to the problems under discussion. But if one does try to do it then in my opinion it is not too difficult to give what looks like an intuitively acceptable solution.[3]

1. First-order languages (without equality symbol and without function symbols) are given relative to a set of logical symbols (we will take $\neg$, $\vee$, $\bigvee$ as primitive symbols), a countable set V of variables (denoted by $v, v', \ldots$) and a class of constants among which individual constants (denoted by $a, a', \ldots$) must be distinguished from relation constants, also called predicates (denoted by $P, P', \ldots$). If K is a set of constants, then I will designate by $T(K)$ the set of K-terms, i.e., the set which contains the individual constants of K and the variables. Formulas (denoted by $\varphi, \psi, \ldots$) are constructed in the usual manner. A formula φ is called a K-formula if the only constants which occur in φ belong to K. The set of K-formulas is denoted by $F(K)$. A formula φ is called a statement if no variable occurs free in φ; the set of K-statements is denoted by $A(K)$. A K-structure is an ordered pair $S = \langle X, I \rangle$, where $X \neq 0$ and I is a mapping which interprets each constant of K in the usual manner. Elements of X^V are called assignment functions and are denoted by $\alpha, \alpha', \ldots$. I shall write $S \models_\alpha \varphi$ for: φ is valid under α in S (see e.g., Bell and Slomson, 1974); specifically $S \models \varphi$ means that φ is a statement and that φ is valid in S (i.e., $S \models_\alpha \varphi$ for any α). A

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first-order language L (of classical type) consists of a set $K(L)$ of constants and a set of $K(L)$-structures. These structures are also called L-structures and instead of $A(K(L))$, $F(K(L))$ and $T(K(L))$ I shall write $A(L)$, $F(L)$ and $T(L)$ respectively.

1.1. DEFINITION: Let L and L' be first-order languages. L' is an extension of L iff

(1) $K(L) \subset K(L')$,
(2) there is a bijective mapping f from the set of L-structures to the set of L'-structures such that $f(S) \upharpoonright K(L) = S$ for any L-structure S.[4]

1.2. DEFINITION: Let L' be a first-order language, and let W be a monadic predicate of $K(L')$.
W is a truth predicate in L' iff for every L'-structure $S' = \langle X', I' \rangle$ and for every $\varphi \in A(L')$:

(A_1) if $\varphi \in I'(W)$ then $S' \models \varphi$;
(A_2) if $S' \models \varphi$ and $\varphi \in X'$ then $\varphi \in I'(W)$.

I call (A_1) the correctness condition and (A_2) the completeness condition for W with respect to S'. If W satisfies (A_1) but not necessarily (A_2) for each S', W is called a partial truth predicate in L'.

In a wider sense it follows from Tarski (1935) that it is impossible in general to extend a first-order language L to a first-order language with a truth predicate $W \notin K(L)$. For instance, if there is any L-structure $S = \langle X, I \rangle$ and any individual constant $a \in K(L)$ such that $I(a) = \neg Wa$ ($\neg Wa$ corresponds to the Liar sentence), then such an extension cannot be defined. Conversely, it is clear that such an extension exists if, for example, the only statements which lie in the universe X of any L-structure $S = \langle X, I \rangle$ belong to $A(L)$. It is a remarkable fact that in the latter case it is not always exactly one extension that is allowed by 1.2. For instance, if every L-structure $S = \langle X, I \rangle$ has the property that Wa is the only statement which belongs to X and that $I(a) = Wa$, then, with respect to each structure of the extended language, it is possible both to regard Wa as true and to regard Wa as not true.

Apart from the problem of whether L can be extended to a first-order language with a truth predicate or only with a partial one, it is an important question whether there is a natural way of defining such an extension.

Before dealing with this question I will consider an example. Let $S = \langle X, I \rangle$ be a L-structure, $\varphi \in A(L)$, $S \models \varphi$ and $a \in K(L)$. In order to extend S to a $K(L) \cup \{W\}$-structure $S' = \langle X', I' \rangle$ one has to specify how W is interpreted by I'. In the case of $I(a) = \varphi$ it seems to be quite natural to set $Wa \in I'(W)$. On the other hand, in the case of $I(a) = Wa$ it is not yet clear whether one should assume $Wa \in I'(W)$ or not. The difference between the two cases demonstrates the crucial point concerning an intuitively adequate interpretation of W. In the first case, the decision on $Wa \in I'(W)$ can be reduced to $S \models \varphi$ with (A_1). In the second case, on the other hand, the decision on $Wa \in I'(W)$ is not reducible. Indeed, the attempt to reduce this decision leads back to the initial condition $Wa \in I'(W)$. According to this it seems to be obvious that an interpretation of W by I' is adequate only if for every φ which belongs to $I'(W)$ the decision for $\varphi \in I'(W)$ is reducible in some way or other to validity properties of S. For if one wishes to have W as a truth predicate it is not admissible to regard arbitrary statements as true in S', but only such statements for which this can be justified by recourse to certain validity properties of S.

With the notion of reducibility it will be possible to characterize certain extensions of L as natural. But first of all it is necessary to give this notion a precise definition.

In the following let L be a first-order language, and let W be a monadic predicate such that $W \notin K(L)$. Let $K' = K(L) \cup \{W\}$ and suppose that $S = \langle X, I \rangle$ is any L-structure.

1.3. DEFINITION: The two-place relation $R(S)$ is defined by: For every $\varphi, \psi \in F(K')$, $\alpha \in X^V$, $t \in T(L)$, $v \in V$ and $x \in X$:

(1) if $\varphi \in A(K')$ and $(I \cup \alpha)(t) = \varphi$ then $\langle Wt, \alpha \rangle\, R(S)\, \{\langle \varphi, \alpha \rangle\}$ and $\langle \neg Wt, \alpha \rangle\, R(S)\, \{\langle \neg \varphi, \alpha \rangle\}$;

(2) if $\varphi \notin F(L)$ then $\langle \neg\neg \varphi, \alpha \rangle\, R(S)\, \{\langle \varphi, \alpha \}$;

(3) if $\varphi \vee \psi \notin F(L)$ then $\langle \varphi \vee \psi, \alpha \rangle\, R(S)\, \{\langle \varphi, \alpha \rangle\}$, $\langle \varphi \vee \psi, \alpha \rangle\, R(S)\, \{\langle \psi, \alpha \rangle\}$ and $\langle \neg(\varphi \vee \psi), \alpha \rangle\, R(S)\, \{\langle \neg \varphi, \alpha \rangle, \langle \neg \psi, \alpha \rangle\}$;

(4) if $\bigvee v\varphi \notin F(L)$ then $\langle \bigvee v\varphi, \alpha \rangle\, R(S)\, \{\langle \varphi, \alpha_v^x \rangle\}$ and $\langle \neg \bigvee v\varphi, \alpha \rangle\, R(S)\, \{\langle \neg \varphi, \alpha_v^y \rangle; y \in X\}$, where α_v^y is defined by $\alpha_v^y(v') = (v')$ for $v' \neq v$ and $\alpha_v^y(v) = y$.

$R(S)$ is called the reduction relation with respect to S.
$\langle \varphi, \alpha \rangle\, R(S)\, \{\langle \varphi_j, \alpha_j \rangle; j \in J\}$ means that the decision on the validity of φ under α can be reduced to all the decisions on the validity of φ_j under α_j

for all $j \in J$. It should be remarked here that $R(S)$ is not the only reduction relation which might be defined. For instance, instead of 1.3(3) it would also be reasonable to define $\langle \varphi \vee \psi, \alpha \rangle$ $R(S)$ $\{\langle \varphi, \alpha \rangle, \langle \psi, \varphi \rangle\}$, $\langle \varphi \vee \psi, \alpha \rangle$ $R(S)$ $\{\langle \varphi, \alpha \rangle, \langle \neg \psi, \alpha \rangle\}$, $\langle \varphi \vee \psi, \alpha \rangle$ $R(S)$ $\{\langle \neg \varphi, \alpha \rangle, \langle \psi, \alpha \rangle\}$ and $\langle \neg (\varphi \vee \psi), \alpha \rangle$ $R(S)$ $\{\langle \neg \varphi, \alpha \rangle, \langle \neg \psi, \alpha \rangle\}$ for $\varphi \vee \psi \notin F(L)$. But in this case one gets a relation which leads to an interpretation of W that is weaker than the one which is obtained by the proposed definition. One can show, however, that 1.3 yields the best possible compatibility with the validity concept of first-order languages.

1.4. DEFINITION: An ordered pair $\langle B, m_0 \rangle$ is a tree with the initial point m_0 iff:

(1) B is a two-place relation,
(2) for each element m of the field of B (i.e., the union of the range and the domain of B) there is a finite B-chain[5] which leads from m_0 to m.

A point m of the tree $\langle B, m_0 \rangle$ (i.e., $m = m_0$ or m belongs to the field of B) is called an end point if there is no m' such that $m\ B\ m'$. $\langle B, m_0 \rangle$ is said to be finite if there does not exist an infinite B-chain.

1.5. DEFINITION: Let $M(S)$ be the set of ordered pairs $\langle \varphi, \alpha \rangle$ such that $\varphi \in F(K')$ and $\alpha \in X^V$.
A tree $\langle B, m_0 \rangle$, with $B \subset M(S) \times M(S)$ and $m_0 \in M(S)$, is called a $R(S)$-tree if for each element m of the range of B there is an $M' \subset M(S)$ such that for each m':
$m\ B\ m'$ iff $m' \in M'$ and $m\ R(S)\ M'$.

1.6. DEFINITION: The set $G(S) \subset M(S)$ is defined by:
$\langle \varphi, \alpha \rangle \in G(S)$ iff there is a finite $R(S)$-tree with the initial point $\langle \varphi, \alpha \rangle$ such that $S \models_{\alpha'} \psi$ for every end point $\langle \psi, \alpha' \rangle$ of the tree.

1.7. DEFINITION: Let φ be a statement of $A(K')$. The decision regarding the truth of φ with respect to S is reducible iff there is an α such that $\langle \varphi, \alpha \rangle \in G(S)$. φ is called grounded with respect to S if the decision regarding the truth of φ, or the truth of $\neg \varphi$ with respect to S, is reducible.[6]

It is now clear that the adequateness condition proposed above can be formulated as follows: an adequate extension $S' = \langle X, I' \rangle$ of S must satisfy the condition that for each $\varphi \in I'(W)$ the decision regarding the truth of φ with respect to S is reducible. Therefore the optimal extension which can be defined must also fulfill the condition that each $\varphi \in X$ for which the decision regarding the truth of φ with respect to S is reducible belongs to $I'(W)$.

1.8. THEOREM: Let L be a first-order language, and let W be a monadic predicate, with $W \notin K(L)$. Let L' be defined by:
$K(L') = K(L) \cup \{W\}$;
$S' = \langle X', I' \rangle$ is a L'-structure iff $S' \restriction K(L)$ is an L-structure and $I'(W) = \{\varphi \in X';\ \langle \varphi, \alpha \rangle \in G(S)$ for some $\alpha\}$. Then W is a partial truth predicate in L' and furthermore the following conditions hold:

(1) For every L'-structure $S' = \langle X', I' \rangle$ and for every statement φ which is grounded with respect to $S' \restriction K(L)$ the conditions (A_1) and (A_2) in 1.2 are fulfilled.

(2) Let L'' be an extension of L such that $K(L'') = K(L')$ and that (A_1) and (A_2) are satisfied for every L''-structure $S' = \langle X'', I'' \rangle$ and for every statement φ which is grounded with respect to $S'' \restriction K(L)$. Then for all L'-structures $S' = \langle X', I' \rangle$ and for all L''-structures $S'' = \langle X'', I'' \rangle$, if $S' \restriction K(L) = S'' \restriction K(L)$ then $I'(W) \subset I''(W)$.

This theorem says in particular that L' is in a sense the weakest extension of L such that (1) is satisfied. A proof of 1.8 is sketched out in my 1976.

The definition of $G(S)$ which is given by 1.6 starts from the intuitive idea that a statement should be regarded as true with respect to the extension of S if this decision can be reduced to validity properties of S. Besides this characterisation of the interpretation of W, it is plausible that for the $G(S)$ which determines the interpretation of W there must exist a recursive definition which is based on the repeated application of (A_2). In fact, it is easy to prove the following result.

1.9. THEOREM: $G(S)$ is equal to the set $G'(S)$ defined, under the assumptions of definition 1.3, by:

if $S \models_\alpha \psi$ then $\langle \varphi, \alpha \rangle \in G'(S)$;

if $\{\langle \varphi_j, \alpha_j \rangle; j \in J\} \subset G'(S)$; and
$\langle \varphi, \alpha \rangle\ R(S)\ \{\langle \varphi_j, \alpha_j \rangle; j \in J\}$ then $\langle \varphi, \alpha \rangle \in G'(S)$.

2. In the preceding section the question has been discussed of whether there is a natural way of defining an extension L' with a truth predicate or at

least with a partial truth predicate $W \notin K(L)$ for each first-order language L. I have argued that such an extension L' is adequate only if L' satisfies the condition:

(C) For every L'-structure $S' = \langle X', I' \rangle$ and for every statement $\varphi \in I'(W)$ the decision regarding the truth of φ with respect to $S' \upharpoonright K(L)$ is reducible.

However, in my opinion this condition does not suffice for characterizing L' as adequate. It seems to be necessary to postulate that instead of (C) L' fulfills the following condition.

(C′) For every L'-structure $S' = \langle X', I' \rangle$ and for every ordered pair $\langle \varphi, \alpha \rangle$, if $S' \models_\alpha \varphi$ then $\langle \varphi, \alpha \rangle \in G(S' \upharpoonright K(L))$.

(C′) is a more general condition than (C) (it is easy to show that (C′) and (A_1) imply (C)). What (C′) postulates is that — intuitively speaking — each validity property of a given structure of the extended language L' must be justified by recourse to validity properties of the underlying structure of the restricted language L. The requirement to give such a justification is in my opinion necessary because the introduction of a truth predicate should not have the effect that in any structure of the extended language some statements become valid by accident and the validity is not based on properties of the underlying structure. If one accepts this argument one has to ask under which conditions an extension L' of L which satisfies (C′) can be defined.

It is clear that it is not in general possible to define a first-order extension L' which satisfies (C′). If there are, for instance, an L-structure $S = \langle X, I \rangle$ and an individual constant $a \in K(L)$ such that $I(a) = Wa$ then Wa is not grounded with respect to S. On the other hand either $S' \models Wa$ or $S' \models_\alpha \neg Wa$ is fulfilled for any suitable extension S' of S.

I think that the only conclusion which can be drawn from these facts is that first-order languages of classical type don't give an appropriate frame for a theory of languages with truth predicates. For in an adequate extension L' of L it must be admissible, if necessary, that for some L'-structures $S' = \langle X, I' \rangle$ and for some statements φ neither $S' \models \varphi$ nor $S' \models \neg \varphi$. More exactly, it follows from (C′) that $S' \models \varphi$ or $S' \models \neg \varphi$ can be fulfilled only if φ is grounded with respect to $S' \upharpoonright K(L)$. Therefore, in the case of $\alpha(v) = \varphi$, $S' \models_\alpha Wv$ or $S' \models_\alpha \neg Wv$ must be satisfied only if φ is grounded with respect to $S' \upharpoonright K(L)$. This means, in other words, that it seems to be inadequate to postulate generally that a truth predicate W is defined for every statement of the extended language. Instead of this one should only demand that W is defined for every grounded statement. But if one wants to satisfy the latter postulate one must leave the frame of first-order languages of classical type and proceed to consider first-order languages with partially defined predicates (called PDP-languages in the following).

First I shall generalize the hitherto used notions of structure and validity for the case of PDP-languages.

2.1. DEFINITION: Let K be a set of constants. $S = \langle X, I \rangle$ is a K-structure iff

(1) $X \neq 0$,
(2) I is a function with range K,
(3) I assigns to each individual constant of K an element of X,
(4) I assigns to each n-ary predicate of K an n-place partial relation on X, i.e., an ordered pair $\langle Z_0, Z_1 \rangle$ such that $Z_i \subset X^n$ for $i < 2$ and $Z_0 \cap Z_1 = 0$.[7]

2.2. DEFINITION: Let K be a set of constants, and let $S = \langle X, I \rangle$ be a K-structure. For any n-ary $P \in K$, for $t_0, \ldots, t_{n-1} \in T(K)$, $\alpha \in X^V$, $\varphi, \psi \in F(K)$, $v \in V$ and for $x \in X$ we set:

(1) if $\langle (I \cup \alpha)(t_0), \ldots, (I \cup \alpha)(t_{n-1}) \rangle \in (I(P))_1$ then $S \models_\alpha Pt_0 \ldots t_{n-1}$;
if $\langle (I \cup \alpha)(t_0), \ldots, (I \cup \alpha)(t_{n-1}) \rangle \in (I(P))_0$ then $S \models_\alpha \neg Pt_0 \ldots t_{n-1}$;
(2) if $S \models_\alpha \varphi$ then $S \models_\alpha \neg\neg\varphi$;
(3) if $S \models_\alpha \varphi$ or $S \models_\alpha \psi$ then $S \models_\alpha \varphi \vee \psi$;
if $S \models_\alpha \neg\varphi$ and $S \models_\alpha \neg\psi$ then $S \models_\alpha \neg(\varphi \vee \psi)$;
(4) if $S \models_{\alpha^x_v} \varphi$ then $S \models_\alpha \bigvee v\varphi$;
if $S \models_{\alpha^y_v} \neg\varphi$ for each $y \in X$ then $S \models_\alpha \neg \bigvee v\varphi$.

In 2.2 I have chosen Kleene's strong interpretation of the logical symbols (see Kleene, 1952, p. 334), the only one which is compatible with 1.3.

2.3. DEFINITION: A PDP-language L consists of a set $K(L)$ of constants and a set of K-structures in the sense of 2.1.

First-order languages of classical type can be regarded as special PDP-languages.

2.4. DEFINITION: A PDP-language is a first-order language of classical type iff for every L-structure $S = \langle X, I \rangle$ and for every n-ary predicate $P \in K(L)$:
$(I(P))_0 \cup (I(P))_1 = X^n$.

In the following I shall use, as far as possible, the notations and definitions which I have introduced in Section 1 also for PDP-languages.

At first glance the logic of PDP-languages seems to have the disadvantage that the completeness theorem does not hold any longer with respect to the classical predicate calculus.[8] However, it is easy to see that this theorem holds if one generalize the notion of logical consequence as follows.

2.5. DEFINITION: Let K be a set of constants, $\varphi \in A(K)$ and $\Phi \subset A(K)$. φ is a logical consequence of Φ iff $S \vDash \varphi$ for every K-structure $S = \langle X, I \rangle$ and for every $\alpha \in X^V$ with the property that S is defined for φ and α (i.e., $S \underset{\alpha}{\vDash} \varphi \vee \neg \varphi$) and that $S \underset{\alpha}{\vDash} \psi$ for every $\psi \in \Phi$.

2.6. THEOREM: Let K be a set of constants, $\varphi \in A(K)$ and $\Phi \subset A(K)$. φ is deducible from Φ iff φ is a logical consequence of Φ.

For the proof of the nontrivial 'only if' part one uses a calculus without cut rule.

In the following we deal with the problem of introducing truth predicates into PDP-languages. For this we must first generalize the definition 1.2 and, in particular, decide under which conditions a statement φ should belong to $(I'(W))_0$ for any L'-structure $S' = \langle X', I' \rangle$. I will choose the generalisation which best preserves the properties of truth predicates in first-order languages of classical type.

2.7. DEFINITION: Let L' be a PDP-language, and let W be a monadic predicate such that $W \in K(L')$.
W is a truth predicate in L' iff for every L'-structure $S' = \langle X', I' \rangle$ and for every $\varphi \in A(L')$:

(B_1) if $\varphi \in (I'(W))_0$ then $S' \vDash \neg \varphi$;
if $\varphi \in (I'(W))_1$ then $S' \vDash \varphi$;
(B_2) if $S' \vDash \neg \varphi$ and $\varphi \in X'$ then $\varphi \in (I'(W))_0$;
if $S' \vDash \varphi$ and $\varphi \in X'$ then $\varphi \in (I'(W))_1$.

2.8. DEFINITION: Let L, L' and L'' be PDP-languages such that $K(L') = K(L'')$ and such that L' and L'' are extensions of L.
L' is a weaker extension of L than L'' iff for every L'-structure $S' = \langle X', I' \rangle$ and for every L''-structure $S'' = \langle X'', I'' \rangle$, if $S' \restriction K(L) = S'' \restriction K(L)$ then:

(1) $I'(a) = I''(a)$ for each individual constant $a \in K(L')$;
(2) $(I'(P))_i \subset (I''(P))_i$ for each predicate $P \in K(L')$ and for each $i < 2$.

2.9. THEOREM: Let L be a PDP-language, and let W be a monadic predicate such that $W \notin K(L)$.
Then there is exactly one PDP-language L' such that

(1) L' is an extension of L and $K(L') = K(L) \cup \{W\}$,
(2) W is a truth predicate in L',
(3) L' fulfills the condition (C').

The language guaranteed by 2.9 is called the natural extension of L with respect to the introduction of W as truth predicate. There are some other characterisations of L' which I will now state.

2.10. THEOREM: Let L and L' be PDP-languages, and let W be a monadic predicate such that $W \notin K(L)$. Suppose that L' is an extension of L, with $K(L') = K(L) \cup \{W\}$.
Then the following conditions are equivalent:

(1) L' is the natural extension of L with respect to the introduction of W as truth predicate.
(2) For every L'-structure $S' = \langle X', I' \rangle$, for every $\varphi \in F(L')$ and for every $\alpha \in X'^V$,
$\langle \varphi, \alpha \rangle \in G(S' \restriction K(L))$ iff $S' \models_{\alpha} \varphi$.
(3) For every L'-structure $S' = \langle X', I' \rangle$,
$(I'(W))_0 = \{\varphi \in X';\ \langle \neg\varphi, \alpha \rangle \in G(S' \restriction K(L))$ for some $\alpha\}$,
$(I'(W))_1 = \{\varphi \in X';\ \langle \varphi, \alpha \rangle \in G(S' \restriction K(L))$ for some $\alpha\}$.
(4) L' is the weakest extension of L such that W is a truth predicate in L'.

For proofs of 2.9 and 2.10 see my 1976 paper. Finally I will present a characterisation of L' which gives a rather simple and plausible construction for L'.[9] It should be remarked, however, that this construction and the determination of the interpretation of W defined thereby are based essentially on the generalized concept of structure introduced in 2.1. In contrast to this, the determination of W defined via $G(S)$ is independent of this concept.

2.11. THEOREM: Let L be a PDP-language, and let W be a monadic predicate such that $W \notin K(L)$. Let L' be the extension of L defined as follows (for sets Φ of formulas let $\bar{\phi} = \{\neg\varphi; \varphi \in \Phi\}$):

(1) $K(L') = K(L) \cup \{W\}$.

(2) To each L-structure $S = \langle X, I \rangle$ is assigned an L'-structure S' via $S' = \langle X, I \cup \{\langle W, \langle Z_0, Z_1 \rangle\rangle\}\rangle$, where $Z_0 = \overline{Z_1}$ and Z_1 is recursively defined by:
if $\varphi \in A(L') \cap X, \Phi \subset Z_1$ and $\langle X, I \cup \{\langle W, \langle \bar{\Phi}, \Phi \rangle\rangle\}\rangle \models \varphi$ then $\varphi \in Z_1$.

Then L' is the natural extension of L with respect to the introduction of W as truth predicate.

The construction of L' given in this theorem follows closely the idea of how — I suppose — one would intuitively say that a language extension with the aim of introducing a truth predicate W must proceed: For a given structure S of the underlying language one defines successively the extension and the anti-extension of W in the extended structure in such a way that they are closed under (B_2).

3. I will conclude this paper with a few general remarks about some of the consequences of the above discussion. This discussion may have made clear why the problems of the Liar paradox and of the introduction of truth predicates could have been controversial for such a long time. In my opinion, this was because the problems of language extensions had not been analyzed strictly enough. There are, however, different types of language extensions. For example, no difficulties attach to the case where a new n-ary predicate Q is introduced into a PDP-language L relative to the variables $v_0, \ldots, v_{n-1}$ and to the formula $\psi \in F(L)$ as follows.

Each L-structure $S = \langle X, I \rangle$ is extended to a $K(L) \cup \{Q\}$-structure $S' = \langle X, I' \rangle$ such that:

(B) for every $\alpha \in X^V$,
$S' \models_\alpha Qv_0 \ldots v_{n-1}$ iff $S' \models_\alpha \psi$,
$S' \models_\alpha \neg Qv_0 \ldots v_{n-1}$ iff $S' \models_\alpha \neg\psi$.

S' is uniquely determined by this condition and hence the definition of the extended language L' can be represented concisely by postulating that (B) holds for each L'-structure. In the special case where L is of classical type it is possible to replace (B) by the condition

$$S' \models \bigwedge v_0 \ldots \bigwedge v_{n-1}(Qv_0 \ldots v_{n-1} \leftrightarrow \psi).$$

Therefore the definition of L' can be represented in this case by postulating that $\bigwedge v_0 \ldots \bigwedge v_{n-1} (Qv_0 \ldots v_{n-1} \leftrightarrow \psi)$ is valid in L'. In other words, Q is definable in L and the language extension considered here is of the well known type of extensions by definitions, where $\bigwedge v_0 \ldots \bigwedge v_{n-1}(Qv_0 \ldots v_{n-1} \leftrightarrow \psi)$ is the defining axiom.

In contrast to this the language extensions for introducing truth predicates are of a more general type which can be described as follows.

A new n-ary predicate Q is introduced into a PDP-language L relative to the variables $v_0, \ldots, v_{n-1}$ and to the function f: $Y^n \to F(K(L) \cup \{Q\})$ so that each L-structure $S = \langle X, I \rangle$ is extended to a $K(L) \cup \{Q\}$)-structure $S' = \langle X, I' \rangle$ which satisfies the condition

(B′) for all $\varphi \in F(K(L) \cup \{Q\})$, for all $y_0, \ldots, y_{n-1} \in Y$ and for all $\alpha \in X^V$, if $f(y_0, \ldots, y_{n-1}) = \varphi$ and $\alpha(v_i) = y_i$ for each $i < n$ then:
$S' \models_\alpha Qv_0 \ldots v_{n-1}$ iff $S' \models_\alpha \varphi$,
$S' \models_\alpha \neg Qv_0 \ldots v_{n-1}$ iff $S' \models_\alpha \neg\varphi$.

In general, (B′) does not determine uniquely one extension S' and hence it is not sufficient for a characterisation of an extension L' of L to postulate that (B′) holds for every L'-structure. According to the results of our discussion in Section 2 it seems to be reasonable, however, to regard the weakest extension L' of L such that every L'-structure fulfills (B′) as the natural extension.

An important difference between the two types of language extensions is the fact that the first, but not the second, type has the following elimination property:

> There is a function e: $F(L') \to F(L)$ which can be defined in a natural way with respect to (B) such that for every L'-structure $S' = \langle X', I' \rangle$, for every $\varphi \in F(L')$ and for every $\alpha \in X'^V$, $S' \models_\alpha \varphi$ iff $S' \restriction K(L) \models_\alpha e(\varphi)$.

The second type has instead only the following reduction property:

> For every L-structure S there is a relation $R(S)$ which can be

defined in a natural way with respect to (B′) such that for every L'-structure $S' = (X', I')$, for every $\varphi \in F(L')$ and for every $\alpha \in X'^V$,

$S' \models_\alpha \varphi$ iff there is a finite $R(S' \restriction K(L))$-tree with the initial point $\langle \varphi, \alpha \rangle$ such that $S' \restriction K(L) \models_\alpha \psi$ for every end point $\langle \psi, \alpha' \rangle$ of the tree.

The loss of the elimination property for the second type is compensated by an essential gain in expressibility. For the example of a truth predicate this gain consists, e.g., in the possibility of expressing the proposition that there are true statements, a proposition which could not be expressed otherwise, not even by an infinite disjunction of statements. I think it is an important task to investigate which theoretically or empirically relevant predicates can be introduced after the second type of language extension (the predicate 'heterological', e.g., on which the antinomy of Grelling is based can be introduced correctly in this way). In addition to this a more general discussion of the problem of language extensions seems to be necessary. Two questions, in particular, should be dealt with:

First, what types of language extensions can or should be distinguished on empirical or theoretical grounds?

Second, what problems arise if several language extensions are carried out successively, especially with regard to the compatibility and extendibility of the respective new notions?

These questions will have to await further investigation; my aim in this section of the present paper has simply been to point out the need for such research.

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NOTES

* I would like to thank M. Pätzold and D. Segal for helping with the English.

[1] The basic ideas underlying my investigations were first presented in a talk on the occasion of a conference at the University of Bielefeld, December 1974. Kripke's research in this field and his paper, November 1975, and also the article of Martin and Woodruff 1976 were unknown to me during my work on the first draft of my 1976. Kripke's approach is based on the same idea and reaches the same main results. There are, however, several differences in matters of presentation, explicitness, emphasis and in some particular results.

[2] Meanwhile I have learned from Feferman (1976) that — besides the approaches of Kripke and Martin and Woodruff — there are also investigations which deal with related problems concerning type-free mathematical theories and propose similar solutions. By the way, the

method of truth and validity definition described in this paper is not essentially new. In particular, I have already applied this method in a general form to the theory of dialogue games although I did not explicitly handle the case of languages with truth predicates (cf. Kindt, 1972).

[3] In contrast to Kripke I am convinced that the given solution is the best justifiable one.

[4] If $S = \langle X, I \rangle$ is a K-structure, then $S \upharpoonright K'$ is defined by $S \upharpoonright K' := \langle X, I \upharpoonright K' \rangle$, where $I \upharpoonright K'$ is the restriction of I to K'.

[5] A sequence f is a B-chain iff $f_j B f_{j+1}$ whenever j and $j+1$ belong to the range of f.

[6] For the notion of groundedness see also Kripke (1975, p. 706). In contrast to Kripke I have defined this notion in a way which is based directly on the idea of reducibility and which is independent of the later discussion dealing the question of what type of languages gives an appropriate frame for the introduction of truth predicates. This way has in my opinion the advantage of showing more clearly why it is natural to restrict the application of truth predicates exactly to the set of grounded statements.

[7] In the terminology of Kripke Z_1 is called the extension and Z_0 the anti-extension of the predicate.

[8] More exactly, this calculus seems to be complete but not correct.

[9] Cf. Kripke (1975, pp. 702–705); the construction of L' given by Kripke is based on a definition by transfinite induction. It is, however, not necessary to make use of the theory of ordinals and the method of transfinite induction if one does not have the need to discriminate different levels in the construction of L'.

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