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The Numerical Computation of Connecting Orbits in Dynamical Systems

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Structural changes in dynamical systems are often related to the appearance or disappearance of orbits connecting two stationary points (either heteroclinic or homoclinic). Homoclinic orbits typically arise in one-parameter problems when on a branch of periodic solutions the periods tend to infinity (e.g. Guckenheimer & Holmes, 1983). We develop a direct numerical method for the computation of connecting orbits and their associated parameter values. We employ a general phase condition and truncate the boundary-value problem to a finite interval by using on both ends the technique of asymptotic boundary conditions; see, for example, de Hoog & Weiss (1980), Lentini & Keller (1980). The approximation error caused by this truncation is shown to decay exponentially. Based on this analysis and additional numerical investigations we set up an adaptive strategy for choosing the truncation interval.

1. Introduction

WE CONSIDER parametrized dynamical systems of the form

$$\dot{x} = f(x, \lambda), \qquad x(t) \in \mathbb{R}^m, \qquad \lambda \in \mathbb{R}^p. \tag{1.1}$$

Global changes in the phase portraits of such systems are often related to the appearance or disappearance of separatrices. These separatrices consist of orbits which connect one or several stationary points. We will call a solution $\bar{x}(t)$, for $-\infty < t < \infty$, of (1.1) at $\lambda = \bar{\lambda}$ a connecting orbit if the limits

$$x_- = \lim_{t \to -\infty} \bar{x}(t), \qquad x_+ = \lim_{t \to \infty} \bar{x}(t)$$

exist. For continuous f these must be stationary points, i.e.

$$f(x_-, \bar{\lambda}) = f(x_+, \bar{\lambda}) = 0.$$

In the case $x_+ = x_-$ we have a homoclinic and in the case $x_+ \neq x_-$ a heteroclinic orbit. The pair $(\bar{x}, \bar{\lambda})$ will be called a *connecting orbit pair* (COP).

Homoclinic orbits typically arise as limiting cases of periodic orbits which attain infinite period but stay bounded in phase space (see, for example, Guckenheimer & Holmes (1983), Glendinning (1988) for the theoretical background and Doedel & Kernevez (1986), Rinzel (1985) for some interesting applications to chemical and biological systems). Further interesting phenomena may occur near homoclinic pairs, such as the appearance of strange attracting sets in the Lorenz system and in Shil'nikov-type systems (Sparrow, 1982; Shil'nikov, 1965; Glendinning, 1988). Finally, we mention the significance of connecting orbits for determining the shape and speed of travelling waves in parabolic systems (see, for example, Doedel & Kernevez (1986: §8), Fife (1979)).

The subject of this paper is the numerical computation of connecting orbit pairs. Since this involves the solution of a boundary-value problem on the real line, we have to truncate the problem to a finite interval in order to make standard codes applicable. At both ends of the interval we use the asymptotic boundary conditions which have been developed for the semi-infinite case by de Hoog & Weiss (1980) and Lentini & Keller (1980) (see also the recent account in Ascher, Mattheij, & Russell (1988)). The so-called projection conditions (de Hoog & Weiss, 1980) prove to be the most effective type of boundary conditions both theoretically and numerically (see Sections 3 and 4). Several applications will be presented in Section 4. In particular, we set up a strategy for an adaptive choice of the truncation interval. This strategy is based on a careful theoretical as well as numerical study of the approximation error. Further, some details of the implementation are discussed, such as the choice of phase condition and the realization of smooth projection conditions, and a comparison is made with the method of Doedel & Kernevez (1986) who approximate homoclinic orbits by periodic orbits of large but fixed period.

It is also our concern to develop a sound theoretical basis for our numerical approach. For that purpose we introduce in Section 2 the notion of a *nondegenerate connecting orbit pair*. This pair may then be regarded as a regular (or isolated) solution of a suitable operator equation. This equation includes a phase fixing condition of a very general type as has been discussed for periodic orbits by Keller & Jepson (1984).

In Section 3 we analyse the error introduced by the truncation of the interval and by the asymptotic boundary conditions. Our results are strongly related to (but do not directly follow from) those of de Hoog & Weiss (1980) and Markowich (1982). Also, in our proofs (given in the Appendices) we prefer to utilize the theory of exponential dichotomies (Coppel, 1978), in particular the results of Palmer (1984). This theory provides the appropriate tools for the treatment of connecting orbits.

Some preliminary results of this paper were communicated in Beyn (1987). We would also like to mention a related approach to connecting orbits in recent papers by Doedel & Friedman (1990a,b).

2. Nondegenerate connecting orbits

We start with some arguments on the number of parameters in (1.1) which are necessary for a connecting orbit to occur. First consider a parameter independent system in $\mathbb{R}^m$

$$\dot{x} = f(x). \tag{2.1}$$

Let x_-, x_+ be stationary points with stable manifolds $\mathcal{M}_s(x_-)$, $\mathcal{M}_s(x_+)$ and unstable manifolds $\mathcal{M}_u(x_-)$, $\mathcal{M}_u(x_+)$. Let the dimensions of these manifolds be

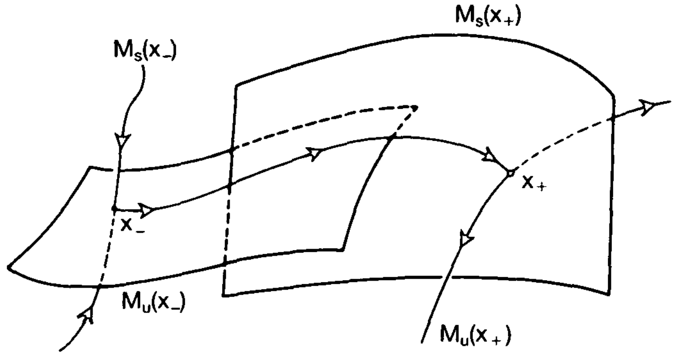

FIG. 1. A heteroclinic orbit in $\mathbb{R}^3$ which satisfies condition (2.2).

m_{-s}, m_{+s} and m_{-u}, m_{+u} where $m_{\pm u} = m - m_{\pm s}$ (see Fig. 1 and Guckenheimer & Holmes (1983: p. 50)). Whenever $\mathcal{M}_u(x_-)$ and $\mathcal{M}_s(x_+)$ intersect in at least one point they intersect in at least one orbit by uniqueness of solutions to the initial value problem. Therefore, we expect their intersection to be exactly one orbit, if the sum of their dimensions is $m+1$, i.e.

$$m_{-u} + m_{+s} = m + 1. \tag{2.2}$$

If $m_{-u} + m_{+s} > m + 1$ then the separatrix $\mathcal{M}_u(x_-) \cap \mathcal{M}_s(x_+)$ has dimension $m_{-u} + m_{+s} - m \geq 2$ and we have to introduce further conditions for parametrizing it. This case will not be considered here. Rather we are interested in the case $m_{-u} + m_{+s} < m + 1$. Then a connecting orbit is not a structurally stable phenomenon for the system (2.1) but we can stabilize it by introducing

$$p = m + 1 - m_{-u} - m_{+s} = m_{-s} - m_{+s} + 1 \tag{2.3}$$

parameters into the system. We obtain this relation if we inflate the stable and unstable manifolds by the parameter space $\mathbb{R}^p$ and write down the relation (2.2) in the new state space $\mathbb{R}^{m+p}$. We notice that $p = 1$ in the homoclinic case $x_+ = x_-$.

The relation (2.3) (including the case $p = 0$) will be our basic assumption in order to recognize a COP as a regular solution of a suitable operator equation. For that purpose we use the Banach spaces

$$\mathcal{X}^k = \{x \in C^k(\mathbb{R}, \mathbb{R}^m) : \lim_{t \to \pm\infty} x^{(j)}(t) \text{ exist for } j = 0, \dots, k\},$$

$$\|x\|_k = \sum_{j=0}^{k} \sup_{t \in \mathbb{R}} \|x^{(j)}(t)\| \quad \text{with } \|\circ\| \text{ some norm in } \mathbb{R}^m.$$

We notice that, by the mean value theorem, any function x in $\mathcal{X}^k$ satisfies $x^{(j)}(t) \to 0$ $(j = 1, \dots, k)$ as $t \to \pm\infty$.

As a preparation for the nonlinear case, we collect several results on linear differential operators with an exponential dichotomy (see Coppel, 1978; Palmer, 1984).

Consider a linear operator

$$L : \mathcal{X}^1 \to \mathcal{X}^0, \qquad Lx = \dot{x} - A(\circ)x, \tag{2.4}$$

where $A(t)$ for $t \in \mathbb{R}$ is a continuous $m \times m$ matrix function such that the following limits exist

$$A(t) \to A_\pm \quad \text{as } t \to \pm\infty. \tag{2.5}$$

As we will see below, differential operators of this type arise when linearizing about a connecting orbit.

Let $Y(t)$ for $t \in \mathbb{R}$ be the fundamental matrix of L which satisfies $Y(0) = I$.

L is said to have an *exponential dichotomy* on some interval $\mathcal{J} \subset \mathbb{R}$ if there exist $K, \alpha > 0$ and a projection P in $\mathbb{R}^m$ such that, for $s, t \in \mathcal{J}$,

$$\|Y(t)PY(s)^{-1}\| \leq Ke^{-\alpha(t-s)} \quad \text{for } s \leq t, \tag{2.6}$$

$$\|Y(t)(I-P)Y(s)^{-1}\| \leq Ke^{-\alpha(s-t)} \quad \text{for } t \leq s. \tag{2.7}$$

In the case $\mathcal{J} = [0, \infty)$ the range $\mathcal{R}(P)$ of P is given by

$$\mathcal{R}(P) = \{x_0 \in \mathbb{R}^m : Y(t)x_0 \text{ is bounded for } t \geq 0\} \tag{2.8}$$

and any projector with the same range has properties (2.6) and (2.7). Similarly, if L has an exponential dichotomy on $(-\infty, 0]$ with projector Q then

$$\mathcal{N}(Q) = \{x_0 \in \mathbb{R}^m : Y(t)x_0 \text{ is bounded for } t \leq 0\}$$

and any projector with the same null space satisfies (2.6) and (2.7).

If $A(t) = A_0$ is a constant matrix then L has an exponential dichotomy on $[0, \infty)$ iff A_0 is hyperbolic, i.e. A_0 has no eigenvalues on the imaginary axis. In this case we may take P (resp. $I - P$) to be the projector onto the stable (resp. unstable) subspace of A_0.

The following lemma, due to Palmer (1984: Lemma 3.4), shows that the exponential dichotomy carries over to the nonconstant case provided the limit matrices $A_\pm$ from (2.5) are hyperbolic.

LEMMA 2.1 *Let* (2.5) *be satisfied with hyperbolic matrices* A_+ *and* A_-. *Then* L *has an exponential dichotomy on both* $[0, \infty)$ *and* $(-\infty, 0]$. *If* P *and* Q *are corresponding projectors then*

$$\lim_{t\to\infty} Y(t)PY(t)^{-1} = P_{+\mathrm{s}}, \qquad \lim_{t\to-\infty} Y(t)(I-Q)Y(t)^{-1} = P_{-\mathrm{u}}, \tag{2.9}$$

where $P_{\pm\mathrm{s}}$ *and* $P_{\pm\mathrm{u}} = I - P_{\pm\mathrm{s}}$ *are the projectors onto the stable and unstable subspaces of* $A_\pm$.

As a simple consequence of (2.9) we can determine the rank of the projector P as

$$\dim \mathcal{R}(P) = \dim \mathcal{R}(P_{+\mathrm{s}}) =: m_{+\mathrm{s}}, \tag{2.10}$$

and similarly

$$\dim \mathcal{N}(Q) = \dim \mathcal{R}(P_{-\mathrm{u}}) =: m_{-\mathrm{u}}. \tag{2.11}$$

The exponential dichotomies allow us to set up a linear Fredholm theory for the differential operator L in (2.4).

LEMMA 2.2 *Under the assumptions of Lemma* 2.1 *the* (*adjoint*) *differential operator*

$$L^* : \mathcal{X}^1 \to \mathcal{X}^0, \qquad L^*x = \dot{x} + A^{\mathrm{T}}(\cdot)x$$

also has exponential dichotomies on $[0, \infty)$ *(resp.* $(-\infty, 0]$*) and suitable projectors are* $I - P^{\mathrm{T}}$ *(resp.* $I - Q^{\mathrm{T}}$*). Moreover*

$$\mathsf{N}(L) = \{Y(t)x_0 : x_0 \in \mathsf{R}(P) \cap \mathsf{N}(Q)\}, \tag{2.12}$$

$$\dim \mathsf{N}(L) = \dim \mathsf{N}(L^*) + m_{+s} - m_{-s}, \tag{2.13}$$

$$y \in \mathsf{R}(L) \Leftrightarrow \int_{-\infty}^{\infty} \varphi^{\mathrm{T}}(t) y(t)\, \mathrm{d}t = 0 \quad \forall\, \varphi \in \mathsf{N}(L^*), \tag{2.14}$$

and $L : \mathcal{X}^1 \to \mathcal{X}^0$ *is Fredholm of index* $m_{+s} - m_{-s} = m_{-u} - m_{+u}$.

If we consider L and L^* as operators on spaces of bounded functions, then this lemma holds verbatim and it follows from Palmer (1984: Lemma 4.2), (2.10), and (2.11). For the special spaces $\mathcal{X}^1$ and $\mathcal{X}^0$ chosen here, some supplements are necessary and these will be given in Appendix A.

Now we consider the nonlinear case and assume that $(\bar{x}, \bar{\lambda})$ is a COP of the system (1.1). Then $\bar{x}$ is at most determined up to a phase shift. This is reflected by the fact that $\dot{\bar{x}}$ solves the variational equation

$$\dot{y} = f_x(\bar{x}, \bar{\lambda}) y \quad \text{in } \mathbb{R}, \qquad y(t) \to \mathbf{0} \text{ as } t \to \pm\infty.$$

Here and in what follows partial derivatives are denoted by lower indices whereas a prime is used for the total derivative. Our formulation of nondegeneracy includes a certain converse of the above statement.

DEFINITION 2.1 A connecting orbit pair $(\bar{x}, \bar{\lambda})$ of the system (1.1) with f in C^1 is called *nondegenerate* if the matrices

$$A_{\pm} = \lim_{t \to \pm\infty} f_x(\bar{x}(t), \bar{\lambda}) = f_x(x_{\pm}, \bar{\lambda}) \tag{2.15}$$

are hyperbolic, if their numbers of stable and unstable eigenvalues relate to the number of parameters as in (2.3), and if the following condition holds:

$$\dot{y} = f_x(\bar{x}, \bar{\lambda}) y + f_\lambda(\bar{x}, \bar{\lambda}) \mu \quad y \in \mathcal{X}^1, \ \mu \in \mathbb{R}^p \quad \Rightarrow$$
$$\mu = \mathbf{0} \quad \text{and} \quad y = c\dot{\bar{x}} \quad \text{for some } c \in \mathbb{R}. \tag{2.16}$$

With the aid of Lemma 2.2 we have the following characterization of the nondegeneracy.

PROPOSITION 2.1 *Let* $(\bar{x}, \bar{\lambda})$ *be a COP of the system* (1.1) *with hyberbolic matrices,* $A_{\pm}$ *as in* (2.15). *Then* $(\bar{x}, \bar{\lambda})$ *is nondegenerate if and only if the linear operator*

$$L = \mathrm{d}/\mathrm{d}t - f_x(\bar{x}, \bar{\lambda}) : \mathcal{X}^1 \to \mathcal{X}^0$$

has the properties

(i) $\dim \mathsf{N}(L) = 1$, $\dim \mathsf{N}(L^*) = p$,

(ii) *the* $p \times p$ *matrix*

$$E = \int_{-\infty}^{\infty} \Phi^{\mathrm{T}}(t) f_\lambda(\bar{x}(t), \bar{\lambda})\, \mathrm{d}t$$

is nonsingular, where the columns $\varphi_i \in \mathcal{X}^1$ *of* $\Phi = (\varphi_1, \ldots, \varphi_p)$ *form a basis of* $\mathsf{N}(L^*)$.

Proof. Assuming that $(\bar{x}, \bar{\lambda})$ is nondegenerate, we obtain from (2.16)

$$\mathcal{N}(L) = \operatorname{span}\{\dot{\bar{x}}\}, \qquad \dim \mathcal{N}(L) = 1, \tag{2.17}$$

and further $\dim \mathcal{N}(L^*) = p$ from (2.13) and (2.3). Moreover, (2.16) implies that $f_\lambda(\bar{x}, \bar{\lambda})\mu \in \mathcal{R}(L)$ only if $\mu = 0$, hence $E\mu = 0$ only if $\mu = 0$ by (2.14) and E is nonsingular.

Conversely, assume that (i) and (ii) hold. Then (2.13) yields the relation (2.3) and by the use of the Fredholm alternative (2.14) the nondegeneracy condition is an easy consequence of (2.17) and assumption (ii). □

As a final prerequisite for the central result of this section we need the following elementary functional analytic lemma. (See Appendix B for a proof.)

LEMMA 2.3 (Bordering lemma) *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and consider the operator*

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{L}[\mathcal{X} \times \mathbb{R}^p, \mathcal{Y} \times \mathbb{R}^q],$$

with bounded linear operators $A \in \mathbb{L}[\mathcal{X}, \mathcal{Y}]$, $B \in \mathbb{L}[\mathbb{R}^p, \mathcal{Y}]$, $C \in \mathbb{L}[\mathcal{X}, \mathbb{R}^q]$, $D \in \mathbb{L}[\mathbb{R}^p, \mathbb{R}^q]$. If A is Fredholm of index N then S is Fredholm of index $N + p - q$.

Our nondegeneracy definition is independent of any phase-fixing condition. But in order to write down a well-posed problem for a COP we must have such a condition. We use the general form

$$\Psi(x, \lambda) = 0, \tag{2.18}$$

where Ψ is any functional in $C^1(\mathcal{X}^0 \times \mathbb{R}^p, \mathbb{R})$ and $\mathcal{X}^0 \times \mathbb{R}^p$ is equipped with the standard product norm.

THEOREM 2.1 *Let $(\bar{x}, \bar{\lambda})$ be a nondegenerate* COP *of the system* (1.1) *with $f \in C^1(\mathbb{R}^{m+p}, \mathbb{R}^m)$ and let a functional $\Psi \in C^1(\mathcal{X}^0 \times \mathbb{R}^p, \mathbb{R})$ be given such that*

$$\Psi(\bar{x}, \bar{\lambda}) = 0, \qquad \Psi_x(\bar{x}, \bar{\lambda})\dot{\bar{x}} \neq 0. \tag{2.19}$$

Then $(\bar{x}, \bar{\lambda}) \in \mathcal{X}^1 \times \mathbb{R}^p$ is a regular solution of the operator equation

$$F(x, \lambda) = \big(\dot{x} - f(x, \lambda),\ \Psi(x, \lambda)\big) = 0. \tag{2.20}$$

Proof. By *regular* we mean that the operator $F : \mathcal{X}^1 \times \mathbb{R}^p \to \mathcal{X}^0 \times \mathbb{R}$, as defined by (2.20), has a total derivative $F'(\bar{x}, \bar{\lambda})$ at $(\bar{x}, \bar{\lambda})$ which is a linear homeomorphism from $\mathcal{X}^1 \times \mathbb{R}^p$ onto $\mathcal{X}^0 \times \mathbb{R}$ (with the norms always understood to be the product norm of the single Banach spaces). In fact, F is easily seen to be continuously differentiable with

$$F'(\bar{x}, \bar{\lambda}) = \begin{bmatrix} L & -f_\lambda(\bar{x}, \bar{\lambda}) \\ \Psi_x(\bar{x}, \bar{\lambda}) & \Psi_\lambda(\bar{x}, \bar{\lambda}) \end{bmatrix}, \qquad Lx = \dot{x} - f_x(\bar{x}, \bar{\lambda})x.$$

By Lemma 2.1, L is Fredholm of index $m_{+s} - m_{-s}$ and an application of Lemma 2.3 shows that $F'(\bar{x}, \bar{\lambda})$ is Fredholm of index $m_{+s} - m_{-s} + p - 1$ which is zero by (2.3). Hence it is sufficient to show that $F'(\bar{x}, \bar{\lambda})(y, \mu) = 0$ has only the trivial solution. Writing down this equation, we first find $\mu = 0$, $y = c\dot{\bar{x}}$ from (2.16) and then $y = 0$ from (2.19). □

We conclude this section with several remarks on the nondegeneracy conditions used in Theorem 2.1.

First, we notice that one can show a certain converse of Theorem 2.1. If $(\bar{x}, \bar{\lambda}) \in X^1 \times \mathbb{R}^p$ is a regular solution of (2.20) and if the matrices $f_x(x_\pm, \bar{\lambda})$ are hyperbolic, then (2.19) holds and the COP is nondegenerate. It may even be that the Fredholm property of $F'(\bar{x}, \bar{\lambda})$ implies the hyperbolicity of $f_x(x_\pm, \bar{\lambda})$ (see Rodrigues & Silveira (1988) for a result in this direction).

Second, in the two-dimensional homoclinic case ($m_{+s} = m_{-u} = 1$, $p = 1$) the nondegeneracy condition (2.16) turns out to be equivalent to the key assumption for the saddle connection bifurcation (e.g. Guckenheimer & Holmes, 1983: Thm. 6.1.1.) which states that the stable and unstable manifolds of the saddle pass each other on a transverse segment with nonvanishing velocity (with respect to λ).

We also notice that so far we have fixed the number p of parameters through its relation (2.3) to the dimensions of the stable and unstable manifolds. If there are more than p parameters in the system (p given by (2.3)) then we may select a particular subset of p parameters as unknowns whereas the remaining parameters are used for continuation. This is the situation in examples 1 to 3 in Section 4.

Further, we mention that our nondegeneracy conditions exclude a direct application to Hamiltonian systems, where, for example, a homoclinic orbit usually occurs in a system without parameter. However, it is easy to introduce an artificial parameter such that the Hamiltonian structure is destroyed and the nondegeneracy condition satisfied. For example, let $\bar{x}(t) = (\bar{p}(t), \bar{q}(t)) \in \mathbb{R}^{2n}$, where $m = 2n$, be a connecting orbit, with $m_{-s} = m_{+s}$, of a Hamiltonian system

$$\dot{p} = -H_q^{\mathrm{T}}(p, q), \qquad \dot{q} = H_p^{\mathrm{T}}(p, q),$$

and assume $\dim N(L) = 1$ for the linearization L at $\bar{x}$. Then $(\bar{x}, \bar{\lambda} = \mathbb{0})$ is a nondegenerate COP of the following extended system (see Schmidt, 1976, for this construction)

$$\dot{p} = -H_q^{\mathrm{T}}(p, q) + \lambda H_p^{\mathrm{T}}(p, q), \qquad \dot{q} = H_p^{\mathrm{T}}(p, q) + \lambda H_q^{\mathrm{T}}(p, q).$$

With (2.13) we verify that $N(L^*) = \mathrm{span}\,\{(-\dot{\bar{q}}, \dot{\bar{p}})\}$ and we can apply Proposition 2.1 since

$$\begin{aligned} E &= \int_{-\infty}^{\infty} [-H_p(\bar{p}, \bar{q})\dot{\bar{q}}(t) + H_q(\bar{p}, \bar{q})\dot{\bar{p}}(t)]\,\mathrm{d}t \\ &= -\int_{-\infty}^{\infty} [\dot{\bar{q}}(t)^{\mathrm{T}}\dot{\bar{q}}(t) + \dot{\bar{p}}(t)^{\mathrm{T}}\dot{\bar{p}}(t)]\,\mathrm{d}t < 0. \end{aligned}$$

Likewise, in parametrized Hamiltonian systems we can introduce the stabilizing parameter λ as above and use the given parameters for continuation.

Finally, we comment on the choice of the phase condition (2.18) and the assumption (2.19). Usually, in applications we assume that some approximation $(\hat{x}, \hat{\lambda}) \in X^1 \times \mathbb{R}^p$ of the COP is known, either from an initial value problem or from a previous solution on a branch of COPs. In the simplest case we take the classical phase condition

$$\Psi_0(x) = \dot{\hat{x}}(0)^{\mathrm{T}}[x(0) - \hat{x}(0)] = 0, \tag{2.21}$$

which fixes $x(0)$ in a hyperplane through $\hat{x}(0)$ and orthogonal to $\dot{\hat{x}}(0)$. The nondegeneracy in (2.19) then requires

$$\dot{\hat{x}}(0)^{\mathrm{T}}\dot{x}(0) \neq 0. \tag{2.22}$$

For numerical purposes the following integral condition (following an idea of Doedel (1981)) is much more reliable

$$\Psi_\infty(x) = \int_{-\infty}^{\infty} \dot{\hat{x}}(t)^{\mathrm{T}}[x(t) - \hat{x}(t)]\,\mathrm{d}t = 0. \tag{2.23}$$

Of course we assume here that $\dot{\hat{x}}(t)$ decays sufficiently fast at $\pm\infty$. The motivation for (2.23) is as follows. In the homoclinic case and in the heteroclinic case with stationary points independent of λ there exists a constant $c \in \mathbb{R}^m$ such that $\hat{x}(t) + c$ and the unknown solution $x(t)$ have the same limits at $\pm\infty$ (take $c = 0$ in the heteroclinic case). Then (2.23) is the necessary condition for a local minimum of the L_2-distance

$$d(\tau) = \int_{-\infty}^{\infty} \|\hat{x}(t) + c - x(t+\tau)\|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} \|\hat{x}(t-\tau) + c - x(t)\|^2\,\mathrm{d}t$$

as a function of the time shift τ.

For Ψ_∞ the nondegeneracy in (2.19) reads

$$\int_{-\infty}^{\infty} \dot{\hat{x}}(t)^{\mathrm{T}}\dot{x}(t)\,\mathrm{d}t \neq 0. \tag{2.24}$$

3. Truncation to a finite interval

For the numerical computation of a COP we have to solve the following parametrized boundary-value problem on the real line

$$\dot{x} = f(x, \lambda) \quad \text{for } t \in \mathbb{R}, \tag{3.1a}$$

$$x(t) \text{ converges as } t \to +\infty \text{ and } t \to -\infty, \tag{3.1b}$$

$$\Psi(x, \lambda) = 0. \tag{3.1c}$$

We replace this problem by a boundary-value problem on a finite interval, say $\mathcal{J} = [T_-, T_+]$, with certain boundary conditions at T_- and T_+,

$$\dot{x} = f(x, \lambda) \quad \text{for } t \in [T_-, T_+], \tag{3.2a}$$

$$b_+(x(T_+), \lambda) = 0, \qquad b_-(x(T_-), \lambda) = 0, \tag{3.2b}$$

$$\Psi_{\mathcal{J}}(x, \lambda) = 0. \tag{3.2c}$$

Here and in what follows we assume

$$f \in \mathrm{C}^2(\mathbb{R}^{m+p}, \mathbb{R}^m), \qquad b_\pm \in \mathrm{C}^1(\mathbb{R}^{m+p}, \mathbb{R}^{n_\pm}),$$

where the numbers n_+ and n_- will be specified later on. We will also use the spaces $\mathcal{X}^k(\mathcal{J}) = \mathrm{C}^k(\mathcal{J}, \mathbb{R}^m)$, for $k \geq 0$, with the standard C^k-norm denoted by $\|\bullet\|_k$. For $x \in \mathcal{X}^k$ we have that the restriction to $\mathcal{J}$, denoted by $x\restriction_{\mathcal{J}}$, is in $\mathcal{X}^k(\mathcal{J})$. For the phase condition $\Psi_{\mathcal{J}}$ we assume $\Psi_{\mathcal{J}} \in \mathrm{C}^1(\mathcal{X}^0(\mathcal{J}) \times \mathbb{R}^p, \mathbb{R})$, the total derivative being $\Psi'_{\mathcal{J}}(x, \lambda) = (\Psi_{\mathcal{J},x}(x, \lambda), \Psi_{\mathcal{J},\lambda}(x, \lambda))$. In view of (2.21) and (2.23) the standard choices for $\Psi_{\mathcal{J}}$ are

$$\Psi_{\mathcal{J}}(x, \lambda) = \dot{\hat{x}}(0)^{\mathrm{T}}[x(0) - \hat{x}(0)] \quad (\text{if } 0 \in J), \tag{3.3}$$

$$\Psi_{\mathcal{J}}(x, \lambda) = \int_{T_-}^{T_+} \dot{\hat{x}}(t)^{\mathrm{T}}[x(t) - \hat{x}(t)]\,\mathrm{d}t. \tag{3.4}$$

Following Lentini & Keller (1980) we will call (3.2b) the *asymptotic boundary conditions*. In the semi-infinite case, the correct setting up of these conditions as well as an error analysis was carried out by de Hoog & Weiss (1980), Lentini & Keller (1980), and Markowich (1982). We essentially adapt their approach to our specific problem, i.e. we take into account the two-sided infinity, the special role of the parameters and the possibly nonlocal phase condition.

Suppose we have a nondegenerate COP $(\bar{x}, \bar{\lambda})$ which connects the stationary points $\bar{x}_-$ and $\bar{x}_+$, and suppose $m_{\pm s}$ and $m_{\pm u}$ are the numbers of stable and unstable eigenvalues of $f_x(\bar{x}_\pm, \bar{\lambda})$. Then, in a neighbourhood of $\bar{x}_\pm$, there exists a smooth manifold of hyperbolic stationary points of (3.1a) with the same numbers of stable and unstable eigenvalues.

More precisely, by the implicit function theorem, we have functions $x_\pm \in C^2(\Lambda, \mathbb{R}^m)$ on some neighbourhood Λ of $\bar{\lambda}$ such that $x_\pm(\bar{\lambda}) = \bar{x}_\pm$ and $f(x_\pm(\lambda), \lambda) = 0$. Moreover, there exist matrix-valued functions $E_{\pm s} \in C^1(\Lambda, \mathbb{R}^{m, m_{\pm s}})$, $E_{\pm u} \in C^1(\Lambda, \mathbb{R}^{m, m_{\pm u}})$ such that the columns of $E_{\pm s}(\lambda)$ (resp. $E_{\pm u}(\lambda)$) span the stable subspace $\mathcal{V}_{\pm s}(\lambda)$ (resp. the unstable subspace $\mathcal{V}_{\pm u}(\lambda)$) of the linearization $A_\pm(\lambda) = f_x(x_\pm(\lambda), \lambda)$ (see Appendix C). Writing down the matrices

$$E_\pm(\lambda) = (E_{\pm u}(\lambda)\ E_{\pm s}(\lambda)), \qquad E_\pm(\lambda)^{-1} = \begin{pmatrix} L_{\pm u}(\lambda) \\ L_{\pm s}(\lambda) \end{pmatrix},$$

we find that the rows of $L_{\pm s}(\lambda)$ and $L_{\pm u}(\lambda)$ form a basis for the stable and unstable subspaces of $f_x(x_\pm(\lambda), \lambda)^{\mathrm{T}}$, hence

$$\mathcal{V}_{\pm u}(\lambda) = \mathcal{R}(E_{\pm u}(\lambda)) = \mathcal{N}(L_{\pm s}(\lambda)), \qquad \mathcal{V}_{\pm s}(\lambda) = \mathcal{R}(E_{\pm s}(\lambda)) = \mathcal{N}(L_{\pm u}(\lambda)). \tag{3.5}$$

The *projection conditions* derived by de Hoog & Weiss (1980) for the parameter independent case generalize to

$$b_+(x, \lambda) = L_{+u}(\lambda)[x - x_+(\lambda)], \qquad b_-(x, \lambda) = L_{-s}(\lambda)[x - x_-(\lambda)]. \tag{3.6}$$

In this case we obtain from (3.5) that the boundary conditions (3.2b) take the form

$$x(T_-) \in x_-(\lambda) + \mathcal{V}_{-u}(\lambda), \qquad x(T_+) \in x_+(\lambda) + \mathcal{V}_{+s}(\lambda).$$

This is a very natural condition since the connecting orbit must leave $x_-(\lambda)$ via its unstable manifold $\mathcal{M}_{-u}(\lambda)$, which is tangent to $x_-(\lambda) + \mathcal{V}_{-u}(\lambda)$, and must approach $x_+(\lambda)$ via its stable manifold $\mathcal{M}_{+s}(\lambda)$ which is tangent to $x_+(\lambda) + \mathcal{V}_{+s}(\lambda)$ (see Fig. 2).

The projection conditions are natural for a second reason. The numbers of asymptotic boundary conditions are then given by $n_+ = m_{+u}$, $n_- = m_{-s}$. Therefore, the system (3.2) has $m_{+u} + m_{-s} + 1$ boundary conditions for the $m + p$ unknowns x and λ, and these numbers match precisely under the condition (2.3).

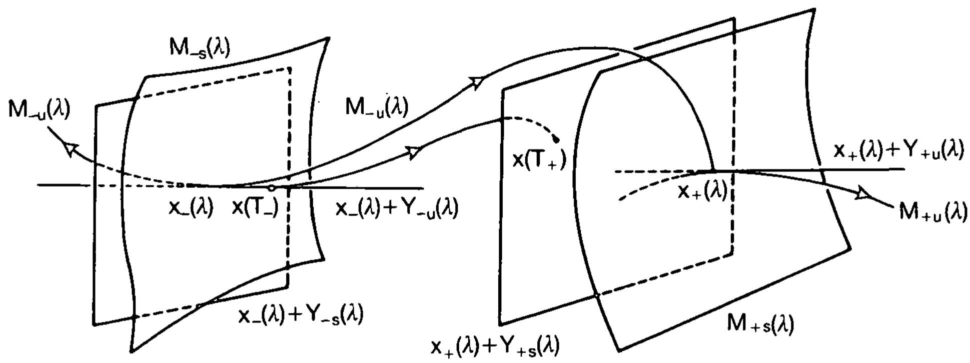

FIG. 2. Illustration of the projection boundary conditions: the endpoints $x(T_-)$, $x(t_+)$ are required to lie on the tangent spaces to the unstable and stable manifolds.

Next we investigate the approximation error of solutions to (3.2). We use a well-known linearization technique (see, for example, Spijker, 1971; Keller, 1975; Vainikko, 1976). The basic tool for analysing the nonlinear problem locally is the following perturbation lemma. See Vainikko (1976: §3) for a proof.

LEMMA 3.1. *Let $F: B_\delta(y_0) \to \mathcal{Z}$ be a C^1-mapping from some ball of radius δ in a Banach space $\mathcal{Y}$ into some Banach space $\mathcal{Z}$. Assume that $F'(y_0)$ is a homeomorphism and that for some constants κ, σ we have*

$$\|F'(y) - F'(y_0)\| \leq \kappa < \sigma \leq \|F'(y_0)^{-1}\|^{-1} \quad \forall y \in B_\delta(y_0), \tag{3.7}$$

$$\|F(y_0)\| \leq (\sigma - \kappa)\delta. \tag{3.8}$$

Then F has a unique zero $\bar{y}$ in $B_\delta(y_0)$ and

$$\|\bar{y} - y_0\| \leq (\sigma - \kappa)^{-1} \|F(y_0)\|, \tag{3.9}$$

$$\|y_1 - y_2\| \leq (\sigma - \kappa)^{-1} \|F(y_1) - F(y_2)\| \quad \forall y_1, y_2 \in B_\delta(y_0). \tag{3.10}$$

The following approximation theorem contains a typical asymptotic convergence result as in de Hoog & Weiss (1980) and Markowich (1982). We consider asymptotic boundary conditions of the general form and we will write $\mathcal{J} \to \mathbb{R}$ instead of $\mathcal{J} = [T_-, T_+]$ and $T_\pm \to \pm\infty$.

THEOREM 3.1 *Let $(\bar{x}, \bar{\lambda})$ be a nondegenerate COP with endpoints $\bar{x}_-$ and $\bar{x}_+$. Assume $f \in C^2(\mathbb{R}^{m+p}, \mathbb{R}^m)$, $b_+ \in C^1(\mathbb{R}^{m+p}, \mathbb{R}^{m_{+u}})$, $b_- \in C^1(\mathbb{R}^{m+p}, \mathbb{R}^{m_{-s}})$, $\Psi_\mathcal{J} \in C^1(\mathcal{X}^0(\mathcal{J}) \times \mathbb{R}^p, \mathbb{R})$, and let the following conditions hold.*

(i) *$b_\pm(\bar{x}_\pm, \bar{\lambda}) = 0$ and the quadratic matrices*

$$b_{+x}(\bar{x}_+, \bar{\lambda})E_{+u}(\bar{\lambda}), \qquad b_{-x}(\bar{x}_-, \bar{\lambda})E_{-s}(\bar{\lambda})$$

are nonsingular

(ii) $$\Psi_\mathcal{J}(\bar{x}\restriction_\mathcal{J}, \bar{\lambda}) \to 0 \quad \text{as } \mathcal{J} \to \mathbb{R}, \qquad |\Psi_{\mathcal{J},x}(\bar{x}\restriction_\mathcal{J}, \bar{\lambda})\dot{\bar{x}}\restriction_\mathcal{J}| \geq d$$

for some $d > 0$ and $\mathcal{J}$ sufficiently large, $\Psi'_\mathcal{J}(\bar{x}\restriction_\mathcal{J}, \bar{\lambda})$ is uniformly bounded and $\Psi'_\mathcal{J}$ is equicontinuous at $(\bar{x}\restriction_\mathcal{J}, \bar{\lambda})$ (see remark).

Then there exist ρ, with $C > 0$, such that, for sufficiently large $\mathcal{J}$, the finite *boundary-value problem* (3.2) *has a unique solution* $(x_{\mathcal{J}}, \lambda_{\mathcal{J}})$ *in the tube*

$$K_\rho = \{(x, \lambda) \in X^1(\mathcal{J}) \times \mathbb{R}^p : \|x - \bar{x}\restriction_{\mathcal{J}}\|_1 + \|\lambda - \bar{\lambda}\| \leq \rho\},$$

and the following estimate holds

$$\|\bar{x}\restriction_{\mathcal{J}} - x_{\mathcal{J}}\|_1 + \|\bar{\lambda} - \lambda_{\mathcal{J}}\| \\ \leq C(\|b_+(\bar{x}(T_+), \bar{\lambda})\| + \|b_-(\bar{x}(T_-), \bar{\lambda})\| + |\Psi_{\mathcal{J}}(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda})|). \quad (3.11)$$

Moreover, there is a unique phase shift $\tau_{\mathcal{J}} \in [-\rho, \rho]$ such that $\bar{y}(t) = \bar{x}(t + \tau_{\mathcal{J}})$ satisfies $\Psi_{\mathcal{J}}(\bar{y}, \bar{\lambda}) = 0$, and we have $\tau_{\mathcal{J}} \to 0$ as $\mathcal{J} \to \mathbb{R}$ as well as the improved estimate

$$\|\bar{y} - x_{\mathcal{J}}\|_1 + \|\bar{\lambda} - \lambda_{\mathcal{J}}\| \leq C(\|b_+(\bar{y}(T_+), \bar{\lambda})\| + \|b_-(\bar{y}(T_-), \bar{\lambda})\|). \quad (3.12)$$

Remark. The precise meaning of the uniformity in assumption (ii) is that there exists some $C > 0$ such that $\|\Psi'_{\mathcal{J}}(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda})\| \leq C$ for $\mathcal{J}$ sufficiently large and that to any $\varepsilon > 0$ we find some $\delta > 0$, $T > 0$, such that

$$\|\Psi'_{\mathcal{J}}(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda}) - \Psi'_{\mathcal{J}}(x, \lambda)\| \leq \varepsilon \quad \text{for } \|x - \bar{x}\restriction_{\mathcal{J}}\|_1 + \|\lambda - \bar{\lambda}\| \leq \delta \quad [-T, T] \subset \mathcal{J}.$$

The somewhat technical assumption (ii) is easily seen to be satisfied for the special phase conditions (3.3) and (3.4) provided the phase of the connecting orbit is fixed according to (2.21) and (2.23), and provided the nondegeneracy conditions (2.22) and (2.24) hold. The minor role of the phase condition is also illustrated by the fact that the error introduced by $\Psi_{\mathcal{J}}$ in (3.11) can be removed by a suitable time shift on the connecting orbit itself, i.e. any solution of (3.2) approximates some suitably shifted orbit with an error which is dominated by the asymptotic boundary conditions.

Proof. We apply Lemma 3.1 with the settings $\mathcal{Y} = X^1(\mathcal{J}) \times \mathbb{R}^p$, $\mathcal{Z} = X^0(\mathcal{J}) \times \mathbb{R}^{m_{+u}+m_{-s}+1}$, $y_0 = (\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda})$, and

$$F(x, \lambda) = (\dot{x} - f(x, \lambda), b_+(x(T_+), \lambda), b_-(x(T_-), \lambda), \Psi_{\mathcal{J}}(x, \lambda)).$$

The inhomogeneous equation $F'(y_0)(x, \lambda) = (y, r_+, r_-, \varepsilon) \in \mathcal{Z}$ turns out to be the following variational equation

$$\dot{x} - A(\circ)x - V(\circ)\lambda = y \quad \text{in } \mathcal{J}, \quad (3.13a)$$

$$K_\pm x(T_\pm) + M_\pm \lambda = r_\pm, \quad (3.13b)$$

$$u(x) + v\lambda = \varepsilon, \quad (3.13c)$$

where

$$A(t) = f_x(\bar{x}(t), \bar{\lambda}), \qquad V(t) = f_\lambda(\bar{x}(t), \bar{\lambda}), \qquad K_\pm = b_{\pm,x}(\bar{x}(T_\pm), \bar{\lambda}),$$
$$M_\pm = b_{\pm,\lambda}(\bar{x}(T_\pm), \bar{\lambda}), \qquad u = \Psi_{\mathcal{J},x}(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda}), \qquad v = \Psi_{\mathcal{J},\lambda}(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda}).$$

In Appendix D we show that there exists a constant $\sigma > 0$ such that for all $\mathcal{J}$ sufficiently large the solution of (3.13) can be estimated as follows

$$\|x\|_1 + \|\lambda\| \leq \sigma^{-1}(\|y\|_0 + \|r_+\| + \|r_-\| + |\varepsilon|). \quad (3.14)$$

Since (3.13) is a linear boundary-value problem of dimension $m_{+u} + m_{-s} + 1 = m + p$ (see (2.3)), we obtain a uniform bound σ^{-1} for $\|F'(y_0)^{-1}\|$. Now we use the smoothness properties of f, $b_\pm$, and $\Psi_{\mathcal{J}}$ to find a $\delta > 0$ such that (3.7) holds with $\kappa = \frac{1}{2}\sigma$ for all $\mathcal{J}$. Finally, by assumptions (i) and (ii)

$$\|F(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda})\| = \|b_+(\bar{x}(T_+), \bar{\lambda})\| + \|b_-(\bar{x}(T_-), \bar{\lambda})\| + |\Psi_{\mathcal{J}}(\bar{x}\restriction_{\mathcal{J}}, \bar{\lambda})| \to 0 \quad \text{as } \mathcal{J} \to \mathbb{R} \tag{3.15}$$

and (3.8) holds for $\mathcal{J}$ sufficiently large. Thus F has a unique zero $\bar{y} = (x_{\mathcal{J}}, \lambda_{\mathcal{J}})$ in K_δ and the estimate (3.11) follows from (3.9) and (3.15). The special phase shift $\tau_{\mathcal{J}}$ is constructed by solving the scalar equation $\Psi_{\mathcal{J}}(\bar{x}(\circ + \tau), \bar{\lambda}) = 0$ for τ.

Again we use Lemma 3.1 but now with $y_0 = \mathbf{0}$. The estimate (3.12) is then obtained by an application of (3.10) to $y_1 = (\bar{x}(\circ + \tau_{\mathcal{J}}), \bar{\lambda})$ and $y_2 = (x_{\mathcal{J}}, \lambda_{\mathcal{J}})$. □

Assumption (i) in Theorem 3.1 is the crucial one. As for numerical computations, it shows that we must be able to evaluate the stationary solution manifolds $x_\pm(\lambda)$ (either explicitly or via an iterative method) and we must know the corresponding numbers of stable and unstable eigenvalues. For the projection conditions, assumption (i) is obviously satisfied. But we also see that it is not necessary to know the stable and unstable subspaces $\mathcal{Y}_{\pm s}(\lambda)$ and $\mathcal{Y}_{\pm u}(\lambda)$ exactly, rather one could pick almost any pair of matrices $L_{+u}(\lambda)$ and $L_{-s}(\lambda)$ in a linear condition of type (3.6) provided that condition (i) is not violated.

The special merit of the projection conditions is that they minimize the approximation error caused by the asymptotic boundary conditions. For the linear case this was first observed by de Hoog & Weiss (1980). In the nonlinear case considered here, we use the fact that the connecting orbit approaches the stationary points tangentially to $\bar{x}_- + \mathcal{Y}_{-u}(\bar{\lambda})$ and $\bar{x}_+ + \mathcal{Y}_{+s}(\bar{\lambda})$. More precisely, if $0 < \mu_+ < -\mathrm{Re}\,\mu$ for all stable eigenvalues μ of $f_x(\bar{x}_+, \bar{\lambda})$ then we find a $T_1 > 0$ such that

$$\bar{x}(t) = \bar{x}_+ + s(t) + u(t),$$

where $s(t) \in \mathcal{Y}_{+s}(\bar{\lambda})$, $u(t) \in \mathcal{Y}_{+u}(\bar{\lambda})$, and $s(t) = O(e^{-\mu_+ t})$, $u(t) = O(e^{-2\mu_+ t})$ uniformly for $t \geq T_1$. The estimate for $s(t)$ follows from the corresponding estimate $\bar{x}(t) - \bar{x}_+ = O(e^{-\mu_+ t})$ (see, for example, Hartmann, 1964: IX Corollary 5.2) and $u(t) = O(\|s(t)\|^2)$ is a consequence of the tangential property (Hartmann, 1964: IX Lemma 5.1).

By a Taylor expansion we obtain

$$\begin{aligned} b_+(\bar{x}(t), \bar{\lambda}) &= b_+(\bar{x}_+, \bar{\lambda}) + b_{+x}(\bar{x}_+, \bar{\lambda})[\bar{x}(t) - \bar{x}_+] + O(\|\bar{x}(t) - \bar{x}_+\|^2) \\ &= b_{+x}(\bar{x}_+, \bar{\lambda})s(t) + O(e^{-2\mu_+ t}). \end{aligned}$$

In general, this term is only $O(e^{-\mu_+ t})$, but it will be $O(e^{-2\mu_+ t})$ if we assume

$$b_{+x}(\bar{x}_+, \bar{\lambda})E_{+s}(\bar{\lambda}) = \mathbf{0}. \tag{3.16}$$

This is satisfied for the projection conditions (3.6), and for asymptotic boundary conditions which are linear in x, these are the only ones with the property (3.16) (up to some trivial scalings). We note, however, that (3.16) is also satisfied by the nonlinear projection conditions $b_+(x, \lambda) = L_{+u}(\lambda)f(x, \lambda)$ which are suggested by the work of Lentini & Keller (1980).

In an analogous way we obtain $b_-(\bar{x}(-t), \bar{\lambda}) = O(e^{-2\mu_- t})$ if $0 < \mu_- < \mathrm{Re}\,\mu$ for all unstable eigenvalues μ of $f_x(\bar{x}_-, \bar{\lambda})$ and if

$$b_{-x}(\bar{x}_-, \bar{\lambda})E_{-u}(\bar{\lambda}) = \mathbf{0}. \tag{3.17}$$

Thus we have shown the following.

COROLLARY 3.1 *Under the assumptions of Theorem* 3.1 *we have, for* $\mathcal{J} = [T_-, T_+]$ *sufficiently large and with* $\bar{y}(t) = \bar{x}(t + \tau_{\mathcal{J}})$,

$$\|\bar{y} - x_{\mathcal{J}}\|_1 + \|\bar{\lambda} - \lambda_{\mathcal{J}}\| \leq C \exp\left(-\min\{\mu_+ T_+, \mu_- |T_-|\}\right). \quad (3.18)$$

If, in addition, (3.16) *and* (3.17) *are satisfied, then a factor* 2 *can be inserted in the exponent.*

4. Numerical implementation and applications

In all examples below we solved the finite boundary value problem (3.2) with a code from the NAG-library, Oxford (DO2RAF which is actually based on PASVA3 by Pereyra (1979)). Further we used some standard transformations (see Ascher, Mattheij, & Russell, 1988) in order to put (3.2) into two-point boundary form.

Classical phase condition (3.3). With $\zeta = \dot{\hat{x}}(0)$, $\gamma = \zeta^{\mathrm{T}} \hat{x}(0)$, $y(t) = x(tT_+)$, and $z(t) = x(tT_-)$, solve the $(2m + p)$-dimensional system

$$\dot{y} = T_+ f(y, \lambda), \qquad \dot{z} = T_- f(z, \lambda), \qquad \dot{\lambda} = 0 \text{ in } [0, 1], \quad (4.1a)$$

$$b_+(y(1), \lambda(1)) = 0, \qquad b_-(z(1), \lambda(1)) = 0, \quad (4.1b)$$

$$y(0) - z(0) = 0, \qquad \zeta^{\mathrm{T}} y(0) - \gamma = 0. \quad (4.1c)$$

Integral phase condition (3.4). With

$$\zeta(t) = \dot{\hat{x}}(t), \qquad \gamma = \int_{T_-}^{T_+} \zeta(t)^{\mathrm{T}} \hat{x}(t)\,\mathrm{d}t, \qquad \eta(t) = \zeta(T_- + (T_+ - T_-)t),$$

$$y(t) = x(T_- + (T_+ - T_-)t), \qquad v(t) = (T_+ - T_-)\int_0^t \eta(s)^{\mathrm{T}} y(s)\,\mathrm{d}s,$$

solve the $(m + p + 1)$-dimensional system

$$\dot{y} = (T_+ - T_-) f(y, \lambda), \qquad \dot{v} = (T_+ - T_-)\eta^{\mathrm{T}} y, \qquad \dot{\lambda} = 0 \text{ in } [0, 1], \quad (4.2a)$$

$$b_+(y(1), \lambda(1)) = 0, \qquad b_-(y(0), \lambda(0)) = 0, \quad (4.2b)$$

$$v(0) = 0, \qquad v(1) = \gamma. \quad (4.2c)$$

Usually some rough starting values $(\hat{x}(t), \hat{\lambda})$ were obtained from an initial value problem and $\zeta(t) = f(\hat{x}(t), \hat{\lambda})$ was taken instead of $\dot{\hat{x}}(t)$. For the integral phase condition it was also necessary to interpolate ζ smoothly from the given discrete values since it enters into a nonautonomous term on the right-hand side of (4.2a).

As for the projection conditions, we have to supply basis vectors of stable and unstable subspaces which vary smoothly with λ. If this could not be done explicitly (as in example 4) we used the following device. For λ close to the initial guess $\hat{\lambda}$, let the rows of $C(\lambda) \in \mathbb{R}^{m_{+u}, m}$ form a basis of the unstable subspace of $f_x^{\mathrm{T}}(x_+(\lambda), \lambda)$. $C(\lambda)$ may be nonsmooth and computed by any method. Then we set

$$L_{+u}(\lambda) = D(\lambda) C(\lambda), \quad (4.3)$$

where $D(\lambda) \in \mathbb{R}^{m_{+u}, m_{+u}}$ is obtained from the linear system

$$D(\lambda)[C(\lambda)C(\hat{\lambda})^{\mathrm{T}}] = C(\hat{\lambda})C(\hat{\lambda})^{\mathrm{T}}. \tag{4.4}$$

In this way the normalization $L_{+u}(\lambda)C(\hat{\lambda})^{\mathrm{T}} = C(\hat{\lambda})C(\hat{\lambda})^{\mathrm{T}}$ is achieved and $L_{+u}(\lambda)$ is as smooth as f_x. This follows from Appendix C with

$$A(\lambda) = f_x^{\mathrm{T}}(x_+(\lambda), \lambda), \qquad B(\lambda) = C(\lambda)^{\mathrm{T}}.$$

We did not implement more sophisticated methods for computing $L_{+u}(\lambda)$ because the dimension of the system (4.4) is low compared to any discretization of the boundary-value problem and hence it does not significantly contribute to the overall computational work. We start with a two-dimensional homoclinic example taken from Guckenheimer & Holmes (1983: Ch. 6.1).

EXAMPLE 1.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_1^2 + \lambda x_2 + \mu x_1 x_2.$$

This system has stationary points $(0,0)$ and $(1,0)$. For fixed $\mu > 0$, a supercritical Hopf bifurcation from $(1,0)$ occurs at $\lambda = -\mu$ and the periodic orbits turn into a homoclinic orbit with saddle $(0,0)$ at some $\lambda = \lambda_c(\mu) > -\mu$ (for the bifurcation diagram compare Beyn (1987) and for the typical changes in the phase portrait see Guckenheimer & Holmes (1983: Ch. 6)).

In what follows we solved (3.2) via the transformed system (4.1) with the settings

$$\mu = 0{\cdot}5, \qquad \zeta = (0, 1), \qquad \gamma = 0.$$

Figure 3 shows the homoclinic orbit in phase space obtained with $T_+ = -T_- = 10$ and with projection conditions. The λ-value is $\bar{\lambda} = -0{\cdot}429505849$ and $f_x(0, \bar{\lambda})$ has

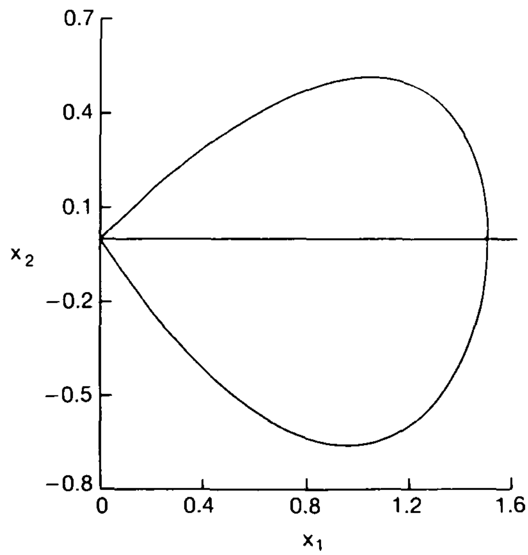

FIG. 3. The homoclinic orbit of example 1 in phase space ($\mu = 0{\cdot}5$) approximated on $[-10, 10]$.

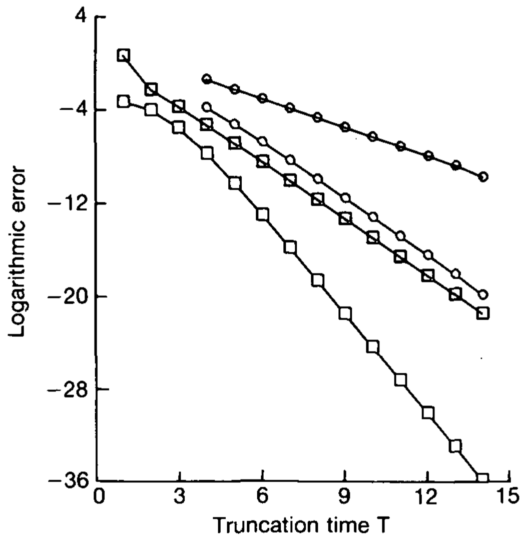

FIG. 4. Natural logarithm of errors vs. truncation time T.

O—O $e_x(T)$ } periodic boundary conditions
O—O $e_\lambda(T)$ }

□—□ $e_x(T)$ } projection boundary conditions
□—□ $e_\lambda(T)$ }

eigenvalues

$$\mu_- = 0{\cdot}808046576, \qquad -\mu_+ = -1.237552425.$$

In Fig. 4 we compare the errors introduced by the projection and the periodic boundary conditions (i.e. $z(1) = y(1)$ in (4.1b)). The quantities

$$e_x(T) = \ln \|\bar{x} \restriction_{[-T,T]} - x_{[-T,T]}\|_0 \qquad \|\circ\| = \text{Euclidean norm},$$
$$e_\lambda(T) = \ln |\bar{\lambda} - \lambda_{[-T,T]}|$$

are displayed in both cases. Actually, the solution on $[-15, 15]$ was taken as the exact solution and all finite boundary-value problems were solved at high accuracy ($\sim 10^{-15}$) so that discretization errors do not spoil the error arising from truncation to a finite domain.

For the projection conditions, e_x shows the expected slope $-2\mu_-$ (see Corollary 3.1) while e_λ shows some superconvergence with a slope of approximately $-2\mu_- - \mu_+$. A similar phenomenon is obtained for periodic boundary conditions but now with slopes $-\mu_-$ and $-2\mu_-$. A theoretical explanation of these effects (which can also be found in many other examples) needs a more detailed error analysis than that in Corollary 3.1 and will be given elsewhere. Nevertheless, Fig. 4 shows that, as a rule of thumb, for a given accuracy the projection conditions require only half of the time interval for periodic boundary conditions.

In Fig. 5 we demonstrate the importance of determining the correct projection conditions. We fix $T = 6$ and the projection condition at T but vary the

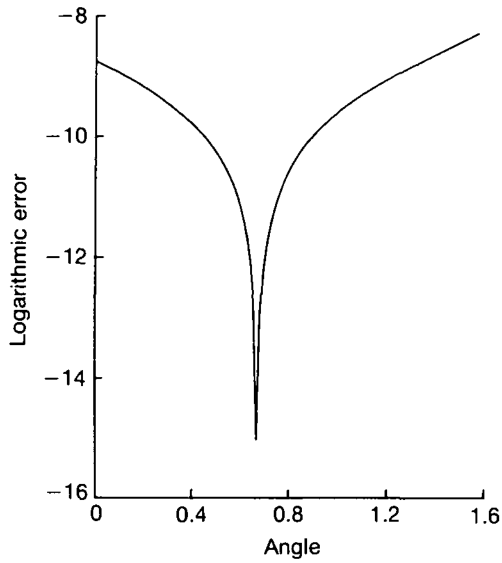

FIG. 5. Error $e_\lambda(T)$ vs. projection angle φ at $-T$ $(T=6)$.

left boundary condition

$$b_-(x(-T), \lambda) = [\cos\varphi, \sin\varphi]^T x(-T) = 0.$$

Then the error e_λ exhibits a sharp minimum at the angle φ which belongs to the projection condition at $-T$.

In the next step we discuss the choice of the truncation interval $[T_-, T_+]$. For the linear semi-infinite case there is a proposal by Mattheij (1987) where the right boundary is adjusted during a stabilized multiple shooting procedure. This approach, however, depends on the intervals of the boundary-value solver. In our case we prefer to solve the finite boundary-value problem (3.2) by any method, but try to estimate the error to the infinite solution by varying the underlying time interval (for a corresponding remark see Ascher, Mattheij, & Russell (1988: 11.4.2)). We also want to make use of the known exponential decay towards the stationary points.

As a consequence of our previous calculations we choose the projection conditions and take into consideration only the x-error given by

$$e(T_-, T_+, t) = \|\bar{x}(t) - x_{[T_-, T_+]}(t)\| \quad \text{for } T_- \leq t \leq T_+. \tag{4.5}$$

Figure 6 shows the logarithm of this error for $T_+ = 6$ and varying T_- with t scaled to $[-1, 1]$.

Clearly, by extending T_- only, the error cannot pass below a certain error function which has its maximum at T_+. But the error at $t = T_-$ decreases over a wide range precisely at the rate $2\mu_-$. Similarly, the rate $2\mu_+$ is found at $t = T_+$ if T_- is fixed and T_+ varies over a certain range. This is summarized in Fig. 7 which compares the local and global errors.

These observations motivated the following algorithm which was very reliable

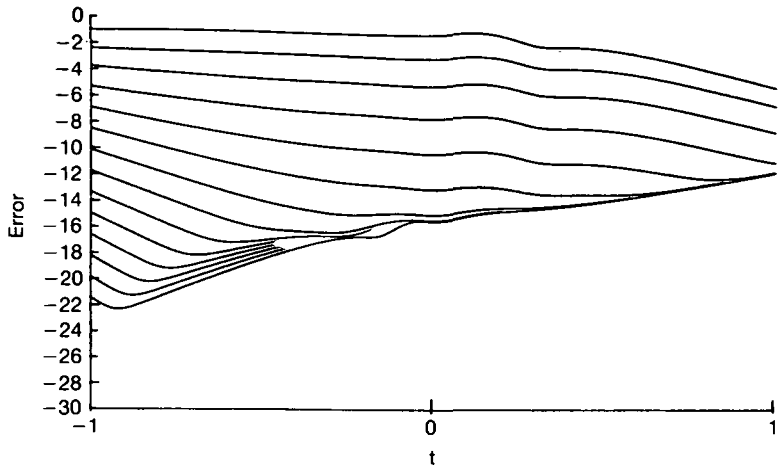

FIG. 6. Variation of error $\ln[e(T_-, T_+, t)]$ by fixing $T_+ = 6$ and varying T_- from -1 to -14, t was scaled to $[-1, 1]$.

in all examples considered below.

Step 1: Choose a tolerance tol, an interval $[T_-, T_+]$, and an initial approximation

$$(\hat{x}, \hat{\lambda}) \in X^1 \times \mathbb{R}^p \quad \text{for the COP.}$$

Compute stationary points $\hat{x}_\pm$ from $f(x, \hat{\lambda}) = 0$ and

$$\mu_\pm = \min\{|\mathrm{Re}\,\mu| : \mu \text{ eigenvalue of } f_x(\hat{x}_\pm, \hat{\lambda}) \text{ and } \mathrm{Re}\,\mu \lesseqgtr 0\}$$

$$\sigma_\pm = \min\{|\mathrm{Re}\,\sigma| : \sigma \text{ eigenvalue of } f_x(\hat{x}_\pm, \hat{\lambda}) \text{ and } \mathrm{Re}\,\sigma \gtreqless 0\}$$

Step 2: Using the initial approximations from the previous step compute the solution (x, λ) of the b.v.p. (3.2), (3.6) (via the system (4.1) or (4.2)) on the interval $[T_-, T_+]$ with tolerance $\frac{1}{3}$tol.

Step 3: Let $\Delta_\pm = 1/\min\{\mu_\pm, \sigma_\pm\}$ and $\tilde{T}_\pm = T_\pm \pm \Delta_\pm$. Repeat step 2 on the interval $[\tilde{T}_-, \tilde{T}_+]$ and obtain $(\tilde{x}, \tilde{\lambda})$.

Step 4: Compute $\varepsilon_\pm = \|x(T_\pm) - \tilde{x}(T_\pm)\|$. If $\max(\varepsilon_+, \varepsilon_-) \le \frac{2}{3}$tol, then accept the interval $[T_-, T_+]$ and the solution (x, λ) as COP. Otherwise continue with step 5.

Step 5: Use (x, λ) as initial approximations and update $\hat{x}_\pm$, $\mu_\pm$, $\sigma_\pm$. Set

$$\Delta_\pm^* = \ln\left(\frac{\text{tol}}{3\varepsilon_\pm}\right) \Big/ (2\mu_\pm), \qquad T_\pm := T_\pm \pm \Delta_\pm^*$$

and continue with step 2.

Step 4 provides approximations $\varepsilon_\pm$ for the quantities $e(T_-, T_+, T_\pm)$ from (4.5). Based on the assumption

$$e(T_-, T_+, T_\pm) = \text{const}_\pm \exp(-2\mu_\pm |T_\pm|),$$

we then try to compute a new interval $[T_-, T_+]$ in step 5 such that $e(T_-, T_+, T_\pm) \le \frac{1}{3}$tol.

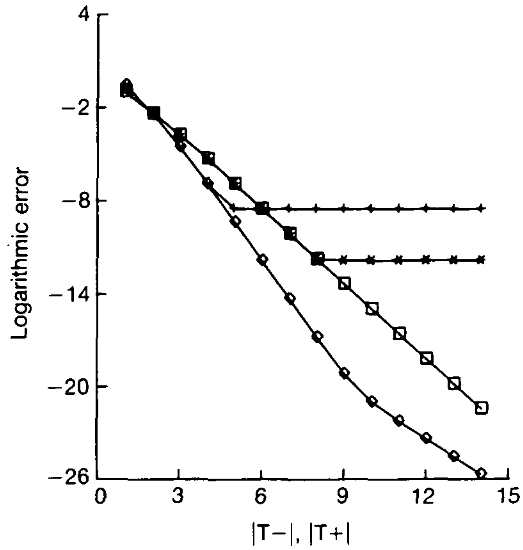

FIG. 7. Local and global errors obtained by varying one end of the interval $[T_-, T_+]$ only. Natural logarithm of

□—□ $e(T_-, 6, T_-)$

×—× $\max\{e(T_-, 6, t) : T_- \leq t \leq 6\}$

◇—◇ $e(-6, T_+, T_+)$

+—+ $\max\{e(-6, T_+, t) : -6 \leq t \leq T_+\}$

Altogether, the algorithm tries to equilibrate the errors caused by the truncation to the finite interval and by the discretization on the finite interval.

In most applications the successful exit in step 4 was reached after one correction of $[T_-, T_+]$ in step 5, i.e. a total of 4 b.v.p. had to be solved. Table 1 illustrates this in the case of example 1 for various initial intervals and tol $= 10^{-6}$. The actual final error

$$e(T_-, T_+) = \max\{e(T_-, T_+, t) : t \in [T_-, T_+]\}$$

is also shown. During the adaptation of $[T_-, T_+]$, the function ζ, which enters into the phase condition (see (4.1), (4.2)), was kept fixed.

TABLE 1

primary				secondary				
T_-	T_+	ε_-	ε_+	T_-	T_+	ε_-	ε_+	$e(T_-, T_+)$
−3	3	2·24E-2	1·12E-2	−9·89	7·20	3·79E-7	3·71E-7	3·79E-7
−6	3	3·17E-4	1·04E-2	−10·24	7·19	2·17E-7	3·86E-7	3·86E-7
−3	6	2·21E-2	1·46E-4	−9·89	8·45	3·79E-7	1·83E-7	3·79E-7
−6	6	1·96E-4	7·13E-6	−9·95	7·23	3·48E-7	3·42E-7	3·48E-7

Next we use the above algorithm during the continuation of a branch of connecting orbits. Then the initial approximations in step 1 are obtained from the predictor. Moreover, even after a successful exit from step 4 we execute step 5 in order to estimate the interval $[T_-, T_+]$ for the next solution on the branch. This accounts for shrinking time intervals along the branch.

We also employ the integral phase condition, i.e. we use (4.2). For periodic boundary value problems this condition allows for rather large continuation steps (see Doedel & Kernevez, 1986) and the same behaviour is found in our setting. Figure 8(a) shows some COPs from such a branch for example 1 with μ taken as parameter (tol $= 10^{-6}$). Actually it turned out that, for the required accuracy, it was not necessary to compute orbits which come very close to the stationary point on the negative time axis. In Fig. 8(b) the variation of $[T_-, T_+]$ along the branch generated by our algorithm is displayed.

EXAMPLE 2. (Lorenz equations, cf. Sparrow (1982))

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \lambda x - y - xz, \qquad \dot{z} = -\mu z + xy.$$

First, at the Lorenz values $\sigma = 10$ and $\mu = \frac{8}{3}$, we computed the homoclinic orbit connecting the origin with itself and the λ-value ($= 13.926557$). It is well known that a strange invariant set is created at the COP for this system (see Sparrow (1982)). Then we followed a branch of COPs by varying μ. Figure 9(a) shows the values of λ, T_+, and $|T_-|$ along the branch as obtained by our procedure. Note that the time interval shrinks with increasing μ while the phase curve continuously grows (Fig. 9(b)).

EXAMPLE 3. (Travelling waves, e.g. Fife (1979))

$$u_t = u_{xx} + u(u - \mu)(1 - u) \quad \text{for } x \in \mathbb{R}, t \geq 0.$$

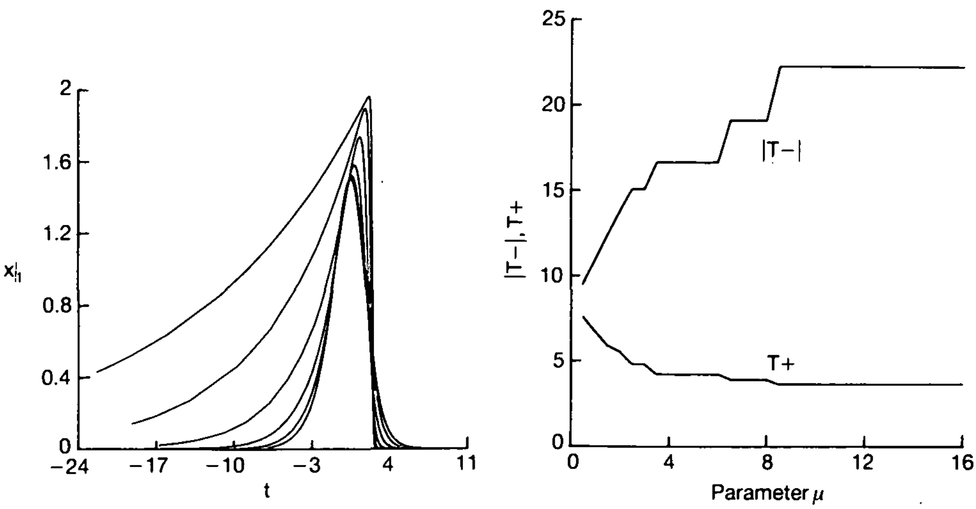

FIG. 8(a). First component of homoclinic orbit for example 1, $\mu = 0{\cdot}5$, 1, 2, 4, 8, 16. (b) Variation of time interval $[T_-, T_+]$ for example 1 as obtained by the adaptive algorithm.

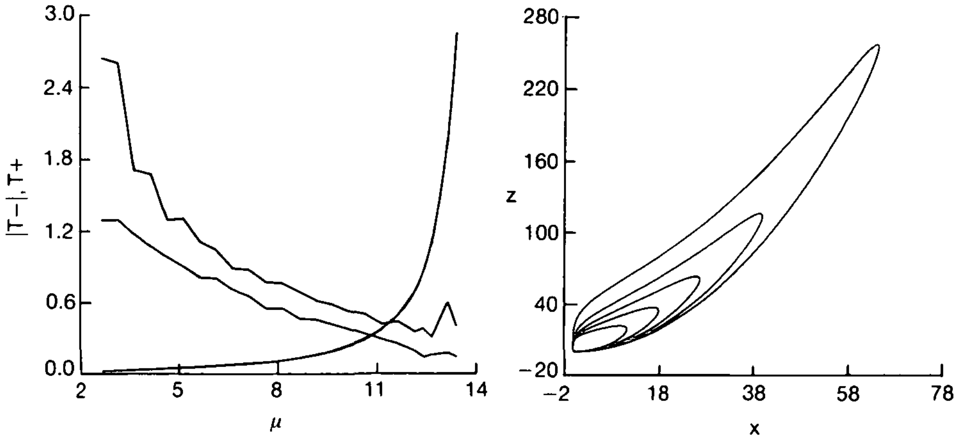

FIG. 9(a). $[T_-, T_+]$ and $\frac{1}{500}\lambda$ for a branch of homoclinic orbits in the Lorenz equations. (b) Projection of homoclinic orbits into xz plane, Lorenz equations $3\mu = 8, 14, 20, 26, 32, 38$.

Travelling waves of the form $u(x, t) = w(x - \lambda t)$ are found as heteroclinic orbits connecting the saddles $(0, 0)$ and $(1, 0)$ of the system

$$w' = z, \qquad z' + \lambda z + w(w - \mu)(1 - w) = 0.$$

Again, this system was rather easy to solve with our algorithm and with the integral phase condition. A typical wave front $w(x)$ at $\mu = \frac{1}{3}$ and the truncation intervals are shown in Fig. 10(a) and (b).

EXAMPLE 4. (Modification by Chay & Keizer (1983) of the Hodgkin–Huxley model for nerve excitation.) This four-dimensional system models the time course of voltage oscillations across the membrane of pancreatic β-cells for a given intracellular calcium concentration Ca [μM] (= parameter λ below). Actually, it occurs as a fast subsystem of a singularly perturbed five-dimensional system (see Rinzel (1985) for more details and an enlightening discussion). The dimensionless equations are

$$\dot{V} = g_K(V_K - V) + 2g_{Ca,HH}(V_{Ca} - V) + g_L(V_L - V),$$
$$\dot{n} = \Phi[\alpha_n(V)(1 - n) - \beta_n(V)n], \qquad \dot{m} = \Phi[\alpha_m(V)(1 - m) - \beta_m(V)m],$$
$$\dot{h} = \Phi[\alpha_h(V)(1 - h) - \beta_h(V)h],$$

where

$$g_K = 0.09\frac{\lambda}{1 + \lambda} + 12n^4, \qquad g_{Ca,HH} = 6.5m^3h, \qquad g_L = 0.04,$$
$$V_K = -75, \qquad V_{Ca} = 100, \qquad V_L = -40, \qquad \Phi = 3^{(T-6.3)/10},$$
$$\alpha_n(V) = \tfrac{1}{100}(-V - 20)\{\exp[\tfrac{1}{10}(-V - 20)] - 1\}^{-1}, \qquad \beta_n(V) = \tfrac{1}{8}\exp[\tfrac{1}{80}(-V - 30)],$$
$$\alpha_m(V) = \tfrac{1}{10}(-V - 25)\{\exp[\tfrac{1}{10}(-V - 25)] - 1\}^{-1}, \qquad \beta_m(V) = 4\exp[\tfrac{1}{20}(-V - 50)],$$
$$\alpha_h(V) = 0.07\exp[\tfrac{1}{20}(-V - 50)], \qquad \beta_h(V) = \{\exp[\tfrac{1}{10}(-V - 20)] + 1\}^{-1}.$$

At temperature $T = 18{\cdot}5$ (°C) we found the homoclinic orbit shown in Fig. 11

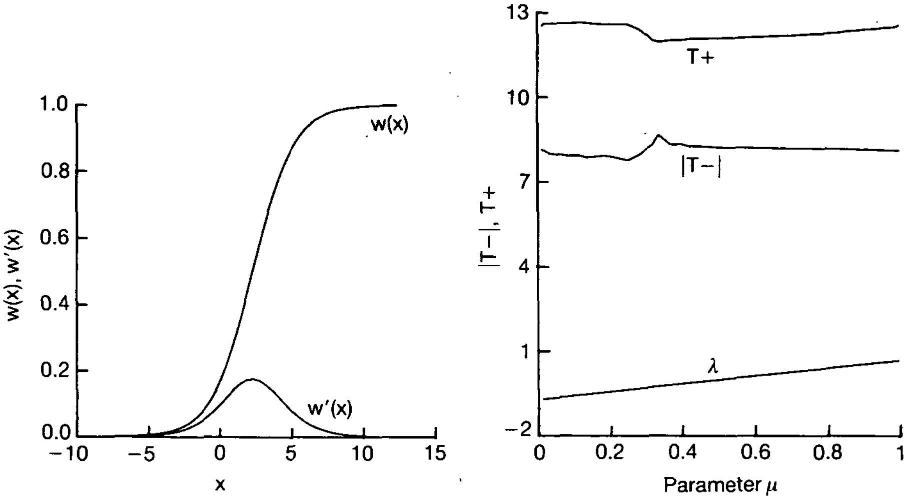

FIG. 10(a). Wavefront $w(x)$ and $w'(x)$ for example 3. (b) Variation of $[T_-, T_+]$ and λ on the branch of heteroclinic orbits for example 3.

(tol = 10^{-6}). It determines the shape of the last spike in a bursting oscillation while $\bar{\lambda}$ ($= 0{\cdot}4576217$) is a threshold value above which the oscillations break down (see Rinzel, 1985). We emphasize that all the quantities $x_+(\lambda)$, $L_{+u}(\lambda)$, and $L_{-s}(\lambda)$ in the projection conditions (3.6) had to be computed numerically. At $\bar{\lambda}$ the stationary point was ($-43{\cdot}578$, $0{\cdot}10926$, $0{\cdot}37003$, $0{\cdot}14264$) with the eigenvalues of f_x being $-12{\cdot}29$, $-0{\cdot}6369$, $-0{\cdot}4454$, $0{\cdot}1031$.

The slow unstable mode actually caused the rather large negative part of the time interval in Fig. 11.

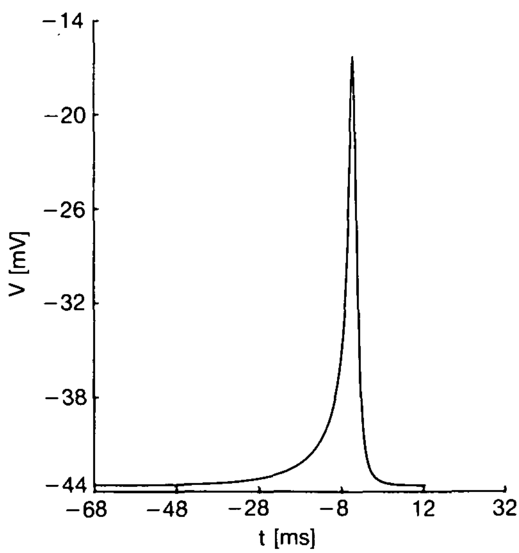

FIG. 11. First component (voltage) of homoclinic orbit for example 4.

Appendix A. Proof of Lemma 2.2

Palmer (1984: Lemma 4.2) gives a proof of Lemma 2.2 when the spaces $\mathcal{X}^k$ are replaced by spaces of bounded functions:

$$\mathcal{B}^k = \{x \in C^k(\mathbb{R}, \mathbb{R}^m) : x^{(j)}(t) \ \ (t \in \mathbb{R}) \text{ bounded for } j = 0,\ldots,k\}.$$

This does not affect $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$, since any bounded function which vanishes under L is already exponentially decreasing in both time directions (see (2.6), (2.7), (2.8)).

Also, the range $L(\mathcal{B}^1)$ contains the range $L(\mathcal{X}^1)$ and the Fredholm property of L follows from (2.13) and (2.14). Therefore, only the backward implication of (2.14) remains to be proved. Let $y \in \mathcal{X}^0$ be given such that

$$\int_{-\infty}^{\infty} \varphi^{\mathrm{T}}(t) y(t)\, \mathrm{d}t = 0 \quad \text{for all } \varphi \in \mathcal{N}(L^*).$$

From Palmer's result we know that $\dot{x} - A(\circ)x = y$ holds for some function $x \in \mathcal{B}^1$ and we show that $x(t)$ converges as $t \to \pm\infty$. For $t \geqslant 0$ a particular bounded solution of the inhomogeneous equation is

$$z(t) = \int_0^t Y(t)PY(s)^{-1}y(s)\,\mathrm{d}s - \int_t^{\infty} Y(t)(I-P)Y(s)^{-1}y(s)\,\mathrm{d}s. \tag{A.1}$$

Then $x(t) - z(t)$ is bounded for $t \geqslant 0$ and satisfies the homogeneous equation, hence $x(t) - z(t) \to 0$ as $t \to \infty$.

Now we replace $y(s)$ in (A.1) by $y(s) - y(\infty) + y(\infty)$ and use the exponential dichotomy (2.6), (2.7) to find

$$\lim_{t\to\infty} [z(t) - C(t)y(\infty)] = 0,$$

where

$$C(t) = \int_0^t Y(t)PY(s)^{-1}\,\mathrm{d}s - \int_t^{\infty} Y(t)(I-P)Y(s)^{-1}\,\mathrm{d}s.$$

We consider

$$\int_0^t Y(t)PY(s)^{-1}\,\mathrm{d}sA_+ = \int_0^t Y(t)PY(s)^{-1}A(s)\,\mathrm{d}s + \int_0^t Y(t)PY(s)^{-1}[A_+ - A(s)]\,\mathrm{d}s,$$

where the last integral vanishes as $t \to \infty$ because of (2.5) and (2.6). Using $(Y^{-1})' = -Y^{-1}A$, (2.6), and (2.9) we find, for the first integral,

$$-\int_0^t Y(t)P(Y^{-1})'(s)\,\mathrm{d}s = Y(t)P\big(Y^{-1}(0) - Y^{-1}(t)\big) \to -P_{+\mathrm{s}} \quad \text{as } t \to \infty.$$

Similarly,

$$\int_t^{\infty} Y(t)(I-P)Y(s)^{-1}\,\mathrm{d}sA_+ \to P_{+\mathrm{u}} \quad \text{as } t \to \infty$$

and therefore

$$C(t)y(\infty) \to (-P_{+\mathrm{s}} - P_{+\mathrm{u}})A_+^{-1}y(\infty) = -A_+^{-1}y(\infty) \quad \text{as } t \to \infty.$$

$x(t)$ has the same limit as $t \to \infty$ and finally $\dot{x}(t) \to 0$ follows from the differential equation. By a reflection of the time axis we also obtain $x(t) \to -A_-^{-1}y(-\infty)$ as $t \to -\infty$. □

Appendix B. Proof of Lemma 2.3

Let us decompose S as follows:

$$S = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$

It is easily seen that the blockdiagonal operator has null space $\mathcal{N}(A) \times \mathcal{N}(D)$ and range $\mathcal{R}(A) \times \mathcal{R}(D)$, hence is Fredholm of index $\operatorname{ind}(A) + \operatorname{ind}(D) = N + p - q$. The operators B and C are compact, so that S is a compact perturbation of its blockdiagonal part. Therefore S is also Fredholm and has the same index, see, for example, Kato (1966: IV Thm 5.26). □

Appendix C. Lemma

Let $A \in C^k(\mathbb{R}^p, \mathbb{R}^{m,m})$ be a matrix valued function and let $\hat{\lambda} \in \mathbb{R}^p$ be given such that $A(\hat{\lambda})$ is hyperbolic with unstable dimension m_u. Then there exists a neighbourhood Λ of $\hat{\lambda}$ such that $A(\lambda)$ is hyperbolic for $\lambda \in \Lambda$ with the same unstable dimension. Moreover, let $B(\lambda) \in \mathbb{R}^{m,m_u}$, *for* $\lambda \in \Lambda$, be given (without any continuity assumption) such that the columns of $B(\lambda)$ span the unstable subspace of $A(\lambda)$. Then, for λ in some neighbourhood of $\hat{\lambda}$, the linear system

$$[B(\hat{\lambda})^T B(\lambda)]\Delta(\lambda) = B(\hat{\lambda})^T B(\hat{\lambda}) \tag{C.1}$$

has a unique solution $\Delta(\lambda) \in \mathbb{R}^{m_u,m_u}$ and the columns of $B(\lambda)\Delta(\lambda)$ form a C^k-smooth basis of the unstable subspace of $A(\lambda)$.

Proof. The stability of the hyperbolicity under small perturbations is classical (see, for example, Kato, 1966: II §1, §5), so we need only consider the special construction of a smooth basis. By our assumption there is a matrix $\hat{U} \in \mathbb{R}^{m_u,m_u}$ such that the spectrum of $\hat{U}$ is the unstable spectrum of $A(\hat{\lambda})$ and such that $A(\hat{\lambda})B(\hat{\lambda}) = B(\hat{\lambda})\hat{U}$. We can now apply the implicit function theorem to the matrix equation

$$G(\lambda, R, U) = \big(A(\lambda)R - RU, B(\hat{\lambda})^T R - B(\hat{\lambda})^T B(\hat{\lambda})\big) = 0$$

and find C^k-smooth solutions $R(\lambda) \in \mathbb{R}^{m,m_u}$, $U(\lambda) \in \mathbb{R}^{m_u,m_u}$, for $\lambda \in \Lambda$, such that $R(\hat{\lambda}) = B(\hat{\lambda})$, $U(\hat{\lambda}) = \hat{U}$. Moreover, the columns of $R(\lambda)$ span the unstable space of $A(\lambda)$ and the spectrum of $U(\lambda)$ consists of the unstable eigenvalues of $A(\lambda)$. Taking Λ sufficiently small we may assume that $B(\hat{\lambda})^T R(\lambda)$, for $\lambda \in \Lambda$, is nonsingular. By our assumption, $B(\lambda) = R(\lambda)T(\lambda)$ holds for some nonsingular $T(\lambda) \in \mathbb{R}^{m_u,m_u}$. Therefore, $B(\hat{\lambda})^T B(\lambda) = B(\hat{\lambda})^T R(\lambda)T(\lambda)$ is also nonsingular and, by the construction of $R(\lambda)$, we find that the unique solution of (C.1) is $\Delta(\lambda) = T(\lambda)^{-1}$. Therefore $B(\lambda)\Delta(\lambda) = R(\lambda)$ has the desired smoothness. □

Appendix D. Proof of (3.14)

We use the matrix $\Phi(t)\in\mathbb{R}^{m,p}$ from Proposition 2.1, the columns of which form a basis of $\mathcal{N}(L^*)$ with $L=\mathrm{d}/\mathrm{d}t-f_x(\bar{x},\bar{\lambda})$. From Lemma 2.2 we have an estimate

$$\|\Phi^{\mathrm{T}}(t)\|\leqslant K\mathrm{e}^{-\alpha|t|}\quad\forall t\in\mathbb{R}. \tag{D.1}$$

Further, by Proposition 2.1 the matrix

$$E=\int_{-\infty}^{\infty}\Phi^{\mathrm{T}}(t)V(t)\,\mathrm{d}t \tag{D.2}$$

is nonsingular. Now we multiply (3.13a) by $\Phi^{\mathrm{T}}(t)$, integrate over $\mathcal{J}$, and find by partial integration

$$\Phi^{\mathrm{T}}(t)x(t)\Big|_{T_-}^{T_+}-\int_{T_-}^{T_+}\Phi^{\mathrm{T}}(t)V(t)\,\mathrm{d}t\lambda=\int_{T_-}^{T_+}\Phi^{\mathrm{T}}(t)y(t)\,\mathrm{d}t.$$

Using (D.1) and (D.2) this implies

$$\|\lambda\|\leqslant c_{\mathcal{J}}\,\|x\|_0+C\,\|y\|_0, \tag{D.3}$$

where C is independent of y, x, and $\mathcal{J}$, whereas $c_{\mathcal{J}}$ is independent of y and x only, but $c_{\mathcal{J}}\to0$ as $\mathcal{J}\to\infty$. By a slight abuse of notation, we write this as $\lambda=o(\|x\|_0)+O(\|y\|_0)$. Our goal is to establish an estimate

$$\|x\|_0=o(\|x\|_0)+O(\|y\|_0+\|r_+\|+\|r_-\|+|\varepsilon|). \tag{D.4}$$

Taking the $o(\|x\|_0)$ term to the left we then obtain

$$\|x\|_0=O(\|y\|_0+\|r_+\|+\|r_-\|+|\varepsilon|)$$

for sufficiently large $\mathcal{J}$, and using (D.3) as well as the differential equation (3.13a) we find the same estimate for $\|x\|_1+\|\lambda\|$. In what follows we will modify the projectors P and Q from Lemma 2.1 slightly without changing $\mathcal{R}(P)$ and $\mathcal{N}(Q)$. By (2.12) and (2.17) we have

$$\mathcal{R}(P)\cap\mathcal{N}(Q)=\operatorname{span}\{\dot{\bar{x}}(0)\}=:\mathcal{Z}_1$$

and we can choose subspaces $\mathcal{Z}_2$, $\mathcal{Z}_3$, and $\mathcal{Z}_4$ such that

$$\mathcal{R}(P)=\mathcal{Z}_1\oplus\mathcal{Z}_2,\qquad \mathcal{N}(Q)=\mathcal{Z}_1\oplus\mathcal{Z}_3,\qquad (\mathcal{R}(P)+\mathcal{N}(Q))\oplus\mathcal{Z}_4=\mathbb{R}^m.$$

This induces a splitting $\mathbb{R}^m=\mathcal{Z}_1\oplus\mathcal{Z}_2\oplus\mathcal{Z}_3\oplus\mathcal{Z}_4$ and we assume P, Q, and Π to be the projectors onto $\mathcal{Z}_1\oplus\mathcal{Z}_2$, $\mathcal{Z}_2\oplus\mathcal{Z}_4$, and $\mathcal{Z}_1$ respectively. We then have

$$\|P\xi-\Pi\xi\|\leqslant C\,\|Q\xi\|,\qquad \|(I-Q)\xi-\Pi\xi\|\leqslant C\,\|(I-P)\xi\|\quad\forall\xi\in\mathbb{R}^m. \tag{D.5}$$

Now we define, for any $y\in\mathcal{X}^0(\mathcal{J})$ (see (A.1)),

$$s_{\mathcal{J}}(y)(t)=$$
$$\begin{cases}\displaystyle\int_0^t Y(t)PY(s)^{-1}y(s)\,\mathrm{d}s-\int_t^{T_+}Y(t)(I-P)Y(s)^{-1}y(s)\,\mathrm{d}s & \text{for } 0\leqslant t\leqslant T_+,\\[2ex] \displaystyle\int_{T_-}^t Y(t)QY(s)^{-1}y(s)\,\mathrm{d}s-\int_t^0 Y(t)(I-Q)Y(s)^{-1}y(s)\,\mathrm{d}s & \text{for } T_-\leqslant t<0.\end{cases}$$

$s_{\mathcal{J}}(y)$ is a bounded solution of $\dot{x} - A(\bullet)x = y$ in $\mathcal{J}\backslash\{0\}$ which may be discontinuous at 0. From (2.6) and (2.7) we find $\|s_{\mathcal{J}}(y)\|_0 = O(\|y\|_0)$. The solution x of (3.13a) may now be written for some $\xi, \eta \in \mathbb{R}^m$ as

$$x(t) = Y(t)\{\eta, \xi\} + r(t), \qquad r = s_{\mathcal{J}}(y + V(\circ)\lambda), \tag{D.6}$$

where $\|r\|_0 = o(\|x\|) + O(\|y\|_0)$ and $\{\eta, \xi\}$ denotes the step function which is ξ for $t \geqslant 0$ and η for $t < 0$. Since x is continuous at 0 we also have

$$\xi - \eta = o(\|x\|_0) + O(\|y\|_0), \qquad \xi = O(\|x\|_0) + O(\|y\|_0). \tag{D.7}$$

Next we set $\xi_+ = Y(T_+)(I - P)\xi$ and use the representation (D.6) in equation (3.13b) to find

$$K_+\xi_+ = r_+ - M_+\lambda - K_+Y(T_+)P\xi - K_+r(T_+) = o(\|x\|_0) + O(\|y\|_0 + \|r_+\|).$$

In order to estimate ξ_+ itself we notice that assumption (i) of Theorem 3.1 implies

$$\|P_{+u}\zeta\| \leqslant C\,\|b_{+x}(\bar{x}_+, \bar{\lambda})P_{+u}\zeta\| \quad \forall\, \zeta \in \mathbb{R}^m.$$

Since $K_+ \to b_{+x}(\bar{x}_+, \bar{\lambda})$ and $Y(T_+)(I - P)Y(T_+)^{-1} \to P_{+u}$ as $\mathcal{J} \to \mathbb{R}$ by Lemma 2.1, the inequality $\|\xi_+\| \leqslant C\,\|K_+\xi_+\|$ holds for sufficiently large $\mathcal{J}$ and we end up with

$$\xi_+ = Y(T_+)(I - P)\xi = o(\|x\|_0) + O(\|y\|_0 + \|r_+\|).$$

Similarly,

$$\eta_- = Y(T_-)Q\eta = o(\|x\|_0) + O(\|y\|_0 + \|r_-\|)$$

and by the exponential dichotomies

$$Y(t)(I - P)\xi = o(\|x\|_0) + O(\|y\|_0 + \|r_+\|) \quad \text{uniformly for } 0 \leqslant t \leqslant T_+, \tag{D.8}$$

$$Y(t)Q\eta = o(\|x\|_0) + O(\|y\|_0 + \|r_-\|) \quad \text{uniformly for } T_- \leqslant t \leqslant 0. \tag{D.9}$$

From the definition of the projector Π we know that $\Pi\xi = \gamma\dot{\bar{x}}(0)$ for some $\gamma \in \mathbb{R}$ and hence $Y(t)\Pi\xi = \gamma\dot{\bar{x}}(t)$. For $0 \leqslant t \leqslant T_+$ we obtain, from (2.6) and (D.5)–(D.9),

$$\begin{aligned} x(t) - \gamma\dot{\bar{x}}(t) &= Y(t)(\xi - \Pi\xi) + r(t) \\ &= Y(t)(I - P)\xi + Y(t)(P\xi - \Pi\xi) + r(t) \\ &= O(\|Q\xi\|) + o(\|x\|_0) + O(\|y\|_0 + \|r_+\|) \\ &= o(\|x\|_0) + O(\|y\|_0 + \|r_+\| + \|r_-\|), \end{aligned}$$

and the same estimate for $T_- \leqslant t \leqslant 0$.

Finally, we use (3.13c) and assumption (ii) of Theorem 3.1

$$\begin{aligned} d\,|\gamma| \leqslant |u(\gamma\dot{\bar{x}})| &\leqslant |u(\gamma\dot{\bar{x}} - x)| + |u(x)| \\ &= o(\|x\|_0) + O(\|y\|_0 + \|r_+\| + \|r_-\| + |\varepsilon|). \end{aligned}$$

Combining this with the estimate of $x - \gamma\dot{\bar{x}}$ gives the desired result (D.4). □

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