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On Invariant Closed Curves for One-Step Methods

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Summary. We show that a one-step method as applied to a dynamical system with a hyperbolic periodic orbit, exhibits an invariant closed curve for sufficiently small step size. This invariant curve converges to the periodic orbit with the order of the method and it inherits the stability of the periodic orbit. The dynamics of the one-step method on the invariant curve can be described by the rotation number for which we derive an asymptotic expression. Our results complement those of [2, 3] where one-step methods were shown to create invariant curves if the dynamical system has a periodic orbit which is stable in either time direction or if the system undergoes a Hopf bifurcation.

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1. Introduction

We are interested in the longtime behaviour of one-step methods with constant step size h

(1.1) $$x(t+h)=x(t)+hf_h(x(t)), \qquad t=0, h, \ldots$$

if applied to an autonomous system

(1.2) $$x'=f(x), \qquad x(0)=x_0\in\mathbb{R}^m.$$

Since standard estimates of discrete and continuous trajectories yield error bounds which grow exponentially with time it is an important question if and in what sense the longtime behaviour of (1.2) is reflected by the recursion (1.1). In particular, what happens to the limit sets of (1.2) under discretization?

The case of stable stationary points is contained in the analysis of Stetter [14], Chap. 3.5. In [1] we have shown in which sense the discrete and continuous trajectories stay close in the neighborhood of hyperbolic stationary points.

The general case of a stable attracting set for (1.2) has been discussed by Kloeden and Lorenz [12]. There it is shown that, under some uniform stability condition, the one-step method (1.1) has a stable attracting set (not necessarily unique) which converges in the Hausdorff distance as $h \to 0$. However, in this generality no results on the order of convergence and on the minimality of the discrete attracting set seem to be possible. If however the continuous attractor is an asymptotically stable periodic orbit then the result of [3] shows that the discrete attracting set may be taken as an invariant curve.

In this paper we analyze the case where (1.2) has a hyperbolic periodic orbit of period T, i.e., 1 is a simple Floquet multiplier and the only multiplier on the unit circle (cf. [11], Chap. 5). We show that, for h sufficiently small, the one-step method (1.1) has a closed invariant curve γ_h, i.e., $(I+hf_h)\,(\gamma_h)=\gamma_h$. γ_h converges to the periodic orbit of (1.2) with the order r of the method.

Moreover, γ_h may be parametrized as

$$\gamma_h = \{x_h(t) \colon t \in \mathbb{R}\}$$

where $x_h \colon \mathbb{R} \to \mathbb{R}^m$ is T-periodic and the invariance condition may be stated more precisely as

$$(I+hf_h)(x_h(t)) = x_h(\sigma_h(t)), \qquad t \in \mathbb{R}$$

where $\sigma_h - I$ is a T-periodic Lipschitz function satisfying $\sigma_h(t) = t + h + O(h^{r+1})$. The asymptotic expression for σ_h will allow us to draw some conclusions for the dynamics on the invariant curve (see Sect. 4) by using the theory of rotation numbers (cf. [8]).

In Sect. 3 we investigate in more detail the case of an asymptotically stable periodic orbit. Then the invariant curve is stable too [3]. Using our previous theory we give rather precise estimates between the discrete and continuous trajectories on an infinite time interval. In particular, the Hausdorff distance of both trajectories converges to zero as $h \to 0$. It seems that one has to resort to distance measures of this form since the discrete and continuous trajectories certainly run out of phase after some time. It is an open problem whether a result of this type is still valid in the general hyperbolic case provided both trajectories are allowed to have different initial values as in [1].

Finally, we like to note that the existence of invariant closed curves probably carries over to one-step methods with variable step size - although we have not gone through all the analytical details. The basic additional assumption would be that the step size h varies smoothly with the value of x, i.e., the right hand side of (1.1) is given by a mapping $x + h(x) f_{h(x)}(x)$. The role of the constant step size is then taken over by the maximal step-size.

Remarks on the Literature

Our work was largely motivated by the results of Brezzi, Fujii and Ushiki [2]. It is shown in [2] that the one-step method (1.1) creates a family of invariant curves if the system (1.2) passes through a Hopf bifurcation. Under these circumstances the techniques for Hopf bifurcation of maps are appropriate.

Later on we learnt of the work of Braun and Hershenov [3] where invariant curves for (1.1) were shown to exist close to periodic orbits of (1.2) which are asymptotically stable in either time direction. The basic technique in [3] is the use of a moving coordinate system (see Hale [9], Chap. 6) which is also employed in this paper.

After submission of our paper the work of Doan [4] appeared and we were informed of the forthcoming papers of Eirola [5, 6]. In [4] the existence of invariant closed curves is shown for multi-step methods although no precise orders of convergence are given. In the case of one-step methods these orders of convergence for the invariant curves itself are also contained in [6] and a relation between discrete periodic orbits and the invariant curve is established in [5] by a technique different from the use of rotation numbers as in Sect. 4.

2. The Main Result

For some open bounded set $\Omega \subset \mathbb{R}^m$ we assume

(P1) $$f \in C^3(\Omega).$$

(P2) The system (1.2) has a hyperbolic closed orbit

$$\gamma = \{\bar{x}(t) : 0 \leqq t \leqq T\} \qquad \text{of period } T \text{ in } \Omega.$$

Our assumptions on the one-step method are of consistency type:

(P3) $f_h \to f$ and $f_h' \to f'$ uniformly in Ω as $h \to 0$,

f_h' is Lipschitz in Ω uniformly in h

(P4) for some positive constants C, h_0, r

$$\sup\{\|(\bar{x}(h; x_0) - x_0)/h - f_h(x_0)\| : x_0 \in \Omega\} \leqq C h^r \qquad \text{if } h \leqq h_0.$$

Throughout we will denote the solution of (1.2) by $\bar{x}(t; x_0)$. Our principal result is

Theorem 2.1. *Let* (P1)-(P4) *hold. Then there exists an* $h_0 > 0$ *and an open neighborhood* U *of* γ *such that the one-step method* (1.1) *has an invariant closed curve* $\gamma_h \subset U$ *for all* $h \leqq h_0$. *More precisely, there exist T-periodic functions* $\bar{x}_h : \mathbb{R} \to U$, $\sigma_h - I : \mathbb{R} \to \mathbb{R}$ *for* $h \leqq h_0$, *which are uniformly Lipschitz and satisfy*

(2.1) $$(I + h f_h)\, \bar{x}_h(t) = \bar{x}_h(\sigma_h(t)), \qquad t \in \mathbb{R}.$$

(2.2) $$\sigma_h(t) = t + h + O(h^{r+1}) \qquad \textit{uniformly for } t \in \mathbb{R}.$$

The curves $\gamma_h = \{\bar{x}_h(t) : 0 \leqq t \leqq T\}$ *converge to* γ *in a Lipschitz norm, in particular*

(2.3) $$\operatorname{Max}\{\|\bar{x}(t) - \bar{x}_h(t)\| : t \in \mathbb{R}\} = O(h^r).$$

(2.4) $$\sup\{\|(\bar{x} - \bar{x}_h)(t_1) - (\bar{x} - \bar{x}_h)(t_2)\| / |t_1 - t_2| : t_1 \neq t_2\} \to 0 \qquad \text{as } h \to 0.$$

Remark. Our proof will show that, if (P4) is omitted, then our assertions are still valid with $O(h^r)$, $O(h^{r+1})$ replaced by $o(1)$, $o(h)$. We will also see that γ_h is unique within the class of those closed curves which admit a T-periodic parametrization close to $\bar{x}$ in the Lipschitz norm.

The proof of Theorem 2.1 needs some preparations. By $\mathbb{R}^{k,l}$, resp. $\mathbb{C}^{k,l}$ we denote the space of real, resp. complex $k \times l$-matrices. Further, we set $K_\varepsilon = \{x \in \mathbb{R}^{m-1}: \|x\| < \varepsilon\}$ for some norm on $\mathbb{R}^{m-1}$.

We make constant use of the one dimensional manifold $S_T := \mathbb{R}/T \cdot \mathbb{Z}$ and of the calculus on manifolds (e.g., [11], Appendix). The distance of two equivalence classes $\theta_i = [t_i] \in S_T$, $i = 1, 2$ is given by

$$|\theta_1 - \theta_2| = \operatorname{Min}\{|t_1 - t_2 + nT|: n \in \mathbb{Z}\}. \tag{2.5}$$

We will also frequently identify functions defined on the sphere S_T with T-periodic functions defined on $\mathbb{R}$.

Let us first briefly summarize the technique of a moving orthonormal system along γ from [9] and the corresponding transformation of the system (1.2). Because of (P1) and (P2) there exists a $Z \in C^3(S_T, \mathbb{R}^{m, m-1})$ such that $(\bar{x}'(\theta)/\|\bar{x}'(\theta)\|, Z(\theta))$ is orthogonal for all $\theta \in S_T$. Then, for ε sufficiently small, the mapping

$$\Gamma: S_T \times K_\varepsilon \to U, \qquad \Gamma(\theta, w) = \bar{x}(\theta) + Z(\theta) w$$

is a C^3-diffeomorphism onto some open tubular neighborhood U of γ. The transformation $x = \Gamma(y)$, $y = (\theta, w)$ puts (1.2) into the form

$$y' = \Gamma'(y)^{-1} f(\Gamma(y)) =: g(y), \qquad g \in C^2(S_T \times K_\varepsilon, S_T \times \mathbb{R}^{m-1}).$$

In (θ, w)-coordinates this may be written as

$$\theta' = 1 + f_1(\theta, w), \qquad w' = A(\theta) w + f_2(\theta, w) \tag{2.6}$$

where $f_1 \in C^2(S_T \times K_\varepsilon, S_T)$, $f_2 \in C^2(S_T \times K_\varepsilon, \mathbb{R}^{m-1})$, $f_1(\theta, 0) = 0$,

$$f_2(\theta, 0) = 0, \qquad \frac{\partial f_2}{\partial w}(\theta, 0) = 0 \quad \text{and} \quad A(\theta) = Z(\theta)^T (f'(\bar{x}(\theta)) Z(\theta) - Z'(\theta)).$$

Let $\bar{y}(t; y)$ denote the solution of $y' = g(y)$ with $\bar{y}(0; y) = y$. Further, we set

$$\bar{Y}(t; y) = \frac{\partial \bar{y}}{\partial y}(t; y), \qquad \bar{Y}(t) = \bar{Y}(t; 0), \qquad \bar{X}(t; x) = \frac{\partial \bar{x}}{\partial x}(t; x), \qquad \bar{X}(t) = \bar{X}(t; \bar{x}(0)).$$

Then the relation $\bar{X}(T) = \Gamma'(0) \bar{Y}(T) \Gamma'(0)^{-1}$ holds. Moreover, (2.6) implies the representation

$$\bar{Y}(t) = \begin{pmatrix} 1 & \xi^T(t) \\ 0 & Y(t) \end{pmatrix}, \qquad t \in \mathbb{R} \tag{2.7}$$

where $\xi \in C^2(\mathbb{R}, \mathbb{R}^{m-1})$ and $Y \in C^2(\mathbb{R}, \mathbb{R}^{m-1, m-1})$ is the fundamental matrix given by $Y'(t) = A(t) Y(t)$, $Y(0) = I$. Hence the eigenvalues of $Y(T)$ are exactly the Floquet multipliers of the orbit which are different from 1 (cf. [9], Chap. 6.2). For $Y(t)$ we have the Floquet representation [9]

(2.8) $$Y(t)=P(t)\exp(tB), \quad t\in\mathbb{R}$$

where

(2.9) $$P\in C^2(S_T, \mathbb{C}^{m-1,m-1}), \quad B\in\mathbb{C}^{m-1,m-1}, \quad Y(T)=\exp(TB).$$

According to the stable and unstable eigenvalues of B we can decompose $\mathbb{C}^{m-1}$ into B-invariant subspaces $\mathbb{C}^{m-1}=\mathbb{C}_s^{m-1}\oplus\mathbb{C}_u^{m-1}$. Following [11] (4.37), there exist norms on $\mathbb{C}_s^{m-1}$ and $\mathbb{C}_u^{m-1}$ and an $\alpha>0$ such that $B_s:=B|_{\mathbb{C}_s^{m-1}}$, $B_u=B|_{\mathbb{C}_u^{m-1}}$ satisfy

(2.10) $$\operatorname{Max}(\|\exp(tB_s)\|, \|\exp(-tB_u)\|)\leqq\exp(-\alpha t) \quad \forall\, t\geqq 0.$$

The norm on $\mathbb{C}^{m-1}$ is given by $\|z\|=\operatorname{Max}(\|z_s\|, \|z_u\|)$ where $z=(z_s, z_u)\in\mathbb{C}_s^{m-1}\oplus\mathbb{C}_u^{m-1}$.

We use the following notation for functions $w\in C(S_T, \mathbb{R}^{m-1})$ throughout

$$\tilde{w}(\theta)=P(\theta)^{-1}w(\theta), \quad \theta\in S_T, \ \|w\|_\infty=\operatorname{Max}\{\|\tilde{w}(\theta)\|: \theta\in S_T\}.$$

The parametrization of the invariant curve within the moving orthonormal system will be found in the following class of functions

(2.11) $$W(\varepsilon, L)=\{w\in C(S_T, \mathbb{R}^{m-1}): \|w\|_\infty\leqq\varepsilon, \|\tilde{w}(\theta_1)-\tilde{w}(\theta_2)\| \leqq L|\theta_1-\theta_2| \ \forall\, \theta_1, \theta_2\in S_T\}$$

ε and L will be taken sufficiently small during the proof while $\varepsilon\leqq 1$ and $L\leqq 1$ is assumed throughout. The term $P(\theta)^{-1}$ in (2.11) is suggested by the final transformation for the w-equation from (2.6) ([9], Chap. 6.2). We note that $W(\varepsilon, L)$ is closed with respect to $\|\ \|_\infty$ and that it is sufficient to require the Lipschitz condition in (2.11) for $|\theta_1-\theta_2|\leqq b$ and for some $b>0$.

For h and ε sufficiently small we can define the transformed discrete flow by

$$E_h(\theta, u)=\Gamma^{-1}(I+hf_h)\Gamma(\theta, u), \quad \theta\in S_T, \ u\in K_\varepsilon$$

as well as the transformed continuous flow with time step h by

$$F_h(\theta, u)=\Gamma^{-1}\bar{x}(h; \Gamma(\theta, u)), \quad \theta\in S_T, \ u\in K_\varepsilon.$$

To any element $w\in W(\varepsilon, L)$ we associate the functions

(2.12) $$(\sigma_w(\theta), u_w(\theta))=E_h(\theta, w(\theta)), \quad \theta\in S_T.$$

Then $\Gamma(\theta, w(\theta))$ is an invariant closed curve for the one-step method if

(2.13) $$w(\sigma_w(\theta))=u_w(\theta) \quad \forall\, \theta\in S_T.$$

This is the equation we want to solve for w.

Some important properties of the mappings E_h, F_h are summarized in the following lemma.

Lemma 2.2. *The following asymptotic relations hold uniformly for $\theta\in S_T$, $u\in K_\varepsilon$:*

(2.14) $$E_h(\theta,u)=F_h(\theta,u)+O(h^{r+1})=(\theta+h,u)+O(h(\varepsilon+h^r)).$$

(2.15) $$E_h'(\theta,u)=\begin{pmatrix}1 & O(h)\\ 0 & Y(\theta+h)\,Y(\theta)^{-1}\end{pmatrix}+O(h\varepsilon)+o(h).$$

E_h' is uniformly Lipschitz in $S_T\times K_\varepsilon$.

Remark. Some caution has to be taken with the expression $Y(\theta+h)Y(\theta)^{-1}$ since $Y(t)$ is not a T-periodic function but $Y(t+h)\,Y(t)^{-1}$ is.

Proof. The first equality in (2.14) follows immediately from (P4). Then we note that $F_h'(\theta,u)=\bar{Y}(h;(\theta,u))$ and that

$$\bar{Y}(h;(\theta,0))=\bar{Y}(h+\theta)\,\bar{Y}(\theta)^{-1},\qquad \theta\in S_T$$

follows from the differential equation for $\bar{Y}$. Using (2.7) we obtain

$$F_h'(\theta,0)=\begin{pmatrix}1 & O(h)\\ 0 & Y(\theta+h)\,Y(\theta)^{-1}\end{pmatrix},\qquad \theta\in S_T.$$

Now $\dfrac{\partial\bar{Y}}{\partial u}(0;(\theta,u))=0$ holds for all $\theta\in S_T$, $u\in K_\varepsilon$. Since $\bar{Y}$ is of class C^2 this implies

(2.16) $$\bar{Y}(h;(\theta,u))=\bar{Y}(h;(\theta,0))+O(h\,\varepsilon)\quad\text{for }\theta\in S_T,\ u\in K_\varepsilon.$$

In particular, $F_h'(\theta,u)=I+O(h)$ and thus $F_h(\theta,u)=F_h(\theta,0)+(0,u)+O(h\varepsilon)$ which proves (2.14). By (P3) we have

$$\bar{X}(h;x)=\bar{X}(0;x)+h\,\frac{\partial\bar{X}}{\partial t}(0;x)+O(h^2)=I+hf_h'(x)+o(h).$$

Differentiating $E_h(\theta,u)$ and using (2.14) then shows

$$E_h'(\theta,u)=\Gamma'(E_h(\theta,u))^{-1}\,\bar{X}(h;\Gamma(\theta,u))\,\Gamma'(\theta,u)+o(h)=F_h'(\theta,u)+o(h)$$

so that (2.15) is a consequence of (2.16). Finally, the uniform Lipschitz bound for E_h' follows from the uniform Lipschitz bound for f_h'. □

Next we discuss the asymptotic and Lipschitz properties of the functions σ_w, u_w defined by (2.12).

Lemma 2.3. *For all functions $w\in W(\varepsilon,L)$ we have*

(2.17) $$(\sigma_w(\theta),u_w(\theta))=(\theta+h,w(\theta))+O(h\varepsilon)+o(h),\qquad \theta\in S_T$$

and for $\theta_1,\theta_2\in S_T$ with $|\theta_1-\theta_2|\leq h\varepsilon$

(2.18) $$\sigma_w(\theta_1)-\sigma_w(\theta_2)=\theta_1-\theta_2+(O(h(\varepsilon+L))+o(h))\,|\theta_1-\theta_2|,$$

(2.19) $$u_w(\theta_1)-u_w(\theta_2)=P(\theta_2+h)\exp(hB)(\tilde{w}(\theta_1)-\tilde{w}(\theta_2))+(P(\theta_1)-P(\theta_2))\,\tilde{w}(\theta_1)+(O(h\varepsilon)+o(h))\,|\theta_1-\theta_2|.$$

Proof. (2.17) follows directly from Lemma 2.2. Since E_h' is uniformly Lipschitz and w is Lipschitz bounded we obtain the following relation from (2.15) for $|\theta_1 - \theta_2| \leqq h\varepsilon$.

$$\begin{aligned}(\sigma_w(\theta_1), u_w(\theta_1)) &- (\sigma_w(\theta_2), u_w(\theta_2))\\ &= (E_h'(\theta_2, w(\theta_2)) + O(h\varepsilon))(\theta_1 - \theta_2, w(\theta_1) - w(\theta_2))\\ &= \left\{\begin{pmatrix}1 & O(h)\\ 0 & Y(\theta_2+h)\,Y(\theta_2)^{-1}\end{pmatrix} + o(h) + O(h\varepsilon)\right\}\begin{pmatrix}\theta_1-\theta_2\\ w(\theta_1)-w(\theta_2)\end{pmatrix}.\end{aligned}$$

This proves (2.18). With regard to (2.19), we note in addition that $Y(\theta_2+h)\,Y(\theta_2)^{-1} = I + O(h)$ implies

$$\begin{aligned}Y(\theta_2+h)\,Y(\theta_2)^{-1}(w(\theta_1) - w(\theta_2)) &= Y(\theta_2+h)\,Y(\theta_2)^{-1}\,P(\theta_2)(\tilde{w}(\theta_1) - \tilde{w}(\theta_2))\\ &+ (P(\theta_1) - P(\theta_2))\,\tilde{w}(\theta_1) + O(h\varepsilon)\,|\theta_1 - \theta_2|\end{aligned}$$

and $Y(\theta_2+h)\,Y(\theta_2)^{-1}\,P(\theta_2) = P(\theta_2+h)\exp(hB)$ by the Floquet formula (2.8). □

Similar to [10], Chap. 3.1 we use (2.18) and the following lemma to conclude that $\sigma_w\colon S_T \to S_T$ is bilipschitz.

Lemma 2.4. *Let the mapping* $\sigma\colon S_T \to S_T$ *satisfy for some* $0 \leqq \alpha < 1$

(2.20) $$|\sigma(\theta_1) - \sigma(\theta_2) - (\theta_1 - \theta_2)| \leqq \alpha\,|\theta_1 - \theta_2| \qquad \forall\,\theta_1, \theta_2 \in S_T.$$

Then σ *is invertible and for all* $\theta_1, \theta_2 \in S_T$

$$\begin{aligned}|\sigma^{-1}(\theta_1) - \sigma^{-1}(\theta_2)| &\leqq (1-\alpha)^{-1}\,|\theta_1 - \theta_2|,\\ |\sigma^{-1}(\theta_1) - \sigma^{-1}(\theta_2) - (\theta_1 - \theta_2)| &\leqq \alpha(1-\alpha)^{-1}\,|\theta_1 - \theta_2|.\end{aligned}$$

Proof. By (2.20) the mapping $\theta \to \theta - \sigma(\theta) + \theta_0$ is a metric contraction for each $\theta_0 \in S_T$. Hence σ is invertible and (2.20) yields

$$\begin{aligned}|\theta_1 - \theta_2| \geqq\; &|\sigma^{-1}(\theta_1) - \sigma^{-1}(\theta_2)|\\ &- |\sigma(\sigma^{-1}(\theta_1)) - \sigma(\sigma^{-1}(\theta_2)) - (\sigma^{-1}(\theta_1) - \sigma^{-1}(\theta_2))|\\ \geqq\; &(1-\alpha)\,|\sigma^{-1}(\theta_1) - \sigma^{-1}(\theta_2)|\end{aligned}$$

as well as

$$\begin{aligned}|\sigma^{-1}(\theta_1) - \sigma^{-1}(\theta_2) - (\theta_1 - \theta_2)| &\leqq \alpha\,|\sigma^{-1}(\theta_1) - \theta^{-1}(\theta_2)|\\ &\leqq \alpha(1-\alpha)^{-1}\,|\theta_1 - \theta_2|. \quad \square\end{aligned}$$

By Lemma 2.3 and Lemma 2.4 we see that σ_w^{-1} exists for all $w \in W(\varepsilon, L)$ and h sufficiently small and that it satisfies the following estimates

(2.21) $$\begin{aligned}&|\sigma_w^{-1}(\theta_1) - \sigma_w^{-1}(\theta_2)|\\ &\quad \leqq (1 + O(h(\varepsilon + L)) + o(h))\,|\theta_1 - \theta_2| \qquad \forall\,\theta_1, \theta_2 \in S_T,\end{aligned}$$

(2.22) $$\begin{aligned}&|\sigma_w^{-1}(\theta_1) - \sigma_w^{-1}(\theta_2) - (\theta_1 - \theta_2)|\\ &\quad \leqq (O(h(\varepsilon + L)) + o(h))\,|\theta_1 - \theta_2| \qquad \forall\,\theta_1, \theta_2 \in S_T,\end{aligned}$$

(2.23) $$\sigma_w^{-1}(\theta) = \theta - h + O(h\varepsilon) + o(h), \qquad \theta \in S_T.$$

We may now write (2.13) as a fixed point equation for the operator Q_h defined by

(2.24) $$Q_h w(\theta)=u_w(\sigma_w^{-1}(\theta)),\ \theta\in S_T, \qquad w\in W(\varepsilon, L).$$

This operator has the following properties.

Lemma 2.5. *For all $w, v\in W(\varepsilon, L)$ and all $\theta, \theta_1, \theta_2\in S_T$ with $|\theta_1-\theta_2|\leqq h\varepsilon$ we have*

(2.25) $$\widetilde{Q_h w}(\theta_1)-\widetilde{Q_h w}(\theta_2)=\exp(hB)(\tilde{w}(\sigma_w^{-1}(\theta_1))-\tilde{w}(\sigma_w^{-1}(\theta_2))) +(O(h\varepsilon)+o(h))|\theta_1-\theta_2|,$$

(2.26) $$\widetilde{Q_h w}(\theta)-\widetilde{Q_h v}(\theta)=\exp(hB)(\tilde{w}(\sigma_v^{-1}(\theta))-\tilde{v}(\sigma_v^{-1}(\theta))) +(O(h(\varepsilon+L))+o(h))\|v-w\|_\infty,$$

(2.27) $$Q_h w(\theta)=w(\sigma_w^{-1}(\theta))+O(h\varepsilon)+o(h).$$

Proof. (2.27) is obvious from (2.17). Further, (2.19) shows

$$\begin{aligned}\widetilde{Q_h w}(\theta_1)-\widetilde{Q_h w}(\theta_2)&=P(\theta_2)^{-1}(P(\theta_2)-P(\theta_1))\,\widetilde{Q_h w}(\theta_1)\\ &+P(\theta_2)^{-1}\,P(\sigma_w^{-1}(\theta_2)+h)\exp(hB)(\tilde{w}(\sigma_w^{-1}(\theta_1))-\tilde{w}(\sigma_w^{-1}(\theta_2)))\\ &+P(\theta_2)^{-1}(P(\sigma_w^{-1}(\theta_1))-P(\sigma_w^{-1}(\theta_2)))\,\tilde{w}(\sigma_w^{-1}(\theta_1))+(O(h\varepsilon)+o(h))|\theta_1-\theta_2|.\end{aligned}$$

This expression leads to (2.25) if we use (2.27) and the following implications of (2.22), (2.23):

$$P(\sigma_w^{-1}(\theta_1))-P(\sigma_w^{-1}(\theta_2))=P(\theta_1)-P(\theta_2)+O(h)|\theta_1-\theta_2|,$$
$$P(\sigma_w^{-1}(\theta_2)+h)=P(\theta_2)+O(h\varepsilon)+\sigma(h), \qquad P(\sigma_w^{-1}(\theta_1))^{-1}=P(\theta_1)^{-1}+O(h).$$

For the proof of (2.26) we first note that Lemma 2.2 together with (2.8) yields $\sigma_w(\theta)-\sigma_v(\theta)=O(h)\,\|w(\theta)-v(\theta)\|$ and

$$u_w(\theta)-u_v(\theta)=(P(\theta+h)\exp(hB)+o(h)+O(h\varepsilon))(\tilde{w}(\theta)-\tilde{v}(\theta)).$$

In particular, we have

(2.28) $$|\sigma_w(\theta)-\sigma_v(\theta)|=O(h)\,\|w-v\|_\infty, \qquad \theta\in S_T$$

and hence $|\sigma_w^{-1}(\theta)-\sigma_v^{-1}(\theta)|=O(h)\,\|w-v\|_\infty$ by (2.21). Summing up we find

$$\begin{aligned}\widetilde{Q_h w}(\theta)-\widetilde{Q_h v}(\theta)&=P(\theta)^{-1}(u_w(\sigma_w^{-1}(\theta))-u_w(\sigma_v^{-1}(\theta)))\\ &+P(\theta)^{-1}(u_w(\sigma_v^{-1}(\theta))-u_v(\sigma_v^{-1}(\theta)))=(O(h(\varepsilon+L))+o(h))\,\|w-v\|_\infty\\ &+P(\theta)^{-1}\,P(\sigma_v^{-1}(\theta)+h)\exp(hB)(\tilde{w}(\sigma_v^{-1}(\theta))-\tilde{v}(\sigma_v^{-1}(\theta)))\end{aligned}$$

and finally (2.23) proves our assertion. □

Our final conclusion from Lemma 2.2 is

(2.29) $$(\sigma_0(\theta), u_0(\theta))=F_h(\theta, 0)+O(h^{r+1})=(\theta+h, 0)+O(h^{r+1})$$

which implies

$$\|Q_h(0)\|_\infty = O(h^{r+1}). \tag{2.30}$$

We will now complete the proof of Theorem 2.1 if γ is orbitally stable. This is equivalent to the stability of B, i.e., $\mathbb{C}_s^{m-1} = \mathbb{C}^{m-1}$ and $B_s = B$. From Lemma 2.5 and (2.10), (2.21) we obtain

$$\begin{aligned}(2.31)\qquad &\|\widetilde{Q_h w}(\theta_1) - \widetilde{Q_h w}(\theta_2)\| \\ &\leqq (Le^{-\alpha h}(1 + O(h(\varepsilon + L)) + o(h)) + O(h\varepsilon) + o(h))\,|\theta_1 - \theta_2|\end{aligned}$$

for $w \in W(\varepsilon, L)$, $|\theta_1 - \theta_2| \leqq h\varepsilon$ and

$$\|Q_h w - Q_h v\|_\infty \leqq (e^{-\alpha h} + O(h(\varepsilon + L)) + o(h))\,\|w - v\|_\infty \tag{2.32}$$

for $w, v \in W(\varepsilon, L)$, $\theta \in S_T$. A careful inspection of these estimates together with (2.30) shows that there exist $L_0, \varepsilon_0, h_0 > 0$ with the following property: for any $L \in (0, L_0]$ and any $\varepsilon_1 \in (0, \varepsilon_0]$ we find $\varepsilon = \varepsilon(L, \varepsilon_1) \in (0, \varepsilon_1]$, $h = h(L, \varepsilon_1) \in (0, h_0]$ such that Q_h is a contraction on $W(\varepsilon, L)$ with constant $1 - \alpha h/2$ for all $h \leqq h(L, \varepsilon_1)$. We may also assume $\varepsilon(L_0, \varepsilon_0) = \varepsilon_0$, $h(L_0, \varepsilon_0) = h_0$. In particular, Q_h, $h \leqq h_0$, has a unique fixed point $\bar{w}_h$ in $W(\varepsilon_0, L_0)$.

Moreover, for any two functions $w, v \in W(\varepsilon_0, L_0)$ we have the stability inequality

$$\|v - w\|_\infty \leqq \frac{2}{\alpha h}\,\|(I - Q_h)\,v - (I - Q_h)\,w\|_\infty \qquad \forall h \leqq h_0. \tag{2.33}$$

Setting $v = 0$, $w = \bar{w}_h$ in (2.33) yields $\|\bar{w}_h\|_\infty = O(h^r)$. This proves (2.3) for the invariant curve $\bar{x}_h(\theta) = \Gamma(\theta, \bar{w}_h(\theta))$, $\theta \in S_T$. As we saw above, for any ε, L small we have $\bar{w}_h \in W(\varepsilon, L)$ if h is small, thus (2.4) holds. Finally, by (2.28) and (2.29) we find

$$|\sigma_{\bar{w}_h}(\theta) - (\theta + h)| \leqq |\sigma_{\bar{w}_h}(\theta) - \sigma_0(\theta)| + O(h^{r+1}) = O(h^{r+1}).$$

In this way we obtain (2.1), (2.2) if we take σ_h to be the lift of $\sigma_{\bar{w}_h}$ from S_T to $\mathbb{R}$ (see [11], Chap. 2 for this notion).

In the general case of a hyperbolic orbit, we have to modify the operator Q_h appropriately. We introduce the projectors

$$\Pi_s\colon \mathbb{C}^{m-1} \to \mathbb{C}_s^{m-1}, \qquad \Pi_u = I - \Pi_s\colon \mathbb{C}^{m-1} \to \mathbb{C}_u^{m-1}.$$

These are in fact real operators as may be seen from the construction of B via $Y(T) = \exp(TB)$. Moreover, they satisfy

$$\Pi_i \exp(tB) = \exp(tB)\,\Pi_i = \exp(tB_i)\,\Pi_i \qquad \text{for } i \in \{s, u\}. \tag{2.34}$$

Our modified operator is defined for $w \in W(\varepsilon, L)$ by

$$R_h w(\theta) = P(\theta)\,\Pi_s P(\theta)^{-1} Q_h w(\theta) + P(\theta)\,\Pi_u P(\theta)^{-1} K_h w(\theta), \qquad \theta \in S_T \tag{2.35}$$

where

$$K_h w(\theta) = w(\theta) + P(\theta)\exp(-|\sigma_w(\theta) - \theta|\,B)\,P(\sigma_w(\theta))^{-1}(w(\sigma_w(\theta)) - u_w(\theta)).$$

Let us first show that R_h is a real operator. From (2.8) and (2.34) we find that $P(t)\Pi_{s,u}P(t)^{-1}=Y(t)\Pi_{s,u}Y(t)^{-1}$ is real for all $t\geqq 0$.

Moreover, by (2.17) and by the definition of the metric on S_T we obtain that, for h and ε sufficiently small, there exist $t\in[0,T)$, $\tau\in\mathbb{R}$ such that $\theta=[t]$, $\sigma_w(\theta)=[\tau]$ and $|\sigma_w(\theta)-\theta|=\tau-t$. But then

$$P(\theta)\exp(-|\sigma_w(\theta)-\theta|B)P(\sigma_w(\theta))^{-1}=Y(t)Y(\tau)^{-1}$$

is also a real matrix.

We rewrite the definition (2.35) as

$$\Pi_s\widetilde{R_h w}=\Pi_s\widetilde{Q_h w},\qquad \Pi_u\widetilde{R_h w}=\Pi_u\widetilde{K_h w}. \tag{2.36}$$

It is easily seen that w is a fixed point of R_h if and only if (2.13) holds. Now we are going to prove the estimates (2.31), (2.32) with R_h in place of Q_h. The estimates for the stable parts

$$\Pi_s(\widetilde{R_h w}(\theta_1)-\widetilde{R_h w}(\theta_2))\quad\text{and}\quad \Pi_s(\widetilde{R_h w}-\widetilde{R_h v})$$

follow immediately from (2.34), (2.10) and Lemma 2.5.

The second equality of (2.36) may be written as

$$\Pi_u R_h w(\theta)=\exp(-|\sigma_w(\theta)-\theta|B_u)\Pi_u\hat{w}(\theta)$$

where

$$\hat{w}(\theta)=\exp(|\sigma_w(\theta)-\theta|B)\tilde{w}(\theta)+\tilde{w}(\sigma_w(\theta))-\widetilde{Q_h w}(\sigma_w(\theta))=O(\varepsilon)$$

for $\theta\in S_T$. For $\theta_1,\theta_2\in S_T$ satisfying $|\theta_1-\theta_2|\leqq h\varepsilon$ we obtain from Lemma 2.3 and (2.10)

$$\begin{aligned}\|\Pi_u(\widetilde{R_h w}(\theta_1)-\widetilde{R_h w}(\theta_2))\| &\leqq C|\sigma_w(\theta_1)-\theta_1-(\sigma_w(\theta_2)-\theta_2)|\,\|\Pi_u\hat{w}(\theta_1)\|\\ &\quad+\exp(-\alpha|\sigma_w(\theta_2)-\theta_2|)\|\Pi_u(\hat{w}(\theta_1)-\hat{w}(\theta_2))\|\\ &\leqq O(h\varepsilon)|\theta_1-\theta_2|+\exp(-\alpha h/2)\|\hat{w}(\theta_1)-\hat{w}(\theta_2)\|.\end{aligned}$$

The last term can be estimated by Lemma 2.3 and 2.5

$$\begin{aligned}\|\hat{w}(\theta_1)-\hat{w}(\theta_2)\| &\leqq C|\sigma_w(\theta_1)-\theta_1-(\sigma_w(\theta_2)-\theta_2)|\,\|\tilde{w}(\theta_1)\|\\ &\quad+\|(\exp(|\sigma_w(\theta_2)-\theta_2|B)-\exp(hB))(\tilde{w}(\theta_1)-\tilde{w}(\theta_2))\|\\ &\quad+\|\tilde{w}(\sigma_w(\theta_1))-\tilde{w}(\sigma_w(\theta_2))\|+(O(h\varepsilon)+o(h))|\theta_1-\theta_2|\\ &\leqq\{L(1+O(h(\varepsilon+L))+o(h))+O(h\varepsilon)+o(h)\}|\theta_1-\theta_2|.\end{aligned}$$

Similarly, for any two functions $w,v\in W(\varepsilon,L)$ we get from Lemma 2.3, 2.5 and (2.28)

$$\|\Pi_u(\widetilde{R_h w}(\theta)-\widetilde{R_h v}(\theta))\|\leqq O(h\varepsilon)\|w-v\|_\infty+\exp(-\alpha h/2)\|\hat{w}(\theta)-\hat{v}(\theta)\|$$

and finally $\|\hat{w}(\theta)-\hat{v}(\theta)\|\leqq(1+O(h(\varepsilon+L))+o(h))\|w-v\|_\infty$.

We also notice that $\|R_h(0)\|_\infty=O(h^{r+1})$ is satisfied because of (2.29). Then the existence of a closed invariant curve as well as the specific estimates (2.2) to (2.4) follow in the same way as in the stable case. This completes the proof of Theorem 2.1.

3. The Stable Case

In the case of a stable periodic orbit γ, we can, in addition to Theorem 2.1, discuss the behaviour of the discrete trajectories near the invariant curve γ_h. For x in a neighbourhood of γ we use the following notations:

$$x^h(nh;x)=(I+hf_h)^n(x), \qquad \omega_h(x)=\{x^h(nh;x): n\geqq 0\}$$
$$\omega(x)=\{\bar{x}(t;x): t\geqq 0\}.$$

Moreover, we use the Hausdorff distance of two sets A and B given by

$$d(A,B)=\operatorname{Max}(\sup_{x\in A}\operatorname{dist}(x,B), \sup_{y\in B}\operatorname{dist}(y,A)).$$

Theorem 3.1. *Let the assumptions of Theorem* 2.1 *hold and let* γ *be (orbitally) stable. Then there exist* $h_0, \alpha, \beta, C, \rho>0$ *such that for* $h\leqq h_0$ *and* $\operatorname{dist}(x,\gamma_h)\leqq\rho$ *the following holds*

(3.1) $$\operatorname{dist}(x^h(nh;x),\gamma_h)\leqq C\exp(-\alpha nh)\operatorname{dist}(x,\gamma_h), \qquad n\geqq 0.$$

(3.2) $$\operatorname{dist}(x^h(nh;x),\omega(x)) \leqq C(h^r+\operatorname{Min}(h^r\exp(\beta nh),\exp(-\alpha nh))), \qquad n\geqq 0.$$

Moreover, for any $\delta>0$ *there exist* $\rho(\delta)$, $h(\delta)>0$ *such that*

(3.3) $$\sup_{n\geqq 0}\operatorname{dist}(x^h(nh;x),\omega(x)) \leqq Ch^{r-\delta} \quad \textit{for} \quad h\leqq h(\delta), \qquad \operatorname{dist}(x,\gamma_h)\leqq\rho(\delta).$$

Finally, we have uniformly for $\operatorname{dist}(x,\gamma)\leqq\rho$

(3.4) $$d(\omega_h(x),\omega(x))\to 0 \quad \textit{as} \quad h\to 0.$$

Remarks. (3.1) implies the stability of the invariant curve and this result is due to [3]. The more detailed estimate (3.2) shows the approximation of the continuous trajectories by the numerical trajectories, in particular the transition from the classical estimates for moderate nh to the asymptotic case $nh\to\infty$. We also notice that we cannot expect any order of convergence for the Hausdorff-distance $d(\omega_h(x),\omega(x))$, even if $\omega(x)$ is restricted to the times $t=nh$, $n\in\mathbb{N}$. The reason for this is that $\omega_h(x)$ may or may not fill the invariant curve depending on the dynamics on the curve (cf. Sect. 4).

Proof. Our first observation is an inequality

(3.5) $$C_1\|u-\bar{w}_h(\theta)\|\leqq\operatorname{dist}(\Gamma(\theta,u),\gamma_h)\leqq C_2\|u-\bar{w}_h(\theta)\|, \qquad \theta\in S_T,\ u\in K_\varepsilon$$

for sufficiently small h. The upper estimate is obvious whereas the lower one follows from the Lipschitz property of $\bar{w}_h$. For all $\xi\in S_T$ we have

$$\begin{aligned}\|\Gamma(\theta,u)-\Gamma(\xi,\bar{w}_h(\xi))\| &\geqq C\operatorname{Max}(|\theta-\xi|,\|u-\bar{w}_h(\xi)\|)\\ &\geqq C\operatorname{Max}(|\theta-\xi|,\|u-\bar{w}_h(\theta)\|-L_0|\theta-\xi|)\geqq C(1+L_0)^{-1}\|u-\bar{w}_h(\theta)\|.\end{aligned}$$

From the proof of Theorem 2.1 we know $\bar{w}_h \in W(\varepsilon_0, L_0)$ for $h \leqq h_0$ and $\bar{w}_h \in W(\varepsilon_0, L_0/2)$ for $h \leqq h_1$. For a given $x_0 = \Gamma(\theta_0, u_0) \in U$ we then find that the function $w_0(\theta) = w_h(\theta) + u_0 - \bar{w}_h(\theta_0)$ is in $W(\varepsilon_0, L_0)$ for h and $\|u_0 - \bar{w}_h(\theta_0)\|$ sufficiently small. Hence, for some $\alpha > 0$

$$\|Q_h^n w_0 - \bar{w}_h\|_\infty \leqq \exp(-\alpha h n) \|w_0 - \bar{w}_h\|_\infty \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

Writing $(\theta_n, u_n) = \Gamma^{-1} x^h(nh; x_0)$ and using (3.6) we obtain

$$\begin{aligned} \|x^h(nh; x_0) - \bar{x}_h(\theta_n)\| &= \|\Gamma(\theta_n, u_n) - \Gamma(\theta_n, \bar{w}_h(\theta_n))\| \leqq C \|u_n - \bar{w}_h(\theta_n)\| \\ &\leqq C \exp(-\alpha h n) \|u_0 - w_h(\theta_0)\| \end{aligned}$$

which by (3.5) gives the stability of γ_h.

Let β denote a Lipschitz constant for f in some neighborhood U of γ. Then the estimate (3.2) follows immediately from (3.1), (2.3) and the standard estimates of the error $x^h(nh; x) - \bar{x}(nh; x)$. Evaluating the maximum of the right hand side of (3.2) we get an order $r\alpha/(\alpha + \beta)$ of convergence. This will be sufficient for (3.3) if β could be chosen arbitrarily small on small neighborhoods of γ_h. For a result of this type we need the following more subtle analysis.

For any two sequences $v_n, w_n \in W(\varepsilon_0, L_0)$ and $h \leqq h_0$ we have

$$\|v_{n+1} - w_{n+1}\|_\infty \leqq (1 - \alpha h) \|v_n - w_n\|_\infty + \|(v_{n+1} - Q_h v_n) - (w_{n+1} - Q_h w_n)\|_\infty$$

and hence by the discrete Gronwall Lemma [7], Chap. 1.3

$$\begin{aligned} \|v_n - w_n\|_\infty &\leqq \exp(-\alpha n h) \|v_0 - w_0\|_\infty \\ &\quad + (\alpha h)^{-1} \operatorname*{Max}_{j \leqq n-1} \|v_{j+1} - Q_h v_j - (w_{j+1} - Q_h w_j)\|_\infty. \end{aligned} \tag{3.7}$$

For functions $w \in W(\varepsilon_0, L_0)$ we define (cf. (2.12), (2.24))

$$(\tau_w(\theta), y_w(\theta)) = F_h(\theta, w(\theta)), \qquad G_h w(\theta) = y_w(\tau_w^{-1}(\theta)).$$

τ_w and y_w have the same properties as σ_w, u_w and moreover, by (2.21) and Lemma 2.2, we obtain

$$y_w = u_w + O(h^{r+1}), \qquad \tau_w = \sigma_w + O(h^{r+1}), \qquad \tau_w^{-1} = \sigma_w^{-1} + O(h^{r+1}). \tag{3.8}$$

For $x_0 = \Gamma(\theta_0, u_0) \in U$ we define w_0 as above and apply (3.7) to the sequences $v_n = G_h^n w_0$, $w_n = Q_h^n w_0$. By (3.8) we find

$$\|v_n - w_n\|_\infty \leqq (\alpha h)^{-1} \sup_{j \leqq n-1} \|y_{v_j} \circ \tau_{v_j}^{-1} - u_{v_j} \circ \sigma_{v_j}^{-1}\|_\infty = O(h^r).$$

Further, we introduce the functions $\tau_n(\theta)$, $\sigma_n(\theta)$, $\theta \in S_T$ given by

$$\tau_n = \tau_{v_{n-1}} \circ \tau_{n-1}, \tau_0 = I, \qquad \sigma_n = \sigma_{w_{n-1}} \circ \sigma_{n-1}, \sigma_0 = I.$$

Then we have

$$\begin{aligned}&\|\bar{x}(nh; x_0) - x^h(nh; x_0)\| \\ &\quad = \|\Gamma(\tau_n(\theta_0), v_n(\tau_n(\theta_0))) - \Gamma(\sigma_n(\theta_0), w_n(\sigma_n(\theta_0)))\| \\ &\quad \leqq C|\tau_n(\theta_0) - \sigma_n(\theta_0)| + O(h^r).\end{aligned}$$

By (3.8) and Lemma 2.3 the first term can be estimated as follows

$$|\tau_n(\theta) - \sigma_n(\theta)| \leqq (1+\beta h)|\tau_{n-1}(\theta) - \sigma_{n-1}(\theta)| + O(h^{r+1}).$$

Here we have $0 < \beta = O(\varepsilon + L) + o(1)$ provided that $v_n, w_n \in W(\varepsilon, L)$. Gronwall's lemma now implies

$$|\tau_n(\theta) - \sigma_n(\theta)| \leqq C\beta^{-1}(\exp(\beta nh) - 1)h^r \leqq Cnh^{r+1}\exp(\beta nh), \qquad \theta \in S_T.$$

Collecting terms, we find that the right hand side of (3.2) can be replaced by

$$e(\beta, h, n) = C(h^r + \mathrm{Min}(nh^{r+1}\exp(\beta nh), \exp(-\alpha nh)).$$

It is easily seen that $e(\beta, h, n) \leqq Ch^{r-\delta}$ for all $n \in \mathbb{N}$ if $\beta \leqq \delta\alpha/(2r)$ and $|\ln h| h^{\delta/2} r \leqq \alpha$. Our previous proof shows that these conditions can be met by taking h and $\mathrm{dist}(x_0, \gamma_h)$ sufficiently small.

Finally, we consider $b(t) = \mathrm{dist}(\bar{x}(t; x), \omega_h(x))$, $t \geqq 0$, $x \in U$. Let $\eta > 0$ be given, then we may choose $t_0 > 0$ such that $\mathrm{dist}(\bar{x}(t; x), \gamma) \leqq \eta$ for all $t \geqq t_0$, $x \in U$. By the classical theory we have $\mathrm{Max}\{b(t); 0 \leqq t \leqq t_0\} \leqq \eta$ for all $x \in U$ and h sufficiently small. For times $t \geqq t_0$, by (2.3) and (3.1), there exist $N \in \mathbb{N}$ and $t_1, t_2 \geqq 0$ such that $0 < t_1 \leqq t_2 < t_1 + T$ and

$$\|\bar{x}(t; x) - \bar{x}_h(t_2)\| \leqq 2\eta, \qquad \|x^h(Nh; x) - \bar{x}_h(t_1)\| \leqq \eta.$$

We choose $k \in \mathbb{N}$ with $|t_1 + kh - t_2| \leqq h$ and obtain from (2.1)

$$\begin{aligned}\|x^h((N+k)h; x) - \bar{x}_h(t_2)\| &\leqq \|(I + hf_h)^k x^h(Nh; x) - (I + hf_h)^k \bar{x}_h(t_1)\| \\ &\quad + \|\bar{x}_h(\sigma_h^k(t_1)) - \bar{x}_h(t_2)\|.\end{aligned}$$

This leads to our assertion (3.4) since $k \leqq T/h$ and

$$\sigma_h^k(t_1) = t_1 + kh + O(h^r) \qquad \text{by (2.2).} \quad \square$$

For an illustration of Theorem 3.1 we consider the model example [2, 9]

$$(3.9) \qquad x_1' = -x_2 + x_1(1 - r^2), \qquad x_2' = x_1 + x_2(1 - r^2), \qquad r^2 = x_1^2 + x_2^2.$$

In polar coordinates this system has the form

$$\theta' = 1, \qquad r' = r(1 - r^2).$$

This system has a stable periodic orbit given by the circle $r = 1$. By an easy computation one finds that Euler's method applied to (3.9) has a stable invariant circle of radius $r_h = (1 + (1 - (1 - h^2)^{1/2})/h)^{1/2} = 1 + O(h)$ for $h \leqq 1$ (see [2] and note that we are outside the chaotic region discussed in [2]). By Theorem 2.1 any Runge Kutta method of order r for the system (3.9) has invariant curves for sufficiently small h. An additional reasoning shows that these are in fact circles of radius

$$r_h = 1 + O(h^r).$$

For the initial value $x_0 = (0.5, 0)$, Fig. 1 shows the longtime behaviour of the error

$$e_1(n, h, x_0) = \|x^h(nh; x_0) - \bar{x}(nh; x_0)\|, \qquad h = \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}$$

for Euler's method. Clearly, the two solutions run out of phase. From the calculations we find that this happens at times which are roughly proportional to h^{-2}. From then on there is a fixed phase shift at each revolution of the trajectories which leads to the oscillations in Fig. 1.

For moderate times the behaviour of e_1 is shown in Fig. 2 together with the distance

$$e_2(n, h, x_0) = \operatorname{dist}(x^h(nh; x_0), \gamma) = |\,\|x^h(nh; x_0)\| - 1\,|.$$

The minimum $e_3 = \operatorname{Min}(e_1, e_2)$ should be a rather precise upper estimate of $\operatorname{dist}(x^h(nh; x_0), \omega(x_0))$ (see (3.2)). Figure 2 also shows how e_3 changes if x_0 moves towards the unstable origin. In order to test (3.3) we evaluated

$$e_4(h, x_0) = \sup\{e_3(n, h, x_0) \colon n \geqq 0\}$$

and estimated the order of convergence by

$$\operatorname{ord}_h = \ln(e_4(h, x_0)/e_4(h/2, x_0))/\ln(2).$$

The results are displayed in Table 1 for $x_0 = (0.5, 0)$ and in Table 2 for $x_0 = (0.005, 0)$. The latter value is certainly well out of the neighborhood of the periodic orbit in which Theorem 3.1 is valid. Correspondingly, the errors $e_4(h, x_0)$ are larger and it takes smaller step sizes to observe the expected orders of convergence.

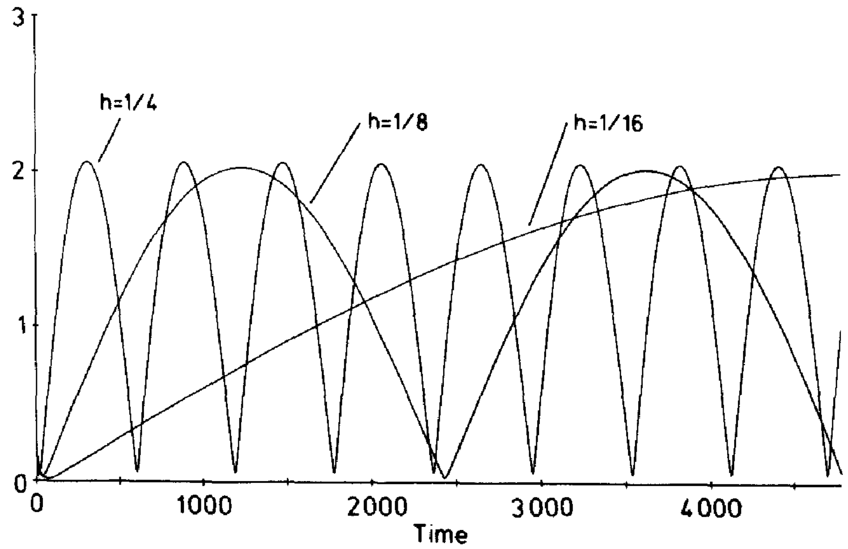

Fig. 1. Error e_1 for Euler's method, $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

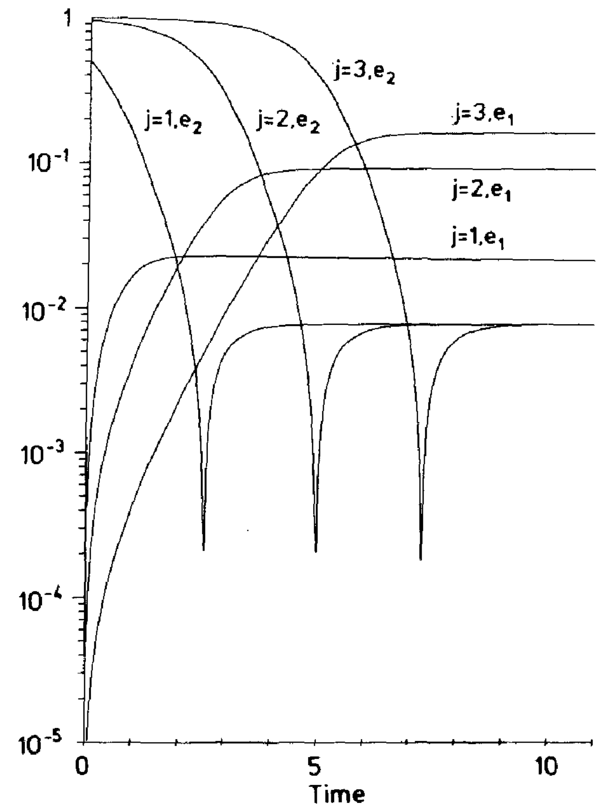

Fig. 2. Errors e_1, e_2 for Euler's method, $h=\frac{1}{32}$, $x_0=(5\cdot 10^{-j}, 0)$

Table 1. $x_0=(0.5, 0)$

h	Euler		Runge Kutta fourth order	
	$e_4(h, x_0)$	ord_h	$e_4(h, x_0)$	ord_h
1	0.414	1.7	0.334 *E*-1	3.74
$\frac{1}{2}$	0.127	0.32	0.250 *E*-2	4.12
$\frac{1}{4}$	0.102	0.46	0.144 *E*-3	3.77
$\frac{1}{8}$	0.0741	0.80	0.105 *E*-4	3.76
$\frac{1}{16}$	0.0427	0.93	0.780 *E*-6	3.77
$\frac{1}{32}$	0.0225		0.574 *E*-7	

Table 2. $x_0=(0.005, 0)$

h	Euler		Runge Kutta fourth order	
	$e_4(h, x_0)$	ord_h	$e_4(h, x_0)$	ord_h
1	0.723	0.44	0.349 *E*-1	2.80
$\frac{1}{2}$	0.534	0.22	0.502 *E*-2	3.52
$\frac{1}{4}$	0.458	0.40	0.437 *E*-3	3.74
$\frac{1}{8}$	0.348	0.60	0.326 *E*-4	3.87
$\frac{1}{16}$	0.230	0.77	0.222 *E*-5	3.94
$\frac{1}{32}$	0.135		0.145 *E*-6	

For the general case of a hyperbolic orbit we have no approximation result on the continuous and discrete trajectories as in Theorem 3.1. It is clear that the estimate (3.3) does not hold in this case because x may lie on the stable manifold of γ but not of γ_h. In view of the results for trajectories near stationary hyperbolic points [1] we make the following conjecture. For any x_0 in a neighborhood U of γ there exists a proper discrete initial value $y_0 = y_0(h, x_0) \in U$ and vice versa such that

$$\sup\{\operatorname{dist}(x^h(nh; y_0), \omega(x_0)) : x^h(jh, y_0) \in U \text{ for } j=0, \ldots, n\} \leqq C h^p$$

for some $0 < p \leqq r$ and $\omega(x_0) = \{\bar{x}(t; x_0) : \bar{x}([0, t]; x_0) \subset U\}$.

In some sense this specifies our intuition that a numerical phase portrait near a hyperbolic periodic orbit is at least graphically correct.

4. The Dynamics on the Invariant Curve

The results of Sect. 2 also provide some information on the behaviour of the one-step method on the invariant curve γ_h. Instead of (P3) we make the slightly stronger assumption

(P3*) $\varphi(h, x) = f_h(x)$ if $h > 0$ and $\varphi(0, x) = f(x)$ for some function $\varphi \in C^2([0, h_0] \times \Omega, \mathbb{R}^m)$.

A careful inspection of the proof of Theorem 2.1 then shows that $R_h w$ from (2.35) depends continuously on $(h, w) \in [0, h_0] \times W(\varepsilon, L)$. Moreover, the operators R_h, $h_1/2 \leqq h \leqq h_1$ are uniform contractions with constant $1 - \alpha h_1/2$ for every small h_1. By the contraction mapping theorem with parameters [11], Appendix we find that the parametrization $\bar{x}_h$ of the invariant curve as well as $\sigma_{\bar{w}_h}$ depend continuously on h.

Now, the homeomorphism

$$\tau_h : S_1 \to S_1, \qquad \tau_h(t) = \sigma_{\bar{w}_h}(tT)/T$$

is Lipschitz and order preserving by Theorem 2.1 and its rotation number ρ_h depends continuously on h (cf. [8], Chap. 6.2). Since σ_h is the lift of $\sigma_{\bar{w}_h}$ and satisfies (2.2) we find for the lift $\tilde{\tau}_h$ of τ_h

$$\tilde{\tau}_h(t) = t + \frac{h}{T} + O(h^{r+1}), \qquad t \in \mathbb{R}. \tag{4.1}$$

We may now use Lemma 3 of [10], Chap. III.3 to conclude

$$\rho_h = \frac{h}{T} + O(h^{r+1}). \tag{4.2}$$

We make ρ_h a continuous function for $h \in [0, h_0]$ by setting $\rho_0 = 0$ which is in accordance with $R_0 = I$ in (2.35). Then the intermediate value theorem shows that any rational or irrational rotation number in $(0, \rho_{h_0}]$ is possible for suitable $h \in (0, h_0]$.

Theorem 4.1. *Let* (P3*) *and the assumptions of Theorem* 2.1 *hold. Then the rotation number* ρ_h *of the one-step function* $I+hf_h$ *with respect to the invariant curve* γ_h *is a continuous function of* $h\in[0, h_0]$ *and satisfies* (4.2). *In particular, there exists* $N_0\in\mathbb{N}$ *and a sequence* h_N, $N\geqq N_0$ *such that* $I+h_N f_{h_N}$ *has an orbit*

$$\{x_i^N=\bar{x}_{h_N}(t_i^N): i=1, \ldots, N\}$$

of period N *on* γ_{h_N}. *The following asymptotic relations hold*

$$(4.3)\quad h_N=\frac{T}{N}+O\left(\left(\frac{T}{N}\right)^{r+1}\right), \qquad t_i^N=t_1^N+(i-1)\frac{T}{N}+O\left(\left(\frac{T}{N}\right)^{r}\right), \qquad i=1, \ldots, N.$$

Proof. We define h_N as a solution of $\rho_h=\frac{1}{N}$. It is well known [8] that τ_{h_N} then has an orbit $s_1^N, \ldots, s_N^N$ of period N in S_1. With $t_i^N=s_i^N T$ we obtain the orbit $x_i^N=\bar{x}_{h_N}(t_i^N)$, $i=1, \ldots, N$ for $I+h_N f_{h_N}$. The first relation of (4.3) follows immediately from (4.2) whereas the second is a consequence of the first and (4.1). The formula for t_i^N also implies that N is in fact the minimal period of the discrete orbit. □

As an example we consider a simple model for a food chain [13].

$$(4.4)\quad x_1'=R(1-x_1)-x_2 f_1(x_1), \qquad x_2'=-Rx_2+x_2 f_1(x_1)-(1-x_1-x_2)f_2(x_2)$$

where $f_i(x)=x/(a_i+b_i x)$, $i=1, 2$. For a large set of parameter values $a_i, b_i>0$, $i=1, 2$ this system shows a Hopf bifurcation with respect to R. We took the values

$$a_1=0.1, \quad b_1=1, \quad a_2=0.5, \quad b_2=0.5, \quad R=0.3$$

at which (4.4) has a stable periodic orbit of period $T\cong 16.7$.

Figure 3 shows the invariant curves for Euler's method with step size $h=0.4$, 0.1, 0.01. These were obtained by starting at $x_1=0.5$, $x_2=0.1$ and plotting the points of the iteration after a 'dark phase' of length 500. After some time these points filled a continuous curve depending on the resolution of the graphics device. This indicates that, for the step sizes chosen, the rotation number was sufficiently irrational.

Figure 4a–e show the faith of these invariant curves for increasing step size. First the invariant curve starts to oscillate, then various finite orbits are observed and at about $h=0.8$ a strange attracting set appears which is shown in Fig. 4e for $h=0.85$. In view of the period given above we are certainly far from the asymptotic behaviour $h\to 0$ and find the spurious solution effect as discussed in [2].

The behaviour of the invariant curves for the standard fourth order Runge Kutta method is completely different (see Fig. 5). With increasing step size the invariant curves shrink towards the unstable stationary point which finally becomes a stable fixed point for the RK-method at about $h=3.75$.

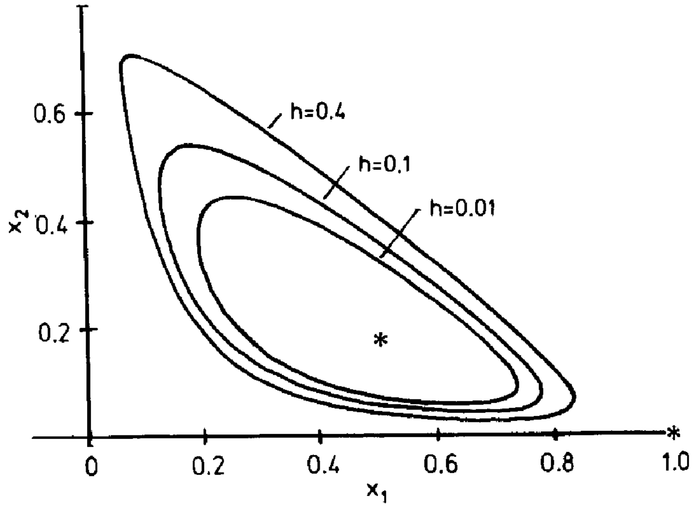

Fig. 3. Euler's method: $h = 0.4, 0.1, 0.01$ (*: stationary points)

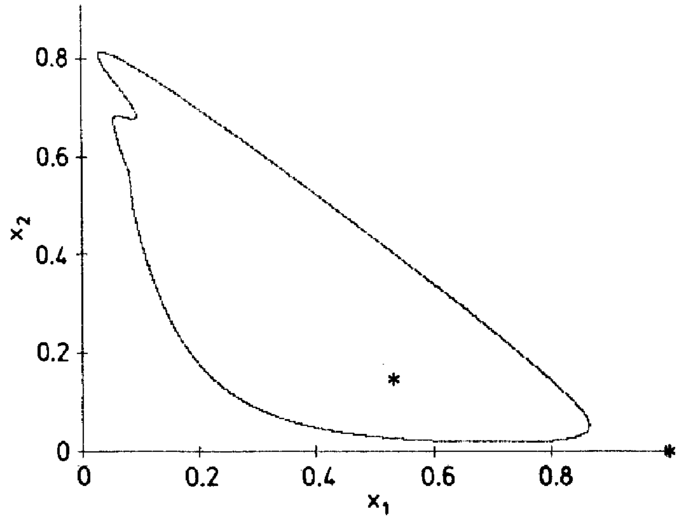

Fig. 4a. $h = 0.6$

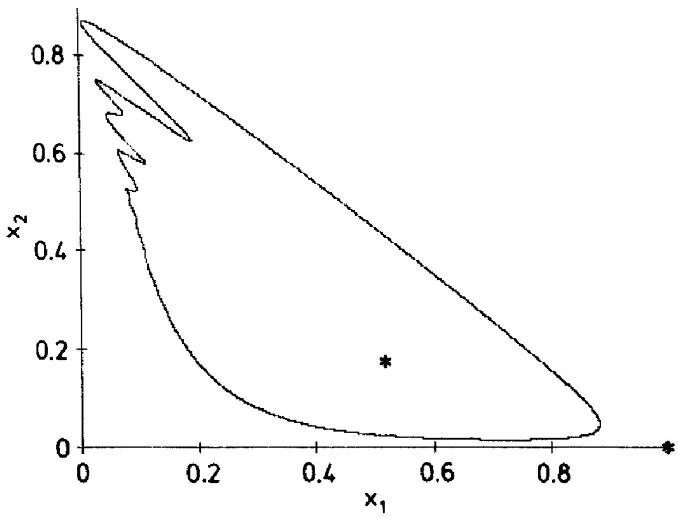

Fig. 4b. $h = 0.7$

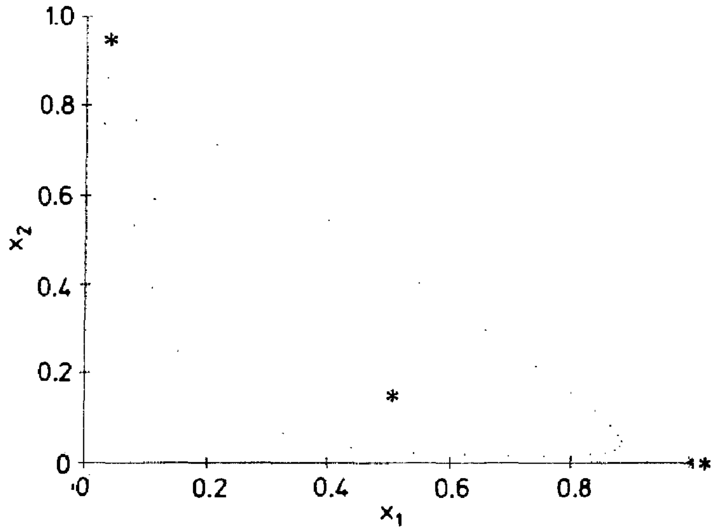

Fig. 4c. $h=0.72$

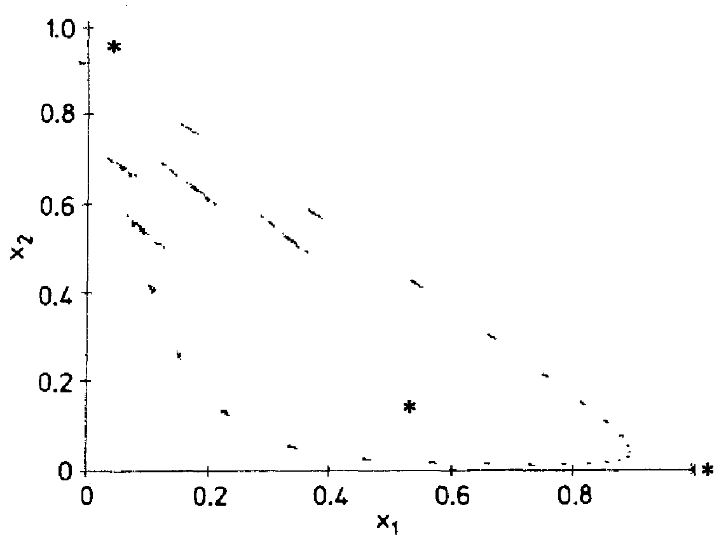

Fig. 4d. $h=0.78$

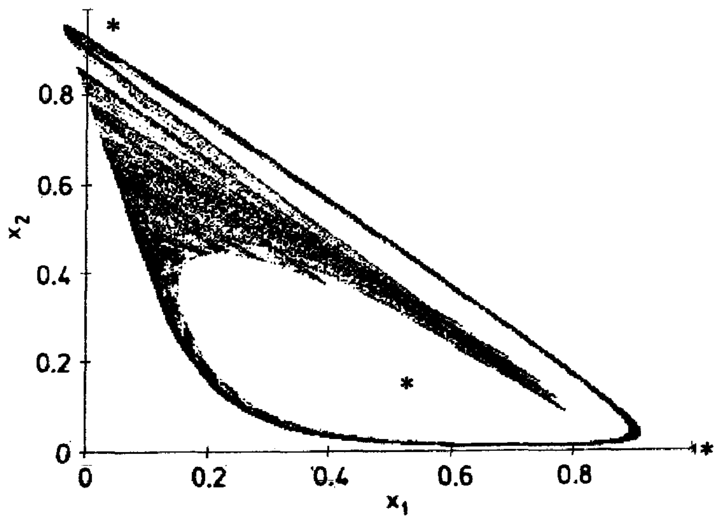

Fig. 4e. $h=0.85$

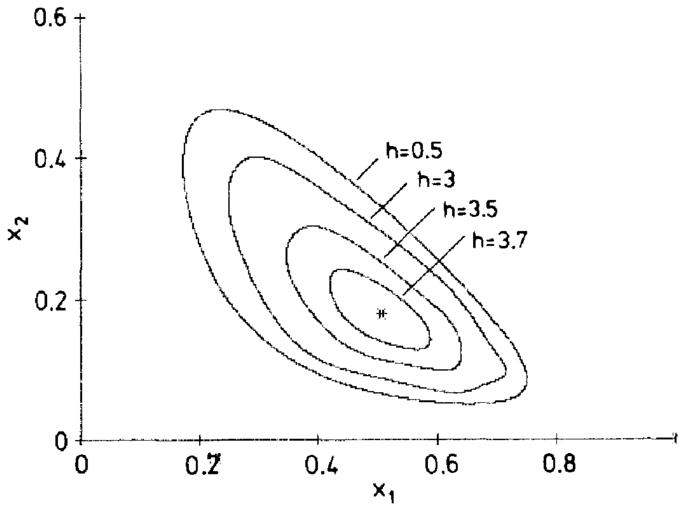

Fig. 5. *RK*-method fourth order: $h=0.5, 3, 3.5, 3.7$

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