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Half-Stable Solution Branches for Ordinary Bifurcation Problems*)

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It is shown that second order bifurcation problems with a positive, autonomous nonlinearity have a smooth branch of positive solutions which tends to infinity. Moreover, this branch satisfies a stability rule saying that the solutions are stable if the branch turns to the right and unstable if it turns to the left.

1 Introduction

The present paper is concerned with the existence and structure of global positive solution branches for the bifurcation problem

$$(1)\qquad \begin{aligned} &-u'' = f(u, u', \lambda) \quad \text{in } [a, b], \\ &\alpha_a u(a) - \beta_a u'(a) = \gamma_a, \; \alpha_b u(b) + \beta_b u'(b) = \gamma_b \end{aligned}$$

where $f \in C^1(\mathbf{R}^3)$ and

$$(2)\qquad \alpha_x, \beta_x, \gamma_x \geqslant 0, \qquad \alpha_x + \beta_x > 0 \quad (x = a, b), \qquad \alpha_a + \beta_b > 0.$$

For positive f one can guarantee the existence of an unbounded continuum of solutions ([18], [9], [1], [21]). However, in order to obtain smooth global branches without secondary bifurcations, usually more restrictive assumptions on f are needed (such as concavity, asymptotic linearity [1]).
Typically, these assumptions are not satisfied for chemical reaction problems with hysteresis where f may be neither concave nor convex with respect to u (cf. [5], [23], [3]). Starting with an idea of Laetsch [14] we will show that the autonomous problems (1) (and slightly more general equations) have smooth global solution branches without imposing growth restrictions on f. A particular result is

Theorem 1 *For* $\lambda = \lambda_0$ *let* (1) *have a unique nonnegative solution* u_0 *which is stable. Assume further that*

$$f(u, v, \lambda) > 0, \qquad f_\lambda(u, v, \lambda) > 0 \qquad \forall u \geqslant 0, v \in \mathbf{R}, \lambda > \lambda_0.$$

*) This paper is an extended version of a part of the author's work [2].

Then there exists a C^1-branch of solutions

$$(u(s), \lambda(s)), \quad s \geqslant 0, \qquad u(0) = u_0, \quad \lambda(0) = \lambda_0$$

with the following properties

(i) $u(s) > 0$ in (a, b), $\lambda(s) > \lambda_0$ *for* $s > 0$,

(ii) $\|u(s)\|_0$ *is monotone increasing and* $\|u_x(s)\|_0 + \|u(s)\|_0 + |\lambda(s)| \to \infty$ *as* $s \to \infty$,

(iii) $(u(s), \lambda(s))$ *is* $\left\{\begin{array}{l} \text{stable} \\ \text{a limit point} \\ \text{unstable} \end{array}\right\} \Leftrightarrow \lambda_s(s) \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} 0.$

Here $\| \ \|_0$ denotes the maximum norm and the subscripts indicate the derivatives with respect to $\lambda \in \mathbf{R}$, $x \in [a, b]$ and $s \geqslant 0$.

The following general result is proved in [14]. If u is a solution of (1) with exactly n zeroes in (a, b) and if $u f(u, u', \lambda) \geqslant 0$ then

$$\varkappa_n(u) < 0 < \varkappa_{n+2}(u)$$

holds, where $\varkappa_k(u)$ denotes the k-th eigenvalue of the linearization of (1) at $u (\varkappa_0(u) := -\infty)$.
In case $n = 0$ this inequality shows that the linearization of (1) at u has at most one nonpositive eigenvalue. We will call solutions of this type half-stable.
In fact, the solutions on the above branch are half-stable. But in contrast to [14] we do not try to avoid critical points (i.e. $\varkappa_1(u) = 0$) on the branch, which leads to severe growth restrictions on f, but rather exclude secondary bifurcations which can be achieved by a positivity condition on f_λ (cf. Theorem 3).
Moreover, a detailed analysis of half-stable solutions in sections 2 and 3 will yield the specific properties (i) – (iii) of the branch. In particular, the stability rule (iii) may be regarded as a global version of a known rule in the neighbourhood of bifurcation points (cf. [8], [20], Ch. III).

2 Preliminaries on linear problems

Throughout the paper let $C^j = C^j\,[a, b]$ $(j \in \mathbf{N})$. For $u, v \in C = C^0$, $g, h \in \mathbf{R}^n$ we set

$$u \leqslant (<)\, v \Leftrightarrow u(x) \leqslant (<)\, v(x) \qquad \forall x \in [a, b],$$
$$u \leqslant (<) v \text{ in } \Omega \Leftrightarrow u(x) \leqslant (<) v(x) \qquad \forall x \in \Omega \subset [a, b],$$
$$g \leqslant (<) h \Leftrightarrow g_i \leqslant (<)\, h_i \qquad \forall i \in \{1, \ldots, n\},$$
$$(u, g) \leqslant (v, h) \Leftrightarrow u \leqslant v \ \text{ and } \ g \leqslant h.$$

In this section we consider a linear differential operator

$$Lu = -u'' + pu' + qu, \; p, q \in C$$

and a boundary operator

$$Bu = (B_a u, B_b u) = (\alpha_a u(a) - \beta_a u'(a), \alpha_b u(b) + \beta_b u'(b)) \tag{3}$$

where

$$\beta_x > 0 \quad \text{or} \quad (\beta_x = 0,\ \alpha_x > 0), \qquad x = a, b.$$

Let us further introduce

$$e(x) = (x-a)^{i_a}(b-x)^{i_b},\ [a,b]_e = \{x \in [a,b] : e(x) > 0\}$$

where $i_x = 0$ if $\beta_x > 0$, $i_x = 1$ if $\beta_x = 0$ $(x = a, b)$.

The operator $P = (L, B) : C^2 \to C \times \mathbf{R}^2$, $Pu = (Lu, Bu)$ is called *inverse monotone* (i.m. for short) iff

$$Pu \geqslant 0, \qquad u \in C^2 \Rightarrow u \geqslant 0.$$

The following well known characterization of this property [17], [22], Ch. II, will be frequently used in the sequel.

Lemma 1 *The following conditions are equivalent*

(i) *P is i.m.,*
(ii) *there exists $u \in C^2$ with $u \geqslant 0$, $Pu \overset{\neq}{\geqslant} 0$ (u is called a majorizing element),*
(iii) $u \in C^2$, $Pu \geqslant 0 \Rightarrow u = 0$ or $u \geqslant Ke$ *for some* $K > 0$,
(iv) *the smallest real eigenvalue λ of $Pu = (\lambda u, 0)$ is positive.*

By the classical Sturm-Liouville theory [4] the eigenvalue problem $Pu = (\lambda u, 0)$ has real simple eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ with corresponding eigenfunctions φ_i which have exactly $(i-1)$ simple zeroes in (a, b), $i \in \mathbf{N}$. Note that φ_1 can be scaled such that $\varphi_1 \geqslant e$.
Lemma 1 characterizes the case $\lambda_1 > 0$ in terms of monotonicity. A corresponding result for the case $\lambda_2 > 0$ is the following

Lemma 2 *$\lambda_2 > 0$ holds if and only if*

(4) *there exists $\bar{x} \in (a, b)$ such that $(Lv, B_a v, v(\bar{x}))$ is i.m. in $[a, \bar{x}]$ and $(Lv, v(\bar{x}), B_b v)$ is i.m. in $[\bar{x}, b]$.*

Proof. Without loss of generality we may assume

(5) $\qquad \varphi_2 > 0 \quad \text{in } (a, \tilde{x}), \qquad \varphi_2 < 0 \text{ in } (\tilde{x}, b) \quad \text{for some } \tilde{x} \in (a, b).$

If $\lambda_2 > 0$ then $L\varphi_2 > 0$ in $(a, \tilde{x})$, $B_a\varphi_2 = 0$, $\varphi_2(\tilde{x}) = 0$ and $L(-\varphi_2) > 0$ in $(\tilde{x}, b)$, $-\varphi_2(\tilde{x}) = 0$, $B_b(-\varphi_2) = 0$. Hence, by Lemma 1 we may put $\bar{x} = \tilde{x}$ in property (4).
On the other hand let (4) and $\lambda_2 \leqslant 0$ be satisfied. In case $\bar{x} \leqslant \tilde{x}$ we have $(Lv, v(\tilde{x}), B_b v)$ is i.m. in $[\tilde{x}, b]$ from (4) and Lemma 1. But this contradicts (5) and

$$L\varphi_2 = \lambda_2\varphi_2 \geqslant 0 \quad \text{in } [\tilde{x}, b], \qquad \varphi_2(\tilde{x}) = 0, \qquad B_b\varphi_2 = 0.$$

Similarly, the case $\bar{x} > \tilde{x}$ leads to a contradiction.

Remark. The result of Lemma 2 is partially contained in [13] where splittings of the interval $[a, b]$ are used to estimate solutions of non inverse monotone problems (see also [22], Ch. IV).

Definition 1 *$P = (L, B)$ is called half inverse monotone or h.i.m. if it satisfies one of the equivalent properties in* Lemma 2.

If P is h.i.m. we consider the non void set

$$J = \{x \in [a, b]_e \colon (Lv, B_a v, v(x)) \text{ is i.m. in } [a, x] \text{ and } (Lv, v(x), B_b v) \text{ is i.m. in } [x, b]\}.$$

Here $(Lv, B_a v, v(x))$ (resp. $(Lv, v(x), B_b v)$) is always assumed to be i.m. in $[a, x]$ if $x = a$ (resp. in $[x, b]$ if $x = b$).
Using Lemma 1 it is easily seen that J is an interval which is open in $[a, b]_e$. We will call J the *splitting interval* of P.

If we use φ_1 as a majorizing element in Lemma 1 we obtain

(6) $\lambda_1 \geqslant 0 \Rightarrow J = [a, b]_e$.

In general, the endpoints x_a, x_b of J can be determined as the largest and smallest zero of certain fundamental solutions of L. More precisely, let v_a and v_b be fundamental solutions of L which satisfy $v(a) = \beta_a$, $v'(a) = \alpha_a$ and $v(b) = \beta_b$, $v'(b) = -\alpha_b$, then $x_a = \inf\{x \in [a, b) \colon v_b > 0 \text{ in } (x, b)\}$, $x_b = \sup\{x \in (a, b] \colon v_a > 0 \text{ in } (a, x)\}$ (cf. [22], IV, Prop. 4.5). Moreover, the theory of initial value problems shows that x_a, x_b depend continuously on the coefficients p, q of L.

The following lemma will be the basis for the stability rule in Theorem 1 (iii).

Lemma 3 *Let $P = (L, B)$ be h.i.m. and let $r \in C$, $r \overset{\neq}{\geqslant} 0$. If $(v, \mu) \in C^2 \times \mathbf{R}$ is a nontrivial solution of*

(7) $Lv - \mu r = 0, \; Bv = 0$

then $v(x) \neq 0 \; \forall x \in J$ and

(8) $\operatorname{sgn}(\lambda_1) = \operatorname{sgn}(\mu) \operatorname{sgn}_J(v)$,

where $\operatorname{sgn}_J(v)$ denotes the sign of v in J.

Proof. Suppose that $v(\bar{x}) = 0$ for some $\bar{x} \in J$. Since it is no restriction to assume $\mu \geqslant 0$ we find

$$(9) \quad \begin{array}{llll} Lv = \mu r \geqslant 0 & \text{in } [a, \bar{x}], & B_a v = 0, & v(\bar{x}) = 0 \\ Lv = \mu r \geqslant 0 & \text{in } [\bar{x}, b], & v(\bar{x}) = 0, & B_b v = 0. \end{array}$$

Therefore, $\bar{x} \in J$ yields $v \geqslant 0$ and $v'(\bar{x}) = 0$ (note that $\beta_{\bar{x}} > 0$, $B_{\bar{x}} v = 0$ in case $\bar{x} \in \{a, b\}$). If $\mu > 0$ then either $\mu r \overset{\neq}{\geqslant} 0$ in $[a, \bar{x}]$ or $\mu r \overset{\neq}{\geqslant} 0$ in $[\bar{x}, b]$ holds. Using (9) and Lemma 1 (iii) we conclude $v'(\bar{x}) \neq 0$, a contradiction. Hence $\mu = 0$ and also $v = 0$ because of (9) and $\bar{x} \in J$. But (v, μ) was a nontrivial solution of (7) by our assumption.
For the proof of (8) it suffices to consider the case $v > 0$ in J and show that

$$\lambda_1 = 0 \Leftrightarrow \mu = 0 \quad \text{and} \quad \lambda_1 > 0 \Leftrightarrow \mu > 0$$

hold.
If $\mu = 0$ then (L, B) has the eigenvalue 0 and $\lambda_1 = 0$ since $\lambda_2 > 0$. If $\lambda_1 = 0$ is assumed we select $\bar{x} \in J$ and let $w = v - v(\bar{x})\, \varphi_1(\bar{x})^{-1} \varphi_1$. The pair (w, μ) is then a solution of (7) satisfying $w(\bar{x}) = 0$ and we conclude $w = 0$, $\mu = 0$.

In the case $\lambda_1 > 0$ the operator P is i.m. and we already know $\mu \neq 0$. Then it follows from (7) and Lemma 1 that $\operatorname{sgn}(\mu) v > 0$ in (a, b) and hence $\mu > 0$. Finally, let $\mu > 0$ be satisfied. We choose $\bar{x} \in J$ and find (9) with "$v(\bar{x}) = 0$" replaced by "$v(\bar{x}) > 0$". Thus $v \geqslant 0$ holds and $Lv = \mu r \gneqq 0$, $Bv = 0$. $\lambda_1 > 0$ is now a consequence of lemma 1.

3 Half-stable solutions

We consider the boundary value problem

$$(10) \qquad T(u, \lambda) = (-u'' + pu' - f(u, u', \lambda), Bu - (\gamma_a, \gamma_b)) = 0$$

where B is given by (3) and satisfies (2). Moreover, we assume

$$(11) \qquad p \in C^1, p' \leqslant 0, \quad \begin{array}{ll} \beta_a p(a) < \alpha_a & \text{if } \alpha_a > 0, \\ \beta_b p(b) > -\alpha_b & \text{if } \alpha_b > 0. \end{array}$$

T is considered as an operator mapping $C^2 \times \mathbf{R}$ into $C \times \mathbf{R}^2$, the respective norms on these spaces are

$$(12) \qquad \begin{array}{ll} \|(u, \lambda)\|_2 = \|u''\|_0 + \|u'\|_0 + \|u\|_0 + |\lambda|, & (u, \lambda) \in C^2 \times \mathbf{R} \\ \|(v, g)\|_0 = \|v\|_0 + \|g\|_0, & (v, g) \in C \times \mathbf{R}^2, \end{array}$$

where $\| \ \|_0$ always denotes the maximum norm.

As usual (cf. [11], [8]), a solution (u, λ) of (10) is called *stable* if the linearization $T_u(u, \lambda)$ has only positive eigenvalues or equivalently if it is i.m. (u, λ) is called *unstable* if $T_u(u, \lambda)$ has a negative eigenvalue.

Definition 2 *A solution (u, λ) of (10) is called half-stable if $T_u(u, \lambda)$ has at most one nonpositive eigenvalue or equivalently if it is h.i.m.*

The significance of this notion is demonstrated by the following theorem the first part of which is due to [14] (for constant p). Although our proof here employs monotonicity methods the essential idea is the same as in [14]: the linearization of (10) can be studied by differentiating the differential equation in (10).

Theorem 2 *Let (u, λ) be a solution of (10) such that $f(u, u', \lambda) \gneqq 0$. Then (u, λ) is half-stable and $u > 0$ in $[a, b]_e$. Moreover, if J denotes the splitting interval of $T_u(u, \lambda)$ there exists $\bar{x} \in [a, b]$ which satisfies $u(\bar{x}) = \operatorname{Max}_{[a,b]} u$ and either $\bar{x} \in J$ or $(\bar{x} \in \{a, b\}, \beta_{\bar{x}} = 0, \gamma_{\bar{x}} > 0)$.*

Proof. Let $(L, B) = T_u(u, \lambda)$, then by differentiating in (10) we obtain

$$Lu' = -u''' + pu'' - f_{u'}(u, u' \lambda) u'' - f_u(u, u', \lambda) u' = -p'u'.$$

The pair $(-D^2 + pD, B)$ is i.m. due to (2). Thus Lemma 1 and (10) yield

$$(13) \qquad u > 0 \quad \text{in } [a, b]_e, \quad \begin{array}{ll} u'(a) > 0 & \text{if } \beta_a = \gamma_a = 0, \\ u'(b) < 0 & \text{if } \beta_b = \gamma_b = 0. \end{array}$$

Moreover, the function

$$\text{(14)} \quad \exp\left(-\int_a^x p(y)\,dy\right) u'(x) \textit{ is monotone decreasing in } [a, b].$$

Next, we choose an $\bar{x} \in [a, b]$ such that $u(\bar{x}) = \operatorname{Max}_{[a,b]} u$.
Because of (14) we may require in addition that

$$\text{(15)} \quad \bar{x} = a \quad \text{if } u'(a) \leqslant 0 \quad \text{and} \quad \bar{x} = b \quad \text{if } u'(b) \geqslant 0$$

holds (note that the case $u'(a) \leqslant 0 \leqslant u'(b)$ implies $u' = 0$ and hence contradicts $f(u, u', \lambda) \neq 0$).
In case $u'(a) > 0$ we find

$$\alpha_a > 0, \bar{x} > a, u' \geqslant 0 \quad \text{in } [a, \bar{x}], \quad Lu' = -p'u' \geqslant 0 \quad \text{in } [a, \bar{x}]$$

and $B_a u' = (\alpha_a - \beta_a p(a))\, u'(a) + \beta_a f(u(a), u'(a), \lambda) > 0$.
Thus $(Lv, B_a v, v(\bar{x}))$ is i.m. in $[a, \bar{x}]$ by Lemma 1.
Similarly, by using $-u'$ as a majorizing element, $(Lv, v(\bar{x}), B_b v)$ turns out to be i.m. in $[\bar{x}, b]$ if $u'(b) < 0$.
Hence (u, λ) is half-stable (cf. (15)) and we may conclude $\bar{x} \in J$ unless $\bar{x} \in \{a, b\}$, $\beta_{\bar{x}} = 0$. However, in the latter case $\gamma_{\bar{x}} > 0$ has been excluded and $\gamma_{\bar{x}} = 0$ is impossible because u cannot attain its maximum in $\bar{x}$ due to (13).

Remark. In general Theorem 2 is false if f depends explicitly on x (cf. [14]). As an easy example consider the linear problem

$$-u'' = \lambda u + (\pi^2 - \lambda) \sin \pi x =: f(x, u, \lambda) \quad \text{in } [0, 1],\ u(0) = u(1) = 0,$$

where $f(x, u, \lambda) > 0$ in $(0, 1)$ at the solution $u(x) = \sin(\pi x)$. Note that the linearization at u can have any fixed number of negative eigenvalues for a suitable choice of λ.
In [15] half-stable solutions are guaranteed by a different method which also works for nonautonomous problems but needs growth conditions for f.

Let N and R resp. denote the null space and range resp. of a linear operator. Further, let $T: U \times \mathbf{R} \to Y$ (U, Y Banach spaces) be a C^1-mapping. Then a solution $z = (u, \lambda)$ of the equation

$$T(u, \lambda) = 0$$

will be called *simple* iff the total derivative $T'(z) = (T_u(z), T_\lambda(z)): U \times \mathbf{R} \to Y$ satisfies

$$\dim N(T'(z)) = 1, \qquad \operatorname{codim} R(T'(z)) = 0.$$

If T is defined by the boundary value problem (10) with norms and spaces as given in (12), then $T_u(u, \lambda)$ is always Fredholm of index 0. Hence (u, λ) is a simple solution of (10) iff either $N(T_u(u, \lambda)) = \{0\}$ or $\dim N(T_u(u, \lambda)) = 1$, $T_\lambda(u, \lambda) \notin R(T_u(u, \lambda))$. (u, λ) is called *isolated* in the first case and a *limit point* in the second case (cf. [12] for these notions).

A branch of solutions of (10) in $C^2 \times \mathbf{R}$ will be called simple if it consists of simple solutions and has a parametrization $z(s)$, $s_0 \leqslant s \leqslant s_1$ which is continuously differentiable w.r.t. the norm (12) and satisfies

$$z_s(s) = \frac{dz}{ds}(s) \neq 0 \quad \text{for } s_0 \leqslant s \leqslant s_1 .$$

With these notations we may summarize some results from [8], [12] in the following

Lemma 4 *For any simple solution $z_0 = (u_0, \lambda_0)$ of (10) there exists an open neighbourhood V of z_0 in $C^2 \times \mathbf{R}$, such that all solutions of (10) in V are given by a simple branch $z(s) = (u(s), \lambda(s))$, $|s| < \delta$, $z(0) = z_0$.*

Let us introduce the norm

$$\|(u, \lambda)\|_1 = \|u\|_0 + \|u'\|_0 + |\lambda|, \qquad (u, \lambda) \in C^1 \times \mathbf{R}.$$

If $z(s)$, $s_0 \leqslant s \leqslant s_1$ is a simple branch of solutions we can reparametrize $z(s)$ in an obvious way as $y(t)$, $t_0 \leqslant t \leqslant t_1$ where $y(t_i) = z(s_i)$ $(i = 0, 1)$ and $\|y_t(t)\|_1 = 1$ for $t_0 \leqslant t \leqslant t_1$.
This will be called the *arc length parametrization.*

Theorem 3 *Assume* (2), (11) *and let (u, λ) be a solution of* (10), *where*

$$f(u, u', \lambda) \underset{\neq}{\geqslant} 0, \quad f_\lambda(u, u', \lambda) \underset{\neq}{\geqslant} 0 .$$

Then (u, λ) is a simple solution and hence a unique simple branch of solutions of (10) *passes through (u, λ).*

Proof. Since (u, λ) is half-stable by theorem 2 it suffices to consider the case where $T_u(u, \lambda)$ has $\lambda_1 = 0$ as its smallest eigenvalue. Let $\varphi_1 \geqslant e$ be the corresponding eigenfunction. If (u, λ) is not a limit point we have $T_u(u, \lambda)\, v = (f_\lambda(u, u', \lambda), 0, 0)$ for some $v \in C^2$. Thus $v + K\varphi_1 \geqslant 0$ holds for some $K > 0$ and $T_u(u, \lambda)\,(v + K\varphi_1) = (f_\lambda(u, u', \lambda), 0, 0) \underset{\neq}{\geqslant} 0$, which by Lemma 1 leads to a contradiction.

4 The existence theorems

Let us consider (10), (2) again. A pair $(u, \lambda) \in C^2 \times \mathbf{R}$ is called a *(strict) subsolution* of (10) if $T(u, \lambda) \leqslant 0$ ($\underset{\neq}{\leqslant} 0$) and it is called a *(strict) supersolution* of (10) if $T(u, \lambda) \geqslant 0$ ($\underset{\neq}{\geqslant} 0$) (cf. [1, § 9]).

In our existence theorem sub- and supersolutions will be used to control the global solution branch as described in the following lemma.

Lemma 5 Let $[\underline{\lambda}, \bar{\lambda}] \subset \mathbf{R}$ and let $(u(s), \lambda(s))$, $s \in [0, 1]$ be a *continuous solution curve of* (10) *which satisfies $\lambda(s) \in [\underline{\lambda}, \bar{\lambda}]$ for $s \in [0, 1]$. If $(v(\lambda), \lambda) \in C^2 \times \mathbf{R}$, $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ is a continuous curve such that each $(v(\lambda), \lambda)$ is either a strict subsolution or a stable solution, then $v(\lambda(0)) \leqslant u(0)$ implies*

(16) $v(\lambda(s)) \leqslant u(s) \quad \forall s \in [0,1]$ *and*
$v(\lambda(s)) < u(s)$ *in* $[a,b]_e$ *if* $v(\lambda(s))$ *is a strict subsolution.*

The same statement holds with "supersolution" instead of "subsolution" and with all inequalities reversed.

Proof. By the mean value theorem and our assumptions we have

(17) $0 \leqslant T(u(s), \lambda(s)) - T(v(\lambda(s)), \quad \lambda(s)) = Q(s)\,(u(s) - v(\lambda(s)))$

with suitable linear operators

$$Q(s) = (-D^2 - p(s)\,D - q(s)\,I, B), \qquad p(s), q(s) \in C \text{ for } s \in [0,1].$$

Let us define

$$\bar{s} = \sup\{s \in [0,1]: v(\lambda(t)) \leqslant u(t)\ \forall t \in [0,s]\}$$

and $\bar{v} = v(\lambda(\bar{s}))$, $\bar{u} = u(\bar{s})$. Then $\bar{v} \leqslant \bar{u}$ holds.
If $(\bar{v}, \lambda(\bar{s}))$ is a strict subsolution we have $Q(\bar{s})\,(\bar{u} - \bar{v}) \overset{\neq}{\geqslant} 0$ and hence $\bar{u} - \bar{v} \geqslant Ke$ for some $K > 0$ by Lemma 1. In case $\bar{s} < 1$ this contradicts the maximality of $\bar{s}$.

Now let $(\bar{v}, \lambda(\bar{s}))$ be a stable solution. If $\bar{u} - \bar{v} \geqslant Ke$ for some $K > 0$ then we obtain $\bar{s} = 1$ as above. Otherwise there exists $\bar{x} \in [a,b]$ such that $\bar{u}(\bar{x}) = \bar{v}(\bar{x})$, $\bar{u}'(\bar{x}) = \bar{v}'(\bar{x})$, which implies $\bar{u} = \bar{v}$. Thus $Q(\bar{s}) = T_u(\bar{u}, \lambda(\bar{s}))$ is i.m. by our assumption and a perturbation argument using Lemma 1 shows that $Q(s)$ is i.m. for $|s - \bar{s}| \leqslant \varepsilon$, $s \in [0,1]$ ($\varepsilon > 0$ sufficiently small).
Then (17) yields

$$u(s) \geqslant v(\lambda(s)) \quad \text{for } |s - \bar{s}| \leqslant \varepsilon,\ s \in [0,1]$$

and hence a contradiction if $\bar{s} < 1$.
The strict inequality in (16) follows easily from (17) and Lemma 1. Finally, the results for supersolutions are obtained from those for subsolutions if we replace $f(u, u', \lambda)$ in (10) by $-f(-u, -u', \lambda)$.

Theorem 4 *Let* (2), (11) *be satisfied. Further, for some* $\lambda_0 \in \mathbf{R}$ *let* $(v(\lambda), \lambda)$ *and* $(w(\lambda), \lambda), \lambda \geqslant \lambda_0$ *be continuous curves of sub- and supersolutions which are strict if* $\lambda > \lambda_0$ *(we also allow for* $w_\lambda \equiv \infty$ *for all* $\lambda \geqslant \lambda_0$*). Assume that* (10) *with* $\lambda = \lambda_0$ *has a unique solution* u_0 *in the interval* $v(\lambda_0) \leqslant u \leqslant w(\lambda_0)$ *and assume that* (u_0, λ_0) *is stable. Further, let f satisfy*

(18) $f(y, v, \lambda) > 0, f_\lambda(y, v, \lambda) > 0$ *for* $m_\lambda < y < M_\lambda$, $v \in \mathbf{R}$, $\lambda > \lambda_0$

where $m_\lambda = \underset{[a,b]}{\mathrm{Min}}\, v(\lambda)$, $M_\lambda = \underset{[a,b]}{\mathrm{Max}}\, w(\lambda)$.
Then there exists a simple solution branch $(u(s), \lambda(s))$, $0 \leqslant s < \infty$ *of* (10) *which satisfies* $(u(0), \lambda(0)) = (u_0, \lambda_0)$, $u(s) > 0$ *in* $[a,b]_e$ *for* $s > 0$ *and which has the following properties.*

(i) $v(\lambda(s)) < u(s) < w(\lambda(s))$ *in* $[a,b]_e$, $\lambda_0 < \lambda(s)$ *for* $s > 0$,
(ii) $\|(u(s), \lambda(s))\|_1 \to \infty$ *as* $s \to \infty$,
(iii) *the solutions* $(u(s), \lambda(s))$, $s \geqslant 0$ *are half-stable and in particular*

$$(u, \lambda)(s) \text{ is } \begin{Bmatrix} \textit{stable} \\ \textit{a limit point} \\ \textit{unstable} \end{Bmatrix} \Leftrightarrow \lambda_s(s) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0,$$

(iv) $\|u(s)\|_0 = \operatorname{Max}_{[a,b]} u(s)$ *is monotone increasing. In addition, it increases strictly if neither* ($\beta_a = 0$, $\gamma_a > 0$) *nor* ($\beta_b = 0$, $\gamma_b > 0$) *holds. Moreover*

$$u_s(s) > 0 \text{ in } [a, b]_e \text{ if } \lambda_s(s) \geqslant 0,\ s > 0.$$

Proof. Since (u_0, λ_0) is a stable solution there exists a stable initial branch of solutions which after a parametrization by the arc length (as in sect. 3) may be written as

$$\text{(19)} \quad \begin{aligned} z(s) &= (u(s), \lambda(s)), \qquad 0 \leqslant s \leqslant \varepsilon_0 \text{ where } \varepsilon_0 > 0 \text{ and} \\ z(0) &= (u_0, \lambda_0),\ \lambda_s(s) > 0 \quad \text{for } 0 \leqslant s \leqslant \varepsilon_0. \end{aligned}$$

Let us now define

$$s^* = \sup\{t > 0\text{: there exists a simple branch of solutions with arc length parametrization } z^t(s),\ 0 \leqslant s \leqslant t \text{ such that } z^t(0) = (u_0, \lambda_0),\ (\lambda^t)_s(0) > 0\}.$$

Clearly, $s^* \geqslant \varepsilon_0 > 0$ holds because of (19). Moreover, any two branches z^{t_1}, z^{t_2} of the above kind coincide in $[0, t_1] \cap [0, t_2]$ as follows from the local uniqueness of simple branches (Lemma 4). Hence we can construct a simple branch of solutions $z(s) = (u(s), \lambda(s)), 0 \leqslant s < s^*$ satisfying $z(0) = (u_0, \lambda_0)$, $\lambda_s(0) > 0$ and this branch coincides with the initial branch (19) in $[0, \varepsilon_0]$. We shall now prove assertions (i) – (iv) for the parameter interval $[0, s^*)$ instead of $[0, \infty)$.

Proof of (i). The first inequalities of (i) will follow from Lemma 5 once we have shown that $\lambda(s) > \lambda_0$, $0 < s < s^*$. In order to prove this, let us assume that there exists a least $\bar{s} \geqslant \varepsilon_0$ where $\lambda(\bar{s}) = \lambda_0$. Then by Lemma 5 and our uniqueness assumption $u(\bar{s}) = u_0$ holds. Hence from Lemma 4 we may conclude that the branches $z(t)$ and $z(\bar{s} - t)$, $0 \leqslant t \leqslant \bar{s}$ are identical, which contradicts the simplicity of the branch. By a similar reasoning it can be seen that the branch $z(s)$, $0 \leqslant s < s^*$ has no double points.

Proof of (iii). Using (i), (18) and Theorem 2 we find that the solutions $(u(s), \lambda(s))$ are half-stable. In addition, we will show

$$\text{(20)} \quad u_s(s) > 0 \quad \text{in } J(s),\ 0 < s < s^*$$

where $J(s)$ denotes the splitting interval associated with $T_u(u(s), \lambda(s)) = (L(s), B)$. For each $s \in (0, s^*)$ the tangent vector $(u_s(s), \lambda_s(s))$ satisfies

$$\text{(21)} \quad L(s)\, u_s(s) - \lambda_s(s) f_\lambda(u(s), u_x(s), \lambda(s)) = 0, \qquad B u_s(s) = 0$$

where $f_\lambda(u(s), u_x(s), \lambda(s)) \overset{\neq}{\geqslant} 0$ holds because of (18) and (i). Now $(L(s), B)$ is i.m. and $\lambda_s(s) > 0$ for $0 \leqslant s \leqslant \varepsilon_0$ and hence we have by Lemma 1

$$u_s(s) > 0 \quad \text{in } [a, b]_e \text{ for } 0 < s \leqslant \varepsilon_0,$$

which proves (20) for $0 < s \leqslant \varepsilon_0$. But then (20) also holds for $0 < s < s^*$ since the endpoints of $J(s)$ depend continuously on s and $u_s(s)$ is of one sign in $J(s)$ by (21) and Lemma 3. Moreover, the sign relation (8) implies the stability rule in (iii).

Proof of (iv). The second assertion follows from (6) and (20). Consider an arbitrary $s \in (0, s^*)$, then Theorem 2 ensures that either $\|u(s)\|_0 = \operatorname{Max}_{[a,b]} u(s) = u(s, \bar{x})$ for some $\bar{x} \in J(s)$ or $\|u(s)\|_0 = u(s, \bar{x})$, $\bar{x} \in \{a, b\}$, $\beta_{\bar{x}} = 0$, $\gamma_{\bar{x}} > 0$.
Using (20) we obtain for $\delta > 0$ sufficiently small $\|u(s)\|_0 = u(s, \bar{x}) < u(s + \delta, \bar{x}) \leqslant \|u(s + \delta)\|_0$ in the first case, and $\|u(s)\|_0 = \gamma_{\bar{x}} = u(s + \delta, \bar{x}) \leqslant \|u(s + \delta)\|_0$ in the second case.

Proof of (ii). Let us assume that $\|u(s_n)\|_1$ is bounded for some sequence $s_n \nearrow s^*$. By a standard compactness argument we have a subsequence $\mathbf{N}' \subset \mathbf{N}$ and a solution (u^*, λ^*) of (10) such that

$$(22) \qquad \|(u(s_n), \lambda(s_n)) - (u^*, \lambda^*)\|_2 \to 0 \quad \text{as } n \to \infty,\ n \in \mathbf{N}'.$$

Moreover, $\lambda_0 \leqslant \lambda^*$, $v(\lambda^*) \leqslant u^* \leqslant w(\lambda^*)$ and hence our assumptions and Theorem 3 guarantee a unique simple branch of solutions with the arc length parametrization $y(t) \in C^2 \times \mathbf{R}$, $|t| < \delta$, $y(0) = (u^*, \lambda^*)$.
From (22) we have $z(s_N) = y(t_N)$ for some $|t_N| < \delta$ and some $N \in \mathbf{N}'$ sufficiently large. After possibly changing the orientation of $y(t)$ we may join the branches $z(s)$, $0 \leqslant s \leqslant s_N$ and $y(t)$, $t_N \leqslant t < \delta$. The maximality of s^* then shows that $y(t)$, $|t| < \delta$ is a subbranch of $z(s)$, $0 \leqslant s < s^*$ and hence that $z(\hat{s}) = (u^*, \lambda^*)$ for some $\hat{s} \in [0, s^*)$.
We thus obtain double points for $z(s)$ near $z(\hat{s})$ by using (22) and the local uniqueness from Lemma 4.
This leads to a contradiction (cf. proof of (i)).
Let us finally note that $s^* = \infty$ follows from (ii) and the estimate

$$\|(u(s), \lambda(s))\|_1 = \left\|\int_0^s u_{x,s}(t)\,dt + u_x(0)\right\|_0 + \left\|\int_0^s u_s(t)\,dt + u(0)\right\|_0$$

$$+ \left|\int_0^s \lambda_s(t)\,dt + \lambda(0)\right| \leqslant \int_0^s \|(u_s(t), \lambda_s(t))\|_1\,dt + \text{const} = s + \text{const}.$$

Remarks. Theorem 1 is obviously a special case of Theorem 4.
If we assume $f_\lambda(y, v, \lambda) < 0$ instead of > 0 in (18) then the assertions (i), (ii), (iii) are still valid whereas $\|u(s)\|_0$ is now (strictly) monotone decreasing.
The special assumptions for the starting point (u_0, λ_0) frequently occur in the applications. However, it is clear how to modify Theorem 4 if we start with an arbitrary solution u_0, $v(\lambda_0) \leqslant u_0 \leqslant w(\lambda_0)$ and continue the branch to both sides as far as the positivity of f and f_λ is guaranteed.
Let us finally note that we can assert

$$(23) \qquad \|u(s)\|_0 + |\lambda(s)| \to \infty \quad \text{as } s \to \infty$$

if f is independent of u' or – more generally – satisfies a Nagumo condition [21] uniformly on bounded λ-intervals.

Theorems 1 and 4 deal with the so called forced case. The case of bifurcation from the zero can be treated in a similar way. To be precise, let us assume that for some $\lambda_0 \in \mathbf{R}$

(24) $f \in C^2(\mathbf{R}^3), f(0,0,\lambda) = 0 \quad \forall \lambda \in \mathbf{R}, \gamma_a = \gamma_b = 0,$

(25) $T_u(0,\lambda_0) = (-D^2 + (p - f_{u'}, (0,0,\lambda_0))\, D - f_u(0,0,\lambda_0)I, B)$ has the smallest eigenvalue 0 (with positive eigenfunction φ),

(26) $T_{u,\lambda}(0,\lambda_0)\, \varphi \notin R\,(T_u(0,\lambda_0))$.

Then a C^1-branch bifurcates from $(0,\lambda_0)$ in the direction (φ,μ) for some $\mu \in \mathbf{R}$ (cf. [7]). Therefore, this branch $(u(s), \lambda(s))$ initially satisfies $u_s(s) > 0$ in $[a,b]_e$ and we can employ a continuation procedure as in the proof of Theorem 4. Although the zero is no longer a strict subsolution now, the branch cannot reattach the trivial branch since $\|u(s)\|_0$ is strictly monotone increasing.

Theorem 5 *For some* $\lambda_0 > \lambda_1$ *let* (2), (11), (24), (25) *hold and assume that* $w(\lambda)$, $\lambda \geqslant \lambda_1$ *is a continuous curve of nonnegative, strict supersolutions* (*or* $w_\lambda \equiv \infty$, $\lambda \geqslant \lambda_1$). *Moreover, for* $\lambda = \lambda_1$ *let* $u = 0$ *be the only solution of* (10) *in the interval* $0 \leqslant u \leqslant w(\lambda_1)$.
If f satisfies

(27) $f(y,v,\lambda) > 0, f_\lambda(y,v,\lambda) > 0$ *for* $0 < y < \operatorname*{Max}_{[a,b]} w(\lambda),\ v \in \mathbf{R},\ \lambda > \lambda_1,$

(28) $f_{u',\lambda}(0,0,\lambda_0)\,\varphi' + f_{u,\lambda}(0,0,\lambda_0)\,\varphi \neq 0,$

then there exists a simple solution branch $(u(s), \lambda(s))$, $0 \leqslant s < \infty$ *of* (10) *with* $(u(0), \lambda(0)) = (0,\lambda_0)$, $\lambda(s) > \lambda_1$ *for* $s \geqslant 0$ *and* $0 < u(s) < w(\lambda(s))$ *in* $[a,b]_e$ *for* $s > 0$.
Moreover, the assertions (ii), (iii) *and* (iv) *of Theorem 4 hold.*

Proof. We merely note that (24), (27) and (28) imply $0 \leqslant \lim_{t \searrow 0} t^{-1} f_\lambda(t\varphi, t\varphi', \lambda_0)$ $= f_{\lambda,u'}(0,0,\lambda_0)\,\varphi' + f_{\lambda,u}(0,0,\lambda_0)\,\varphi \neq 0$ and hence condition (26) by an argument as in the proof of Theorem 3.

Remark. In a neighbourhood of the bifurcation point $(0,\lambda_0)$ the stability rule (iii) of Theorem 5 is known in more general situations (see e.g. [8], Th. 1.16 and note that the quantity $\gamma'(\lambda_0)$ in [8] is negative under our assumptions). If we assume $f_\lambda(y,v,\lambda) < 0$ in (27) then in accordance with [8] we have to reverse the inequalities in (iii) while all the other assertions remain unchanged.

5 Applications

The range of applications of Theorems 4 and 5 is quite clear and we just briefly mention some problems which frequently occur in the literature.

Example 1. (Exothermic catalytic reactions with diffusion and convection [5], [23], [3], Kap. VII, VIII)

$$-u'' + \nu u' = \mu(\beta - u)\exp\left(-\frac{\gamma}{1+u}\right) \text{ in } [0,1],\ u(0) = u(1) = 0,$$

where ν, μ, β, γ are nonnegative constants. The sub- and supersolutions are $u = 0$ and $u \equiv \beta$ and any of the parameters μ, β, γ may be used as the parameter λ in Theorem 4 with $\lambda_0 = 0$. Note that $\|u(s)\|_0$ is strictly monotone increasing if $\lambda = \mu$ or $\lambda = \beta$ but strictly monotone decreasing if $\lambda = \gamma$. Moreover $\lambda(s) \to \infty$ as $s \to \infty$ in any case. Finally, we remark that our results complement those of [23] on the existence of at least three different solutions for this problem. By an additional uniqueness argument it is then possible to rigorously prove the hysteresis effect for certain values of the parameters. Related nonlinearities in thermal ignition problems such as

$$\mu \exp\left(\frac{\gamma u}{\gamma + u}\right), \quad \gamma, \mu \geqslant 0 \quad (\text{cf. } [10])$$

can be handled in much the same way.

Example 2. (Enzyme reactions with diffusion [16], [3], Kap. IX)

$$v'' = \lambda\, vR(v) \text{ in } [0,1],\ R \text{ a nonnegative rational function in } \mathbf{R}_+$$
$$v(a) - \beta v'(a) = \gamma, \quad \alpha v(b) + v'(b) = 0, \quad \text{where } \alpha, \beta \geqslant 0,\ \gamma > 0.$$

By the transformation $u = \gamma - v$ we obtain the standard form

$$-u'' = \lambda(\gamma - u)\,R(\gamma - u) \quad \text{in } [0,1], \qquad u(a) - \beta u'(a) = 0,$$
$$\alpha u(b) + u'(b) = \alpha\gamma.$$

Now Theorem 4 immediately applies with $v(\lambda) = 0$, $w(\lambda) \equiv \gamma$, $\lambda_0 = 0$, $u_0 = 0$.

Example 3. (Buckling of rods [6], Kap. III, [19], [3], Kap. II)

$$-u'' = \lambda \sin u \text{ in } [0,1],\ u'(0) = u(1) = 0$$
$$-u'' = \lambda u(1 + u'^2)^{3/2} \text{ in } [0,1],\ u(0) = u'(1) = 0.$$

In both cases we may apply Theorem 5 with $\lambda_0 = \pi^2/4$, $\lambda_1 = 0$. The supersolutions in the first case are $w(\lambda) \equiv \pi$ whereas $w(\lambda) \equiv \infty$ in the second case.

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