

THE EFFECT OF DISCRETIZATION ON HOMOCLINIC ORBITS

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1. Introduction

The numerical analysis of the longtime behavior of a dynamical system

$$(1) \dot{z} = f(z, \lambda), \quad z \in \mathbb{R}^m, \lambda \in \mathbb{R} \text{ (a parameter)}$$

usually follows two complementary approaches.

I. Direct methods: Set up and solve defining equations for limit sets of (1), such as steady states and periodic orbits, determine their stability characteristics and find singular points with respect to the parameter.

II. Indirect methods: Integrate (1) numerically for various values of $z(0)$ and λ and analyze the longtime behavior of the numerical trajectories.

Although there has been considerable success with direct methods the integration seems to be indispensable in complicated situations where e.g. global stability properties are of interest or where direct methods are not available. For the integration approach it is important to know in which sense the longtime behavior of continuous trajectories is reflected by the numerical trajectories. For simplicity, we will assume the latter to be sequences z^n generated by some one-step method with constant step

size h

$$(2) \quad z^{n+1} = \varphi(z^n, \lambda, h) = z^n + hg(z^n, \lambda, h)$$

It is the purpose of this paper to comment on both approaches in the case where the system (1) has a homoclinic orbit.

2. A direct method for homoclinic orbits

A homoclinic orbit $\bar{z}(t)$, $t \in \mathbb{R}$ is a solution of (1) at some $\lambda = \bar{\lambda}$ which satisfies

$$(3) \quad \bar{z}(t) \rightarrow z_\infty \text{ as } t \rightarrow \pm\infty \text{ and } f(z_\infty, \bar{\lambda}) = 0$$

Homoclinic orbits are typical one parameter phenomena and - as with Hopf points - they mark the begin and end of branches of periodic orbits (see [5,10] for a theoretical and a numerical illustration of this fact). We consider the following two-dimensional model example (see [8],Ch.6.1)

$$(4) \quad \dot{x} = y, \quad \dot{y} = x - x^2 + \lambda y + axy$$

Fig. 1 gives the numerical bifurcation picture for $a=0.5$ as obtained with the Code AUTO by Doedel [5].

The periodic orbits created at the Hopf point $\lambda = -a$ vanish through a homoclinic orbit at $\bar{\lambda} = -0.429505$

(see [8] for the typical changes in the phase

plane near this point). In [5] homoclinic orbits are simply computed as periodic orbits with large period, nevertheless it seems

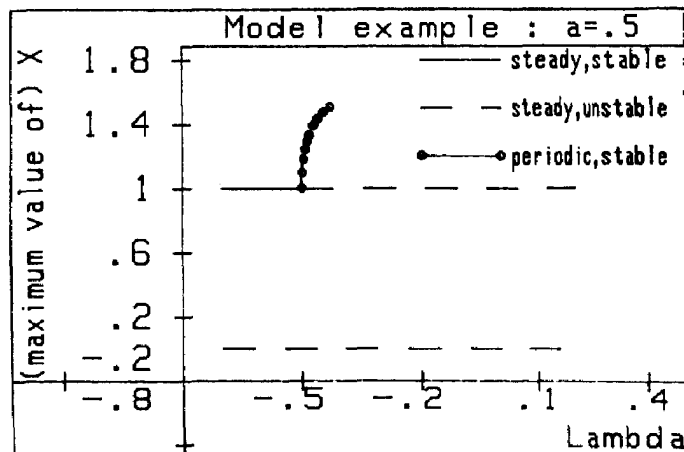


Figure 1

attractive (and probably more efficient) to have a direct method for homoclinic orbits which make use of the base point z_∞ and of the linearization at z_∞ (see below).

We return to the general system (1) and assume that it has some smooth branch $(z_\infty(\lambda), \lambda)$ of steady states such that $A(\lambda) = \frac{\partial f}{\partial z}(z_\infty(\lambda), \lambda)$ has no eigenvalues on the imaginary axis. The computation of a homoclinic orbit with a base point on this branch would then require to find λ and $z(t)$ ($t \in \mathbb{R}$) satisfying (1) and

$$(5a) \quad \lim_{t \rightarrow \infty} z(t) = z_\infty(\lambda), \quad \lim_{t \rightarrow -\infty} z(t) = z_\infty(\lambda)$$

As for periodic orbits we have to add a phase fixing condition which we take in the simple form

$$(5b) \quad \Psi^T z(0) - \alpha = 0 \quad \text{where } \Psi \in \mathbb{R}^m, \alpha \in \mathbb{R}.$$

For numerical purposes we have to replace the infinite interval $(-\infty, \infty)$ by a finite one, say $[-T, T]$, and we have to replace (5a) by appropriate boundary conditions at $-T, T$. For boundary value problems on semi-infinite intervals there is a well developed theory of how to do this [9, 11, 12] and how to estimate the resulting error. This theory easily carries over to the present case. Let $P(\lambda)$ and $Q(\lambda)$ be the projectors onto the invariant subspaces of $A(\lambda)$ associated with the unstable and stable eigenvalues. Then we replace (1), (5a,b) by (5b) and

$$(6a) \quad \dot{z} = f(z, \lambda), \quad -T \leq t \leq T$$

$$(6b) \quad P(\lambda)(z(T) - z_\infty(\lambda)) = 0, \quad Q(\lambda)(z(-T) - z_\infty(\lambda)) = 0.$$

The boundary condition (6b) forces the orbit to leave $z_\infty(\lambda)$ close to the unstable manifold and to approach it again close to the stable manifold. There are many more possible choices for (6b), however (cf. [9]).

Finally, by setting $y(t) = z(-t)$ we obtain a two-point boundary value problem of dimension $2m+1$ on $[0, T]$ to which we can apply a

standard code

$$(7a) \quad \dot{z} - f(z, \lambda) = 0, \quad \dot{y} + f(y, \lambda) = 0, \quad \dot{\lambda} = 0 \quad (0 \leq t \leq T)$$

$$(7b) \quad P(\lambda(T))(z(T) - z_\infty(\lambda(T))) = 0, \quad Q(\lambda(T))(y(T) - z_\infty(\lambda(T))) = 0$$

$$\Psi^T z(0) - \alpha = 0, \quad z(0) = y(0)$$

The choice of Ψ and α is relatively easy if we are following a branch of periodic solutions with increasing period. If $\hat{z}(t)$ is such a large period solution we could use $\Psi = \hat{z}'(0)$, $\alpha = \Psi^T \hat{z}(0)$. Using the theory from [9] we can show the following result. Suppose that $(\bar{z}(t) (t \in \mathbb{R}), \bar{\lambda})$ is a nondegenerate homoclinic pair, i.e. it solves (1), (5a,b) and the linearized problem

$$\dot{y} - \frac{\partial f}{\partial z}(\bar{z}, \bar{\lambda})y - \frac{\partial f}{\partial \lambda}(\bar{z}, \bar{\lambda})\mu = 0, \quad \lim_{t \rightarrow \pm\infty} y(t) = 0, \quad \Psi^T y(0) = 0$$

has only the trivial solution (y, μ) . Then (5b), (6a,b) has a unique solution (z_T, λ_T) for T sufficiently large and

$$\max_{|t| \leq T} \|\bar{z}(t) - z_T(t)\| + |\bar{\lambda} - \lambda_T| \leq C \{ \|Q(\bar{\lambda})(\bar{z}(-T) - z_\infty(\bar{\lambda}))\| + \|P(\bar{\lambda})(\bar{z}(T) - z_\infty(\bar{\lambda}))\| \}$$

In fact, due to the hyperbolicity of $z_\infty(\bar{\lambda})$ the right hand side decays like $\exp(-\beta T)$ for some $\beta > 0$.

The defining system (7a,b) can be modified in an obvious way for heteroclinic orbits connecting two hyperbolic points at which the Jacobian of f has stable subspaces of equal dimension.

Finally, it is interesting to note that in the two-dimensional case the nondegeneracy of the homoclinic pair $(\bar{z}, \bar{\lambda})$ is equivalent to one of the assumptions in the homoclinic bifurcation theorem ([8], Th.6.1.1.(2)). Together with $\text{trace } A(\bar{\lambda}) \neq 0$ this theorem ensures that in a one sided neighborhood of $\bar{\lambda}$ there exists a branch of periodic orbits which turn into the homoclinic orbit at $\bar{\lambda}$.

2. Numerical integration of systems with homoclinic orbits

We now consider the asymptotic behavior $n \rightarrow \infty$ of a one-step method

(2) if h is small and if λ is close to the value $\bar{\lambda}$, for which the system (1) has a homoclinic orbit.

First of all, it is instructive to review the results on (2) if the system (1), for a fixed λ , has a hyperbolic, periodic orbit $z(t)$ with period T . Apart from some smoothness we assume p -th order accuracy of the method (2) in the form

$$(8) \quad \frac{\partial^i \varphi}{\partial h^i}(z, \lambda, 0) = \frac{\partial^i \phi}{\partial h^i}(z, \lambda, 0), \quad i = 0, \dots, p$$

where $\phi(z, \lambda, h)$ denotes the flow of the system (1) with time step h . It was shown in [1, 2, 4, 6, 7] that, for h sufficiently small, the one step method (2) has a closed invariant curve $\{z_h(t) : t \in \mathbb{R}\}$ where z_h is T -periodic and satisfies

$$(9) \quad \text{Max}\{\|z(t) - z_h(t)\| : 0 \leq t \leq T\} = O(h^p)$$

$$(10) \quad \varphi(z_h(t), \lambda, h) = z_h(\sigma_h(t)) \quad (t \in \mathbb{R}) \quad \text{where} \quad \sigma_h(t) = t + h + O(h^{p+1})$$

These results may be generalized to compact branches of hyperbolic periodic orbits, but it is clear that the critical value of h , below which the invariant curve exists, tends to zero if we approach a Hopf point or a homoclinic point. The situation near a Hopf bifurcation was successfully analyzed in [3]. But - as far as we know - there is no precise answer to the question, what happens to the invariant curves of (2) if λ passes the homoclinic point.

Fig. 2 shows some numerical experiments with Euler's method for the model example (4) with $h = 0.4$, $a = 0.5$.

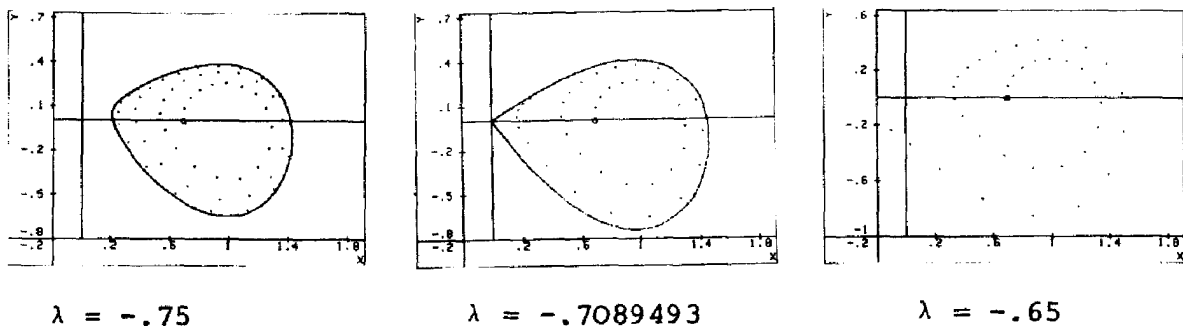


Figure 2

For $\lambda < \bar{\lambda}_h \cong -0.7089493$ the points of the iteration starting at $(0.7, 0)$ filled an invariant curve after some time whereas for $\lambda > \bar{\lambda}_h$ the sequence became unbounded. It seems that, exactly as for the continuous system, there is a critical value $\bar{\lambda}_h$ at which we have an "invariant homoclinic curve". From a generic point of view this is quite surprising (see below) but it has been observed also for smaller step sizes h and different Runge Kutta methods. What we can actually prove is the following. Let the branch $(z_\infty(\lambda), \lambda)$ from section 1 be the trivial one and let $(\bar{z}, \bar{\lambda})$ be a nondegenerate homoclinic pair. Moreover, assume (8) and $\varphi(0, \lambda, h) \equiv 0$ for all λ and h as well as

$$(11) \quad g(z, \lambda, 0) = f(z, \lambda) \text{ (compare (1) and (2)), } g \in C^1(\mathbb{R}^m \times \mathbb{R} \times [0, 1], \mathbb{R}^m)$$

Then, for h sufficiently small, the iteration (2) has a discrete homoclinic pair $(z_h^n (n \in \mathbb{Z}), \lambda_h)$, i.e.

$$(12) \quad z_h^{n+1} = \varphi(z_h^n, \lambda_h, h) \quad (n \in \mathbb{Z}) \text{ and } z_h^n \rightarrow 0 \text{ as } n \rightarrow \pm\infty, \quad \psi^T z_h^0 - \alpha = 0.$$

Moreover, we have the estimate

$$\sup\{\|z_h^n - \bar{z}(nh)\| : n \in \mathbb{Z}\} + |\lambda_h - \bar{\lambda}| = O(h^p).$$

The following table shows a few values of λ_h for our model example (Euler's method) which were obtained by truncating the boundary conditions in (12) in a way analogous to (6b) and using $\psi^T = (0, 1), \alpha = 0$.

h	0.8	0.4	0.2	0.1	0.05
λ_h	-0.9645	-0.7089	-0.5716	-0.5010	-0.4653

, $a = 0.5$

In all cases we found that λ_h coincided with the value at which the invariant curve vanishes.

In the two dimensional case, under our assumptions above, we have that 0 is a hyperbolic point of the mapping $\varphi(\cdot, \lambda_h, h)$. The stable and unstable manifolds M_s^h and M_u^h of this point both contain the discrete homoclinic orbit so that it is crucial to decide whether these manifolds (actually curves) intersect transversely or not.

A transverse intersection would exclude the possibility of a homoclinic curve and would also imply certain chaotic features for the one-step method (2) (e.g. infinitely many discrete periodic orbits and horseshoes [8]).

In fact, following a suggestion by B. Fiedler during this conference, a transversal intersection was observed after adding an artificial perturbation to Euler's method for (4) (actually we added to the first component $10h^3 \varphi((x-1.5)/h) \varphi((y-\frac{h}{4})\frac{5}{h})$ where $\varphi(x) = \exp(-1/(1-x)^2)$ for $|x| < 1$ and 0 otherwise) which did not disturb the discrete homoclinic orbit and which still satisfies our smoothness and consistency requirements. Fig. 3 shows the resulting unstable manifold and the typical oscillation effect due to the transversal intersection. The picture was produced by plotting the points of the iteration (2) by starting randomly on

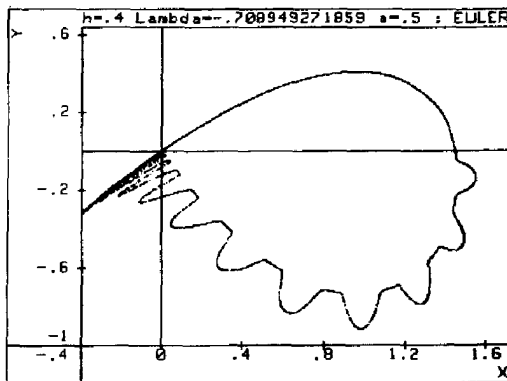


Figure 3

a straight line which was very close to the local unstable manifold. Similar observations were made for smaller step-sizes and different one-step methods.

Let us finally relate our observations for one-step methods to the behavior of the dynamical system (1) under perturbations. It is well known that the nondegenerate homo-

clinic bifurcation in \mathbb{R}^2 is stable with respect to autonomous perturbations. However, nonautonomous perturbations introduce a third dimension and hence may produce transverse homoclinic points. For a periodic perturbation, for example, the stable and unstable manifolds of the stationary point now appear as stable and unstable manifolds of a Poincaré map. Hence they may - and in a generic sense also will - intersect transversely after perturbation (see [13]).

In view of our results above we are therefore led to the following question. Do standard one-step methods - without any artificial perturbation - have some intrinsic property which forces them to

act rather as an autonomous than as a nonautonomous perturbation of the dynamical system itself?

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