

On well-posed problems for connecting orbits in dynamical systems

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ABSTRACT. We develop formulations of well-posed problems for orbits which connect steady states to periodic orbits or periodic orbits to each other in a dynamical system. It turns out that the property of asymptotic phase on the periodic side plays a crucial role for the resulting boundary value problem on the real line. Our approach is closely related to a paper by Hale and Lin [HaLi 86] where Liapunov-Schmidt type methods and associated bifurcation functions have been developed for periodic-to-periodic connections in functional differential equations. In our formulation we avoid any non-autonomous transformation of the independent variable and we keep the periodic orbits as part of the problem. The boundary value problems thus obtained, are directly amenable to numerical approximation schemes on finite intervals.

1. Introduction

In this introduction we consider the general case of two compact invariant manifolds which are connected by an orbit of a given parametrized dynamical system. Our aim here is to establish a relation between the dimensions of the unstable manifolds of the invariant manifolds and the number of parameters for which we expect such a connecting orbit to occur generically. We will also outline how the formulation of a well-posed problem should look like for connecting orbits of this general type.

While these considerations will be mainly nonrigorous the main body of the paper is designed to provide the analytical details for the special case in which the connected manifolds are stationary points or periodic orbits. We are particularly interested in the formulation of well-posed boundary value problems which can be tackled numerically.

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Consider a parametrized dynamical system

$$(1.1) \quad \dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \mathbb{R}^p$$

where $f : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ is assumed to be sufficiently smooth. In many instances it will be convenient to work with the variables $z = (x, \lambda) \in \mathbb{R}^{m+p}$ and to rewrite (1.1) as

$$(1.2) \quad \dot{z} = g(z), \quad g(x, \lambda) = (f(x, \lambda), 0).$$

Any compact invariant set $M \subset \mathbb{R}^{m+p}$ of this system is trivially foliated

$$(1.3) \quad M = \bigcup_{\lambda \in \Lambda} (M(\lambda) \times \{\lambda\})$$

where $\Lambda \subset \mathbb{R}^p$ is compact and the $M(\lambda)$ are compact invariant sets of the system (1.1).

Let $M_+, M_- \subset \mathbb{R}^{m+p}$ be two such compact invariant sets and let $z(t) = (x(t), \lambda(t))$, $t \in \mathbb{R}$ be a solution of (1.2). Then the orbit $\gamma = \{(x(t), \lambda(t)) : t \in \mathbb{R}\}$ will be called a **connecting orbit from M_- to M_+** if

$$(1.4) \quad \text{dist}(z(t), M_{\pm}) \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

In particular, the α - and ω -limit sets satisfy

$$(1.5) \quad \alpha(\gamma) \subset M_-, \quad \omega(\gamma) \subset M_+.$$

In case $\alpha(\gamma) = \omega(\gamma)$ we call γ a **homoclinic orbit** and a **heteroclinic orbit** otherwise.

Of course we may rephrase (1.4) as

$$\text{dist}(x(t), M_{\pm}(\lambda(0))) \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty$$

where $M_{\pm}(\lambda)$ are the sets in the decomposition of M_{\pm} corresponding to (1.3).

Now let us assume in addition, that M_{\pm} are smooth invariant manifolds of dimension $m_{\pm c} + p$ where $m_{\pm c}$ is the dimension of the manifolds $M_{\pm}(\lambda)$ in the decomposition (1.3). We want to determine the number p of parameters for which we expect the connecting orbits above to be isolated and stable phenomena in the system (1.1). In this case we also expect the connecting orbits to occur in a generic sense.

Let $M_{\pm}(\lambda)$, $\lambda \in \Lambda$, have stable and unstable manifolds $M_{\pm s}(\lambda)$, $M_{\pm u}(\lambda)$ which are of dimension $m_{\pm c} + m_{\pm s}$ and $m_{\pm c} + m_{\pm u}$ respectively and let these be independent of $\lambda \in \Lambda$ (e.g. let $M_{\pm}(\lambda)$ have a hyperbolic structure, see [Irw 80], [HPS 77]). Then we have $m = m_{\pm c} + m_{\pm s} + m_{\pm u}$ and

$$\gamma \subset M_{-u} \cap M_{+s} \quad \text{where } M_{-u} = \bigcup_{\lambda} (M_{-u}(\lambda) \times \{\lambda\}), \quad M_{+s} = \bigcup_{\lambda} (M_{+s}(\lambda) \times \{\lambda\}).$$

M_{-u} and M_{+s} are in fact center-unstable resp. center-stable manifolds for the system (1.2).

We expect γ to be an isolated connecting orbit if

$$(1.6) \quad T_{z(t)}M_{-u} \cap T_{z(t)}M_{+s} = T_{z(t)}\gamma = \text{span} \{\dot{z}(t)\} \quad \text{for all } t \in \mathbb{R}$$

holds for the tangent spaces at $z(t)$. Moreover, the connecting orbit should persist in p -parameter systems if the intersection is transversal, i.e.

$$(1.7) \quad T_{z(t)}M_{-u} + T_{z(t)}M_{+s} = \mathbb{R}^{m+p}.$$

Counting dimensions we obtain from (1.6) and (1.7)

$$p + m_{-u} + m_{-c} + m_{+s} + m_{+c} + p - 1 = m + p$$

and thus

$$(1.8) \quad p = m_{+u} - m_{-u} - m_{-c} + 1.$$

We notice that this relation also makes sense in the framework of partial differential evolution equations provided the center and center-unstable manifolds are finite dimensional. Further, it can easily be seen, that any of the three statements (1.6) to (1.8) is implied by the other two.

In the steady case we have $m_{-c} = m_{+c} = 0$ and (1.8) is the relation discussed in [Bey 90a]. In cases where periodics are involved the right hand side of (1.8) turns out to be the negative of the index as defined in [HaLi 86]. In the general case we define the index of the connecting orbit as

$$\text{ind}(\gamma) = m_{-u} + m_{-c} - 1 - m_{+u}$$

and this turns out to be the Fredholm index of some suitable linearization around γ (see Proposition 2.8).

If $\text{ind}(\gamma) + p = 0$, i.e. (1.8) holds, then we set up a well-posed problem for γ . In the case $\text{ind}(\gamma) + p < 0$ we should add another $-\text{ind}(\gamma) - p$ parameters in order to have a well-posed problem while in case $\text{ind}(\gamma) + p > 0$ there is a $(p + \text{ind}(\gamma))$ -dimensional manifold of connecting orbits and we may add $p + \text{ind}(\gamma)$ conditions for its parametrization.

In the following let us assume that $z(t) = (x(t), \lambda(t))$ is an orbit connecting M_- to M_+ and let (1.6)–(1.8) hold. In view of the general theory of asymptotic stability with rate constants ([Fen 74], [Fen 77]) we look for solutions y_- and y_+ of

$$(1.9) \quad \dot{y}_{\pm} = f(y_{\pm}, \lambda)$$

which lie on $M_{\pm}(\lambda)$ and which have the same **asymptotic phase** as $x(t)$, i.e. for some $\epsilon > 0$

$$(1.10) \quad \|x(t) - y_{\pm}(t)\| = O(e^{-\epsilon|t|}) \text{ as } t \rightarrow \pm\infty.$$

Altogether, (1.2) and (1.9) comprises a set of $3m + p$ differential equations for which we need an appropriate number of boundary conditions. We use a scalar condition which fixes the phase of the triple (x, y_-, y_+)

$$(1.11) \quad \Psi(x, y_-, y_+, \lambda) = 0$$

and some equation which specifies that $(y_{\pm}(0), \lambda)$ is in M_{\pm}

$$(1.12) \quad N_{\pm}(y_{\pm}(0), \lambda) = 0.$$

The last are $m - m_{-c} + m - m_{+c}$ equations. In (1.10) we require that $x(0)$ lies in the fiber which is in asymptotic phase with $y_{\pm}(0)$ and which has dimension m_{+s} and m_{-u} respectively. Therefore we get a total of

$$m_{-s} + m_{-c} + m_{+u} + m_{+c} + 1 + 2m - m_{-c} - m_{+c} = 3m + m_{+u} - m_{-u} - m_{-c} + 1$$

boundary conditions which is precisely $3m + p$ by (1.8).

In general it can be very difficult to set up the equation (1.12) because it requires the knowledge of the manifolds $M_{\pm}(\lambda)$. In this paper we are mainly interested in the case where y_- is a steady state and $y_+ = y$ is a periodic orbit. For appropriate $\epsilon > 0$ define the spaces

$$Z_0(\epsilon) = \{(x, y) : x \in C(\mathbb{R}, \mathbb{R}^m) \text{ is bounded, } y \in C(\mathbb{R}, \mathbb{R}^m) \text{ is 1-periodic and } \|x(t) - y(t)\| \leq C e^{-\epsilon t} \text{ for } t \geq 0\},$$

$$Z_1(\epsilon) = \{(x, y) : (x, y) \in Z_0, (\dot{x}, \dot{y}) \in Z_0\}$$

with norms

$$\|(x, y)\|_{Z_0} = \sup_{t \in \mathbb{R}} \|x(t)\| + \sup_{t \in \mathbb{R}} \|y(t)\| + \sup_{t \geq 0} e^{\epsilon t} \|x(t) - y(t)\|,$$

$$\|(x, y)\|_{Z_1} = \|(x, y)\|_{Z_0} + \|(\dot{x}, \dot{y})\|_{Z_0}.$$

We include the period T as an unknown in the connecting orbit problem and write it in the following form

$$(1.13) \quad 0 = F(x, y, T, \lambda) = \begin{pmatrix} \dot{x} - Tf(x, \lambda) \\ \dot{y} - Tf(y, \lambda) \\ \Psi(x, y, T, \lambda) \end{pmatrix} = 0$$

where F maps $Z_1(\epsilon) \times \mathbb{R} \times \mathbb{R}^p$ into $Z_0(\epsilon) \times \mathbb{R}$.

In the case of two periodic orbits y_- and y_+ we have to determine both periods T_- and T_+ and we split up x into x_- on \mathbb{R}_- and x_+ on \mathbb{R}_+ . In some appropriate

spaces we then consider the equation

$$(1.14) \quad 0 = F(x_-, y_-, x_+, y_+, T_-, T_+, \lambda) = \begin{bmatrix} \dot{x}_- - T_- f(x_-, \lambda) \\ \dot{y}_- - T_- f(y_-, \lambda) \\ \dot{x}_+ - T_+ f(x_+, \lambda) \\ \dot{y}_+ - T_+ f(y_+, \lambda) \\ x_+(0) - x_-(0) \\ \Psi(x_-, y_-, x_+, y_+, T_-, T_+, \lambda) \end{bmatrix}.$$

For both cases we will show in Section 3 that regular solutions of these operator equations (i.e. solutions at which the linearization is homeomorphic) are directly related to the transversal intersection of the manifolds M_{-u} and M_{+s} above. In Section 2 we provide the necessary preparations about linear systems, in particular a result on a special type of exponential trichotomies (compare [HaLi 86], [ChLi 90]).

Finally, we treat an example of a point-to-periodic connection from the Lorenz system ([Spa 82]). It shows that it is rather straightforward to set up and solve a boundary value problem which approximates (1.13) on a finite interval.

However, a detailed analysis of the errors involved in this approximation will be deferred to a subsequent paper.

2. Preliminaries on linear systems

We consider linear differential operators

$$Lx = \dot{x} - A(t)x$$

where $A \in C(J, \mathbb{R}^{m,m})$, $x \in C^1(J, \mathbb{R}^m)$ and $J \subset \mathbb{R}$ is some interval. By $S(t, s)$, $t, s \in J$ we denote the solution operator of L , i.e. $x(t) = S(t, s)\xi$ is the solution of $Lx = 0$, $x(s) = \xi$.

In [HaLi 86] the notion of an exponential dichotomy [Cop 78] was generalized to an exponential trichotomy where a certain center part was allowed to grow or decay at an exponential rate close to an intermediate value (see also [ChLi 90]). For the linearizations about point-to-periodic connections this center part will in fact be bounded in both time directions as in an ordinary dichotomy ([Cop 78], p. 10). This motivates the following definition in which the center part is required to have an exact exponential behaviour.

DEFINITION 2.1.

L has an **ordinary exponential trichotomy** on J with exponents $\alpha < \nu < \beta$ if there exists a constant $K > 0$ and projectors $P_\kappa(t)$, $t \in J$, $\kappa = s, c, u$ of rank m_κ such that $P_s + P_c + P_u = I$ in J and such that the following conditions hold for all $t \geq s$ in J

$$(2.1) \quad S(t, s)P_\kappa(s) = P_\kappa(t)S(t, s), \quad \kappa = s, c, u$$

$$(2.2) \quad \begin{aligned} \|S(t, s)P_s(s)\| &\leq Ke^{\alpha(t-s)}, \quad \|S(s, t)P_u(t)\| \leq Ke^{-\beta(t-s)} \\ \|S(t, s)P_c(s)\| &\leq Ke^{\nu(t-s)}, \quad \|S(s, t)P_c(t)\| \leq Ke^{-\nu(t-s)}. \end{aligned}$$

As in [HaLi 86] we speak of a **shifted exponential dichotomy** if the center part is trivial ($P_c = 0$) and of an **exponential dichotomy** if in addition $\alpha < 0 < \beta$.

An easy calculation shows that (2.1) holds if and only if P_κ satisfies Liapunov's equation

$$(2.3) \quad P_\kappa = AP_\kappa - P_\kappa A \text{ in } J.$$

Further, if $P_\kappa(0) = X(0)\Phi(0)^T$ holds for some matrices $X(0), \Phi(0) \in \mathbb{R}^{m, m_\kappa}$ then we also have

$$(2.4) \quad P_\kappa(t) = X(t)\Phi(t)^T \text{ where } X(t) = S(t, 0)X(0), \Phi(t) = S(0, t)^T\Phi(0).$$

Finally, we notice that if L has an ordinary exponential trichotomy, then so has the adjoint $L^* = \frac{d}{dt} + A(t)^T$ with solution operator $S^*(t, s) = S(s, t)^T$, exponents $-\beta < -\nu < -\alpha$ and projectors $P_s^*(t) = P_u(t)^T, P_c^*(t) = P_c(t)^T, P_u^*(t) = P_s(t)^T$.

Shifting the indices in a trichotomy is described in the following lemma which is straightforward to verify.

LEMMA 2.2.

For $\gamma \in \mathbb{R}$ let $L_\gamma x = Lx + \gamma x$ with solution operator $S_\gamma(t, s)$. Then the following holds

- (i) $e^{\gamma t}L_\gamma x = L(e^{\gamma t}x)$ for $t \in J, x \in C^1(J, \mathbb{R}^m)$
- (ii) $S_\gamma(t, s) = e^{\gamma(s-t)}S(t, s)$ for $t, s \in J$
- (iii) If L has an ordinary exponential trichotomy on J with exponents $\alpha < \nu < \beta$ then L_γ has an ordinary exponential trichotomy on J with exponents $\alpha - \gamma < \nu - \gamma < \beta - \gamma$ and the same projectors.

Another important tool is the roughness theorem for exponential dichotomies ([Cop78], p. 34) to which we add some estimates on the projectors which are useful later on.

PROPOSITION 2.3.

Let L have a shifted exponential dichotomy on $J = [\tau, \infty)$ with exponents $\alpha < \beta$, with constant K and with projectors $P_s(t), P_u(t), t \in J$. Assume that $B \in C(J, \mathbb{R}^{m, m})$ satisfies

$$(2.5) \quad \frac{8K^2b_\infty}{\beta - \alpha} < 1 \text{ where } b_\infty = \sup_{t \geq \tau} \|B(t)\|.$$

Then the perturbed operator $\tilde{L}x = Lx - B(t)x$ has a shifted exponential dichotomy on J with exponents

$$\tilde{\alpha} = \alpha + 2b_{\infty}K < \tilde{\beta} = \beta - 2b_{\infty}K,$$

with constant $\tilde{K} = \frac{5}{2}K^2$ and with projectors $\tilde{P}_s(t), \tilde{P}_u(t), t \in J$ which satisfy

$$(2.6) \quad \|\tilde{P}_{\kappa}(t) - P_{\kappa}(t)\| \leq 2K\tilde{K} \int_{\tau}^{\infty} e^{-(\tilde{\beta}-\alpha)|t-s|} \|B(s)\| ds, \quad \kappa = s, u.$$

PROOF.

It is sufficient to prove this for an exponential dichotomy with $\alpha = -\beta$. In the general case we first shift by $\gamma = \frac{1}{2}(\alpha + \beta)$, apply Lemma 2.2 and the known results and then shift back by $\gamma = -\frac{1}{2}(\alpha + \beta)$.

In the case $\alpha = -\beta$ the result is proved in [Cop78], p. 34 with the exception of (2.6). In the proof there, $Y_1(t) = \tilde{S}(t, \tau)\tilde{P}_s(\tau)$ is constructed as the solution of the integral equation

$$(2.7) \quad Y_1(t) = (GY_1)(t) + S(t, \tau)P_s(\tau), t \geq \tau.$$

where

$$(GY_1)(t) = \int_{\tau}^t S(t, s)P_s(s)B(s)Y_1(s)ds - \int_t^{\infty} S(t, s)P_u(s)B(s)Y_1(s)ds.$$

Then $Y_1(\tau) = \tilde{P}_s(\tau)$ turns out to be a projector satisfying

$$(2.8) \quad N(\tilde{P}_s(\tau)) = N(P_s(\tau))$$

We set $\tilde{P}_s(t) = \tilde{S}(t, \tau)\tilde{P}_s(\tau)\tilde{S}(\tau, t) = Y_1(t)\tilde{S}(\tau, t)$, $\tilde{P}_u(t) = I - \tilde{P}_s(t)$ for $t \in J$ and find

$$(2.9) \quad \tilde{P}_s(t) - P_s(t) = P_u(t)\tilde{P}_s(t) - P_s(t)\tilde{P}_u(t).$$

Using (2.7) and (2.1) we obtain

$$\begin{aligned} P_u(t)\tilde{P}_s(t) &= P_u(t)Y_1(t)\tilde{S}(\tau, t) = - \int_t^{\infty} S(t, s)P_u(s)B(s)Y_1(s)ds\tilde{S}(\tau, t) \\ &= - \int_t^{\infty} S(t, s)P_u(s)B(s)\tilde{S}(s, t)\tilde{P}_s(t)ds \end{aligned}$$

and thus by the exponential dichotomies

$$\|P_u(t)\tilde{P}_s(t)\| \leq K\tilde{K} \int_t^{\infty} e^{(\tilde{\beta}-\alpha)(s-t)} \|B(s)\| ds.$$

For the second term in (2.9) we use the formula

$$(2.10) \quad P_s(t)\tilde{P}_u(t) = \int_{\tau}^t S(t,s)P_s(s)B(s)\tilde{P}_u(s)\tilde{S}(s,t)ds$$

and again the dichotomies to get (2.6). For the proof of (2.10) we notice that both sides have the value 0 at $t = \tau$ due to (2.8) and both satisfy the same differential equation as may be easily seen with the help of (2.3). ■

Similar to [Pal 84] we now replace the smallness assumption (2.5) by an asymptotic estimate on the perturbation

$$(2.11) \quad \|B(t)\| \leq Ce^{-\varepsilon t} \text{ for } t \geq 0 \text{ and some } \varepsilon > 0.$$

This will be the interesting case with our applications but we notice that it is easy to derive generalized statements under the assumption $B(t) \rightarrow 0$ as $t \rightarrow \infty$.

PROPOSITION 2.4.

Let L have a shifted exponential dichotomy on $J = [\tau, \infty)$ with exponents $\alpha < \beta$, with constant K and with projectors $P_s(t), P_u(t), t \in J$ and let $B \in C(J, \mathbb{R}^{m,m})$ satisfy (2.11) for some $0 < \varepsilon < \beta - \alpha$.

The the perturbed operator $\tilde{L}x = Lx - B(t)x$ has a shifted exponential dichotomy on J with exponents $\tilde{\alpha} < \tilde{\beta}$, which may be chosen arbitrarily close to $\alpha < \beta$, and with constants \tilde{K} depending on $\tilde{\alpha}, \tilde{\beta}$. For the projectors $\tilde{P}_s(t), \tilde{P}_u(t)$, and the solution operator $\tilde{S}(t,s)$ the following estimates hold for all $t \geq s$ in J

$$(2.12) \quad \|\tilde{P}_\kappa(t) - P_\kappa(t)\| \leq Ce^{-\varepsilon t}, \kappa = s, u$$

$$(2.13) \quad \|\tilde{S}(t,s)\tilde{P}_s(s) - S(t,s)P_s(s)\| \leq C \left(e^{-\varepsilon t + \tilde{\alpha}(t-s)} + e^{-\varepsilon s + \alpha(t-s)} \right)$$

$$(2.14) \quad \|\tilde{S}(s,t)\tilde{P}_u(t) - S(s,t)P_u(t)\| \leq C \left(e^{-\varepsilon s - \tilde{\beta}(t-s)} + e^{-\varepsilon t - \beta(t-s)} \right).$$

PROOF.

We choose $\tau_1 > \tau$ such that $b_\infty = \sup_{t \geq \tau_1} \|B(t)\|$ satisfies (2.5) and $2b_\infty K < \text{Min}(\varepsilon, \beta - \alpha - \varepsilon)$. From Proposition 2.3 we obtain the shifted exponential dichotomy for \tilde{L} on $J_1 = [\tau_1, \infty)$. Moreover, by (2.6) and (2.11)

$$\|\tilde{P}_\kappa(t) - P_\kappa(t)\| \leq C \left[\int_{\tau_1}^t e^{-(\tilde{\beta}-\alpha)(t-s)-\varepsilon s} ds + \int_t^\infty e^{-(\tilde{\beta}-\alpha)(s-t)-\varepsilon s} ds \right] \leq Ce^{-\varepsilon t}.$$

For the estimate (2.13) we use the identity

$$\begin{aligned} \tilde{S}(t, s)\tilde{P}_s(s) - S(t, s)P_s(s) &= P_s(t)(\tilde{S}(t, s) - S(t, s))\tilde{P}_s(s) \\ &+ (\tilde{P}_s(t) - P_s(t))\tilde{S}(t, s)\tilde{P}_s(s) + P_s(t)S(t, s)(\tilde{P}_s(s) - P_s(s)). \end{aligned}$$

The last two terms can be estimated by the dichotomies and (2.12). The first term is zero at $t = s$ and satisfies a linear inhomogenous differential equation which upon integration gives

$$P_s(t)(\tilde{S}(t, s) - S(t, s))\tilde{P}_s(s) = \int_s^t S(t, \sigma)P_s(\sigma)B(\sigma)\tilde{S}(\sigma, s)\tilde{P}_s(s)d\sigma.$$

Again the dichotomies and (2.11) yield the desired result. The proof of (2.14) proceeds in an analogous way. Finally, we notice that the shifted exponential dichotomy as well as the estimates (2.12)–(2.14) easily carry over from J_1 to J . ■

Unlike the case of exponential trichotomies ([HaLi 86], Lemma 4.3) the exact exponential behaviour of the center part in an ordinary exponential trichotomy is generally not preserved under perturbation. The following lemma describes a specialized situation where this is the case.

LEMMA 2.5.

Let L have an ordinary exponential trichotomy on $J = [0, \infty)$ with exponents $\alpha < \nu < \beta$ and projectors $P_\kappa(t)$, $\kappa = s, c, u$. Further assume that $P_c(t)$ is of rank $m_c = 1$ and has the form $P_c(t) = z(t)\psi(t)^T$ where $Lz = 0$ and

$$(2.15) \quad C_1 e^{\nu t} \leq \|z(t)\| \leq C_2 e^{\nu t}, t \in J, \quad C_1, C_2 > 0$$

Let $B \in C(J, \mathbb{R}^{m, m})$ satisfy (2.11) for some $\varepsilon < \text{Min}(\nu - \alpha, \beta - \nu)$, and assume that $\tilde{L}x = Lx - B(t)x = 0$ has a solution \tilde{z} which satisfies

$$(2.16) \quad \|z(t) - \tilde{z}(t)\| \leq C e^{(\nu - \varepsilon)t}, t \in J.$$

Then \tilde{L} has an ordinary exponential trichotomy on J with exponents $\tilde{\alpha} < \nu < \tilde{\beta}$ where $\tilde{\alpha}, \tilde{\beta}$ may be chosen arbitrarily close to α, β . Moreover, the estimates (2.12) for $\kappa = s, c, u$ and (2.13), (2.14) hold and $\tilde{P}_c(t)$ can be represented as $\tilde{P}_c(t) = \tilde{z}(t)\tilde{\psi}(t)^T$ for some $\tilde{\psi}$ with $\tilde{L}^*\tilde{\psi} = 0$ and

$$(2.17) \quad \|\psi(t) - \tilde{\psi}(t)\| \leq C e^{(\nu - \varepsilon)t}, t \in J.$$

PROOF.

In a first step we apply Prop. 2.4 to the two shifted exponential dichotomies of L with exponents $\alpha < \nu$, $\nu < \beta$ and projectors $Q_s = P_s$, $Q_u = P_c + P_u$ and $R_s = P_s + P_c$, $R_u = P_u$ respectively. Let $\tilde{Q}_\kappa, \tilde{R}_\kappa$ be the corresponding projectors

for \tilde{L} . We define $\tilde{P}_s(t) = \tilde{Q}_s(t)$, $\tilde{P}_u(t) = \tilde{R}_u(t)$ so that (2.13), (2.14) and (2.12) for $\kappa = s, u$ have been proved. Moreover, the range of $\tilde{R}_s(0)$ is uniquely determined and we have $R(\tilde{Q}_s(0)) \subset R(\tilde{R}_s(0))$ with the codimension being $m_c = 1$.

Next we show $\tilde{z}(0) \notin R(\tilde{Q}_s(0))$. If this is false then

$$\begin{aligned} \|z(0)\| &= \|S(0, t)P_c(t)z(t)\| \leq Ke^{-\nu t}(\|z(t) - \tilde{z}(t)\| + \|\tilde{S}(t, 0)\tilde{z}(0)\|) \\ &\leq Ke^{-\nu t}(Ce^{(\nu-\varepsilon)t} + \tilde{K}e^{\tilde{\alpha}t}\|\tilde{z}(0)\|) \rightarrow 0 \text{ as } t \rightarrow \infty \end{aligned}$$

and we arrive at a contradiction.

Thus we have the decomposition

$$\mathbb{R}^m = R(\tilde{Q}_s(0)) \oplus \text{span}\{\tilde{z}(0)\} \oplus R(\tilde{R}_u(0))$$

and we let $\tilde{P}_c(0)$ be the projector onto $\text{span}\{\tilde{z}(0)\}$. If we write $\tilde{P}_c(0) = \tilde{z}(0)\tilde{\psi}(0)^T$ where $\tilde{\psi}(0)^T\tilde{z}(0) = 1$ and let $\tilde{\psi}(t) = \tilde{S}(0, t)^T\tilde{\psi}(0)$ then $\tilde{P}_c(t) + \tilde{P}_u(t) + \tilde{P}_s(t) = I$ holds for $\tilde{P}_c(t) = \tilde{z}(t)\tilde{\psi}(t)^T$ (see (2.4)). Therefore the estimate (2.12) is also valid for $\kappa = c$.

Finally, we notice that (e. g. in Euclidean norm)

$$\|P_c(t)\| = \|z(t)\psi(t)^T\| = \|z(t)\| \|\psi(t)\|$$

so that the estimates in (2.2) are equivalent to

$$(2.18) \quad \|z(t)e^{-\nu t}\|, \|\psi(s)e^{\nu s}\| \leq C \text{ for } t, s \in J.$$

From (2.15) and (2.16) we then find $\|e^{-\nu t}\tilde{z}(t)\| \geq c > 0$ and hence by (2.12), (2.16)

$$\begin{aligned} C\|\psi(t) - \tilde{\psi}(t)\| &\leq \|e^{-\nu t}\tilde{z}(t)(\psi(t) - \tilde{\psi}(t))^T\| \\ &= \|e^{-\nu t}(P_c(t) - \tilde{P}_c(t)) + e^{-\nu t}(\tilde{z}(t) - z(t))\psi(t)^T\| \\ &\leq Ce^{(-\nu-\varepsilon)t}. \quad \blacksquare \end{aligned}$$

We apply Lemma 2.5 to the periodic case. Assume that $A(t)$, $t \in \mathbb{R}$ is 1-periodic and that $Lx = \dot{x} - A(t)x$ has the simple Floquet multiplier 1 and no further multipliers on the unit circle. By classical Floquet theory we have a fundamental matrix of the form

$$(Z_s(t) \ z(t) \ Z_u(t)) \exp \left(t \begin{pmatrix} B_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_u \end{pmatrix} \right)$$

where Z_s, z, Z_u are 1-periodic and where the Floquet exponents, given by the spectra of B_s, B_u , satisfy for some $\alpha < \beta$

$$(2.19) \quad \text{Re } \sigma(B_s) < \alpha < 0 < \beta < \text{Re } \sigma(B_u).$$

Setting $(\Psi_s(t) \ \psi(t) \ \Psi_u(t)) = (Z_s(t) \ z(t) \ Z_u(t))^{T-1}$ we find for the solution operator

$$(2.20) \quad S(t, s) = Z_s(t)e^{(t-s)B_s}\Psi_s(s)^T + z(t)\psi(s)^T + Z_u(t)e^{(t-s)B_u}\Psi_u(s)^T.$$

Therefore L has an ordinary exponential trichotomy on any interval $J \subset \mathbb{R}$ with exponents $\alpha < \nu = 0 < \beta$ and with projectors

$$P_\kappa(t) = Z_\kappa(t)\Psi_\kappa(t)^T, \ \kappa = s, u, \ P_c(t) = z(t)\psi(t)^T.$$

We notice that the projectors P_s, P_u are always real operators but the matrices $Z_\kappa, \Psi_\kappa, \kappa = s, u$ may be complex in general.

Lemma 2.5 then yields the following

COROLLARY 2.6.

Let $Lx = \dot{x} - A(t)x$ be as above and assume that $\tilde{L}x = \dot{x} - \tilde{A}(t)x$, $\tilde{A} \in C([0, \infty), \mathbb{R}^{m,m})$ satisfies

$$(2.21) \quad \|A(t) - \tilde{A}(t)\| \leq Ce^{-\epsilon t} \text{ for some } 0 < \epsilon < \text{Min}(-\alpha, \beta)$$

and that there exists a solution \tilde{z} of $\tilde{L}\tilde{z} = 0$ such that

$$(2.22) \quad \|z(t) - \tilde{z}(t)\| \leq Ce^{-\epsilon t}, \ t \geq 0.$$

Then \tilde{L} has an ordinary exponential trichotomy on $[0, \infty)$ with exponents $\tilde{\alpha} < 0 < \tilde{\beta}$ arbitrarily close to α and β . The projectors $\tilde{P}_s, \tilde{P}_c, \tilde{P}_u$ satisfy the estimates (2.12) — (2.14) and we have $\tilde{P}_c(t) = \tilde{z}(t)\tilde{\psi}(t)^T$ where $\tilde{L}^*\tilde{\psi} = 0$ and

$$(2.23) \quad \|\psi(t) - \tilde{\psi}(t)\| \leq Ce^{-\epsilon t} \text{ for } t \geq 0.$$

In the situation of this corollary we consider the differential operator $\Gamma : Z_1^+(\epsilon) \rightarrow Z_0^+(\epsilon)$ defined by

$$\Gamma(x, y) := (\tilde{L}x, Ly)$$

with spaces and norms given by

$$Z_0^+(\epsilon) = \{(x, y) : x \in C([0, \infty), \mathbb{R}^m) \text{ bounded, } y \in C([0, \infty), \mathbb{R}^m) \text{ 1-periodic} \\ \|x(t) - y(t)\| \leq Ce^{-\epsilon t} \text{ for } t \geq 0\},$$

$$\|(x, y)\|_{Z_0^+} := \sup_{t \geq 0} \|x(t)\| + \sup_{t \geq 0} \|y(t)\| + \sup_{t \geq 0} e^{\epsilon t} \|x(t) - y(t)\|,$$

$$Z_1^+(\epsilon) = \{(x, y) \in Z_0^+(\epsilon) : (\dot{x}, \dot{y}) \in Z_1^+(\epsilon)\}$$

$$\|(x, y)\|_{Z_1^+} := \|(x, y)\|_{Z_0^+} + \|(\dot{x}, \dot{y})\|_{Z_0^+}.$$

PROPOSITION 2.7.

Under the assumptions of Corollary 2.6 the operator

$$\Gamma : Z_1^+(\epsilon) \rightarrow Z_0^+(\epsilon), \quad \Gamma(x, y) = (\tilde{L}x, Ly)$$

is Fredholm of index m_{+s} , which is the number of stable Floquets multipliers of L . Moreover, for the range and nullspace we have

$$(2.24) \quad R(\Gamma) = \{(r, \rho) \in Z_0^+(\epsilon) : \int_0^1 \psi(t)^T \rho(t) dt = 0\}$$

$$(2.25) \quad N(\Gamma) = \text{span} \{(\tilde{z}, z), (\tilde{S}(t, 0)\xi, 0) \text{ where } \xi \in R(\tilde{P}_s(0))\}.$$

REMARK.

In case $(r, \rho) \in R(\Gamma)$ we give a representation of a special solution (\hat{x}, \hat{y}) of $\Gamma(x, y) = (r, \rho)$ which will be used later on.

PROOF.

It is enough to show (2.24), (2.25) since this implies $\text{codim } R(\Gamma) = 1$ and $\dim N(\Gamma) = 1 + m_{+s}$.

For $(r, \rho) = (\tilde{L}x, Ly) \in R(\Gamma)$ we obtain in the standard way

$$\int_0^1 \psi(t)^T \rho(t) dt = \int_0^1 \psi(t)^T Ly(t) dt = - \int_0^1 (L^* \psi(t))^T y(t) dt + [\psi(t)^T y(t)]_0^1 = 0.$$

Now suppose that $(r, \rho) \in Z_0^+(\epsilon)$ and $\int_0^1 \psi(t)^T \rho(t) dt = 0$ hold. Then there exists a 1-periodic solution \hat{y} of $Ly = \rho$. For example, we may define \hat{y} by

$$(2.26) \quad P_s(t)\hat{y}(t) = S(t, 0)\xi + \int_0^t S(t, s)P_s(s)\rho(s) ds$$

$$(2.27) \quad P_c(t)\hat{y}(t) = z(t) \int_0^t \psi(s)^T \rho(s) ds$$

$$(2.28) \quad P_u(t)\hat{y}(t) = - \int_t^\infty S(t, s)P_u(s)\rho(s) ds$$

where ξ is the unique solution in $R(P_s(0))$ of the linear system

$$(2.29) \quad (I - S(1, 0))\xi = \int_0^1 S(1, s)P_s(s)\rho(s)ds = P_s(0) \int_0^1 S(1, s)\rho(s)ds.$$

Equation (2.29) guarantees that $P_s(t)\hat{y}(t)$ is 1-periodic. For $P_c(t)\hat{y}(t)$ the periodicity follows from our assumption while for $P_u(t)\hat{y}(t)$ it is a consequence of the relation

$$S(t + 1, s + 1) = S(t, s).$$

For the solution \hat{x} of $\tilde{L}x = r$ we use an analogous set of formulae

$$(2.30) \quad \tilde{P}_s(t)\hat{x}(t) = \int_0^t \tilde{S}(t, s)\tilde{P}_s(s)r(s)ds$$

$$(2.31) \quad \tilde{P}_c(t)\hat{x}(t) = \omega\tilde{z}(t) + \tilde{z}(t) \int_0^t \tilde{\psi}(s)^T r(s)ds$$

$$(2.32) \quad \tilde{P}_u(t)\hat{x}(t) = - \int_t^\infty \tilde{S}(t, s)\tilde{P}_u(s)r(s)ds,$$

where ω will be determined so that

$$\hat{x}(t) - \hat{y}(t) = O(e^{-\epsilon t}).$$

For the center part we find

$$\begin{aligned} P_c(t)\hat{y}(t) - \tilde{P}_c(t)\hat{x}(t) &= (z(t) - \tilde{z}(t)) \int_0^t \psi(s)^T \rho(s)ds \\ &+ \tilde{z}(t) \left[\int_0^t (\psi(s)^T \rho(s) - \tilde{\psi}(s)^T r(s))ds - \omega \right]. \end{aligned}$$

The first term behaves like $O(e^{-\epsilon t})$ by (2.22) and our assumption and so does the second if we set

$$(2.33) \quad \omega = \int_0^\infty (\psi(s)^T \rho(s) - \tilde{\psi}(s)^T r(s))ds.$$

Notice that (2.33) and $r(t) - \rho(t) = O(e^{-\epsilon t})$ yield

$$\psi(s)^T \rho(s) - \tilde{\psi}(s)^T r(s) = (\psi(s) - \tilde{\psi}(s))^T \rho(s) + \tilde{\psi}(s)^T (\rho(s) - r(s)) = O(e^{-\epsilon s})$$

and therefore

$$|\omega - \int_0^t (\psi(s)^T \rho(s) - \tilde{\psi}(s)^T r(s)) ds| \leq C \int_t^\infty e^{-\epsilon s} ds = Ce^{-\epsilon t}.$$

Thus we have shown

$$P_c(t)\hat{y}(t) - \tilde{P}_c(t)\hat{x}(t) = O(e^{-\epsilon t}).$$

In particular $\tilde{P}_c\hat{x}$ is bounded and by the exponential trichotomy this is also true for $\tilde{P}_s\hat{x}$ and $\tilde{P}_u\hat{x}$. Hence \hat{x} is a bounded solution of $\tilde{L}x = r$.

Furthermore, by using (2.13) we obtain

$$\begin{aligned} \|P_s(t)\hat{y}(t) - \tilde{P}_s(t)\hat{x}(t)\| &\leq \|S(t,0)\xi\| + \\ &\int_0^t \|S(t,s)P_s(s)\| \|\rho(s) - r(s)\| + \|S(t,s)P_s(s) - \tilde{S}(t,s)\tilde{P}_s(s)\| \|r(s)\| ds \\ &\leq C\{e^{\alpha t} + \int_0^t (e^{-\epsilon s + \alpha(t-s)} + e^{-\epsilon t + \tilde{\alpha}(t-s)}) ds\} \leq Ce^{-\epsilon t}. \end{aligned}$$

In a similar way we use (2.14) to show

$$\|P_u(t)\hat{y}(t) - \tilde{P}_u(t)\hat{x}(t)\| \leq Ce^{-\epsilon t}.$$

For the proof of (2.25) we first notice that

$$(\tilde{z}, z), (\tilde{S}(t,0)\xi, 0) \in N(\Gamma) \text{ for } \xi \in R(\tilde{P}_s(0))$$

is obvious. Suppose that $(x, y) \in N(\Gamma)$, then $Ly = 0$ and $y = cz$ for some $c \in \mathbb{R}$ follows. But the trichotomy of \tilde{L} implies

$$x(t) = \tilde{S}(t,0)\xi + \beta\tilde{z} \text{ for some } \xi \in R(\tilde{P}_s(0)), \beta \in \mathbb{R}$$

and

$$(c - \beta)z(t) = y(t) - x(t) + \beta(\tilde{z}(t) - z(t)) + \tilde{S}(t,0)\xi = O(e^{-\epsilon t}).$$

Hence $c = \beta$ and we have the form

$$(x, y) = c(\tilde{z}, z) + (\tilde{S}(t,0)\xi, 0).$$

■

Lemma 2.7 will be used for the linearization of initial value problems.

The corresponding result for the boundary value case which uses the spaces $Z_1(\epsilon)$, $Z_0(\epsilon)$ from the introduction is given in the following

PROPOSITION 2.8.

Let the differential operators $Lx = \dot{x} - A(t)x$ and $\tilde{L}x = \dot{x} - \tilde{A}(t)x$, $t \in \mathbb{R}$ satisfy the assumptions of Corollary 2.6. Further assume that $\tilde{z}(t)$ is bounded for $t \leq 0$ and

$$(2.34) \quad \tilde{A}(t) \rightarrow A_- \text{ as } t \rightarrow -\infty$$

where A_- is hyperbolic with stable subspace of dimension m_{-s} and unstable subspace of dimension $m_{-u} = m - m_{-s}$.

Then the operator

$$\Gamma : Z_1(\epsilon) \rightarrow Z_0(\epsilon), \Gamma(x, y) = (\tilde{L}x, Ly)$$

is Fredholm of index $m_{-u} - m_{+u} - 1$ where m_{+u} is the number of unstable Floquet multipliers for L .

The operator \tilde{L} has an exponential dichotomy on \mathbb{R}_- with projectors $Q_s(t)$, $Q_u(t)$ and

$$(2.35) \quad N(\Gamma) = \{(\tilde{S}(t, 0)\xi, 0) + c(\tilde{z}(t), z(t)) : c \in \mathbb{R}, \xi \in R(\tilde{P}_s(0)) \cap R(Q_u(0))\}.$$

Moreover, with $\Gamma^*(x, y) = (\tilde{L}^*x, L^*y)$ we have that $(r, \rho) \in Z_0(\epsilon)$ is in $R(\Gamma)$ if and only if the following two conditions are satisfied

$$(2.36) \quad \int_0^1 \varphi(t)^T \rho(t) dt = 0 \text{ for all } \varphi \in N(L^*)$$

$$(2.37) \quad \int_{-\infty}^0 \tilde{\varphi}(t)^T r(t) dt + \int_0^{\infty} (\tilde{\varphi}(t)^T r(t) - \varphi(t)^T \rho(t)) dt = 0 \text{ for all } (\tilde{\varphi}, \varphi) \in N(\Gamma^*).$$

PROOF.

The exponential dichotomy on \mathbb{R}_- with projectors Q_s, Q_u of rank m_{-s} and m_{-u} follows from (2.34) and the roughness theorem. Since \tilde{z} is a bounded solution of $\tilde{L}x = 0$ on \mathbb{R} the representation (2.35) is easily obtained as in the previous proposition.

Now suppose that $(r, \rho) = \Gamma(x, y) \in R(\Gamma)$. Then (2.36) is clear and we consider some $(\tilde{\varphi}, \varphi) \in N(\Gamma^*)$. It follows that

$$\begin{aligned} \tilde{\varphi}(0) &\in R(\tilde{P}_s^*(0) + \tilde{P}_c^*(0)) \cap R(Q_u^*(0)) \\ &= R(\tilde{P}_u(0)^T + \tilde{P}_c(0)^T) \cap R(Q_s(0)^T) \\ &= (R(\tilde{P}_u(0)^T) \oplus \text{span} \{\tilde{\psi}(0)\}) \cap R(Q_s(0)^T). \end{aligned}$$

From $\varphi \in N(L^*)$ we obtain $\varphi = c\psi$ for some $c \in \mathbb{R}$ and hence by (2.23)

$$(\tilde{\varphi} - c\tilde{\psi})(t) = (\tilde{\varphi} - \varphi)(t) + c(\psi - \tilde{\psi})(t) = O(e^{-\epsilon t}).$$

Integration by parts then yields

$$\begin{aligned} \int_{-\infty}^0 \tilde{\varphi}(t)^T r(t) dt &= \tilde{\varphi}(0)^T x(0) = (\tilde{\varphi}(0) - c\tilde{\psi}(0))^T x(0) + c\tilde{\psi}(0)^T x(0) \\ &= - \int_0^{\infty} (\tilde{\varphi}(t) - c\tilde{\psi}(t))^T r(t) dt + c\tilde{\psi}(0)^T x(0). \end{aligned}$$

By Proposition 2.7 we may assume (see (2.26)–(2.32))

$$y = \hat{y}, \quad x(t) = \hat{x}(t) + \tilde{S}(t, 0)\xi \quad \text{for some } \xi \in R(\tilde{P}_s(0)) \text{ and } t \geq 0.$$

From (2.31) and (2.33) we then find

$$\tilde{\psi}(0)^T x(0) = \omega \tilde{\psi}(0)^T \tilde{z}(0) = \omega$$

and hence the assertion (2.37)

$$\begin{aligned} \int_{-\infty}^0 \tilde{\varphi}(t)r(t)dt &= - \int_0^{\infty} (\tilde{\varphi}(t) - c\tilde{\Psi}(t))^T r(t)dt + c \int_0^{\infty} (\Psi(s)^T \rho(s) - \tilde{\Psi}(s)^T r(s))ds \\ &= - \int_0^{\infty} (\tilde{\varphi}(t)^T r(t) - \varphi(t)^T \rho(t))dt. \end{aligned}$$

For the converse statement let us assume that $(r, \rho) \in Z_0^+(\epsilon)$ satisfies (2.36), (2.37). We seek a solution (x, y) of $\Gamma(x, y) = (r, \rho)$ in the form

$$y = \hat{y}, \quad x(t) = \begin{cases} \hat{x}(t) + \tilde{S}(t, 0)Q_u(0)\xi, & t < 0 \\ \hat{x}(t) + \tilde{S}(t, 0)\tilde{P}_s(0)\xi, & t \geq 0 \end{cases}$$

where \hat{y} is given by (2.26)–(2.29) and $\hat{x}(t)$ is defined for $t \geq 0$ through (2.30)–(2.33). For $t < 0$ we set

$$\hat{x}(t) = \int_{-\infty}^t \tilde{S}(t, s)Q_s(s)r(s)ds - \int_t^0 \tilde{S}(t, s)Q_u(s)r(s)ds$$

and ξ will be determined so that x is continuous at 0, i.e.

$$(2.38) \quad (Q_u(0) - \tilde{P}_s(0))\xi = \hat{x}(0+) - \hat{x}(0-).$$

This equation has a solution if for any η satisfying $\eta^T(Q_u(0) - \tilde{P}_s(0)) = 0$ we can show $\eta^T(\hat{x}(0+) - \hat{x}(0-)) = 0$.

Let $c = \eta^T \tilde{z}(0)$ and define

$$\hat{\psi}(t) = \begin{cases} \tilde{S}^*(t, 0)Q_u^*(0)\eta & \text{for } t < 0 \\ \tilde{S}^*(t, 0)\tilde{P}_s^*(0)\eta + c\tilde{\psi}(t) & \text{for } t \geq 0 \end{cases}$$

$\widehat{\psi}$ is in fact continuous at 0 since

$$\begin{aligned}\widehat{\psi}(0+)^T - \widehat{\psi}(0-)^T &= \eta^T(\widetilde{P}_u(0) - Q_s(0)) + c\widetilde{\psi}(0)^T \\ &= \eta^T(I - \widetilde{P}_s(0) - \widetilde{P}_c(0) - (I - Q_u(0) + \widetilde{z}(0))\widetilde{\psi}(0)^T) = 0.\end{aligned}$$

Therefore, $\widehat{\psi}$ is a bounded solution of $\widetilde{L}^*\widehat{\psi} = 0$ and $(\widehat{\psi}, c\psi) \in N(\Gamma^*)$. An application of (2.37) yields

$$\begin{aligned}0 &= \int_{-\infty}^0 \widehat{\psi}(t)^T r(t) dt + \int_0^{\infty} \widetilde{\psi}(t)^T r(t) - c\psi(t)^T \rho(t) dt \\ &= \eta^T \left[\int_0^{\infty} \widetilde{z}(0)\widetilde{\psi}(t)^T r(t) + \widetilde{P}_u(0)\widetilde{S}(0, t)r(t) - \widetilde{z}(0)\psi(t)^T \rho(t) dt + \right. \\ &\quad \left. \int_{-\infty}^0 Q_s(0)\widetilde{S}(0, t)r(t) dt \right] \\ &= \eta^T(\widehat{x}(0-) - \widehat{x}(0+)).\end{aligned}$$

Let us finally compute the Fredholm index of Γ . From the ordinary exponential trichotomy and (2.24) we get

$$\dim N(\Gamma) = \dim (R(\widetilde{P}_s(0) + \widetilde{P}_c(0)) \cap R(Q_u(0))).$$

Since $\widetilde{z}(0) \in R(\widetilde{P}_c(0)) \cap R(Q_u(0))$ by our assumption we find

$$\dim N(\Gamma) = \dim (V \cap W) + 1 \quad \text{where } V = R(\widetilde{P}_s(0)), \quad W = R(Q_u(0)).$$

Similarly

$$\dim N(\Gamma^*) = \dim (R(\widetilde{P}_u(0)^T + \widetilde{P}_c(0)^T) \cap R(Q_s(0)^T)) = \dim (V^\perp \cap W^\perp).$$

The formulae (2.36), (2.37) show that $R(\Gamma)$ is the null space of a space of linear functionals which has dimension $\dim N(\Gamma^*) + 1$.

Then we conclude as in [Pal 84]

$$\begin{aligned}\text{ind}(\Gamma) &= \dim(V \cap W) - \dim(V^\perp \cap W^\perp) \\ &= \dim(V \cap W) - \dim((V + W)^\perp) \\ &= \dim(V \cap W) - m + \dim(V + W) \\ &= \dim V + \dim W - m = m_{+s} + m_{-u} - m = m_{-u} - m_{+u} - 1.\end{aligned}$$

■

3. Characterizations of well-posedness

Throughout this section we assume that we have a \bar{T} -periodic hyperbolic orbit

$$\bar{\gamma} = \{\hat{y}(t) : t \in \mathbb{R}\}$$

of the system (1.1) at some $\lambda = \bar{\lambda}$. Let m_{+s} and m_{+u} denote the number of its stable and unstable Floquet multipliers respectively.

In our first step we repeat the construction of the local stable manifold and its foliation induced by the asymptotic phase by using the results of section 2 and the implicit function theorem. This approach is similar to [Irw 80, Ch. 6, II] where the time \bar{T} -map is employed but different from the more standard approach via the Poincaré map (cf. [Har 64, Ch. IX], [Hal 69, Ch. VI]) and we take some care in relating these two approaches.

The function $\bar{y}(t) = \hat{y}(t \bar{T})$ is a 1-periodic solution of

$$(3.1) \quad \dot{x} = \bar{T} f(x, \bar{\lambda})$$

and the linear operator

$$L = \frac{d}{dt} - A(t), \quad A(t) = \bar{T} \frac{\partial f}{\partial x}(\bar{y}(t), \bar{\lambda})$$

has an ordinary exponential trichotomy with projectors $P_\kappa(t)$, $\kappa = s, c, u$ and solution operator $S(t, s)$.

Let C_1^k ($k \geq 0$) denote the 1-periodic C^k -functions from \mathbb{R} to \mathbb{R}^m .

We can continue $(\bar{y}(\cdot), \bar{T})$ to $(y(\cdot, \lambda), T(\lambda)) \in C_1^1 \times \mathbb{R}$, λ in some neighbourhood $U(\bar{\lambda})$, by an application of the implicit function theorem to the equation

$$(3.2) \quad F_1(y, T, \lambda) = \begin{pmatrix} \dot{y} - T f(y, \lambda) \\ \chi(y) \end{pmatrix} = 0.$$

Here F_1 maps $C_1^1 \times \mathbb{R}^{p+1}$ into $C_1^0 \times \mathbb{R}$ and $\chi : C_1^1 \rightarrow \mathbb{R}$ is a C^1 -phase condition satisfying

$$(3.3) \quad \chi(\bar{y}) = 0, \quad \chi'(\bar{y}) \dot{\bar{y}} \neq 0.$$

Let $\varphi^t(\cdot, \lambda)$ denote the t -flow of the scaled system

$$(3.4) \quad \dot{x} = T(\lambda) f(x, \lambda)$$

and let $\Phi^t(x, \lambda) = (\varphi^t(x, \lambda), \lambda)$ denote the t -flow in \mathbb{R}^{m+p} obtained by adding $\dot{\lambda} = 0$ (see (1.2)).

For a suitable neighbourhood \bar{V} of $\bar{\gamma}$ the local stable set of $\bar{\gamma}$

$$M_{+s}(\bar{V}, \bar{\gamma}) = \{x \in \bar{V} : \varphi^t(x, \bar{\lambda}) \in \bar{V} \\ \text{for } t \geq 0 \text{ and } \text{dist}(\varphi^t(x, \bar{\lambda}), \bar{\gamma}) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

is known to be an $(m_{+s} + 1)$ -dimensional smooth manifold (cf. [Hal 69, Ch. VI]). Then the local stable manifolds of the periodic orbits $\gamma(\lambda) = \{y(t, \lambda) : t \in \mathbb{R}\}$, $\lambda \in U(\bar{\lambda})$ can be put together to form an $(m_{+s} + 1 + p)$ -dimensional manifold

$$M_{+s}(V) = \{(x, \lambda) \in V = \bar{V} \times U(\bar{\lambda}) : \varphi^t(x, \lambda) \in \bar{V} \text{ for } t \geq 0 \text{ and} \\ \text{dist}(\varphi^t(x, \lambda), \gamma(\lambda)) \rightarrow \infty \text{ as } t \rightarrow \infty\}$$

after possibly adjusting \bar{V} and $U(\bar{\lambda})$.

Usually, the differentiable structure on $M_{+s}(V)$ is defined via the Poincaré map P with respect to a transversal section

$$\Sigma = \{\bar{y}(0) + \eta : \eta \in U(0) \subset R(P_s(0) + P_u(0))\}.$$

We have a representation

$$M_{+s}(V) \cap (\Sigma \times U(\bar{\lambda})) = \{(\bar{y}(0) + \xi + h(\xi, \lambda), \lambda) : \xi \in U(0) \subset R(P_s(0)), \lambda \in U(\bar{\lambda})\}$$

where h is a smooth mapping into $R(P_u(0))$ which satisfies $h(0, \bar{\lambda}) = 0$.

The charts on $M_{+s}(V)$ are then given by the inverses of the local parametrizations

$$(3.5) \quad \pi(t, \xi, \lambda) = \Phi^t(\bar{y}(0) + \xi + h(\xi, \lambda), \lambda)$$

where $\xi \in U(0) \subset R(P_s(0))$, $\lambda \in U(\bar{\lambda})$ and t is in some open interval $J \subset \mathbb{R}_+$ of length less than 1.

THEOREM 3.1.

Let $\bar{\gamma}$ be a hyperbolic periodic orbit as above and let $\epsilon < \text{Min}(-\alpha, \beta)$ where α, β are bounds on the real parts of the Floquet exponents as in (2.19).

Then, for $\xi \in R(P_s(0))$ and $\lambda - \bar{\lambda}$ sufficiently small the operator equation

$$(3.6) \quad F_+(x, y, T, \xi, \lambda) = \begin{pmatrix} \dot{x} - T f(x, \lambda) \\ \dot{y} - T f(y, \lambda) \\ P_s(0)(x(0) - \bar{y}(0)) - \xi \\ \chi(y) \end{pmatrix} = 0$$

has a unique solution $(x_+(\cdot, \xi, \lambda), y(\cdot, \lambda), T(\lambda))$ in some neighbourhood of $(\bar{y}(\cdot), \bar{y}(\cdot), \bar{T})$ in $Z_1^+(\epsilon) \times \mathbb{R}$ satisfying

$$x_+(t, 0, \bar{\lambda}) = \bar{y}(t).$$

The local inverses of the mappings

$$(3.7) \quad \beta(t, \xi, \lambda) = (x_+(t, \xi, \lambda), \lambda), \quad t \in U(0), \quad \xi \in U(0) \subset R(P_s(0)), \quad \lambda \in U(\bar{\lambda})$$

are admissible charts of the local stable manifold $M_{+s}(V)$.

PROOF.

Clearly, $F_+(\bar{y}, \bar{y}, \bar{T}, 0, \bar{\lambda}) = 0$ and

$$F_+ : Z_1^+(\epsilon) \times \mathbb{R} \times R(P_s(0)) \times \mathbb{R}^p \rightarrow Z_0^+(\epsilon) \times R(P_s(0)) \times \mathbb{R}$$

is a smooth operator with

$$(3.8) \quad K := \frac{\partial F_+}{\partial(x, y, T)}(\bar{y}, \bar{y}, \bar{T}, 0, \bar{\lambda}) = \begin{bmatrix} \Gamma & -f(\bar{y}, \bar{\lambda}) \\ P_s(0)E_0 & 0 \\ 0 & \chi'(\bar{y}) \end{bmatrix},$$

where $\Gamma(x, y) = (Lx, Ly)$ and $E_0x = x(0)$. By Proposition 2.7 Γ has Fredholm index m_{+s} and hence by the bordering lemma [Bey 90b] K has Fredholm index

$$\text{ind}(K) = \text{ind}(\Gamma) + 1 - (m_{+s} + 1) = 0.$$

From (2.24) we find

$$\begin{pmatrix} f(\bar{y}, \bar{\lambda}) \\ f(\bar{y}, \bar{\lambda}) \end{pmatrix} = \begin{pmatrix} \dot{\bar{y}} \\ \dot{\bar{y}} \end{pmatrix} \notin R(\Gamma)$$

and moreover from (2.25) and (3.3)

$$\begin{pmatrix} P_s(0)E_0 & 0 \\ 0 & \chi'(\bar{y}) \end{pmatrix} : N(\Gamma) \rightarrow R(P_s(0)) \times \mathbb{R}$$

is nonsingular. Hence K is a linear homeomorphism and the implicit function theorem applies to (3.6). Since (3.2) has locally unique solutions the solutions of (3.6) are of the form

$$(x_+(\cdot, \xi, \lambda), y(\cdot, \lambda), \bar{T}(\lambda)) \text{ with } \xi \in U(0) \subset R(P_s(0)), \lambda \in U(\bar{\lambda}).$$

By the construction of x_+ and by making $U(0), U(\bar{\lambda})$ sufficiently small we may assume that $\beta(t, \xi, \lambda) = (x_+(t, \xi, \lambda), \lambda) \in M_{+s}(V)$ holds for all $t \geq 0$.

Moreover, by implicit differentiation we obtain

$$(3.9) \quad \frac{\partial x_+}{\partial \xi}(t, 0, \bar{\lambda}) = S(t, 0)P_s(0)$$

and hence

$$(3.10) \quad \frac{\partial \beta}{\partial(t, \xi, \lambda)}(0, 0, \bar{\lambda}) = \begin{pmatrix} \dot{\bar{y}}(0) & I_{|R(P_s(0))} & \frac{\partial \beta}{\partial \lambda}(0, 0, \bar{\lambda}) \\ 0 & 0 & I_p \end{pmatrix}$$

has rull rank $m_{+s} + 1 + p$ and β is a local immersion.

Using the rank theorem (e.g. [Die 60, Ch. X]) we may write

$$\beta = \sigma_2 \circ E \circ \sigma_1$$

where σ_1 is a diffeomorphism from a neighbourhood $\tilde{U} \subset \mathbb{R} \times R(P_s(0)) \times \mathbb{R}^p$ of $(0, 0, \bar{\lambda})$ onto the open unit ball in $\mathbb{R}^{m+s+1+p}$, σ_2 is a diffeomorphism from the open unit ball in \mathbb{R}^{m+p} onto $\beta(\tilde{U})$ and

$$E(x_1, \dots, x_{m+s+1+p}) = (x_1, \dots, x_{m+s+1+p}, 0, \dots, 0) \in \mathbb{R}^{m+p}.$$

Therefore, we have a smooth representation of $\beta^{-1}\pi$ on the common domain of existence as

$$\beta^{-1}\pi(t, \xi, \lambda) = \sigma_2^{-1} \circ E^T \circ \sigma_1^{-1} \circ \pi(t, \xi, \lambda).$$

Furthermore, let $\tau(x, \lambda)$ be the unique return time in $-J$ for points of the form (3.5), then we find the smooth representation

$$\pi^{-1}\beta(t, \xi, \lambda) = (\tau \circ \beta(t, \xi, \lambda), P_s(0)(\varphi^{\tau \circ \beta(t, \xi, \lambda)}(\beta(t, \xi, \lambda)) - \bar{y}(0)), \lambda).$$

Thus β^{-1} is an admissible chart of $M_{+s}(V)$ near $(\bar{y}(0), \bar{\lambda})$. ■

REMARK.

We used the implicit function theorem to construct the foliation of the stable manifold by asymptotic phase. This has some computational advantages. E.g. we can compute derivatives of

$$g(\xi) = x_+(0, \xi, \bar{\lambda}), \quad g(0) = \bar{y}(0)$$

which parametrizes the fiber which is in asymptotic phase with $\bar{y}(0)$. Taking implicit derivatives in (3.6) and using (2.26)–(2.31) one finds

$$\begin{aligned} g'(0) &= I_{|R(P_s(0))}, \quad P_s(0)g''(0) = 0 \quad \text{and} \\ P_\kappa(0)g''(0)(\xi_1, \xi_2) &= -\bar{T} \int_0^\infty P_\kappa(0)S(0, t)f_{xx}(\bar{y}(t), \bar{\lambda})(S(t, 0)\xi_1, S(t, 0)\xi_2) dt \\ &= - \int_0^\infty \hat{P}_\kappa(0)\hat{S}(0, t)f_{xx}(\hat{y}(t), \bar{\lambda})(\hat{S}(t, 0)\xi_1, \hat{S}(t, 0)\xi_2) dt \end{aligned}$$

for $\kappa = c, u$ and $\xi_1, \xi_2 \in R(P_s(0))$.

Here \hat{S} and \hat{P}_κ denote the solution operator and projectors for the unscaled linearized problem

$$(3.11) \quad \dot{y} = \frac{\partial f}{\partial x}(\hat{y}(t), \bar{\lambda})y$$

We now consider the analogue of Theorem 3.1 for the global stable manifold.

Let $\hat{x}(t)$, $t \geq 0$ be a solution of (1.1) at $\lambda = \bar{\lambda}$ such that

$$\text{dist}(\hat{x}(t), \bar{\gamma}) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

We set $\bar{x}(t) = \hat{x}(t\bar{T})$ and find for some \bar{t} sufficiently large

$$(\bar{x}(\bar{t}), \bar{\lambda}) \in M_{+s}(V).$$

Taking V sufficiently small we can cover $M_{+s}(V)$ by finitely many charts of the type (3.7), hence for some t_0

$$(3.12) \quad \|\bar{x}(t) - \bar{y}(t + t_0)\| + \|\dot{\bar{x}}(t) - \dot{\bar{y}}(t + t_0)\| = O(e^{-\epsilon t})$$

where $\epsilon < \text{Min}(-\alpha, \beta)$ as in (2.19), (2.21).

Without loss of generality we will assume $t_0 = 0$. All the assumptions of Proposition 2.7 are then satisfied with the settings

$$(3.13) \quad z = \dot{\bar{y}}, \quad L = \frac{d}{dt} - \bar{T} \frac{\partial f}{\partial x}(\bar{y}, \bar{\lambda}), \quad \tilde{z} = \dot{\bar{x}}, \quad \tilde{L} = \frac{d}{dt} - \bar{T} \frac{\partial f}{\partial x}(\bar{x}, \bar{\lambda}).$$

In the following theorem we make use of the solution operator \tilde{S} and the projectors \tilde{P}_κ , $\kappa = s, c, u$ associated with \tilde{L} .

THEOREM 3.2.

Under the assumptions above $(\bar{x}, \bar{y}, \bar{T}, 0, \bar{\lambda}) \in Z_1^+(\epsilon) \times \mathbb{R} \times R(\tilde{P}_s(0)) \times \mathbb{R}^p$ is a solution of the operator equation

$$(3.14) \quad F_+(x, y, T, \xi, \lambda) = \begin{pmatrix} \dot{x} - Tf(x, \lambda) \\ \dot{y} - Tf(y, \lambda) \\ \tilde{P}_s(0)(x(0) - \bar{x}(0)) - \xi \\ \chi(y) \end{pmatrix} = 0.$$

This equation has a unique solution

$$(x_+(\cdot, \xi, \lambda), y(\cdot, \lambda), T(\lambda)) \in Z_1^+(\epsilon) \times \mathbb{R}$$

in some neighbourhood of $(\bar{x}, \bar{y}, \bar{T})$ which depends smoothly on (ξ, λ) in some neighbourhood of $(0, \bar{\lambda})$.

The local inverse of

$$(3.15) \quad \beta(t, \xi, \lambda) = (x_+(t, \xi, \lambda), \lambda), \quad t \in U(0), \quad \xi \in U(0) \subset R(\tilde{P}_s(0)), \quad \lambda \in U(\bar{\lambda})$$

is an admissible chart of the global stable manifold

$$(3.16) \quad M_{+s} = \{(x, \lambda) \in \mathbb{R}^m \times U(\bar{\lambda}) : \text{dist}(\varphi^t(x, \lambda), \gamma(\lambda)) \rightarrow 0 \text{ as } t \rightarrow \infty\}.$$

PROOF.

We are brief here because the implicit function theorem applies as in the proof of Theorem 3.1 with (\bar{y}, \bar{y}) replaced by (\bar{x}, \bar{y}) and P_s by \tilde{P}_s .

The analogue of (3.10) is

$$\frac{\partial \beta}{\partial(t, \xi, \lambda)}(0, 0, \bar{\lambda}) = \begin{pmatrix} \dot{\bar{x}}(0) & I_{|R(\tilde{P}_s(0))} & \frac{\partial \beta}{\partial \lambda}(0, 0, \bar{\lambda}) \\ 0 & 0 & I_p \end{pmatrix}$$

so that β is again a local immersion. The same is then true for $\tilde{\beta} = \Phi^{\bar{t}} \circ \beta$ where we take \bar{t} so large that the image of $\tilde{\beta}$ is in $M_{+s}(V)$. As in the proof Theorem 3.1 $\tilde{\beta}$ is an admissible chart of the local stable manifold. Therefore, $\beta^{-1} = \tilde{\beta}^{-1} \circ \Phi^{\bar{t}}$ is

admissible for the global stable manifold, because the differentiable structure on M_{+s} is defined by the charts of the local stable manifold transferred backwards in time by the flow. ■

Now let $(\widehat{x}(t), \bar{\lambda})$, $t \in \mathbb{R}$ be an orbit connecting a hyperbolic steady state $y_-(\bar{\lambda})$ with stability indices m_{-s} , m_{-u} to the hyperbolic periodic orbit $\bar{\gamma}$.

Again, $y_-(\bar{\lambda})$ is contained in a smooth manifold of steady states

$$\{y_-(\lambda) : \lambda \in U(\bar{\lambda})\}$$

which have the same stability indices and we can form the global unstable manifold

$$(3.17) \quad M_{-u} = \{(x, \lambda) \in \mathbb{R}^m \times U(\bar{\lambda}) : \varphi^t(x, \lambda) \rightarrow y_-(\lambda) \text{ as } t \rightarrow -\infty\}.$$

We set $\bar{x}(t) = \widehat{x}(t\bar{T})$, $t \in \mathbb{R}$ and

$$\tilde{L} = \frac{d}{dt} - \frac{\partial f}{\partial x}(\bar{x}, \bar{\lambda})$$

so that \tilde{L} has an exponential dichotomy on \mathbb{R}_- with projectors Q_s, Q_u and an ordinary exponential trichotomy on \mathbb{R}_+ with projectors P_s, P_c, P_u (see Proposition 2.8).

Similar to Theorem 3.2 we may parametrize M_{-u} near $(\bar{x}(0), \bar{\lambda})$ by

$$\beta(t, \eta, \lambda) = (x_-(0, \eta, \lambda), \lambda), \quad \eta \in U(0) \subset R(Q_u(0)), \quad \lambda \in U(\bar{\lambda}).$$

Here $x_-(\cdot, \eta, \lambda)$ is the solution of

$$(3.18) \quad F_-(x, \eta, \lambda) = \begin{pmatrix} \dot{x} - T(\lambda)f(x, \lambda) \\ Q_u(0)(x(0) - \bar{x}(0)) - \eta \end{pmatrix} = 0$$

obtained from the implicit function theorem in a neighbourhood of $x = \bar{x}|_{\mathbb{R}_-}$, $\eta = 0$, $\lambda = \bar{\lambda}$ in the space of bounded C^1 -functions (cf. [Bey 90b]).

Using $x_-(\cdot, \eta, \lambda)$ and $x_+(\cdot, \xi, \lambda)$ from Theorem 3.2 we define the $(m \times p)$ matrices

$$(3.19) \quad E_{\pm}(t) = \frac{\partial x_{\pm}}{\partial \lambda}(t, 0, \bar{\lambda}), \quad t \in \mathbb{R}_{\pm}.$$

These satisfy the variational equations

$$(3.20) \quad \dot{E}_{\pm} - \bar{T} \frac{\partial f}{\partial x}(\bar{x}, \bar{\lambda})E_{\pm} = f(\bar{x}, \bar{\lambda})T'(\bar{\lambda}) + \bar{T} \frac{\partial f}{\partial \lambda}(\bar{x}, \bar{\lambda}) \text{ in } \mathbb{R}_{\pm}$$

subject to

$$Q_u(0)E_-(0) = 0, \quad \tilde{P}_s(0)E_+(0) = 0$$

and

$$E_+(t) = \frac{\partial y}{\partial \lambda}(t, \bar{\lambda}) + O(e^{-\epsilon t}) \text{ as } t \rightarrow \infty.$$

Here $T'(\bar{\lambda})$ and $Y(\cdot) = \frac{\partial y}{\partial \lambda}(\cdot, \bar{\lambda})$, can be obtained from (3.2) by implicit differentiation

$$(3.21) \quad LY = f(\bar{y}, \bar{\lambda})T'(\bar{\lambda}) + \bar{T} \frac{\partial f}{\partial \lambda}(\bar{y}, \bar{\lambda}), \quad \chi'(\bar{y})Y = 0.$$

The tangent spaces at $\bar{z}(0) = (\bar{x}(0), \bar{\lambda})$ can now be written as

$$(3.22a) \quad T_{\bar{z}(0)}M_{-u} = \{(\eta + E_-(0)\lambda, \lambda) : \eta \in R(Q_u(0)), \lambda \in \mathbb{R}^p\}$$

$$(3.22b) \quad T_{\bar{z}(0)}M_{+s} = \{(\xi + E_+(0)\lambda + c \dot{\bar{x}}(0), \lambda) : \xi \in R(\tilde{P}_s(0)), \lambda \in \mathbb{R}^p, c \in \mathbb{R}\}.$$

With these preparations we can state the main result.

THEOREM 3.3.

As above let $\bar{\gamma} = \{\bar{y}(t) = \hat{y}(t\bar{T}) : t \in \mathbb{R}\}$ be a \bar{T} -periodic hyperbolic orbit at $\lambda = \bar{\lambda}$ and let $\bar{z}(t) = (\bar{x}(t), \bar{\lambda}) = (\hat{x}(t\bar{T}), \bar{\lambda})$ be an orbit connecting a hyperbolic steady state $y_-(\bar{\lambda})$ to $\bar{\gamma}$. Further, let $\Psi \in C^1(Z_1(\epsilon) \times \mathbb{R}^{p+1}, \mathbb{R})$ be given such that

$$(3.23) \quad \Psi(\bar{x}, \bar{y}, \bar{T}, \bar{\lambda}) = 0, \quad \frac{\partial \Psi}{\partial x}(\bar{x}, \bar{y}, \bar{T}, \bar{\lambda}) \dot{\bar{x}} + \frac{\partial \Psi}{\partial y}(\bar{x}, \bar{y}, \bar{T}, \bar{\lambda}) \dot{\bar{y}} \neq 0.$$

Then the following conditions are equivalent.

- (i) The manifolds M_{-u} and M_{+s} intersect transversely in the strong sense that for all $t \in \mathbb{R}$

$$(3.24) \quad T_{\bar{z}(t)}M_{-u} + T_{\bar{z}(t)}M_{+s} = \mathbb{R}^{m+p}, \quad T_{\bar{z}(t)}M_{-u} \cap T_{\bar{z}(t)}M_{+s} = \text{span} \{\dot{\bar{z}}(t)\}.$$

- (ii) The linear mapping

$$(3.25) \quad B(\eta, \xi, \lambda) = \xi - \eta + (E_+(0) - E_-(0))\lambda$$

is a bijection from $R(Q_u(0)) \times R(\tilde{P}_s(0)) \times \mathbb{R}^p$ into \mathbb{R}^m .

- (iii) $(\bar{x}, \bar{y}, \bar{T}, \bar{\lambda}) \in Z_1(\epsilon) \times \mathbb{R}^{p+1}$ is a regular solution of the system (1.13), i.e. $F'(\bar{x}, \bar{y}, \bar{T}, \bar{\lambda}) : Z_1(\epsilon) \times \mathbb{R}^p \rightarrow Z_0(\epsilon) \times \mathbb{R}$ is a linear homeomorphism.

PROOF. (i) \Rightarrow (ii)

From (3.24) we obtain the relation (see (1.8))

$$(3.26) \quad p = m_{+u} - m_{-u} + 1$$

and this holds iff B is given by a quadratic matrix. Suppose $B(\eta, \xi, \lambda) = 0$ for some $\eta \in R(Q_u(0))$, $\xi \in R(\tilde{P}_s(0))$, $\lambda \in \mathbb{R}^p$. Then

$$(\xi + E_+(0)\lambda, \lambda) = (\eta + E_-(0)\lambda, \lambda) \in T_{\bar{z}(0)}M_{-u} \cap T_{\bar{z}(0)}M_{+s}$$

holds and by (3.24) $\xi = \eta = c \dot{\bar{x}}(0)$ for some $c \in \mathbb{R}$. But $\dot{\bar{x}}(0) \notin R(\tilde{P}_s(0))$ so that $c = 0$ and $\xi = \eta = 0$ follows.

(ii) \Rightarrow (i)

The linearized flow $\frac{\partial \Phi^t}{\partial z}(\bar{z}(t))$ maps the tangent spaces at $\bar{z}(0)$ onto those at $\bar{z}(t)$, hence it is sufficient to prove (3.24) at $t = 0$. Moreover, (3.26) follows from (ii) so that we need only prove the second relation in (3.24).

Suppose $(x, \lambda) \in T_{\bar{z}(0)}M_{-u} \cap T_{\bar{z}(0)}M_{+s}$ holds, then from (3.22) we obtain

$$x = \eta + E_-(0)\lambda = \xi + E_+(0)\lambda + c\dot{\bar{x}}(0)$$

for suitable ξ, η, λ . With $\tilde{\eta} = \eta - c\dot{\bar{x}}(0) \in R(Q_u(0))$ we find $B(\tilde{\eta}, \xi, \lambda) = 0$ and hence $\xi = 0, \lambda = 0, x = \eta = c\dot{\bar{x}}(0)$.

(ii) \Rightarrow (iii)

By our assumptions $(\bar{x}, \bar{y}, \bar{T}, \bar{\lambda})$ is a solution of (1.13). For the derivative at this point (which we abbreviate as (\cdot)) we find

$$F'(\cdot) = \begin{bmatrix} & -f(\bar{x}, \bar{\lambda}) & -\bar{T} \frac{\partial f}{\partial \lambda}(\bar{x}, \bar{\lambda}) \\ \Gamma & -f(\bar{y}, \bar{\lambda}) & -\bar{T} \frac{\partial f}{\partial \lambda}(\bar{y}, \bar{\lambda}) \\ \frac{\partial \Psi}{\partial x}(\cdot) & \frac{\partial \Psi}{\partial y}(\cdot) & \frac{\partial \Psi}{\partial T}(\cdot) & \frac{\partial \Psi}{\partial \lambda}(\cdot) \end{bmatrix}$$

where $\Gamma(x, y) = (\tilde{L}x, Ly)$. By Proposition 2.8 Γ has Fredholm index $m_{-u} - m_{+u} - 1$ and from (3.26) we obtain with the help of the bordering lemma ([Bey 90a]) that $F'(\cdot)$ has Fredholm index 0.

Suppose $F'(\cdot)(x, y, T, \lambda) = 0$ for some $(x, y) \in Z_1(\epsilon), T \in \mathbb{R}, \lambda \in \mathbb{R}^p$. By (3.3) we can choose $c \in \mathbb{R}$ such that $\tilde{y} = y - c\dot{\bar{y}}$ satisfies $\chi'(\tilde{y})\tilde{y} = 0$.

Using this and the equation

$$L\tilde{y} - Tf(\tilde{y}, \bar{\lambda}) = \bar{T} \frac{\partial f}{\partial \lambda}(\tilde{y}, \bar{\lambda})\lambda$$

we find that (\tilde{y}, T) and $(\frac{\partial y}{\partial \lambda}(\cdot, \bar{\lambda})\lambda, T'(\bar{\lambda})\lambda)$ satisfy the same system (see (3.21)) and hence

$$(3.27) \quad y = c\dot{\bar{y}} + \frac{\partial y}{\partial \lambda}(\cdot, \bar{\lambda})\lambda, \quad T = T'(\bar{\lambda})\lambda.$$

We define $\xi = \tilde{P}_s(0)x(0)$ and $\tilde{x}(t) = x(t) - \tilde{S}(t, 0)\xi - c\dot{\bar{x}}(t)$ for $t \geq 0$ so that

$$\tilde{x}(t) - \tilde{y}(t) = x(t) - y(t) - \tilde{S}(t, 0)\xi = O(e^{-\epsilon t})$$

and $\tilde{P}_s(0)\tilde{x}(0) = 0$.

Therefore, $(\tilde{x}, \tilde{y}) \in Z_1^+(\epsilon)$ and we have shown, with the operator F_+ from (3.14), that

$$\frac{\partial F_+}{\partial(x, y, T)}(\cdot)(\tilde{x}, \tilde{y}, \tilde{T}) = -\frac{\partial F_+}{\partial \lambda}(\cdot)\lambda$$

where (\cdot) denotes evaluation at $(\bar{x}, \bar{y}, \bar{T}, 0, \bar{\lambda})$. Thus we obtain $\tilde{x}(t) = E_+(t)\lambda$ and

$$(3.28) \quad x(0) = \xi + c\dot{\bar{x}}(0) + E_+(0)\lambda.$$

In a similar way we set $\eta = Q_u(0)x(0)$, $v(t) = x(t) - \tilde{S}(t,0)\eta$ ($t \leq 0$) and find $v(t) = E_-(t)\lambda$ by using equation (3.18) for $x_-(\cdot, \eta, \lambda)$. Now we combine $x(0) = \eta + E_-(0)\lambda$ and (3.28) to get

$$B(\eta - c\dot{\bar{x}}(0), \xi, \lambda) = 0.$$

From assumption (ii) and (3.27) we conclude $\lambda = 0$, $\xi = 0$, $\eta = c\dot{\bar{x}}(0)$, $T = 0$, $x = c\dot{\bar{x}}$ and $y = c\dot{\bar{y}}$. An application of (3.23) finally yields $c = 0$ and $x = 0$, $y = 0$.

(iii) \Rightarrow (ii)

We know that $F'(\cdot)$ has Fredholm index 0, hence by Proposition 2.8 and the bordering lemma we have

$$0 = \text{ind}(\Gamma) + p = m_{-u} - m_{+u} - 1 + p$$

i.e. (3.26) holds.

Now assume $B(\eta, \xi, \lambda) = 0$ and define

$$T = T'(\bar{\lambda}), \lambda, y(t) = \frac{\partial y}{\partial \lambda}(t, \bar{\lambda})\lambda + c\dot{\bar{y}}(t)$$

$$x(t) = \begin{cases} E_+(t)\lambda + \tilde{S}(t,0)\xi + c\dot{\bar{x}}(t) & \text{for } t \geq 0 \\ E_-(t)\lambda + \tilde{S}(t,0)\eta + c\dot{\bar{x}}(t) & \text{for } t < 0 \end{cases}$$

where c will be determined later on.

Using $B(\eta, \xi, \lambda) = 0$ we see that x is continuous at 0 and in fact $(x, y) \in Z_1(\epsilon)$. Moreover, the equation (3.20) yields

$$\Gamma \begin{pmatrix} x \\ y \end{pmatrix} - T \begin{pmatrix} f(\bar{x}, \bar{\lambda}) \\ f(\bar{y}, \bar{\lambda}) \end{pmatrix} - \bar{T} \begin{pmatrix} \frac{\partial f}{\partial \lambda}(\bar{x}, \bar{\lambda})\lambda \\ \frac{\partial f}{\partial \lambda}(\bar{y}, \bar{\lambda})\lambda \end{pmatrix} = 0.$$

Finally, by (3.23) we can determine c so that

$$0 = \frac{\partial \Psi}{\partial x}(\cdot)x + \frac{\partial \Psi}{\partial y}(\cdot)y + \frac{\partial \Psi}{\partial T}(\cdot)T + \frac{\partial \Psi}{\partial \lambda}(\cdot)\lambda.$$

Then (x, y, T, λ) is in the null space of $F'(\cdot)$ and we conclude $x = 0$, $y = 0$, $T = 0$, $\lambda = 0$. In particular, $\xi + c\dot{\bar{x}}(0) = 0 = \eta + c\dot{\bar{x}}(0)$ and $\xi = 0$, $c = 0$ because $\dot{\bar{x}}(0) \notin R(\tilde{P}_s(0))$. ■

REMARK.

We can easily rewrite the nonsingularity of the matrix B in an equivalent form using the adjoint operator Γ^* from Proposition 2.8. If B is nonsingular then necessarily

$$(3.29) \quad R(Q_u(0)) \cap R(\tilde{P}_s(0)) = \{0\}.$$

Therefore

$$\dim N(\Gamma^*) = -\text{ind}(\Gamma) = m_{+u} - m_{-u} + 1 = p$$

and we can find a basis of the form

$$\begin{pmatrix} \tilde{\psi} \\ \psi \end{pmatrix}, \begin{pmatrix} \varphi_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \varphi_p \\ 0 \end{pmatrix}$$

where $\tilde{L}^* \varphi_i = 0$ and $\varphi_i(t) = O(e^{-\epsilon t})$ for $i = 2, \dots, p$. Setting $\varphi_1 = \tilde{\psi}$ we find from the proof of Proposition 2.8 that the columns of $\Phi(0) := (\varphi_1(0), \dots, \varphi_p(0))$ form a basis of

$(R(\tilde{P}_s(0)) + R(Q_u(0)))^\perp$. Then the nonsingularity of B reduces to the nonsingularity of the $(p \times p)$ -matrix

$$(3.30) \quad \Lambda = \Phi(0)^T (E_+(0) - E_-(0)).$$

This is the derivative with respect to λ of some generalized Melnikov-type function

$$\Phi(0)^T (x_+(0, 0, \lambda) - x_-(0, 0, \lambda)).$$

Of course, for the way back from the nonsingularity of Λ to that of B we need to assume conditions (3.29) and (3.26).

It is now quite straightforward to develop an analogue of Theorem 3.3 for periodic-to-periodic connections. However, there are a few differences due to the fact that there is no simple scaling of the time axis which is suitable for both periodic orbits. This is the reason for the delicate nonautonomous transformations in [Ha Li 86]. Of course, the split formulation (1.14) is also some kind of nonautonomous transformation. But its treatment is more convenient, it is closer to numerical approximation schemes and it is easily covered by the theory developed so far.

Let $(\hat{x}(t), \bar{\lambda})$, $t \in \mathbb{R}$ be an orbit connecting a \bar{T}_- -periodic hyperbolic orbit $\{\hat{y}_-(t) : t \in \mathbb{R}\}$ to a \bar{T}_+ -periodic one $\{\hat{y}_+(t) : t \in \mathbb{R}\}$. We introduce the scaled functions

$$\bar{y}_\pm(t) = \hat{y}_\pm(t\bar{T}_\pm), \quad \bar{x}_\pm(t) = \hat{x}(t\bar{T}_\pm) \quad \text{for } t \in \mathbb{R}_\pm$$

and the differential operators

$$L_\pm = \frac{d}{dt} - \bar{T}_\pm \frac{\partial f}{\partial x}(\bar{y}_\pm, \bar{\lambda}), \quad \tilde{L}_\pm = \frac{d}{dt} - \bar{T}_\pm \frac{\partial f}{\partial x}(\bar{x}_\pm, \bar{\lambda})$$

with solution operators S_\pm and \tilde{S}_\pm respectively. L_- and \tilde{L}_- have ordinary exponential trichotomies on \mathbb{R}_- and we denote the corresponding projectors by Q_κ and \tilde{Q}_κ , $\kappa = s, c, u$.

The unstable manifold M_{-u} is of the form

$$M_{-u} = \{(x, \lambda) \in \mathbb{R}^m \times U(\bar{\lambda}) : \text{dist}(\varphi^t(x, \lambda), \gamma_-(\lambda)) \rightarrow 0 \text{ as } t \rightarrow -\infty\}$$

where $\gamma_-(\lambda) = \{y_-(t, \lambda) : t \in \mathbb{R}\}$ are $T_-(\lambda)$ -periodic orbits and $y_-(\cdot, \bar{\lambda}) = \bar{y}_-$. M_{-u} can be parametrized by $(x_-(t, \eta, \lambda), \lambda)$ in a way analogous to Theorem 3.2.

We will also assume that the phase conditions χ_\pm (compare (3.2)) for $y_\pm(\cdot, \lambda)$ have been chosen in such a way that

$$\bar{x}(t) - \bar{y}_\pm(t) = O(e^{-\epsilon|t|}) \text{ as } t \rightarrow \pm\infty$$

holds. Finally, the spaces $Z_1^-(\epsilon)$, $Z_0^-(\epsilon)$ are defined analogously to $Z_1^+(\epsilon)$, $Z_0^+(\epsilon)$ and ϵ is chosen such that $[-\epsilon, \epsilon]$ contains no real parts of the Floquet exponents of $\gamma_\pm(\bar{\lambda})$.

THEOREM 3.4.

Let $\bar{z}(t) = (\bar{x}(t), \bar{\lambda})$ be a periodic-to-periodic connecting orbit as above and assume that $\Psi \in C^1(Z_1^+(\epsilon) \times Z_1^-(\epsilon) \times \mathbb{R}^{p+2}, \mathbb{R})$ satisfies

$$(3.31) \quad \begin{aligned} &\Psi(\bar{u}) = 0 \text{ for } \bar{u} = (\bar{x}_-, \bar{y}_-, \bar{x}_+, \bar{y}_+, \bar{T}_-, \bar{T}_+, \bar{\lambda}) \\ &\frac{\partial \Psi}{\partial x_-}(\bar{u}) \dot{\bar{x}}_- + \frac{\partial \Psi}{\partial y_-}(\bar{u}) \dot{\bar{y}}_- + \frac{\partial \Psi}{\partial x_+}(\bar{u}) \dot{\bar{x}}_+ + \frac{\partial \Psi}{\partial y_+}(\bar{u}) \dot{\bar{y}}_+ \neq 0. \end{aligned}$$

Then the following conditions are equivalent.

- (i) The manifolds M_{-u} and M_{+s} intersect transversely along $\bar{z}(t)$ in the strong sense (3.24)
- (ii) The linear mapping

$$B(c, \eta, \xi, \lambda) = c \dot{\bar{x}}(0) + \xi - \eta + (E_+(0) - E_-(0))\lambda$$

is a bijection from $\mathbb{R} \times R(\tilde{Q}_u(0)) \times R(\tilde{P}_s(0)) \times \mathbb{R}^p$ into \mathbb{R}^m

- (iii) \bar{u} is a regular solution in $Z_1^+(\epsilon) \times Z_1^-(\epsilon) \times \mathbb{R}^{p+2}$ of the operator equation (1.14).

PROOF.

Since the proof is quite similar to that of Theorem 3.3 we only sketch the main differences.

(3.22a) becomes

$$T_{\bar{z}(0)} M_{-u} = \{(\eta + E_-(0)\lambda + c \dot{\bar{x}}(0), \lambda) : \eta \in R(\tilde{Q}_u(0)), \lambda \in \mathbb{R}^p, c \in \mathbb{R}\}$$

and the relation (3.26) changes to (cf. (1.8))

$$(3.32) \quad p = m_{+u} - m_{-u}.$$

From this the equivalence of (i) and (ii) follows immediately.

Let us assume (ii) and calculate

$$F'(\bar{u}) = \begin{bmatrix} \Gamma_- & 0 & -f(\bar{x}_-, \bar{\lambda}) & 0 & -\bar{T}_- \frac{\partial f}{\partial \lambda}(\bar{x}_-, \bar{\lambda}) \\ & & -f(\bar{y}_-, \bar{\lambda}) & 0 & -\bar{T}_- \frac{\partial f}{\partial \lambda}(\bar{y}_-, \bar{\lambda}) \\ 0 & \Gamma_+ & 0 & -f(\bar{x}_+, \bar{\lambda}) & -\bar{T}_+ \frac{\partial f}{\partial \lambda}(\bar{x}_+, \bar{\lambda}) \\ & & 0 & -f(\bar{y}_+, \bar{\lambda}) & -\bar{T}_+ \frac{\partial f}{\partial \lambda}(\bar{y}_+, \bar{\lambda}) \\ -E_0 & 0 & E_0 & 0 & 0 \\ \frac{\partial \Psi}{\partial(x_-, y_-)}(\bar{u}) & \frac{\partial \Psi}{\partial(x_+, y_+)}(\bar{u}) & \frac{\partial \Psi}{\partial T_-}(\bar{u}) & \frac{\partial \Psi}{\partial T_+}(\bar{u}) & \frac{\partial \Psi}{\partial \lambda}(\bar{u}) \end{bmatrix}$$

where $\Gamma_{\pm}(x, y) = (\tilde{L}_{\pm} x, L_{\pm} y)$. The Fredholm index of $F'(\bar{u})$ is zero due to (3.32) and Theorem 3.2.

Let $u = (x_-, y_-, x_+, y_+, T_-, T_+, \lambda)$ be in the null space of $F'(\bar{u})$. Then we choose c_{\pm} such that

$$\chi'_{\pm}(\bar{y}_{\pm})(y_{\pm} - c_{\pm} \dot{\bar{y}}_{\pm}) = 0$$

and obtain as in (3.27)

$$y_{\sigma} = c \dot{\bar{y}}_{\sigma} + \frac{\partial y_{\sigma}}{\partial \lambda}(\cdot, \bar{\lambda})\lambda, \quad T_{\sigma} = T'_{\sigma}(\bar{\lambda})\lambda \quad \text{for } \sigma = +, -.$$

The analogue of (3.28) is

$$x_{\sigma}(0) = \xi_{\sigma} + c_{\sigma} \dot{\bar{x}}(0) + E_{\sigma}(0)\lambda, \quad \sigma = +, -,$$

where $\xi_+ = \tilde{P}_s(0)x_+(0)$ and $\xi_- = \tilde{Q}_u(0)x_-(0)$.

From the equality $x_+(0) = x_-(0)$ we find

$$B(c_+ - c_-, \xi_-, \xi_+, \lambda) = 0$$

and hence $c_+ = c_-$, $\xi_- = 0$, $\xi_+ = 0$, $\lambda = 0$. Finally, using the last row in $F'(\bar{u})$ and assumption (3.31) we end up with $c_+ = c_- = 0$ and $x_{\sigma} = 0$, $y_{\sigma} = 0$, $T_{\sigma} = 0$ ($\sigma = +, -$).

For the converse statement we notice that (3.32) is a consequence of $\text{ind}(F'(\bar{u})) = 0$ and Theorem 3.2.

Assuming $B(c, \eta, \xi, \lambda) = 0$ we define $T_{\pm} = T'_{\pm}(\bar{\lambda})\lambda$ and

$$\begin{aligned} x_+(t) &= E_+(t)\lambda + \tilde{S}_+(t, 0)\xi + (c + \tilde{c}) \dot{\bar{x}}(t), \quad t \geq 0, \\ y_+(t) &= \frac{\partial y_+}{\partial \lambda}(t, \bar{\lambda})\lambda + (c + \tilde{c}) \dot{\bar{y}}_+(t), \\ x_-(t) &= E_-(t)\lambda + \tilde{S}_-(t, 0)\eta + \tilde{c} \dot{\bar{x}}(t), \quad t \leq 0, \\ y_-(t) &= \frac{\partial y_-}{\partial \lambda}(t, \bar{\lambda})\lambda + \tilde{c} \dot{\bar{y}}_-(t), \end{aligned}$$

where \tilde{c} is determined in such a way that the last row of $F'(\bar{u})$ applied to $u = (x_-, y_-, x_+, y_+, T_-, T_+, \lambda)$ vanishes.

Then $u \in N(F'(\bar{u}))$ and the assertion follows from $u = 0$ and $\dot{\bar{x}}(0) \notin R(\tilde{P}_s(0))$, $\dot{\bar{x}}(0) \notin R(\tilde{Q}_u(0))$. ■

This theorem holds also in the case $p = 0$ which e.g. occurs for a homoclinic periodic connection (see (3.32)). Then there are no parameters in the system (1.1) and all the statements involving λ trivialize in an obvious way.

Suppose in the homoclinic case that we can find a section Σ with Poincaré map P such that successive intersections of the orbit with Σ are obtained by an application of P . Then it can be shown that the transversality conditions of Theorem 3.4 hold if and only if the points of intersection with Σ are transversal homoclinic points of the map P (see e.g. [Pal 88] for an analysis of transversal homoclinic points). By this shooting type approach we have reduced the computation of homoclinic periodic connections to that of homoclinic points of maps. However, in the general homoclinic case it is not clear whether such a reduction is possible and we expect that one still has to tackle infinite boundary value problems of the form (1.14) in a numerical calculation.

4. A numerical example

In the well-posed formulations of section 3 the boundary conditions for the connecting orbits are hidden in the function spaces used.

For a numerical approximation we replace $(-\infty, \infty)$ by some large interval $J = (T_-, T_+)$ and we have to introduce finite boundary conditions at T_- and T_+ .

In the case of stationary connecting orbits it is well-known how to set up these boundary conditions and how to estimate the errors involved, see [Bey 90a, Bey 90b, DoFr 89, FrDo 91, Kuz 90, Sch 93a, Sch 93b]. To our knowledge there are no numerical approaches to the periodic case and we will treat here only the point-to-periodic connection, i.e. equation (1.13).

We consider the following system of differential equations

$$(4.1) \quad \dot{x} = f(x, \lambda), \quad t \in J = [T_-, T_+]$$

$$(4.2) \quad \dot{y} = T f(y, \lambda), \quad t \in [0, 1]$$

$$(4.3) \quad \dot{T} = 0, \quad \dot{\lambda} = 0.$$

Here we have $2m + p + 1$ variables $u = (x, y, T, \lambda)$ and we need the same number of boundary conditions. We assume these to be of the following form

$$(4.4) \quad B_-(x(T_-), \lambda) = 0$$

$$(4.5) \quad B_+(x(T_+), y(0), \lambda) = 0$$

$$(4.6) \quad \psi_J(x, y, T, \lambda) = 0$$

$$(4.7) \quad y(1) - y(0) = 0$$

where $B_- : \mathbb{R}^{m+p} \rightarrow \mathbb{R}^{m-s}$ and $B_+ : \mathbb{R}^{2m+p} \rightarrow \mathbb{R}^{m+u+1}$ define the asymptotic boundary conditions at T_- and T_+ and ψ_J acts as a scalar phase condition.

The number of boundary conditions is

$$m_{-s} + m_{+u} + 2 + m = 2m - m_{-u} + m_{+u} + 2$$

which coincides with $2m + p + 1$ under the assumption (3.26) (see Theorem 3.3).

The condition (4.4) requires $x(T_-)$ to lie in some approximation to the unstable manifolds of the steady states $x_-(\lambda)$ and a good choice are projection boundary conditions (see [Bey 90a])

$$(4.8) \quad B_-(x, \lambda) = V_s(\lambda)(x - x_-(\lambda))$$

where the rows of $V_s(\lambda) \in \mathbb{R}^{m-s, m}$ depend smoothly on λ and span the stable subspace of $\frac{\partial f}{\partial x}(x_-(\lambda), \lambda)^T$, i.e.

$$V_s(\lambda) \frac{\partial f}{\partial x}(x_-(\lambda), \lambda) = G_-(\lambda)V_s(\lambda)$$

for some $G_-(\lambda) \in \mathbb{R}^{m-s, m-s}$ with $\text{Re } \sigma(G_-(\lambda)) < 0$.

When $x_-(\lambda)$ is known, the matrix $V_s(\lambda)$ can easily be computed by an eigenvalue solver combined with a normalization procedure ([Bey 90a], [ChKu 93]).

Similarly, we should choose B_+ in such a way that $B_+(\cdot, y(0, \lambda), \lambda) = 0$ is an approximation to the fiber of dimension m_{+s} which is in asymptotic phase with $y(0, \lambda)$. Here, as in section 3, we denote by $y(\cdot, \lambda)$ the 1-periodic solutions of (4.2) with $T = T(\lambda)$ fixed by some suitable phase condition. The analogue of (4.8) then is

$$(4.9) \quad B_+(x(T_+), y(0), \lambda) = V_u(0, \lambda)(x(T_+) - y(0))$$

where $V_u(t, \lambda) \in \mathbb{R}^{m+u+1, m}$ solves the adjoint variational equation

$$(4.10) \quad \dot{V} = -V T(\lambda) \frac{\partial f}{\partial x}(y(\cdot, \lambda), \lambda)$$

$$(4.11) \quad V_u(1, \lambda) = G_+(\lambda)V_u(0, \lambda) \quad \text{for some } G_+(\lambda) \in \mathbb{R}^{m+u+1, m+u+1}$$

with $|\mu| \leq 1$ for all eigenvalues μ of $G_+(\lambda)$.

Notice that the spectrum of $G_+(\lambda)$ includes the trivial Floquet multiplier 1 and that the rows of $V_u(t, \lambda)$ should span the same space as those of

$$V_F(t, \lambda) = \begin{pmatrix} \psi(t, \lambda)^T \\ e^{-t} B_u(\lambda) \Psi_u(t, \lambda)^T \end{pmatrix}$$

which is obtained from the Floquet decomposition (2.20).

Since $y(\cdot, \lambda)$ is itself a result of the computation it is unrealistic to assume that $V_u(\cdot, \lambda)$ is known a-priori. One way out of this dilemma is to attach the variational equation (4.10) to the system (4.1)–(4.3) and to add some boundary conditions for V derived from the invariance condition (4.11). However, this blows up to the dimension of the boundary value problem by $m \cdot (m_{+u} + 1)$ and is thus very costly.

In the example below we will use a simpler device, where (4.9) is replaced by

$$(4.12) \quad B_+(x(T_+), y(0), \lambda) = \widehat{V}_u(x(T_+) - y(0))$$

and $\widehat{V}_u \in \mathbb{R}^{m_{+u}+1, m}$ is an approximation to $V_u(0, \lambda)$ obtained by solving (4.10), (4.11) with a periodic orbit $y(\cdot, \widehat{\lambda})$ for an initial guess $\widehat{\lambda}$. Of course, some accuracy is lost in this approach.

As an example we treat the well-known Lorenz equations (see [Spa 82])

$$(4.13) \quad \dot{x}_1 = \sigma(x_2 - x_1), \quad \dot{x}_2 = \lambda x_1 - x_2 - x_1 x_3, \quad \dot{x}_3 = x_1 x_2 - b x_3.$$

For $\sigma > b + 1$ this system has a subcritical Hopf bifurcation from the nontrivial steady states

$$\xi_{\pm}(\lambda) = (\pm(b(\lambda - 1))^{1/2}, \pm(b(\lambda - 1))^{1/2}, \lambda - 1), \quad \lambda > 1$$

at

$$\lambda = \lambda_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}.$$

The periodic orbits $\gamma_{\pm}(\lambda)$ shrinking to $\xi_{\pm}(\lambda)$ at λ_H have two-dimensional stable manifolds ($m_{+u} = m_{+s} = 1$) and there is apparently a specific value

$$\lambda_A = \lambda_A(\sigma, b) < \lambda_H$$

at which the one-dimensional unstable manifold of the origin meets these stable manifolds, see [Spa 82, pp. 32–47] and in particular [Kuz 91, Lecture 7] for a nice illustration. (3.26) yields $p = 1$ and so we expect this point to periodic connection to be a stable one parameter phenomenon. For $\lambda < \lambda_A$ the unstable manifold is attracted towards $\xi_-(\lambda)$ or $\xi_+(\lambda)$ while for $\lambda > \lambda_A$ it becomes part of the strange attractor. However, it is not clear whether λ_A is the precise value at which the strange invariant set becomes attracting.

Figures 1 and 2 show the different fate of two trajectories started on the linearized unstable manifold of the origin at values

$$\lambda_0 = 24.05 < \lambda_A < \lambda_1 = 24.06, \quad \sigma = 10, \quad b = \frac{8}{3}.$$

The trajectory with initial values

$$\widehat{x}(0) = (0.00435, 0.009, 0)$$

and $T_+ = -T_- = 0.8667$ was taken as initial approximation for the connecting orbit. Then a periodic orbit was computed with values

$$\begin{aligned}\widehat{\lambda} &= 24.06, & \widehat{T} &= 0.677, \\ \widehat{y}(1) &= (-6.116, -4.799, 23.07)\end{aligned}$$

and a suitably normalized \widehat{V}_u was found from (4.10), (4.11) to be

$$\widehat{V}_u = \begin{pmatrix} 0, & 0.769, & 1 \\ 1, & 2.916, & 0 \end{pmatrix}.$$

The phase condition (4.6) was simply

$$(4.14) \quad y_3(0) - (\lambda - 1) = 0.$$

With these initial data the boundary value problem (4.1)–(4.7), (4.12) was solved with tolerance 10^{-8} by the code D02RAF (NAG-library, Oxford). The third component of the solutions x and y is shown in Figure 3 and the λ -value is

$$\lambda_A = 24.05790.$$

A good initial approximation is crucial in this example because the periodic orbit is close to the Hopf point and the periodic boundary value problem allows for the trivial solution $T = 0$, y constant.

Finally, in Figure 4 we show two further point-to-periodic connections obtained by increasing the parameter b . The continuation with respect to this parameter turns out to be rather sensitive. This is a well-known phenomenon due to the fact that the rigid phase condition (4.14) is not well-suited for mesh adaptation. Good alternatives are integral phase conditions [FrDo 91] or the use of the Gauss-Newton method [Deu 84]. In view of the theoretical results of section 3 it is clearly acceptable to have phase conditions which involve both the connecting and the periodic orbit.

REFERENCES

- [Bey 90a] W.-J. Beyn, *The numerical computation of connecting orbits in dynamical systems*, IMA J. Numer. Anal. **9** (1990), 379–405.
- [Bey 90b] _____, *Global bifurcations and their numerical computation in: Continuation and Bifurcations Numerical Techniques and Applications* (D. Roose, A. Spence, B. De Dier Eds.) (1990), Kluwer, Dordrecht, 169–181.
- [CHM 80] S.N. Chow, J.K. Hale, J. Mallet-Paret, *An example of bifurcation to homoclinic orbits*, J. Differential Equations **37** (1980), 351–373.
- [ChKu 93] A.R. Champneys, Yu. A. Kuznetsov, *Numerical detection and continuation of codimension-two homoclinic bifurcations*, Report AM-R 9308, (1993), Amsterdam.
- [ChLi 90] S.-N. Chow, X.-B. Lin, *Bifurcation of a homoclinic orbit with a saddle-node equilibrium*, Differential and Integral Equations **3** (1990), 435–466.

- [Cop 78] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics **629** (1978).
- [Deu 84] P. Deuffhard, *Computation of periodic solutions of nonlinear ODEs*, BIT **24** (1984), 456–466.
- [Die 60] J. Dieudonné, *Foundations of Modern Analysis* (1960), Academic Press, New York.
- [DoFr 89] E.J. Doedel, M. Friedman, *Numerical computation of heteroclinic orbits*, J. Comp. Appl. Math. **26** (1989), 159–170.
- [Fen 74] N. Fenichel, *Asymptotic stability with rate constants*, Indiana Univ. Math. J. **23** (1974), 1109–1137.
- [Fen 77] _____, *Asymptotic stability with rate constants II*, Indiana Univ. Math. J. **26** (1977), 81–93.
- [FrDo 91] M. Friedman, E.J. Doedel, *Numerical computation of invariant manifolds connecting fixed points*, SIAM J. Numer. Anal. **28** (1991), 789–808.
- [Hal 69] J.K. Hale, *Ordinary Differential Equations* (1969), Wiley & Sons, New York.
- [HaLi 86] J. Hale, X.-B. Lin, *Heteroclinic orbits for retarded functional differential equations*, J. Differential Equations **65** (1986), 175–202.
- [Har 64] Ph. Hartman, *Ordinary Differential Equations* (1964), Wiley & Sons, New York.
- [HPS 77] M.W. Hirsch, C.C. Pugh, M. Shub, *Invariant Manifolds* **583** (1977), Springer Lecture Notes in Mathematics.
- [Irw 80] M.C. Irwin, *Smooth dynamical systems* (1980), Academic Press, New York.
- [Kuz 90] Y. A. Kuznetsov, *Computation of invariant manifold bifurcations in: Continuation and Bifurcations: Numerical Techniques and Applications* (1990), (D. Roose, A. Spence, B. De Dier Eds.), Kluwer, Dordrecht.
- [Kuz 91] _____, *Elements of Nonlinear Dynamics* (1991), Lectures delivered at the Dipartimento di Elettronica di Politecnico di Milano, Italy.
- [Pal 84] K.J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Diff. Eqn. **55** (1984), 225–256.
- [Pal 88] _____, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points in: Dynamics Reported, Volume I* (1988), (U. Kirchgraber, H.O. Walther Eds.), Teubner, 265–306.
- [Sch 93a] S. Schecter, *Numerical computation of saddle-node homoclinic bifurcation points*, To appear in SIAM J. Numer. Anal.
- [Sche 93b] _____, *Rate of convergence of numerical approximations to homoclinic bifurcation points*, Preprint, North Carolina State University (1993).
- [Spa 82] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (1982), Springer, New York.

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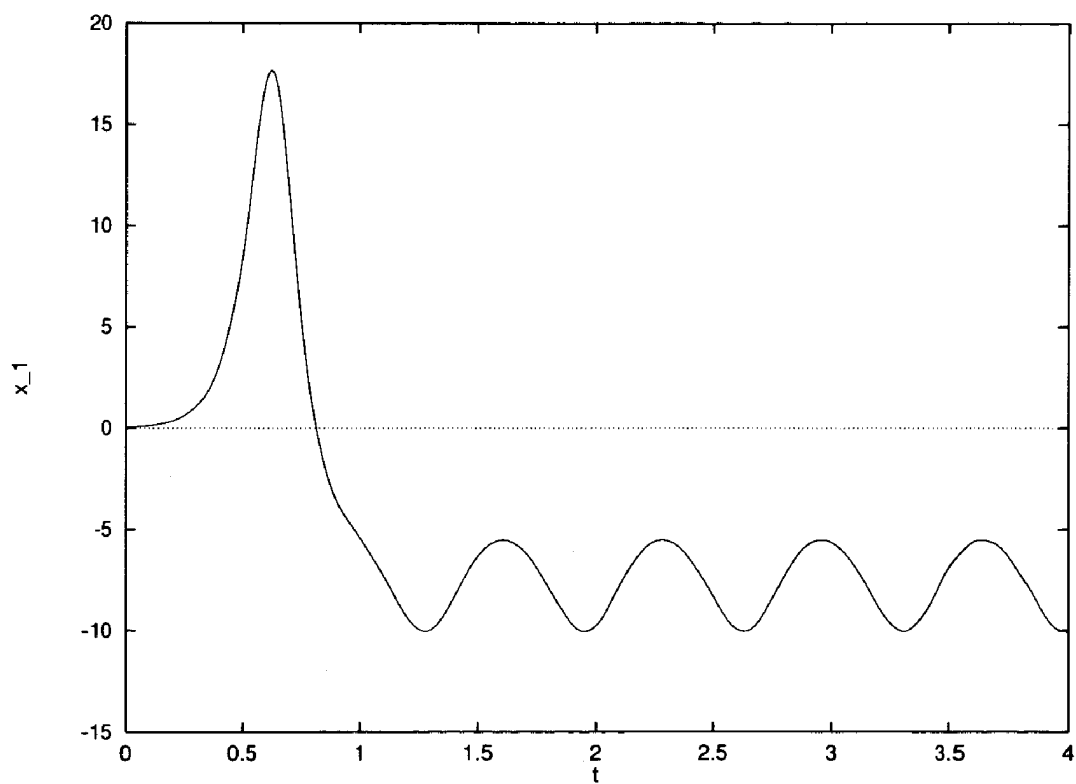


FIGURE 1A. Unstable manifold of the origin at $\lambda_0 = 24.05$, first component plotted over the time interval $[0, 4]$

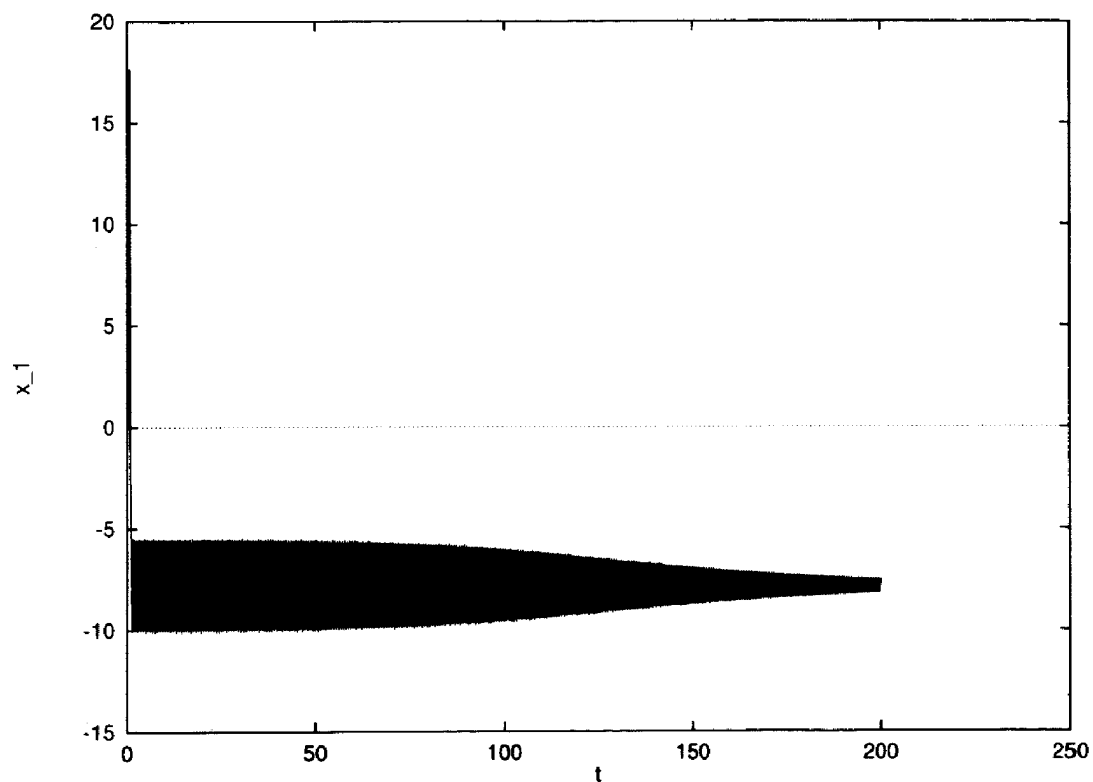


FIGURE 1B. The same as Figure 1a but with time interval $[0, 200]$

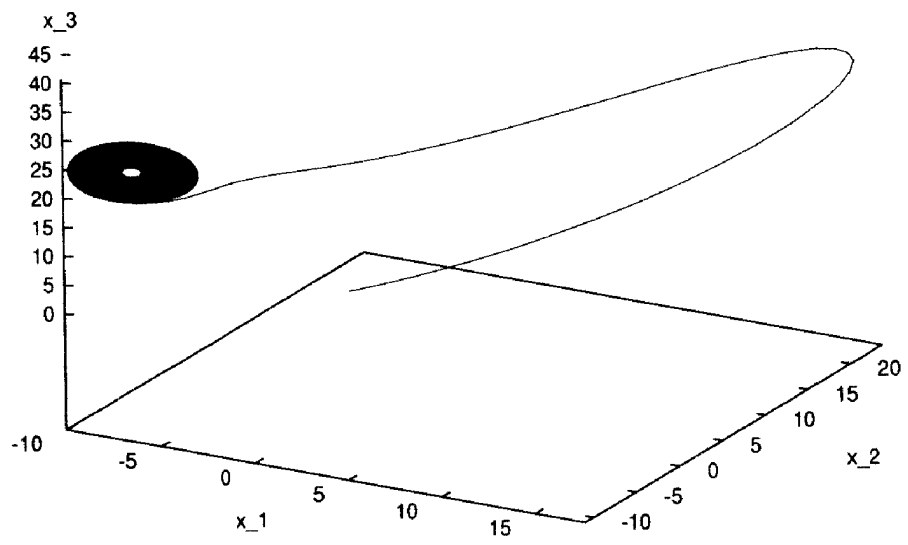


FIGURE 1C. The same as Figure 1b but plotted in phase space

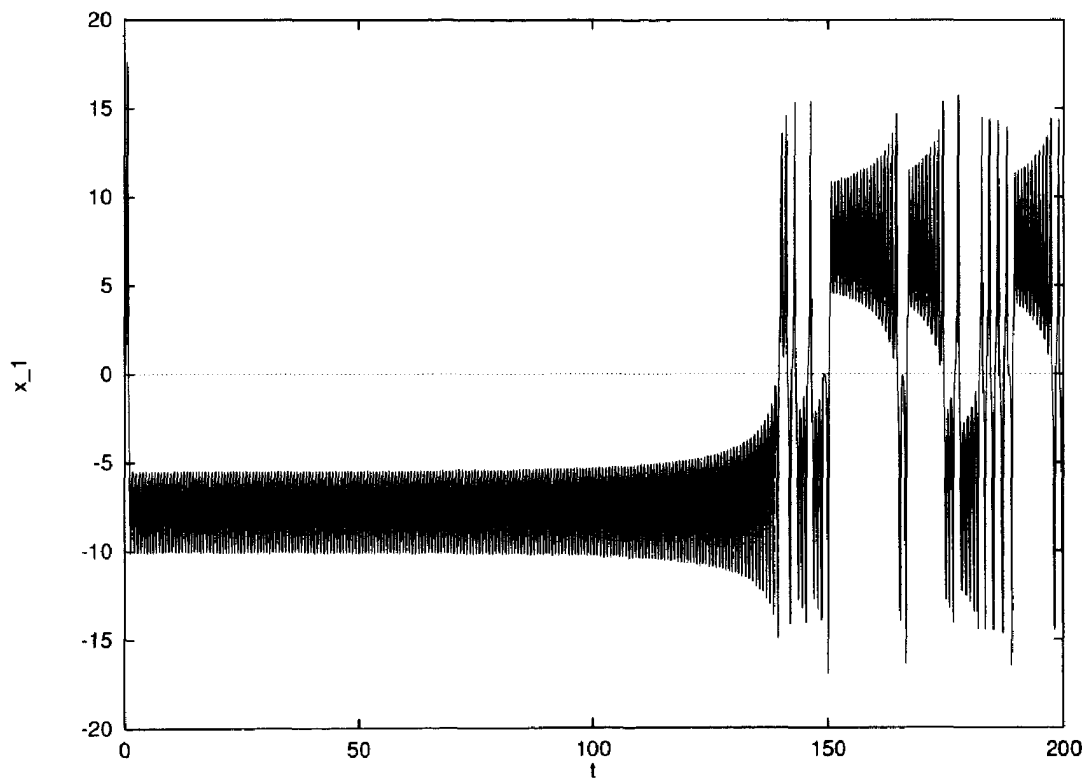


FIGURE 2A. Unstable manifold of the origin at $\lambda_1 = 24.06$, first component plotted over the time interval $[0, 200]$

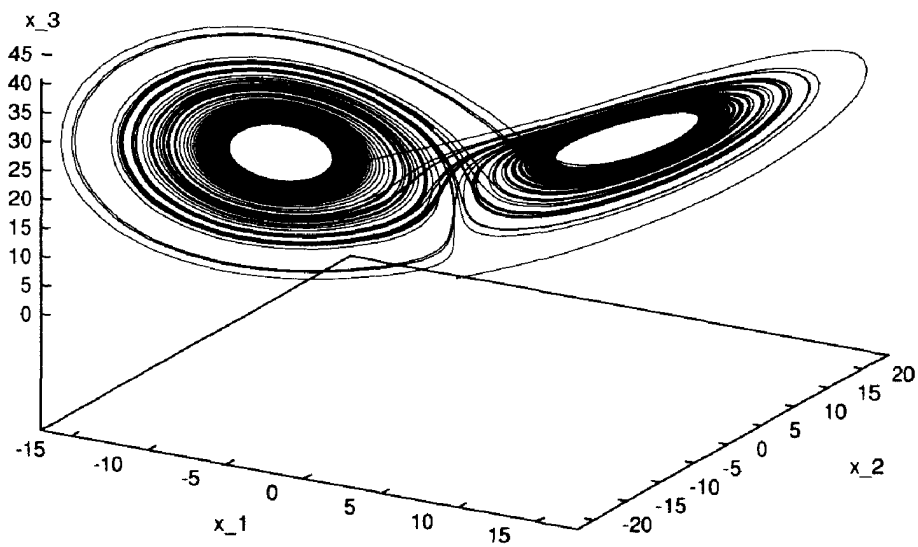


FIGURE 2B. The same as Figure 2a but plotted in phase space

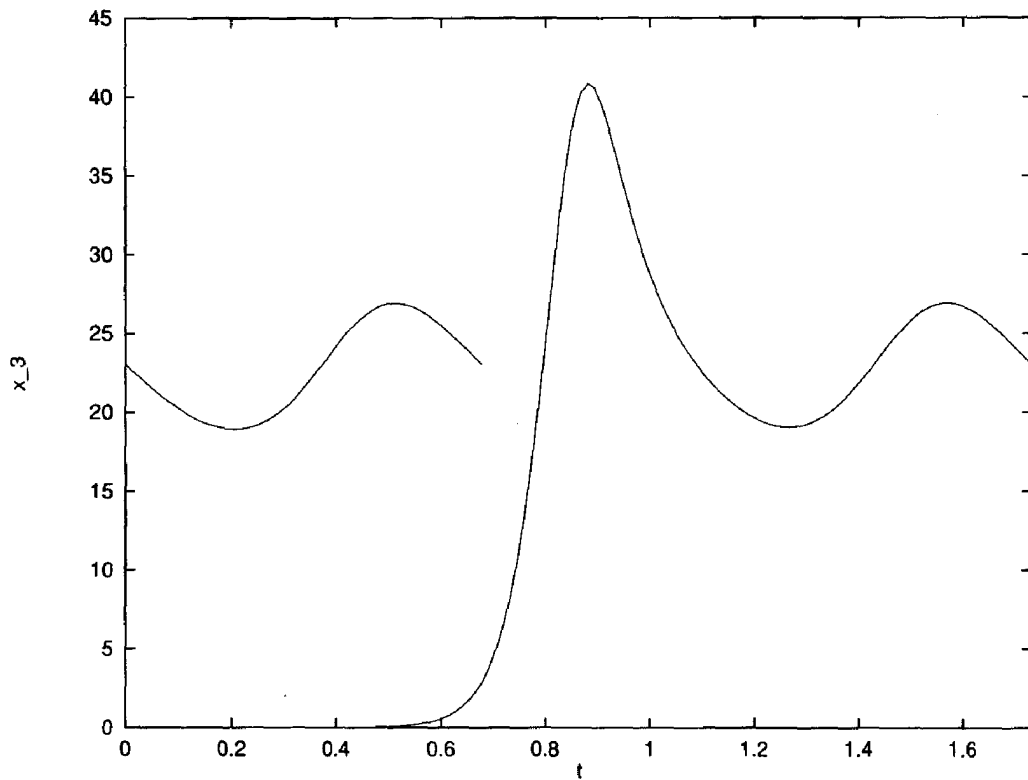


FIGURE 3. Third component of connecting and periodic orbit obtained numerically at $\lambda_A = 24.05790$

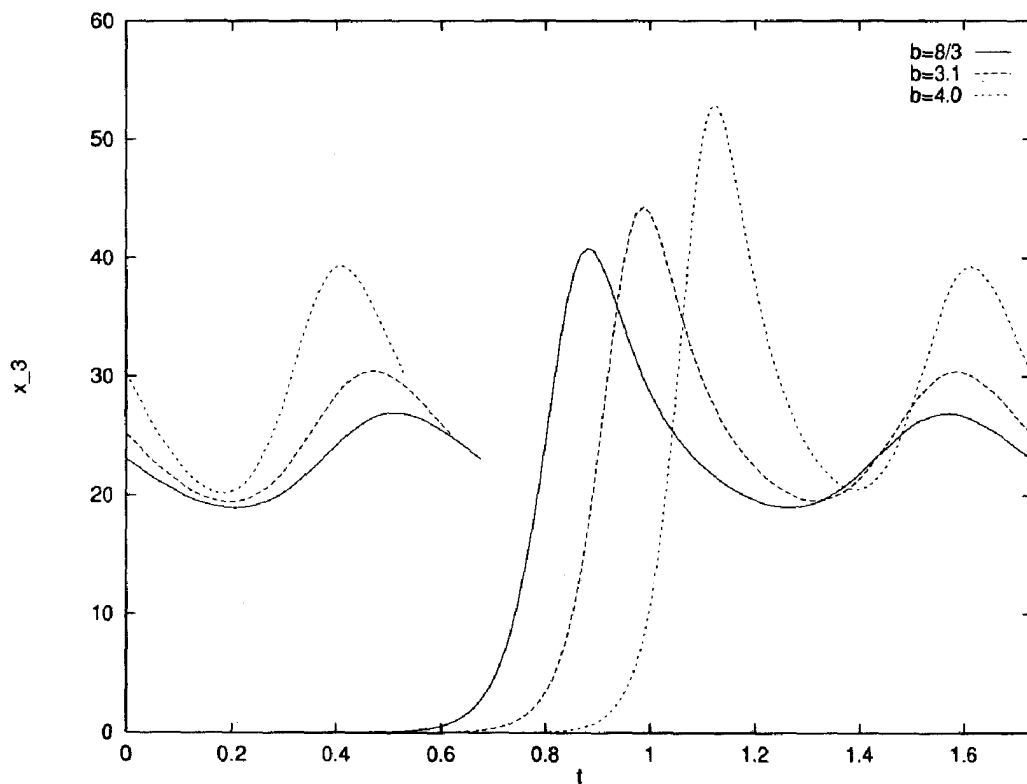


FIGURE 4. Third component of connecting and periodic orbits by continuation with respect to the parameter b .

$(b, \lambda_A) = (\frac{8}{3}, 24.05790), (3.1, 26.16990), (4.0, 31.29453)$