GLOBAL BIFURCATIONS AND THEIR NUMERICAL COMPUTATION

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ABSTRACT. Global bifurcations in dynamical systems often occur from homoclinic or heteroclinic orbits. The best known effect is the termination of a branch of periodic orbits at a homoclinic orbit. In this paper we extend our numerical approach to connecting orbits and the error analysis developed in [1]. The basic nondegeneracy condition is characterized by a geometric transversality condition. Further, the analysis of the error obtained by truncating to a finite interval is generalized in order to include periodic boundary conditions and to explain the superconvergence phenomenon with respect to the parameter as observed in [1].

1. INTRODUCTION

Global changes in the asymptotic regime of a parametrized dynamical system

$$\dot{x} = f(x,\lambda), \quad f \in C^1(\mathbb{R}^{m+p}, \mathbb{R}^m) \tag{1.1}$$

are often related to the appearance or disappearance of connecting orbits. Here we call a solution $(\bar{x}(t)(t\in\mathbb{R}), \bar{\lambda})$ of (1.1) a *connecting orbit pair* (COP) if

$$\bar{x}(t) \to \bar{x}_{\pm} \text{ as } t \to \pm\infty \quad \text{and } f(\bar{x}_{\pm}, \bar{\lambda}) = 0. \tag{1.2}$$

For the homoclinic case (i.e. $\bar{x}_- = \bar{x}_+$) the most important effect is the birth of periodic orbits or more complicated invariant sets (see e.g. Shil'nikov [12], Sparrow [13], Guckenheimer & Holmes [9], Glendinning [8] and section 3 below). Moreover, both homoclinic and heteroclinic ($\bar{x}_+ \neq \bar{x}_-$) orbits occur when determining the shape and speed of traveling waves in parabolic systems, (see e.g. Fife [6]).

In Beyn [1] we have introduced the notion of *nondegenerate connecting orbit pairs*. these turn out to be regular solutions of the infinite boundary value problem (1.1), (1.2) if a suitable phase condition is added:

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$$\Psi(x,\lambda) = 0 \tag{1.3}$$

In this paper we characterize this nondegeneracy by the transversal intersection of certain stable and unstable manifolds.

Moreover, we extend our numerical approach from [1], which was based on earlier work of de Hoog & Weiß [3], Keller & Lentini [10]. Here we consider approximating finite boundary value problems of the following general type

$$\dot{x} = f(x,\lambda) \text{ on } J = [T_-,T_+] \tag{1.4a}$$

$$B(x(T_-),\ x(T_+),\lambda) = 0 \tag{1.4b}$$

$$\Psi_J(x,\lambda) \qquad = 0 \tag{1.4c}$$

We assume (1.4c) to be a scalar condition and (1.4b) to be a set of $m+p-1$ so called *asymptotic boundary conditions*. The whole system (1.4) is then a boundary value problem of dimension $m+p$ for the unknowns (x,λ). The general form (1.4b) includes the most efficient projection conditions (see [1]) as well as the periodic b.c. which are convenient for the homoclinic one-parameter case. For the solutions of (1.4) and (1.1) - (1.3) we present a detailed error analysis. This will explain the superconvergence phenomenon for the parameter as observed in [1] and it will give us a theorem on the bifurcation of periodic orbits from a homoclinic orbit.

A related approach to connecting orbits has been developed by Friedman & Doedel [5],[7]. The main difference in their approach is that the stationary points $\bar{x}_\pm$ as well as certain eigenvalues and eigenvectors of the linearizations $f_x(\bar{x}_\pm,\bar{\lambda})$ are introduced as new unknowns into the system. This simplifies the implementation of the boundary conditions but also increases the dimension of the system and requires some a-priori knowledge about the structure of the spectrum. Other differences relate to the integral phase condition and the use of weighted Banach spaces in [5],[7] whereas here we employ the theory of exponential dichotomies Coppel [2], Palmer [11].

2. A WELL-POSED PROBLEM FOR CONNECTING ORBITS

The theory of exponential dichotomies [2],[11] turns out to be a useful tool when dealing with linearizations at connecting orbits. Therefore we start with some results on linear differential operators

$$Lx = \dot{x} - A(t)x,\quad x \in C^1(J,\mathbb{R}^m),\quad A \in C^0(J,\mathbb{R}^{m,m}) \tag{2.1}$$

where J is an open interval (T_-,T_+). We include the cases where

$T_- = -\infty$ or $T_+ = +\infty$ or both.

L is said to have an *exponential dichotomy* on J, if there is a fundamental solution matrix $Y(t)$, $t \in J$, a projector P in $\mathbb{R}^m$ and constants $K, \alpha_1, \alpha_2 > 0$ such that for all $t, s \in J$

$$\| Y(t) P Y(s)^{-1} \| \leq K \exp(-\alpha_1(t-s)) \quad \forall t \geq s \qquad (2.2a)$$

$$\| Y(t)(I-P) Y(s)^{-1} \| \leq K \exp(-\alpha_2(s-t)) \quad \forall s \geq t \qquad (2.2b)$$

For later estimates it is useful to introduce the *critical exponents* (α_s, α_u) of L given by

$$\alpha_s = \sup\{\alpha_1 > 0 : \text{ there exist } P, Y, \| \ \|, \alpha_2, K \text{ with } (2.2)\}$$

$$\alpha_u = \sup\{\alpha_2 > 0 : \text{ there exist } P, Y, \| \ \|, \alpha_1, K \text{ with } (2.2)\}$$

It is easy to show (see [2], p. 16) that for any given $\| \ \|$, $Y(t)$ and any $\varepsilon > 0$ there exists a projector P and a constant K such that (2.2a) and (2.2b) hold with $\alpha_1 = \alpha_s - \varepsilon$, $\alpha_2 = \alpha_u - \varepsilon$. For the constant coefficient case $Lx = \dot{x} - Ax$ with a hyperbolic matrix A, we may take $J = \mathbb{R}$ and find the critical exponents

$$\alpha_s = \text{Min}\{-\text{Re}\lambda : \lambda \text{ is an eigenvalue of } A \text{ with } \text{Re}\lambda < 0\} \qquad (2.3a)$$

$$\alpha_u = \text{Min}\{\text{Re}\lambda : \lambda \text{ is an eigenvalue of } A \text{ with } \text{Re}\lambda > 0\} \qquad (2.3b)$$

where $P = P_s$ resp. $I - P = P_u$ are the projectors onto the stable resp. unstable subspaces of A.

A close inspection of the roughness theorem [2] and Lemma 3.4 in [11] also shows that the exponential dichotomy on an interval (t_0, ∞) as well as the critical exponents are invariant under perturbations of $A(t)$ which vanish as $t \to \infty$. Combining these results gives us the first part of the following Lemma.

<u>Lemma 2.1</u>

For a linear operator $Lx = \dot{x} - A(t)x$ assume that $A(t) \to A_\pm$ as $t \to \pm\infty$ with hyperbolic matrices $A_\pm$ and let $\alpha_{\pm s}$, $\alpha_{\pm u}$ be the corresponding spectral bounds (2.3). Then, for any $t_0 \in \mathbb{R}$, L has an exponential dichotomy on both $(-\infty, t_0)$ and (t_0, ∞) with critical exponents $(\alpha_{-s}, \alpha_{-u})$ and $(\alpha_{+s}, \alpha_{+u})$. Moreover, if P_- and P_+ are projectors for which the dichotomy estimates (2.2) hold on $(-\infty, t_0)$ and (t_0, ∞) then

$$Y(t)P_{\pm}Y(t)^{-1} \to P_{\pm s} \text{ as } t \to \pm\infty \tag{2.4}$$

where $P_{\pm s}$ denotes the projector onto the stable subspace of $A_{\pm}$. Finally, the adjoint operator $L^*x = \dot{x} + A^T(t)x$ also has exponential dichotomies on $(-\infty, t_0)$ and (t_0, ∞) with projectors $I - P_-^T$ and $I - P_+^T$. The critical exponents are $(\alpha_{-u}, \alpha_{-s})$ and $(\alpha_{+u}, \alpha_{+s})$.

The last statement follows immediately from (2.2) and the fact that $Y^{-1T}(t)$ is a fundamental matrix of L^*.

In our next step we introduce the Banach spaces

$$X^k(J) = \{x \in C^k(J, \mathbb{R}^m) : x^{(j)}(t) \text{ converges as } t \to T_{\pm} \text{ for } j = 0, \ldots, k\}$$

$$\|x\|_k = \sum_{j=0}^{k} \sup\{\|x^{(j)}(t)\| : t \in J\}$$

The solutions of linear inhomogenous initial value problems in these spaces are described in the following Lemma (see [2] and [1], Appendix for a proof).

Lemma 2.2

Under the assumptions of Lemma 2.1 the general solution $x \in X^1(t_0, \infty)$ of $Lx = r \in X^0(t_0, \infty)$ is given by $x(t) = Y(t)Y(t_0)^{-1}\xi + (G_+ r)(t)$ where $\xi \in R(P_+)$ and $(G_+ r)(t) = \int_{t_0}^{t} Y(t)PY(s)^{-1}r(s)ds - \int_{t}^{\infty} Y(t)(I-P)Y(s)^{-1}r(s)ds$

Finally we recall from [1] that a COP $(\bar{x}, \bar{\lambda}) \in X^1(\mathbb{R}) \times \mathbb{R}^p$ of the system (1.1) is called *nondegenerate* if the following conditions hold

$$A_{\pm} := f_x(\bar{x}_{\pm}, \bar{\lambda}) \text{ is hyperbolic with stable dimension } m_{\pm s} \tag{2.5}$$

$$p = m_{-s} - m_{+s} + 1 \tag{2.6}$$

the only solutions $(y, \mu) \in X^1(\mathbb{R}) \times \mathbb{R}^p$ of the variational system $\dot{y} = f_x(\bar{x}, \bar{\lambda})y + f_{\lambda}(\bar{x}, \bar{\lambda})\mu$ are $y = c\dot{\bar{x}} (c \in \mathbb{R}),\ \mu = 0$ (2.7)

A nondegenerate COP $(\bar{x}, \bar{\lambda})$ was shown in [1] to be a regular solution of the operator equation

$$\dot{x} - f(x, \lambda) = 0,\ \Psi(x, \lambda) = 0 \tag{2.8}$$

provided the phase condition $\Psi \in C^1(X^0(\mathbb{R}) \times \mathbb{R}^p, \mathbb{R})$ satisfies

$$\Psi(\bar{x}, \bar{\lambda}) = 0,\ \Psi_x(\bar{x}, \bar{\lambda})\dot{\bar{x}} \neq 0 \tag{2.9}$$

The following Theorem gives a geometrical equivalent of the nondegeneracy condition (2.7). First we apply the implicit function theorem to obtain locally unique stationary points $x_{\pm}(\lambda)$ of (1.1) such that

$x_\pm(\bar{\lambda}) = \bar{x}_\pm$.

Theorem 2.3

Let $(\bar{x},\bar{\lambda}) \in X^1(\mathbb{R}) \times \mathbb{R}^P$ be a COP of the system (1.1) which satisfies (2.5) and (2.6). Then $(\bar{x},\bar{\lambda})$ is nondegenerate if and only if the immersed stable and unstable manifolds

$$M_{+s} = \{(x,\lambda) : \|\lambda - \bar{\lambda}\| < \varepsilon,\ \Phi(t,x,\lambda) \to x_+(\lambda) \text{ as } t \to \infty\}$$
$$M_{-u} = \{(x,\lambda) : \|\lambda - \bar{\lambda}\| < \varepsilon,\ \Phi(t,x,\lambda) \to x_-(\lambda) \text{ as } t \to -\infty\} \quad (2.10)$$

intersect transversely at any point $(\bar{x}(t_0),\bar{\lambda})$, $t_0 \in \mathbb{R}$, of the COP.

Remark: In (2.10) we have just given the set definition of M_{+s}, M_{-u} by using the t-flow $\Phi(t,\cdot,\lambda)$ of (1.1). The manifold structure will be made precise in the following proof. For a two-dimensional saddle-saddle connection the transverse intersection is illustrated in Figure 1.

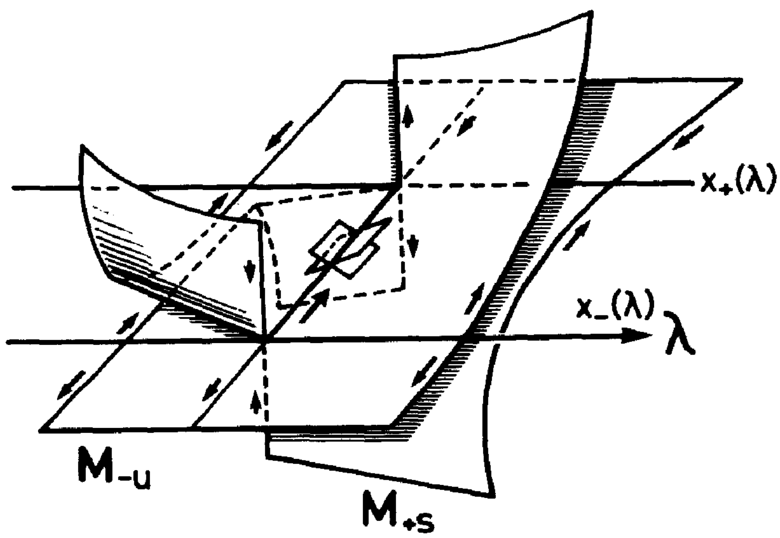

Figure 1. Illustration of transversal intersection for the two-dimensional heteroclinic case.

Proof: We fix some $t_0 \in \mathbb{R}$ and apply Lemma 2.1 to the linearization $Lx = \dot{x} - f_x(\bar{x},\bar{\lambda})x$. The parametrization of M_{+s} near $(\bar{x}(t_0),\bar{\lambda})$ will be obtained from the initial value problem

$$\dot{y} = f(y,\lambda) \text{ in } [t_0,\infty),\quad P_+(y(t_0) - \bar{x}(t_0)) = \xi \in R(P_+) \quad (2.11)$$

In fact, this may be written as an operator equation for $(y,\xi,\lambda) \in X^1(t_0,\infty) \times R(P_+) \times \mathbb{R}^P$. Using Lemma 2.2 we can invoke the implicit function theorem to obtain solutions $y(\cdot,\xi,\lambda) \in X^1(t_0,\infty)$ with ξ and $\lambda - \bar{\lambda}$ close to zero such that $y(t,0,\bar{\lambda}) = \bar{x}(t)$. M_{+s} is then locally parametrized by $(y(t_0,\xi,\lambda),\lambda)$ with tangent space

$$T_1 := T_{(\bar{x}(t_0),\bar{\lambda})}M_{+s} = \{(\xi + z_+(t_0)\lambda,\lambda) : \xi \in R(P_+), \lambda \in \mathbb{R}^P\}$$

where $z_+(t) = (G_+ f_\lambda(\bar{x},\bar{\lambda}))(t)$. Similarly, we find

$$T_2 := T_{(\bar{x}(t_0),\bar{\lambda})}M_{-u} = \{(\eta + z_-(t_0)\lambda,\lambda) : \eta \in N(P_-),\ \lambda \in \mathbb{R}^P\}$$

where
$$z_-(t) = (G_- f_\lambda(\bar{x},\bar{\lambda}))(t) = \int_{-\infty}^{t} Y(t)P_-Y(s)^{-1} f_\lambda(\bar{x}(s),\bar{\lambda})ds - \int_t^{t_0} Y(t)(I - P_-)Y(s)^{-1} f_\lambda(\bar{x}(s),\bar{\lambda})ds.$$

From $\dot{\bar{x}} \in N(L)$ we conclude $\dot{\bar{x}}(t_0) \in R(P_+) \cap N(P_-)$, hence $(\dot{\bar{x}}(t_0),0) \in T_1 \cap T_2$ and $\dim (T_1 \cap T_2) \geq 1$. From (2.6) we obtain $\dim (T_1 + T_2) = \dim(T_1) + \dim(T_2) - \dim(T_1 \cap T_2) = m_{+s} + p + m - m_{-s} + p - \dim(T_1 \cap T_2) = m + p + 1 - \dim(T_1 \cap T_2) \leq m + p$. Thus the transversality condition $T_1 + T_2 = \mathbb{R}^{m+p}$ is equivalent to

$$T_1 \cap T_2 = \operatorname{span}\{(\dot{\bar{x}}(t_0),0)\} \tag{2.12}$$

Let us now assume that $(\bar{x},\bar{\lambda})$ is nondegenerate and consider an element

$$(\xi + z_+(t_0)\mu,\mu) = (\eta + z_-(t_0)\mu,\mu) \in T_1 \cap T_2 \tag{2.13}$$

where $\xi \in R(P_+)$, $\eta \in N(P_-)$. Then the function

$$y(t) = \begin{cases} z_+(t)\mu + Y(t)Y(t_0)^{-1}\xi\ , & t \geq t_0 \\ z_-(t)\mu + Y(t)Y(t_0)^{-1}\eta\ , & t < t_0 \end{cases} \tag{2.14}$$

is continuous at t_0 and satisfies $Ly = f_\lambda(\bar{x},\bar{\lambda})\mu$. Hence $\mu = 0$ and $y = c\dot{\bar{x}}$ for some $c \in \mathbb{R}$ by (2.7) and we have $(\xi + z_+(t_0)\mu,\mu) = (c\dot{\bar{x}}(t_0),0)$. Conversely, assume (2.12) and let $y \in X^1(t_0,\infty)$, $\mu \in \mathbb{R}$ satisfy $Ly = f_\lambda(\bar{x},\bar{\lambda})\mu$. Then by Lemma 2.2 and the analogous result on $(-\infty,t_0)$ we may write y as in (2.14) for some $\xi \in R(P_+)$, $\eta \in N(P_-)$. The continuity of y now yields (2.13) and we find $\mu = 0$, $y(t_0) = c\dot{\bar{x}}(t_0)$ from our assumption. Finally, (2.14) gives us $y(t) = c\ Y(t)Y(t_0)^{-1}\dot{\bar{x}}(t_0) = c\dot{\bar{x}}(t)$ for all $t \in \mathbb{R}$ □

3. THE APPROXIMATION ERROR

We consider numerical approximations of a COP obtained by solving the finite boundary value problem (1.4). Condition (1.4c) is a finite phase condition, such as

$$\Psi_J(x,\lambda) = \int_{T_-}^{T_+} \dot{x}_0(t)^T(x(t) - x_0(t))dt, \tag{3.1}$$

where x_0 is an initial approximation (e.g. given by the last solution on a branch of COP's). For more general integral conditions see [5], [7].

A very efficient type of asymptotic boundary conditions (1.4b) are the so called *projection boundary conditions* which have been analyzed in detail in [1],[3]. They are defined by

$$B(x(T_-),\ x(T_+),\lambda) = B_-(\lambda)(x(T_-) - x_-(\lambda)),\ B_+(\lambda)(x(T_+) - x_+(\lambda))) \tag{3.2}$$

where the rows of $B_-(\lambda) \in \mathbb{R}^{m_{-s},m}$ resp. $B_+(\lambda) \in \mathbb{R}^{m_{+u},m}$ form a basis of the stable subspace of $f_x^T(x_-(\lambda),\lambda)$ resp. the unstable subspace of $f_x^T(x_+(\lambda),\lambda)$. Notice that $m_{+u} + m_{-s} = m + p - 1$ follows from (2.6) and that the projection conditions force the endpoints $x(T_\pm)$ to lie in the linearized stable and unstable manifolds of $x_\pm(\lambda)$. Of course, (3.2) requires to compute the stationary solutions $x_\pm(\lambda)$ as well as smooth bases of stable and unstable subspaces (see [1] for more details on the implementation).

In the homoclinic case ($p = 1$) a simple alternative are periodic boundary conditions

$$B(x(T_-),\ x(T_+),\lambda) = x(T_+) - x(T_-) \tag{3.3}$$

and this has been extensively used by Doedel & Kernevez [4]. However, the periodicity condition introduces larger errors than the projection conditions.

For an illustration of these errors we consider the following two-dimensional system from [1]

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_1 - x_1^2 + \lambda x_2 + \mu x_1 x_2 \tag{3.4}$$

For fixed μ this system has at some $\bar{\lambda} = \bar{\lambda}(\mu)$ a homoclinic orbit based at the origin. For $\mu = 0.5$ and the simple phase condition $x_2(0) = 0$, table 1 shows the errors

$e_x(T) = \sup\{\|\bar{x}(t) - x_T(t)\| : |t| \le T\}$, $e_\lambda(T) = |\bar{\lambda} - \lambda_T|$

where x_T, λ_T denotes the solution of (1.4) on [-T,T]. Actually, the solution on [-15,15] was taken as exact solution and all finite boundary value problems were solved at high accuracy ($\sim 10^{-15}$) so that the error arising from truncation to a finite interval becomes dominant. Anticipating an error behaviour $O(e^{-\delta T})$ we also display the terms $\delta_{x,\lambda}(T) = \ln e_{x,\lambda}(T-1) - \ln e_{x,\lambda}(T)$.

T	$e_x(T)$	$\delta_x(T)$	$e_\lambda(T)$	$\delta_\lambda(T)$	
5	1.01 E-3	1.59	3.24 E-5	2.57	
6	2.02 E-4	1.61	2.13 E-6	2.72	projection
7	4.03 E-5	1.61	1.31 E-7	2.79	b.c.
8	8.03 E-6	1.61	7.79 E-9	2.82	
9	1.60 E-6	1.62	4.57 E-10	2.84	
5	1.02 E-1	0.863	5.15 E-3	1.45	
6	4.53 E-2	0.813	1.11 E-3	1.53	periodic
7	2.02 E-2	0.806	2.30 E-4	1.58	b.c.
8	9.03 E-3	0.806	4.66 E-5	1.60	
9	4.03 E-3	0.807	9.33 E-6	1.61	

Table 1

We note that $\bar{\lambda} = -0.429505849$ and that $f_x(0,\bar{\lambda})$ has eigenvalues $-\alpha_s = -1.237552425$, $\alpha_u = 0.808046576$. Table 1 then suggests that the exponents for the x-error are $2\alpha_u$ for the projection conditions and α_u for the periodic conditions. The λ-error, however, shows a superconvergence with exponents $2\alpha_u + \alpha_s$ and $2\alpha_u$ respectively. These effects will be explained by the following approximation theorem which is a generalization of the corresponding result [1], Theorem 3.2. We will write $J \to \mathbb{R}$ for $J = (T_-, T_+)$, $T_\pm \to \pm\infty$ and we use the phrase 'J sufficiently large' correspondingly.

<u>Theorem 3.1</u>

Let $(\bar{x},\bar{\lambda})$ be a nondegenerate COP of (1.1) with endpoints $\bar{x}_\pm$ and assume $f \in C^2(\mathbb{R}^{m+p}, \mathbb{R}^m)$. Further assume

A1: $B \in C^2(\mathbb{R}^{2m+p}, \mathbb{R}^{m+p-1})$, $B(\bar{x}_-, \bar{x}_+, \bar{\lambda}) = 0$ and the matrix $(\bar{B}_- C_{-s}, \bar{B}_+ C_{+u}) \in \mathbb{R}^{m+p-1, m+p-1}$ is nonsingular. Here $\bar{B}_\pm = \frac{\partial B}{\partial x(T_\pm)}(\bar{x}_-, \bar{x}_+, \bar{\lambda})$ and the columns of the matrices $C_{+u} \in \mathbb{R}^{m,m+u}$ resp. $C_{-s} \in \mathbb{R}^{m,m-s}$ form a basis of the unstable resp. stable subspace of $f_x(\bar{x}_\pm, \bar{\lambda})$.

A2: $\Psi_J \in C^2(X^o(J) \times \mathbb{R}^P, \mathbb{R})$, $\Psi_J(\bar{x}_{|J}, \bar{\lambda}) \to 0$ as $J \to \mathbb{R}$, $|\Psi_{J,x}(\bar{x}_{|J}, \bar{\lambda})\dot{\bar{x}}_{|J}| \geq g > 0$ for all large J and the derivatives Ψ_J', Ψ_J'' are bounded uniformly in J in some tube K_δ, $\delta > 0$ (see (3.5)).

Then there exists a constant $\rho \leq \delta$ such that the boundary value problem (1.4) has a unique solution (x_J, λ_J) in

$$K_\rho = \{(x,\lambda) \in X^1(J) \times \mathbb{R}^P : \|x - \bar{x}_{|J}\|_1 + \|\lambda - \bar{\lambda}\| \leq \rho\} \tag{3.5}$$

for sufficiently large J. Moreover, there exist $\tau_J \to 0$ as $J \to \mathbb{R}$ and to any $\varepsilon > 0$ a constant C_ε such that the following estimates hold with $d = 1$ and $\bar{y}_J(t) = \bar{x}(t + \tau_J)$

$$\|\bar{y}_J - x_J\|_1 \leq C_\varepsilon \{\exp(-(d\alpha_{+s} - \varepsilon)T_+) + \exp(-(d\alpha_{-u} - \varepsilon)|T_-|)\} \tag{3.6}$$

$$\begin{aligned} \|\bar{\lambda} - \lambda_J\| &\leq C_\varepsilon \{\exp(-(\delta_+ - \varepsilon)T_+) + \exp(-(\delta_- - \varepsilon)|T_-|)\}, \\ \delta_+ &= \mathrm{Min}(2d\alpha_{+s}, d\alpha_{+s} + \alpha_{+u}), \delta_- = \mathrm{Min}(2d\alpha_{-u}, d\alpha_{-u} + \alpha_{-s}). \end{aligned} \tag{3.7}$$

Here $\alpha_{\pm s}$ and $\alpha_{\pm u}$ are the spectral bounds for $f_x(\bar{x}_\pm, \bar{\lambda})$ as in (2.3). Finally, for the projection conditions the above estimates hold with $d = 2$.

Remark: The phase shift τ_J is constructed in such a way that $\Psi_J(\bar{y}_J, \bar{\lambda}) = 0$ holds. If we replace $\bar{y}_J$ by $\bar{x}_{|J}$ in (3.6) then error terms depending on Ψ_J will appear (see [1]).

Before proceeding to the proof we notice some important consequences of Theorem 3.1. Assumption A1 is satisfied for the projection conditions if f is in C^3 and A2 is a rather mild though technical assumption on Ψ_J

which is satisfied in standard cases. For our example above we have $\alpha_u < \alpha_s < 2\alpha_u$, thus (3.6) gives the exponent $2\alpha_u$ and (3.7) the exponent $2\alpha_u + \alpha_s$.

Similarly, A1 is automatically satisfied in the homoclinic case (p=1) for periodic boundary conditions and we obtain the exponents α_u and $2\alpha_u$ for our example as in Table 1. In the periodic case, Theorem 3.1 also yields a general result on bifurcation of periodic orbits from a homoclinic orbit. We take $J = (-T,T)$ and the simple phase condition

$$\Psi_J(x,\lambda) = \dot{\bar{x}}(0)^T(x(0) - \bar{x}(0)) \tag{3.8}$$

Then A2 is satisfied and we also have $\tau_J = 0$ since $\Psi_J(\bar{x}|_{[-T,T]},\bar{\lambda}) = 0$

<u>Corollary 3.2</u>

Let $(\bar{x},\bar{\lambda})$ be a nondegenerate homoclinic orbit pair of a one-parameter dynamical system (1.1) with f in C^2. Then there exists a $T_0 > 0$ and a branch of 2T-periodic orbits $x_T \in X^1(-T,T), \lambda_T \in \mathbb{R}$ $(T \geq T_0)$ which after a suitable choice of phase satisfy the estimates

$$\|\bar{x}|_{[-T,T]} - x_T\|_1 \leq C_\varepsilon \exp(-(\mathrm{Min}(\alpha_s,\alpha_u) - \varepsilon)T)$$

$$|\bar{\lambda} - \lambda_T| \leq C_\varepsilon \exp(-(2\mathrm{Min}(\alpha_s,\alpha_u) - \varepsilon)T)$$

where α_s,α_u are the spectral bounds for $f_x(\bar{x}_+,\bar{\lambda})$ according to (2.3).

Even in the two-dimensional case this result is more general than the standard global bifurcation theorems. For example, the saddle connection bifurcation in [9], Ch. 6 requires a nonvanishing trace in addition to our nondegeneracy condition (which is formulated there in a geometric way similar to Theorem 2.3).

<u>Proof of Theorem 3.1.</u> We merely sketch the basic steps of the proof which is very similar to that of Theorem 3.2 in Beyn [1]. Let us write (1.4) as an operator equation

$$F_J(x,\lambda) = 0 \tag{3.9}$$

where $F_J : X^1(J) \times \mathbb{R}^p \to X^0(J) \times \mathbb{R}^{m+p}$ is defined by the left hand sides of (1.4). The unique solvability of (3.9) in some K_ρ follows from a local contraction theorem (see [1], Lemma 3.1) provided we have a uniform bound for the inverse of the Frechet derivative $F_J'(\bar{x}|_J,\bar{\lambda})$. For that purpose we set

$$Lx = \dot{x} - f_x(\bar{x}|_J,\bar{\lambda})x,\ \bar{B}_\pm = \frac{\partial B}{\partial x(T_\pm)}(\bar{x}(T_-),\ \bar{x}(T_+),\bar{\lambda})$$

$$\bar{B}_\lambda = \frac{\partial B}{\partial \lambda}(\bar{x}(T_-),\ \bar{x}(T_+),\bar{\lambda}),\ \bar{f}_\lambda = f_\lambda(\bar{x}|_J,\bar{\lambda})$$

and consider the variational equations

$$Ly - \bar{f}_\lambda \mu = r \in X^0(J) \tag{3.10a}$$

$$\bar{B}_- y(T_-) + \bar{B}_+ y(T_+) + \bar{B}_\lambda \mu = b \in \mathbb{R}^{m+p-1} \tag{3.10b}$$

$$\Psi_J'(\bar{x}|_J,\bar{\lambda})(y,\mu) = \beta \in \mathbb{R} \tag{3.10c}$$

For these we show an estimate uniformly in J

$$\begin{pmatrix} \|y\|_1 \\ \|\mu\| \end{pmatrix} \le C_\varepsilon \begin{pmatrix} 1 & 1 \\ 1 & \gamma(\varepsilon,J) \end{pmatrix} \begin{pmatrix} \|r\|_0 \\ \|b\| + |\beta| \end{pmatrix} \tag{3.11}$$

where $\gamma(\varepsilon,J) = \exp(-(\alpha_{+u}-\varepsilon)\ T_+) + \exp(-(\alpha_{-s}-\varepsilon)|T_-|)$.

First we take a matrix function $\Phi = (\varphi_1,\ldots,\varphi_p)$ such that the columns $\varphi_i \in X^1(\mathbb{R})$ form a basis of $N(L^*)$(see [1] Prop. 2.4). By Lemma 2.1 these satisfy an estimate

$$\|\varphi_i(t)\| \le C_\varepsilon \begin{cases} \exp(-(\alpha_{-s}-\varepsilon)t), & t \le 0 \\ \exp(-(\alpha_{+u}-\varepsilon)t), & t \ge 0 \end{cases} \tag{3.12}$$

We multiply (3.10a) by $\Phi^T(t)$, integrate over J and find by partial integration

$$\Phi^T(t)y(t)\Big|_{T_-}^{T_+} - \int_{T_-}^{T_+}\Phi^T(t)\bar{f}_\lambda(t)dt\mu = \int_{T_-}^{T_+}\Phi^T(t)r(t)dt$$

Using (3.12) and the nondegeneracy of $(\bar{x},\bar{\lambda})$([1], Prop. 2.4) this gives us the estimate

$$\|\mu\| \le C_\varepsilon\ (\|r\|_0 + \gamma(\varepsilon,J)\ \|y\|_0) \tag{3.13}$$

which is the key to the superconvergence phenomenon. In much the same way as in ([1], Appendix) we use A1 to find an estimate $\|y\|_1 \le C(\|r\|_0 + \|b\| + |\beta|)$. Combining this with (3.13) yields (3.11).

In the next step we replace $\gamma(\varepsilon,J)$ by 1 in (3.11) and then obtain an inequality

$$\|\bar{y}_J - x_J\|_1 + \|\bar{\lambda} - \lambda_J\| \le C_\varepsilon \|B(\bar{y}_J(T_-),\ \bar{y}_J(T_+),\ \bar{\lambda})\| =: \sigma(\varepsilon,J) \tag{3.14}$$

as in ([1], Theorem 3.2). Using A1 and the exponential approach of the connecting orbit towards the stationary points we find

$$\sigma(\varepsilon,J) = O(\exp(-(d\alpha_{+s} - \varepsilon)T_+) + \exp(-(d\alpha_{-u} - \varepsilon)|T_-|)) \qquad (3.15)$$

which proves (3.6).

For the refined estimate (3.7) we first notice that (3.11) also holds with $\bar{y}_J$ in place of $\bar{x}_{|J}$. We then apply (3.11) with $y = \bar{y}_J - x_J$, $\mu = \bar{\lambda} - \lambda_J$ and find for the right hand sides by the C^2-smoothness of f,B and Ψ_J

$$\|r\|_0 = \|f(\bar{y}_J,\bar{\lambda}) - f(x_J,\lambda_J) - f_x(\bar{y}_J,\bar{\lambda})(\bar{y}_J - x_J) - f_\lambda(\bar{y}_J,\bar{\lambda})(\bar{\lambda} - \lambda_J)\|_0$$

$$\le C_\varepsilon(\|\bar{y}_J - x_J\|_0 + \|\bar{\lambda} - \lambda_J\|)^2 \le C_\varepsilon \sigma(\varepsilon,J)(\|\bar{y}_J - x_J\|_0 + \|\bar{\lambda} - \lambda_J\|)$$

$$\|b\| \le \|B(\bar{y}_J(T_+),\bar{y}_J(T_-),\bar{\lambda})\| + C_\varepsilon(\|\bar{y}_J - x_J\|_0 + \|\bar{\lambda} - \lambda_J\|)^2$$

$$\le \sigma(\varepsilon,J)(1 + C_\varepsilon(\|\bar{y}_J - x_J\|_0 + \|\bar{\lambda} - \lambda_J\|))$$

$$|\beta| \le C_\varepsilon \sigma(\varepsilon,J)(\|\bar{y}_J - x_J\|_0 + \|\bar{\lambda} - \lambda_J\|).$$

Inserting this into (3.11) and taking the terms involving $\|\bar{y}_J - x_J\|_0 + \|\bar{\lambda} - \lambda_J\|$ to the left we end up with

$$\begin{pmatrix} \|\bar{y}_J - x_J\|_1 \\ \|\bar{\lambda} - \lambda_J\| \end{pmatrix} \le C_\varepsilon \begin{pmatrix} 1 & 1 \\ 1 & \sigma(\varepsilon,J) + \gamma(\varepsilon,J) \end{pmatrix} \begin{pmatrix} 0 \\ \sigma(\varepsilon,J) \end{pmatrix} = \begin{pmatrix} \sigma(\varepsilon,J) \\ (\sigma^2 + \sigma\gamma)(\varepsilon,J) \end{pmatrix}$$

Now the exponential terms for σ and γ yield the final estimate

□

We remark that further numerical examples with connecting orbits in spaces of dimension greater than two appear in [1]. There we have employed the projection boundary conditions (3.2) and the integral phase condition (3.1). Moreover, in that paper we developed an adaptive strategy for choosing the finite interval which partially was based on qualitative error estimates as in Theorem 3.1 and partially on further numerical observations of the error behaviour at the boundary.

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