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Discrete Green's Functions and Strong Stability Properties of the Finite Difference Method

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For a general class of finite difference equations approximating linear two-point boundary value problems we prove the convergence of the discrete Green's functions to the continuous Green's function. Using this result we obtain various strong stability properties of the finite difference method, such as the convergence of stability constants for linear problems and of contraction constants for nonlinear problems.

1. INTRODUCTION

There is a well developed theory of stability ([2, 9, 13, 14, 15]) for finite difference approximations to linear two-point boundary value problems

$$\begin{aligned} Lx &= x^{(k)} + \sum_{j=0}^{k-1} p_j x^{(j)} = r \text{ in } [a,b] \qquad (k \geqq 1), \\ (Rx)_i &= \sum_{j=0}^{k-1} (\alpha_{ij} x^{(j)}(a) + \beta_{ij} x^{(j)}(b)) = 0 \qquad (i = 1, \ldots, k), \end{aligned} \tag{1.1}$$

where $r, p_j \in C[a,b] (j = 0, \ldots, k-1)$.

If we write the discrete equations in the form (see Section 2 for details)

$$L_h x_h = r_h, \ R_h x_h = 0 \qquad (h = mesh\ size) \tag{1.2}$$

then typical stability inequalities are

$$\|x_h\|_{k-1} \leqq C\ (\|L_h x_h\|_* + \|R_h x_h\|_0) \quad \text{for all} \quad x_h, \tag{1.3}$$

where $\|\ \|_0$ denotes the maximum norm and $\|\ \|_{k-1}$ the maximum norm which involves the difference quotients up to the order $k-1$. The weakest norm $\|\ \|_*$—known so far—for which (1.3) holds is the discrete L_1-norm (see Esser [9, 11]).

If (1.1) has a unique solution it may be represented as

$$x(t) = \int_a^b G(t, s) r(s) ds \tag{1.4}$$

where G is Green's function. Similarly the solution of (1.2) may be written as

$$x_h(t) = h \sum_s G_h(t, s) r_h(s). \tag{1.5}$$

whereby the discrete Green's function G_h is defined.

The purpose of this paper is to establish the convergence

$$G_h(t, s) \to G(t, s) \quad \text{as} \quad h \to 0 \tag{1.6}$$

with respect to various norms. In particular, for each fixed $s \in (a, b)$ this convergence is uniform in t (if $k \geqq 2$). The precise statements will be given in Section 3.

The problems encountered in the proof of (1.6) become apparent from the fact that $y(t) = G(t, s)$ may be regarded as a solution of the weak boundary value problem

$$Ly = \delta_s, Ry = 0 \ \ (\delta_s = \delta\text{-distribution in } s), \tag{1.7}$$

whereas $y_h(t) = G_h(t, s)$ is a solution of

$$L_h y_h = \delta_s^h,\ R_h y_h = 0 \ \ \left(\delta_s^h(t) = \begin{cases} h^{-1} & \text{if} \quad t = s \\ 0 & \text{if} \quad t \neq s \end{cases}\right). \tag{1.8}$$

Indeed, the stability inequalities (1.3) mentioned above, even with the weakest norm $\|\ \|_*$ (see [5]), are not sufficient to prove that (1.8) is a convergent discretization of (1.7). As we will show in Sections 4 and 5 these difficulties can be overcome by the use of suitable weighted maximum norms on both sides of (1.3).

In Section 6 our results will be applied to the convergence of stability constants for linear difference equations and of contraction constants for nonlinear equations. We also give some numerical results for boundary value problems with non smooth solutions such as (1.7). These confirm the convergence result (1.6). However, the question of determining an order of convergence in these problems is still unresolved.

This paper extends some results which were stated without proof in [3].

2. NOTATIONS AND BASIC ASSUMPTIONS

Let H be a null sequence of mesh sizes. For each $h \in H$ we assume an equidistant grid

$$J_h = \{a_h, a_h + h, \ldots, b_h - h, b_h\}$$

which satisfies

$$|a_h - a| + |b_h - b| \leq \bar{C}h \quad \text{for all} \quad h \in H \text{ and some } \bar{C} > 0. \tag{2.1}$$

Moreover, we have an interior grid

$$J_h^0 = \{a_h^0 = a_h + k_1 h, a_h^0 + h, \ldots, \ b_h^0 - h, \ b_h^0 = b_h - k_2 h\},$$

where $k_1, k_2 \in \mathbb{N}$ are independent of h and satisfy $k_1 + k_2 = k$. Let X_h and X_h^0 denote the space of real valued functions on the grids J_h and J_h^0 respectively.

Then our difference equations (1.2) are determined by $r_h \in X_h^0$ and by linear operators

$$L_h\colon X_h \to X_h^0, \ R_h\colon X_h \to \mathbb{R}^k.$$

Hence (1.2) is a system of $|J_h|$ linear equations for the $|J_h|$ unknowns $x_h(t)$, $t \in J_h$.

The discrete Green's function $G_h(t, s)$, $t \in J_h$, $s \in J_h^0$ — if it exists—is then a $|J_h| \times (|J_h| - k)$ matrix defined by (1.5). Let us now choose an interval $[\bar{a}, \bar{b}]$ such that

$$J_h \cup [a, b] \subset (\bar{a}, \bar{b}) \quad \text{for all} \quad h \in H \tag{2.2}$$

and let us extend the coefficients p_j in (1.1) continuously to $[\bar{a}, \bar{b}]$. We then have an extended operator

$$L = \partial^k + M\colon C^k[\bar{a}, \bar{b}] \to C[\bar{a}, \bar{b}], \quad \text{where} \quad M = \sum_{j=0}^{k-1} p_j \partial^j. \tag{2.3}$$

If the Green's function G for (1.1) exists, then a careful inspection of [8, p. 190] shows that G may be extended to $[\bar{a},\bar{b}]\times[a,b]$ with the following properties:

$\partial_t^j G\in C([\bar{a},\bar{b}]\times[a,b])$ for $j=0,\dots,k-2$
(∂_t = partial derivative with respect to t), the restriction of G to the domain $\bar{a}\leqq t\leqq s\leqq b$, $a\leqq s$ (resp. $a\leqq s\leqq t\leqq\bar{b}$, $s\leqq b$) has continuous derivatives $\partial_t^{k-1}G$ and $\partial_t^k G$,

$$\partial_t^{k-1}G(t+0,t)-\partial_t^{k-1}G(t-0,0)=1 \quad \forall t\in[a,b],$$

$$\left.\begin{array}{ll} LG(\cdot,s)=0 & \text{in } [\bar{a},\bar{b}]\setminus\{s\} \\ RG(\cdot,s)=0 & \end{array}\right\} \text{ for } s\in[a,b]. \tag{2.4}$$

Note that $\partial_t^{k-1}G(a-,a)$ resp. $\partial_t^{k-1}G(b+,b)$ has to be inserted when evaluating $RG(\cdot,a)$ resp. $RG(\cdot,b)$.

Before we can state our main assumptions on the difference scheme (1.2) we need some further notation. For real functions

$$x: D(x)\subset\mathbb{R}\to\mathbb{R}$$

we set

$$\begin{aligned}
[x]_h &= \text{restriction of } x \text{ to } D(x)\cap J_h,\\
[x]_h^0 &= \text{restriction of } x \text{ to } D(x)\cap J_h^0,\\
Ex(t) &= x(t+h) \quad \text{if } t+h\in D(x),\\
\Delta^i x(t) &= h^{-i}(E-I)^i x(t) \quad \text{if } t,t+h,\dots,t+ih\in D(x),\\
\|x\|_{j,\Omega} &= \sum_{i=0}^{j} \sup\{|\Delta^i x(t)| : t\in\mathbb{R} \text{ such that } t,\ t+h,\dots,t+ih\in\Omega\}
\end{aligned}$$

and $\|x\|_j = \|x\|_{j,D(x)}$ if $\Omega\subset D(x)\subset J_h$.

By analogy to (2.3) we assume a decomposition

$$L_h=P_h E^{-k_1}\Delta^k+M_h \tag{2.5}$$

with linear operators $P_h: X_h^0\to X_h^0$, $M_h: X_h\to X_h^0$ (cf. [2]). Our assumptions on (1.1) and (1.2) read (cf. [4])

A) (boundary value problem)
the homogeneous equation (1.1) has only the zero solution,

B) (boundary conditions)
B1: R_h is consistent with R, i.e.

$$\|R_h[x]_h - Rx\|_0 \to 0 \quad \text{as} \quad h \to 0, \quad \forall x \in C^{k-1}[\bar{a}, \bar{b}],$$

B2: R_h is locally $\| \ \|_{k-1}$-bounded, i.e. $\exists N \in \mathbb{N}$, $C > 0$ s.t.

$$\|R_h x_h\|_0 \leqq C\|x_h\|_{k-1, \{a_h, \ldots, a_h + Nh, b_h - Nh, \ldots, b_h\}}$$
$$\forall x_h \in X_h, \ h \in H,$$

C) (terms of lower order)

C1: M_h is consistent with M, i.e.

$$\|M_h[x]_h - [Mx]_h^0\|_0 \to 0 \ \text{ as } h \to 0, \ \forall x \in C^{k-1}[\bar{a}, \bar{b}],$$

C2: M_h is locally $\| \ \|_{k-1}$-bounded, i.e. $\exists K \in \mathbb{N}$, $C > 0$ s.t.

$$|M_h x_h(t)| \leqq C\|x_h\|_{k-1, J_h \cap \{t-Kh, \ldots, t+Kh\}}$$
$$\forall t \in J_h^0, \ x_h \in X_h, h \in H,$$

D) (principal term)

D1: P_h may be written as

$$P_h x_h(t) = \begin{cases} q_j(E)x_h(a_h^0), \ t = a_h^0 + jh, \ j = 0, \ldots, \alpha - 1 \\ P(E)x_h(t), \ t = a_h^0 + \alpha h, \ldots, b_h^0 - \bar{\alpha} h \\ \bar{q}_j(E^{-1})x_h(b_h^0), \ t = b_h^0 - jh, \ j = 0, \ldots, \bar{\alpha} - 1 \end{cases}$$

where $P(E) = \sum_{i=-\alpha}^{\bar{\alpha}} \xi_i E^i (\alpha, \bar{\alpha} \in \mathbb{N})$ and $q_j, \bar{q}_j$

are polynomials independent of h which satisfy the consistency relations $P(1) = 1$ and

$$q_j(1) = \bar{q}_i(1) = 1 \quad \text{for} \quad j = 0, \ldots, \alpha - 1, \quad i = 0, \ldots, \bar{\alpha} - 1,$$

D2: the root conditions hold:

$$P(z) \neq 0 \quad \forall z \in \mathbb{C}, |z| = 1;$$

let σ_j, $j = 1, \ldots, d$ (resp. $\bar{\sigma}_j, j = 1, \ldots, \bar{d}$) denote the roots of $z^\alpha P(z)$ (resp. $z^{\bar{\alpha}} P(z^{-1})$) inside the unit circle with multiplicities v_j (resp. $\bar{v}_j$), then

$$\alpha=\sum_{j=1}^{d} v_j,\ \bar{\alpha}=\sum_{j=1}^{\bar{d}} \bar{v}_j \text{ and the matrices}$$

$$Q=\left(q_i^{(\nu-1)}(\sigma_j):\ \begin{matrix} i=0,\dots,\alpha-1 \\ \nu=1,\dots,v_j, j=1,\dots,d\end{matrix}\right),$$

$$\bar{Q}=\left(\bar{q}_i^{(\nu-1)}(\bar{\sigma}_j):\ \begin{matrix} i=0,\dots,\bar{\alpha}-1 \\ \nu=1,\dots,\bar{v}_j, j=1,\dots,\bar{d}\end{matrix}\right)$$

are nonsingular.

These somewhat technical assumptions become more transparent upon noting that (B), (C) and (D1) are satisfied for almost every reasonable difference scheme (with the exception of schemes which use the so called boundary extrapolation [9, 15]). Moreover, B2 and C2 will follow from B1 and C1 by an application of the uniform boundedness theorem if we assume that $R_h x_h$ (resp. $M_h x_h(t)$) only depends on the values

$$x_h(s), s\in\{a_h,\dots,a_h+Nh,\ b_h-Nh,\dots,b_h\}$$

$$(\text{resp. } x_h(s),\ s\in\{t-Kh,\dots,t+Kh\}).$$

Finally, the crucial condition D2 has been verified in [3, 10] for a broad class of schemes.

3. THE MAIN RESULT

Let us briefly consider the special case $Lx=x''$ where the difference scheme (1.2) is generated by $a_h=a$, $b_h=b$, $k_1=k_2=1$ and the formulas

$$x''(t)\sim h^{-2}(x(t-h)-2x(t)+x(t+h)),$$

$$x'(a)\sim h^{-1}(x(a+h)-x(a)),\ x'(b)\sim h^{-1}(x(b)-x(b-h)).$$

Then it is easily seen (cf. [12, 16, 17]) that

$$G_h(t,s)=G(t,s)\quad \forall t\in J_h,\ s\in J_h^0 \tag{3.1}$$

holds. However, as Lorenz [16] has shown, if one uses a difference formula of higher order for $x'(a)$ instead, then

$$|G_h(t,a+2h)-G(t,a+2h)|\geqq C>0 \qquad \forall t\in J_h,\ h\in H$$

occurs while (3.1) is still valid for points $a+Nh \leqq s \leqq b-Nh$ (N large).

Hence there is no convergence for the columns of the discrete Green's function which are close to the boundary and the best result we can expect is for columns $G_h(\cdot, s)$, $s \in \tilde{J}_h$ where

$$\tilde{J}_h = \{s \in J_h^0 : |s-a|, |s-b| \geqq Ch\},$$

$C>0$ sufficiently large.

Our main theorem only states the convergence for the slightly smaller set

$$J_h^* = \{s \in J_h^0 : |s-a|, |s-b| \geqq h \ln(h^{-1})\}.$$

Here the $\ln(h^{-1})$ term is caused by the fundamental solutions of the difference operator P_h from $D1$ in our stability theorem below. In fact, if the difference scheme is compact i.e. P_h = identity for all h, then some modifications of our proofs show that J_h^* may be replaced by $\tilde{J}_h$. But we do not know whether this is true in general.

THEOREM 3.1 *Under the assumptions (A), (B), (C) and (D) there holds*

$$\operatorname*{Max}_{s \in J_h^*} \|G_h(\cdot, s) - [G(\cdot, s)]_h\|_{k-1,s} \to 0 \quad \text{as} \quad h \to 0, \tag{3.2}$$

where

$$\|x_h\|_{k-1,s} = \|x_h\|_{k-2} + \|\Delta^{k-1} x_h\|_{w_s} \quad \text{for} \quad x_h \in X_h,$$

$$\|y_h\|_{w_s} = \operatorname{Max}\left\{\frac{|y_h(t)|}{w_s(t)} : t \in \Omega\right\} \quad \text{for} \quad y_h : \Omega \subset J_h \to \mathbb{R}.$$

Furthermore the weight function w_s is of the form

$$w_s(t) = 1 + h^{-\kappa} \sigma^{h^{-1}|t-s|}, \quad t \in \mathbb{R}$$

where κ, $\sigma>0$ are constants which satisfy (compare D2)

$$|\sigma_i|,\ |\bar{\sigma}_j| < \sigma < 1 \quad \textit{for} \quad i=1,\ldots,d,\ j=1,\ldots,\bar{d}$$

$$0 < \kappa \leqq \operatorname{Min}(1, -\ln(\sigma)). \tag{3.3}$$

As an immediate consequence of Theorem 3.1 we have the following.

COROLLARY *Let (A)–(D) be satisfied, then*

$$\operatorname*{Max}_{s\in J_h^*}\|G_h(\cdot,s)-[G(\cdot,s)]_h\|_{k-2}\to 0 \quad \text{as} \quad h\to 0\,(k\geqq 2) \tag{3.4}$$

holds and for every $\varepsilon>0$

$$\operatorname*{Max}_{s\in J_h^*}\|G_h(\cdot,s)-[G(\cdot,s)]_h\|_{k-1,J_h\setminus[s-\varepsilon,s+\varepsilon]}\to 0 \quad \text{as} \quad h\to 0. \tag{3.5}$$

(3.5) follows from Theorem 3.1 and the estimate

$$w_s(t)\leqq 1+h^{-\kappa}\sigma^{h^{-1}\varepsilon}\leqq C \quad \text{for} \quad |t-s|\geqq\varepsilon, \quad h\in H.$$

The proof of Theorem 3.1 will be based on special consistency and stability theorems for Eq. (1.8) where $\delta_s^h\in X_h^0$. For a grid function $y_h:\{a_h,\ldots,b_h-\nu h\}\to\mathbb{R}$ let us define the summation operator Σ (cf. [4]) by

$$(\Sigma y_h)(a_h)=0$$

$$(\Sigma y_h)(t)=h\sum_{\substack{\tau\in J_h\\ \tau\leqq t-h}} y_h(\tau) \quad \text{for} \quad t=a_h+h,\ldots,b_h-(\nu-1)h \tag{3.6}$$

and the weighted norm

$$\|y_h\|_{\Sigma,s}=\|\Sigma y_h\|_{w_s} \tag{3.7}$$

with $\|\ \|_{w_s}$ as given in Theorem 3.1. Note that $\|\ \|_{\Sigma,s}$ is a norm weaker than any discrete L_p-norm ($1\leqq p\leqq\infty$).

THEOREM 3.2 (Consistency) *Let (B), (C) and D1 hold, then*

$$\operatorname*{Max}_{s\in J_h^*}\{\|L_h[G(\cdot,s)]_h-\delta_s^h\|_{\Sigma,s}+\|R_h[G(\cdot,s)]_h\|_0\}\to 0 \tag{3.8}$$

$$\text{as} \quad h\to 0.$$

THEOREM 3.3 (Stability) *Assume (A), (B), (C) and (D), then there exist* $C>0$, $h_0>0$ *such that*

$$\|x_h\|_{k-1,s}\leqq C\{\|L_hx_h\|_{\Sigma,s}+\|R_hx_h\|_0\} \tag{3.9}$$

holds for all $x_h\in X_h$, $s\in J_h^*$, $h\leqq h_0$.

Theorem. 3.1 now follows from Theorems 3.2 and 3.3 when (3.9) is applied to $x_h=[G(\cdot,s)]_h-G_h(\cdot,s)$.

4. PROOF OF THE CONSISTENCY THEOREM

Our first auxiliary lemma shows that the discretization error of the difference operators P_h, M_h, R_h in (1.2) tends to zero uniformly for a certain class of functions.

Let us write $x\in \bar{C}^j(\Omega)$ for real functions

$$x:[\bar{a},\bar{b}]\to\mathbb{R} \quad \text{with} \quad x|_\Omega\in C^j(\Omega),$$

where Ω is a finite union of closed subintervals of $[\bar{a},\bar{b}]$ and let

$$\|x\|_{j,\Omega}=\sum_{v=0}^{j}\sup\{|x^{(v)}(t)|:t\in\Omega\},\ \|x\|_j=\|x\|_{j,[\bar{a},\bar{b}]}. \tag{4.1}$$

Usually there should be no confusion with the corresponding norms for grid functions as defined in Section 2.

LEMMA 4.1 *Let (B), (C) and D1 be satisfied. Then there exists a null sequence ρ_h $(h\in H)$ and $C>0$, $\tilde{N}\in\mathbb{N}$ such that*

$$|Mx(t)-M_h[x]_h(t)|\leqq\rho_h\|x\|_{k,[t-Kh,t+Kh]}$$

$$\forall x\in\bar{C}^k[t-Kh,t+Kh],\ t\in J_h^0,\ h\in H,\ \textit{with } K \textit{ from } C2,$$

$$\|Rx-R_h[x]_h\|\leqq\rho_h\|x\|_{k,\Omega_h}\ \forall x\in\bar{C}^k(\Omega_h),\ h\in H \tag{4.2}$$

where $\Omega_h=[\mathrm{Min}(a,a_h),\ \mathrm{Max}(a,a_h)+Nh]\cup$

$$[\mathrm{Min}(b,b_h)-Nh,\ \mathrm{Max}(b,b_h)]\ \textit{with } N \textit{ from } B2, \tag{4.3}$$

$$|x^{(k)}(t)-P_hE^{-k_1}\Delta^k[x]_h(t)|\leqq C\sup_{|\tau-t|\leqq\tilde{N}h}|x^{(k)}(t)-x^{(k)}(\tau)|$$

$$\forall x\in\bar{C}^k[t-\tilde{N}h,t+\tilde{N}h],\ t\in J_h^0,\ h\in H. \tag{4.4}$$

Proof Let us first prove

$$\|[Mx]_h^0-M_h[x]_h\|_0\leqq\rho_h\|x\|_k \quad \forall x\in C^k[\bar{a},\bar{b}],\ h\in H. \tag{4.5}$$

(4.2) will then follow from C2 and from an application of (4.5) to a smooth extension of $x \in \bar{C}^k[t-Kh, t+Kh]$.

Let us assume on the contrary to (4.5) that there exists a subsequence $H' \subset H$ and a sequence $x^h \in C^k[\bar{a}, \bar{b}], h \in H'$, such that

$$\|x^h\|_k = 1, \|[Mx^h]_h^0 - M_h[x^h]_h\|_0 \geqq C > 0 \quad \forall h \in H'.$$

Then we have

$$\|x - x^h\|_{k-1} \to 0 \ (h \in H'' \subset H') \text{ for some } x \in C^{k-1}[\bar{a}, \bar{b}]$$

and by C1, C2

$$\|M_h[x^h]_h - [Mx^h]_h^0\|_0 \leqq \|M_h[x^h - x]_h\|_0$$

$$+ \|M_h[x]_h - [Mx]_h^0\|_0 + \|[M(x - x^h)]_h^0\|_0$$

$$\leqq C\{\|[x^h - x]_h\|_{k-1} + \|M_h[x]_h - [Mx]_h^0\| + \|x - x^h\|_{k-1}\} \to 0$$

a contradiction.

(4.3) follows along the same lines from B1, B2.

The proof of (4.4) will only be given for $t \in \{a_h^0 + \alpha h, \ldots, b_h^0 - \bar{\alpha} h\}$. With suitable η_i, $|t - \eta_i| \leqq \tilde{N} h$ we obtain from D1

$$|x^k(t) - P_h E^{-k_1} \Delta^k [x]_h(t)| = |x^{(k)}(t) - \sum_{i=-\alpha}^{\bar{\alpha}} \xi_i x^{(k)}(\eta_i)|$$

$$\leqq \sum_{i=-\alpha}^{\bar{\alpha}} |\xi_i| \; |x^{(k)}(t) - x^{(k)}(\eta_i)|. \qquad \text{Q.E.D.}$$

The following lemma is an easy consequence of the mean value property of Δ^i. Note that the norm (4.1) can be extended in an obvious way to C^{j-1} functions x, for which $x^{(j-1)}$ is piecewise continuously differentiable.

LEMMA 4.2 *Let $x \in C^{j-1}[\bar{a}, \bar{b}] (j \geqq 1)$ and $x^{(j-1)}$ be piecewise continuously differentiable, then*

$$\|[x]_h\|_j \leqq j\|x\|_j. \tag{4.6}$$

Proof of (3.8) Using Lemma 4.1 and (2.4) we find

$$\operatorname*{Max}_{s\in J_h^*}\|R_h[G(\cdot,s)]_h\|_0=\operatorname*{Max}_{s\in J_h^*}\|R_h[G(\cdot,s)]_h-RG(\cdot,s)\|_0$$

$$\leqq \rho_h \sup_{s\in J_h^*}\|G(\cdot,s)\|_{k,\Omega_h}\to 0 \quad \text{as} \quad h\to 0,$$

since $\Omega_h \cap J_h^* = \emptyset$ for sufficiently small $h\in H$.

Let $v_h^s = L_h[G(\cdot,s)]_h - \delta_s^h \in X_h^0$, then there exists $\hat{N}\in\mathbb{N}$ such that

$$\operatorname*{Max}_{s\in J_h^*}\|v_h^s\|_{0,J_h^s}\to 0 \quad \text{as} \quad h\to 0,\ J_h^s = J_h^0\backslash[s-\hat{N}h,\ s+\hat{N}h]. \tag{4.7}$$

This follows from Lemma 4.1, (2.4) and the estimate

$$\|v_h^s\|_{0,J_h^s}=\|L_h[G(\cdot,s)]_h-[LG(\cdot,s)]_h^0\|_{0,J_h^s}$$

$$\leqq \|P_h E^{-k_1}\Delta^k[G(\cdot,s)]_h-[\partial_t^k G(\cdot,s)]_h^0\|_{0,J_h^s}$$

$$+\|M_h[G(\cdot,s)]_h-[MG(\cdot,s)]_h^0\|_{0,J_h^s}.$$

Using the behaviour of w_s our assertion will be a consequence of

$$\|\Sigma v_h^s\|_{0,J_h^s}\to 0,\quad \|\Sigma v_h^s\|_{0,\{s-\hat{N}h,\ldots,s+\hat{N}h\}}\leqq C \tag{4.8}$$

uniformly for $s\in J_h^*$.

First, note that $(\Sigma v_h^s)(t)\to 0$ uniformly for $t\leqq s-\hat{N}h$ as follows from (4.7). If $|\tau-s|\leqq \hat{N}h$ then (C), (D) and Lemma 4.2 yield

$$|v_h^s(\tau)|\leqq h^{-1}+\|L_h[G(\cdot,s)]_h\|_0\leqq h^{-1}+C\|[G(\cdot,s)]_h\|_k$$

$$\leqq h^{-1}(1+C\|[G(\cdot,s)]_h\|_{k-1})\leqq h^{-1}(1+C\|G(\cdot,s)\|_{k-1}).$$

Hence for $|t-s|\leqq \hat{N}h$

$$|\Sigma v_h^s(t)|\leqq|\Sigma v_h^s(s-\hat{N}h)|+h\sum_{\tau=s-\hat{N}h}^{t-h}|v_h^s(\tau)|\leqq C.$$

Finally, the convergence $(\Sigma v_h^s)(t)\to 0$ uniformly for $t>s+\hat{N}h$ follows from (4.7) and the above estimates, if

$$\operatorname*{Max}_{s\in J_h^*} h\left|\sum_{\tau=s-\hat{N}h}^{s+\hat{N}h} v_h^s(\tau)\right|\to 0 \quad \text{as} \quad h\to 0 \tag{4.9}$$

holds. For the proof of (4.9) use D1 to obtain

$$\left|h\sum_{\tau=s-\hat{N}h}^{s+\hat{N}h} v_h^s(\tau)\right| \leqq h\sum_{\tau=s-\hat{N}h}^{s+\hat{N}h} \left|M_h[G(\cdot,s)]_h(\tau)\right|$$

$$+\left|\sum_{i=-\alpha}^{\bar{\alpha}} \xi_i\left\{h\sum_{\tau=s-\hat{N}h}^{s+\hat{N}h} \Delta^k G(\tau+(i-k_1)h,s)-1\right\}\right|.$$

The first sum tends to zero as follows from (C) and Lemma 4.2. For the second one we use the jump relation from (2.4) and

$$h\sum_{\tau=s-\hat{N}h}^{s+\hat{N}h} \Delta^k G(\tau+(i-k_1)h,s)=\Delta^{k-1}G(s+(\hat{N}+i-k_1+1)h,s)$$

$$-\Delta^{k-1}G(s-(\hat{N}-i+k_1)h,s). \qquad \text{Q.E.D.}$$

5. PROOF OF THE STABILITY THEOREM

Let us first summarize some known stability inequalities from [2, 9, 11].

LEMMA 5.1 *Under the assumptions (A)–(D) the following stability inequalities hold for sufficiently small $h\in H$*

$$\|x_h\|_k \leqq C(\|L_h x_h\|_0+\|R_h x_h\|_0) \quad \forall x_h\in X_h, \tag{5.1}$$

$$\|x_h\|_{k-1} \leqq C(\|L_h x_h\|_{L_1}+\|R_h x_h\|_0) \quad \forall x_h\in X_h, \tag{5.2}$$

where

$$\|y_h\|_{L_1}=h\sum_{\tau\in J_h^0}|y_h(\tau)|, \quad y_h\in X_h^0.$$

Our proof of (3.9) is a successive reduction to a problem for difference operators with constant coefficients which can be dealt with explicitly.

I. Reduction to initial conditions

Let $R_h^0 x_h=(\Delta^j x_h(a_h): j=0,\dots,k-1)$, then (3.9) follows from the stability inequality

$$\|x_h\|_{k-1,s} \leqq C(\|L_h x_h\|_{\Sigma,s}+\|R_h^0 x_h\|_0) \quad \forall x_h\in X_h,\ s\in J_h^*,\ h\leqq h_0. \tag{5.3}$$

Proof Each $x_h\in X_h$ may be written as $x_h=y_h+z_h$, where y_h, z_h are defined by

$$L_h y_h = L_h x_h,\ R_h^0 y_h = 0$$

$$L_h z_h = 0,\ R_h z_h = R_h(x_h - y_h).$$

Now (5.3) yields

$$\|x_h\|_{k-1,s} \leqq \|y_h\|_{k-1,s} + \|z_h\|_{k-1,s} \leqq C\|L_h x_h\|_{\Sigma,s} + \|z_h\|_{k-1}$$

and by Lemma 5.1 and B2 we obtain

$$\|z_h\|_{k-1} \leqq C(\|R_h x_h\|_0 + \|y_h\|_{k-1,\Gamma_h})$$
$$\leqq C(\|R_h x_h\|_0 + \|y_h\|_{k-2} + \|\Delta^{k-1} y_h\|_{w_s} \|w_s\|_{0,\Gamma_h}),$$

where $\Gamma_h = \{a_h, \ldots, a_h + Nh,\ b_h - Nh, \ldots, b_h\}$.
An elementary calculation using (2.1) and (3.3) shows

$$\|w_s\|_{0,\Gamma_h} \leqq C \quad \forall s \in J_h^*,\ h \in H,$$

which proves our assertion.

II. Reduction to the principal term

(5.3) is a consequence of

$$\|v_h\|_{\Sigma,s} \leqq C\|P_h v_h\|_{\Sigma,s}\ \forall v_h \in X_h^0,\ s \in J_h^*,\ h \leqq h_0. \tag{5.4}$$

Proof We shall use the following properties of the weight function w_s:

$$w_s(t) \leqq \sigma^{-n} w_s(\tau)\ \forall t, \tau \in J_h,\ |\tau - t| \leqq nh,\ s \in J_h^*,\ h \in H, \tag{5.5}$$

$$\|\Sigma w_s\|_0 \leqq \|w_s\|_{L_1} \leqq C\ \forall s \in J_h^*,\ h \in H. \tag{5.6}$$

Let us again write $x_h \in X_h$ as $x_h = y_h + z_h$ where

$$P_h E^{-k_1} \Delta^k y_h = L_h x_h,\ R_h^0 y_h = 0$$

$$L_h z_h = -M_h y_h,\ R_h^0 z_h = R_h^0 x_h. \tag{5.7}$$

By (5.4), (5.5), (5.6) we have the estimates

$$\|\Delta^{k-1}y_h\|_{w_s} \leqq C\|E^{-k_1}\Delta^k y_h\|_{\Sigma,s} \leqq C\|L_h x_h\|_{\Sigma,s},$$

$$\|\Delta^{k-2}y_h\|_0 = \|\Sigma\Delta^{k-1}y_h\|_0 \leqq \|\Delta^{k-1}y_h\|_{w_s}\|\Sigma w_s\|_0$$

$$\leqq C\|\Delta^{k-1}y_h\|_{w_s}$$

and by successive summation $\|y_h\|_{k-1,s} \leqq C\|L_h x_h\|_{\Sigma,s}$. Moreover, using Lemma 5.1 and C2 we obtain

$$\|z_h\|_{k-1} \leqq C(\|M_h y_h\|_{L_1} + \|R_h^0 x_h\|_0)$$

$$\leqq C(\|y_h\|_{k-1,s}\|w_s\|_{L_1} + \|R_h^0 x_h\|_0) \leqq C(\|y_h\|_{k-1,s} + \|R_h^0 x_h\|_0).$$

Combining the estimates for y_h and z_h gives the desired result.

III. Stability of the principal term

The method of proof of (5.4) follows [15] and the techniques are quite similar—although more involved here—to those in [4, Theorem 5] for different weight functions. Let us introduce complex valued grid functions and identify J_h^0 and $\{0,\ldots,n\}$, X_h^0 and $\mathbb{R}^{n+1}$, P_h and the real part of the complex operator

$$(P_n x)(j) = \begin{cases} q_j(E)x(0), \; j=0,\ldots,\alpha-1 \\ P(E)x(j), \;\; j=\alpha,\ldots,n-\bar{\alpha}, \\ \bar{q}_{n-j}(E^{-1})x(n), \;\; j=n-\bar{\alpha}+1,\ldots,n \end{cases} \qquad x\in\mathbb{C}^{n+1},\ n\in\mathbb{N}.$$

Now $s\in J_h^*$ is of the form

$$s = a_h^0 + i_s h \text{ where } i_s, n-i_s \geqq C\ln(h^{-1}).$$

Moreover $w_s(i) = 1 + h^{-\kappa}\sigma^{|i-i_s|}$ and for $x\in\mathbb{C}^{j+1}$

$$\|x\|_{w_s} = \mathrm{Max}\left\{\frac{|x(i)|}{w_s(i)} : i=0,\ldots,j\right\},$$

$$\Sigma x(i) = h\sum_{j=0}^{i-1} x(j) \quad \text{for} \quad i=1,\ldots,j+1, \quad \Sigma x(0)=0.$$

Our proof is divided into three steps.

S1:

$\forall r \in \mathbb{C}^{n+1}\ \exists x \in \mathbb{C}^{n+1}$ s.t. $P_n x(j) = r(j)$ for $j = \alpha, \ldots, n-\bar{\alpha}$

and $\|x\|_{\Sigma,s} \leqq C\|r\|_{\Sigma,s}$.

S2:

$$\|P_n x\|_{\Sigma,s} \leqq C\|x\|_{\Sigma,s}\ \forall x \in \mathbb{C}^{n+1}.$$

S3:

$\forall r \in \mathbb{C}^{n+1}$ with $r(j)=0$ $(j=\alpha,\ldots,n-\bar{\alpha})$ there exists $x \in \mathbb{C}^{n+1}$ such that $P_n x = r$ and $\|x\|_{\Sigma,s} \leqq C\|r\|_{\Sigma,s}$.

These properties hold for $n \geqq n_0$ and the constants n_0 and C are independent of $s \in J_h^*$.

Proof of (5.4) For a given $x \in \mathbb{C}^{n+1}$ determine $y \in \mathbb{C}^{n+1}$ by S1 such that $P_n y = P_n x$ in $\{\alpha,\ldots,n-\bar{\alpha}\}$ and $z \in \mathbb{C}^{n+1}$ by S2 such that $P_n z = P_n(x-y)$.
Since P_n is invertible for n large ([2, Satz 3]) we have $z = x - y$ and

$$\|x\|_{\Sigma,s} \leqq \|y\|_{\Sigma,s} + \|z\|_{\Sigma,s} \leqq C(\|P_n x\|_{\Sigma,s} + \|P_n(x-y)\|_{\Sigma,s})$$

$$\leqq C(\|P_n x\|_{\Sigma,s} + \|y\|_{\Sigma,s}) \leqq C\|P_n x\|_{\Sigma,s}.$$

Proof of S1 By property (D) there exists a factorization

$$P(E) = cE^{\bar{\alpha}-\alpha} \prod_{i=1}^{d} (E - \sigma_i I)^{v_i} \prod_{i=1}^{\bar{d}} (\bar{\sigma}_i I - E^{-1})^{\bar{v}_i},\ c \neq 0.$$

Therefore, it is sufficient to prove that for any $g \in \mathbb{C}^{\{\eta,\ldots,n-\bar{\eta}\}}$ where $\eta, \bar{\eta} \in \mathbb{N}$, $\eta \leqq \alpha$, $\bar{\eta} \leqq \bar{\alpha}$ and any

$$\lambda \in \{\sigma_i (i=1,\ldots,d),\ \bar{\sigma}_j (j=1,\ldots,\bar{d})\}$$

the following implications hold

$$(E - \lambda I)y = g,\ y(\eta) = 0 \Rightarrow \|y\|_{\Sigma,s} \leqq C\|g\|_{\Sigma,s}, \tag{5.8}$$

$$(\lambda I - E^{-1})z = g,\ z(n-\bar{\eta}) = 0 \Rightarrow \|z\|_{\Sigma,s} \leqq C\|g\|_{\Sigma,s}. \tag{5.9}$$

The grid function $u=\Sigma y$ satisfies

$$(E-\lambda I)u=\Sigma g, \quad u(\eta)=0.$$

Hence, with the notation $\Sigma_{\varnothing}=0$ we obtain from (3.3)

$$|u(i)|=\left|\sum_{j=\eta}^{i-1}\lambda^{i-1-j}(\Sigma g)(j)\right|\leqq\|g\|_{\Sigma,s}\sum_{j=\eta}^{i-1}|\lambda|^{i-1-j}w_s(j)$$

$$\leqq\|g\|_{\Sigma,s}\left\{\sum_{j=\eta}^{i-1}\sigma^{i-1-j}+h^{-\kappa}\left(\sum_{j=\eta}^{\mathrm{Min}(i,i_s)-1}\sigma^{i_s+i-1-2j}\right.\right.$$

$$\left.\left.+\sum_{j=i_s}^{i-1}\sigma^{j-i_s}|\lambda|^{i-1-j}\right)\right\}\leqq\|g\|_{\Sigma,s}\left\{C+h^{-\kappa}\sigma^{|i-i_s|}\right.$$

$$\left.\left(\sum_{j=\eta}^{\mathrm{Min}(i,i_s)-1}\sigma^{2(\mathrm{Min}(i,i_s)-j)-1}+\sigma^{-1}\sum_{j=i_s}^{i-1}\left(\frac{|\lambda|}{\sigma}\right)^{i-1-j}\right)\right\}$$

$$\leqq C\|g\|_{\Sigma,s}w_s(i) \quad \text{and}$$

$$\|y\|_{\Sigma,s}=\|u\|_{w_s}\leqq C\|g\|_{\Sigma,s}.$$

Now let $u=\Sigma z$ where z is defined by (5.9). By a change of summation we find for $i=\eta,\ldots,n-\bar{\eta}+1$

$$u(i)=-h\sum_{j=\eta-1}^{i-1}\sum_{v=j+1}^{n-\bar{\eta}}\lambda^{v-j-1}g(v)=\frac{1}{\lambda-1}\{\Sigma g(i)+\hat{g}(i)-\lambda\hat{g}(\eta)\}$$

where $\hat{g}(i)=h\sum_{v=i}^{n-\bar{\eta}}\lambda^{v-i}g(v)$

and by partial summation

$$|\hat{g}(i)|\leqq C\{|\Sigma g(n-\bar{\eta}+1)|+|\Sigma g(i)|+\sum_{v=i+1}^{n-\bar{\eta}}|\lambda|^{v-1-i}|\Sigma g(v)|\}.$$

For the last sum, a calculation very similar to the proof of (5.8) leads to the bound $C\|g\|_{\Sigma,s}w_s(i)$. Collecting the various terms we end up with

$$|\Sigma z(i)|=|u(i)|\leqq C\|g\|_{\Sigma,s}(w_s(i)+w_s(\eta)+w_s(n-\bar{\eta}+1)).$$

Since $w_s(\eta)+w_s(n-\bar{\eta}+1)\leqq C$ holds uniformly in $s\in J_h^*$ (cf. (3.3)) our assertion follows.

Proof of S2 Let $x\in\mathbb{C}^{n+1}$ and consider the case $j\in\{0,\dots,n-\bar{\alpha}+1\}$. For $\bar{N}\in\mathbb{N}$ sufficiently large we obtain from D1 and (5.5)

$$|\Sigma P_n x(j)|\leqq Ch\underset{i=0,\dots,\bar{N}}{\mathrm{Max}}|x(i)|+h\left|\sum_{\mu=-\alpha}^{\bar{\alpha}}\xi_\mu\sum_{i=\alpha}^{j-1}x(i+\mu)\right|$$

$$\leqq Ch\underset{i=0,\dots,\bar{N}}{\mathrm{Max}}|x(i)|+\sum_{\mu=-\alpha}^{\bar{\alpha}}|\xi_\mu|(|\Sigma x(j+\mu)|+|\Sigma x(\alpha+\mu)|)$$

$$\leqq C\left\{\underset{i=0,\dots,\bar{N}}{\mathrm{Max}}|\Sigma x(i)|+\|\Sigma x\|_{w_s}\sum_{\mu=-\alpha}^{\bar{\alpha}}w_s(j+\mu)\right\}$$

$$\leqq C\|x\|_{\Sigma,s}\{w_s(\bar{N})+w_s(j)\}\leqq C\|x\|_{\Sigma,s}w_s(j).$$

Using this result with $j=n-\bar{\alpha}+1$ we can estimate in the case $j\in\{n-\bar{\alpha}+2,\dots,n\}$ as follows:

$$|\Sigma P_n x(j)|\leqq C\left\{\|x\|_{\Sigma,s}w_s(n-\bar{\alpha}+1)+h\underset{i=n-\bar{N},\dots,n}{\mathrm{Max}}|x(i)|\right\}$$

$$\leqq C\left\{\|x\|_{\Sigma,s}w_s(j)+2\|\Sigma x\|_{w_s}\underset{i=n-\bar{N},\dots,n}{\mathrm{Max}}w_s(i)\right\}$$

$$\leqq C\|x\|_{\Sigma,s}w_s(j).$$

Proof of S3 By [4, Theorem 5] we have the stability inequalities

$$\|x\|_e\leqq C\|P_n x\|_e,\ \|x\|_{\tilde{e}}\leqq C\|P_n x\|_{\tilde{e}}\ \forall x\in\mathbb{C}^{n+1},\ n\geqq n_0 \tag{5.10}$$

with the weight functions $e(i)=\sigma^{n-i}$, $\tilde{e}(i)=\sigma^i$.

Given $r\in\mathbb{C}^{n+1}$, $r(j)=0(j=\alpha,\dots,n-\bar{\alpha})$ let us write $r=r^0+r^1$

where $r^0(j)=\begin{cases}r(j), & j=0,\dots,\alpha-1\\ 0 & \text{otherwise}\end{cases}$

Let $P_n x^i=r^i(i=0,1)$, then (5.10) yields

$$\|x^0\|_{\tilde{e}}\leqq C\|r^0\|_{\tilde{e}}\leqq Ch^{-1}\underset{i=0,\dots,\alpha}{\mathrm{Max}}|\Sigma r(i)|$$

$$\leqq Ch^{-1}\|r\|_{\Sigma,s}w_s(\alpha)\leqq Ch^{-1}\|r\|_{\Sigma,s}$$

and similarly $\|x^1\|_e \leqq Ch^{-1}\|r\|_{\Sigma,s}$

Now $x=x^0+x^1$ satisfies $P_n x=r$ and

$$\|x\|_{\Sigma,s} \leqq \|\Sigma x^0\|_{w_s} + \|\Sigma x^1\|_{w_s} \leqq \|\Sigma x^0\|_0 + \|\Sigma x^1\|_0$$

$$\leqq \|x^0\|_e \|\Sigma e\|_0 + \|x^1\|_{\bar{e}} \|\Sigma \bar{e}\|_0 \leqq hC(\|x^0\|_e + \|x^1\|_{\bar{e}}) \leqq C\|r\|_{\Sigma,s}.$$

Q.E.D.

6. CONVERGENCE OF OPERATOR NORMS AND NUMERICAL EXAMPLES

Let us assume for simplicity that

$$J_h^0 \subset [a,b] \; \forall h \in H. \tag{6.1}$$

(6.1) is very natural for common finite difference schemes, since J_h^0 contains the points at which the differential equation in (1.1) is discretized.

Because of (6.1) we may define a $|J_h| \times (|J_h| - k)$-matrix $[G]_h$ by

$$[G]_h(t,s) = G(t,s), \quad t \in J_h, \quad s \in J_h^0.$$

Our first conclusion from Theorem 3.1 is

THEOREM 6.1 *Let (6.1) and (A)–(D) be satisfied. For $j \in \{0,\dots,k-1\}$ let $F_h: X_h \to X_h$ be linear, consistent with ∂^j and locally $\|\ \|_j$-bounded (cf. C2).*

Then the convergence

$$\|F_{h,t}(hG_h) - h[\partial_t^j G]_h\|_0 \to 0 \quad \text{as} \quad h \to 0 \tag{6.2}$$

holds with respect to the operator norm $\|\ \|_0$. Moreover, in the case $j \leqq k-2$ we have for any fixed $s \in (a,b)$ with $s \in J_h^0$ for a.e. $h \in H$

$$\|F_{h,t}G_h(\cdot,s) - [\partial_t^j G(\cdot,s)]_h\|_0 \to 0 \quad \text{as} \quad h \to 0. \tag{6.3}$$

Remark $F_{h,t}$ indicates that F_h is applied with respect to the first variable t.

Proof The proof of (6.3) follows immediately from our corollary in

Section 3 and the estimate

$$\|F_{h,t}G_h(\cdot,s)-[\partial_t^j G(\cdot,s)]_h\|_0 \leqq \|F_{h,t}(G_h(\cdot,s)-[G(\cdot,s)]_h)\|_0$$
$$+\|F_{h,t}[G(\cdot,s)]_h-[\partial_t^j G(\cdot,s)]_h\|_0 \leqq C\|G_h(\cdot,s)-[G(\cdot,s)]_h\|_{k-2}$$
$$+\|F_{h,t}[G(\cdot,s)]_h-[\partial_t^j G(\cdot,s)]_h\|_0.$$

For the proof of (6.2) we use the grid J_h^* from Theorem 3.1 and find

$$\|F_{h,t}(hG_h)-h[\partial_t^j G]_h\|_0 \leqq \operatorname*{Max}_{t\in J_h}\Big\{h\sum_{s\in J_h^0}|\partial_t^j G(t,s)-F_{h,t}[G]_h(t,s)|\Big\}$$
$$+\operatorname*{Max}_{t\in J_h}\Big\{h\sum_{s\in J_h^*}|F_{h,t}([G]_h-G_h)(t,s)|\Big\}$$
$$+h\sum_{s\in J_h^0\setminus J_h^*}\|F_{h,t}([G]_h-G_h)(\cdot,s)\|_0.$$

For the first term, Lemma 4.1 and Lemma 4.2 yield for some $\bar{K}>0$:

$$\leqq \operatorname*{Max}_{t\in J_h}\Big\{h\rho_h\sum_{\substack{s\in J_h^0\\ |s-t|>\bar{K}h}}\|G(\cdot,s)\|_{j+1,[t-\bar{K}h,t+\bar{K}h]}$$
$$+Ch\sum_{\substack{s\in J_h^0\\ |s-t|\leqq\bar{K}h}}\|G(\cdot,s)\|_j\Big\}\leqq C\rho_h+Ch\to 0.$$

Using (5.5), (5.6) and Theorem 3.1 we can estimate the second term by

$$C\operatorname*{Max}_{t\in J_h}\Big\{h\sum_{s\in J_h^*}\|([G]_h-G_h)(\cdot,s)\|_{j,J_h\cap[t-\bar{K}h,t+\bar{K}h]}\Big\}$$
$$\leqq C\operatorname*{Max}_{s\in J_h^*}\|([G]_h-G_h)(\cdot,s)\|_{k-1,s}\operatorname*{Max}_{t\in J_h}\Big\{h\sum_{s\in J_h^*}w_s(t)\Big\}$$
$$\leqq C\operatorname*{Max}_{s\in J_h^*}\|([G]_h-G_h)(\cdot,s)\|_{k-1,s}\to 0 \quad \text{as} \quad h\to 0.$$

As for the last term, note that

$\|G_h(\cdot,s)\|_{k-1}\leqq C\|\delta_s^h\|_{L_1}=C$ from Lemma 5.1 and (1.8) and use Lemma 4.2 to obtain

$$\leqq Ch \sum_{s \in J_h^0 \setminus J_h^*} \|([G]_h - G_h)(\cdot, s)\|_j$$

$$\leqq Ch \sum_{s \in J_h^0 \setminus J_h^*} \{\|G(\cdot, s)\|_j + \|G_h(\cdot, s)\|_j\} \leqq Ch \ln(h^{-1}) \to 0. \qquad \text{Q.E.D.}$$

We will now show the convergence of the operator norms. For that purpose we introduce the operators (cf. (1.4), (1.5))

$$A: C[a,b] \to C^k[a,b],\ Ar(t) = \int_a^b G(t,s) r(s) ds,\ t \in [a,b],$$

$$A_h: X_h^0 \to X_h,\ A_h r_h(t) = h \sum_{s \in J_h^0} G_h(t,s) r_h(s),\ t \in J_h$$

$$B: C[a,b] \times \mathbb{R}^k \to C^k[a,b],\ B(r,\gamma) = Ar + T\gamma$$

$$B_h: X_h^0 \times \mathbb{R}^k \to X_h,\ B_h(r_h, \gamma) = A_h r_h + T_h \gamma$$

where $T\gamma = \sum_{i=1}^k \gamma_i z_i$, $T_h \gamma = \sum_{i=1}^k \gamma_i z_i^h$ and

$$Lz_i = 0 \text{ in } [a,b],\ Rz_i = u^i \ (i\text{-th unit vector in } \mathbb{R}^k), \tag{6.4}$$

$$L_h z_i^h = 0 \text{ in } J_h^0,\ R_h z_i^h = u^i. \tag{6.5}$$

B and B_h are the solution operators for the inhomogeneous equations (1.1), (1.2) with $Rx = \gamma$ and $R_h x_h = \gamma$.

Let us further introduce the norms

$$\|A_h\|_{0,j} = \sup_{\|r_h\|_0 = 1} \|A_h r_h\|_j$$

$$\|B_h\|_{0,j} = \sup_{\|r_h\|_0 + \|\gamma\|_0 = 1} \|B_h(r_h, \gamma)\|_j, \quad j = 0, \ldots, k$$

and $\|A\|_{0,j}$, $\|B\|_{0,j}$ analogously.

THEOREM 6.2 *Under the assumptions of Theorem* 6.1 *we have for each* $j \in \{0, \ldots, k-1\}$

$$\|A_h\|_{0,j} \to \|A\|_{0,j} \quad \text{as} \quad h \to 0, \tag{6.6}$$

$$\|B_h\|_{0,j} \to \|B\|_{0,j} \quad \text{as} \quad h \to 0. \tag{6.7}$$

Proof

$$\big|\, \|A_h\|_{0,j} - \|A\|_{0,j} \big| \leqq \big|\, \|A_h\|_{0,j} - \|h[G]_h\|_{0,j} \big|$$

$$+ \big|\, \|h[G]_h\|_{0,j} - \|A\|_{0,j} \big| \leqq \| hG_h - h[G]_h\|_{0,j} + \big|\, \|h[G]_h\|_{0,j} - \|A\|_{0,j} \big|.$$

For the first term we find

$$= \sup_{\|r_h\|_0 = 1} \left\{ \sum_{\nu=0}^{j} \operatorname{Max}_{t \in J_h} h \left| \sum_{s \in J_h^0} \Delta_t^\nu (G_h - [G]_h)(t,s) r_h(s) \right| \right\}$$

$$\leqq \sum_{\nu=0}^{j} \operatorname{Max}_{t \in J_h} \left\{ h \sum_{s \in J_h^0} |\Delta_t^\nu (G_h - [G]_h)(t,s)| \right\} \to 0$$

by using the last estimates from the proof of Theorem 6.1 with $F_h = \Delta^\nu$.

For the last term

$$\lim_{h \to 0} (\|h[G]_h\|_{0,j} - \|A\|_{0,j}) = 0$$

follows by straightforward but nontrivial estimates which will be omitted here.

Next we prove

$$\lim_{h \to 0} \|T_h\|_{0,j} = \|T\|_{0,j} \tag{6.8}$$

which by the use of

$$\|B\|_{0,j} = \operatorname{Max}\{\|A\|_{0,j}, \|T\|_{0,j}\}, \ \|B_h\|_{0,j} = \operatorname{Max}\{\|A_h\|_{0,j}, \|T_h\|_{0,j}\}$$

yields our assertion.

Now (6.5) is a stable and consistent discretization of (6.4), hence

$$\|z_i^h - [z_i]_h\|_k \to 0 \quad \text{as} \quad h \to 0 \quad (i = 1, \ldots, k). \tag{6.9}$$

Let $\gamma \in \mathbb{R}^k$, $\|\gamma\|_0 = 1$, then (2.1) and (6.9) show

$$\big|\, \|T\gamma\|_j - \|T_h \gamma\|_j \big| \leqq \sum_{\nu=0}^{j} \left\{ \sum_{i=1}^{k} \sup_{\substack{t,s \in [\bar{a},\bar{b}] \\ |t-s| \leqq (k+C)h}} |z_i^{(\nu)}(t) - z_i^{(\nu)}(s)| \right.$$

$$\left. + k \operatorname{Max}_{i=1,\ldots,k} \|[z_i]_h - z_i^h\|_j \right\} \to 0$$

and hence (6.8). Q.E.D.

The convergence of operator norms (6.7) may be interpreted as the convergence of stability constants, i.e. if C^* denotes the smallest constant C such that

$$\|x\|_j \leqq C(\|Lx\|_0 + \|Rx\|_0) \quad \forall x \in C^k[a,b]$$

and C_h^* denotes the smallest constant C_h such that

$$\|x_h\|_j \leqq C_h(\|L_h x_h\|_0 + \|R_h x_h\|_0) \quad \forall x_h \in X_h$$

then $C_h^* \to C^*$ as $h \to 0$.

This gives a stronger result than usual stability inequalities which only state that C_h^* is bounded as $h \to 0$.

Theorem 6.2 also yields the convergence of Lipschitz constants for nonlinear boundary value problems. For example, let us consider the nonlinear problem

$$Lx = f(\cdot, x, \ldots, x^{(j)}) =: Fx \text{ in } [a,b], \; Rx = 0 \tag{6.8}$$

where L, R are defined as in (1.1) and $f \in C([a,b] \times \mathbb{R}^{j+1})$ satisfies

$$|f(t, u_0, \ldots, u_j) - f(t, v_0, \ldots, v_j)| \leqq K \sum_{i=0}^{j} |u_i - v_i|$$

for all $t \in [a,b]$, $u_i, v_i \in \mathbb{R}$.

Then (6.8) may be written as an operator equation

$$x = AFx, \; x \in C^j[a,b],$$

where AF has a Lipschitz constant $K\|A\|_{0,j}$ with respect to $\| \ \|_j$.

Let us assume that (6.8) is discretized as

$$L_h x_h = f(\cdot, D_h^0 x_h, \ldots, D_h^j x_h) =: F_h x_h, \; R_h x_h = 0, \tag{6.9}$$

where $D_h^\nu : X_h \to X_h^0$ are difference operators of the form

$$D_h^\nu x_h(t) = \sum_{i=\rho_1}^{\rho_2} \eta_i \Delta^\nu x_h(t + ih)$$

(ρ_1, ρ_2, η_i may depend on t, h) which satisfy

$$\sum_{i=\rho_1}^{\rho_2} \eta_i = 1, \; \eta_i \geqq 0 \quad (i = \rho_1, \ldots, \rho_2).$$

Under the assumptions of Theorem 6.1 we may then write (6.9) as

$$x_h = A_h F_h x_h,\ x_h \in X_h.$$

It is easily verified that $A_h F_h$ has the Lipschitz constant $K\|A_h\|_{0,j}$ with respect to $\|\ \|_j$.

By Theorem 6.2 we have

$$K\|A_h\|_{0,j} \to K\|A\|_{0,j} \quad \text{as} \quad h \to 0.$$

In particular, let $K\|A\|_{0,j} < 1$ so that (6.8) has a unique solution and the Picard iteration is convergent (for a survey of results of this type in the case $k=2$ see [1,6]), then (6.9) has a unique solution for h sufficiently small. Moreover the discrete Picard iteration is convergent with asymptotically the same rate of convergence. This behaviour is often observed in numerical computations.

It is also obvious how to extend these results to different iteration schemes or to boundary value problems where the contraction is valid only locally or with respect to different norms.

Let us conclude this section with some numerical results. The following tables show the behaviour of the error

$$\varepsilon_h = \|[G(\cdot, \tfrac{1}{2})]_h - G_h(\cdot, \tfrac{1}{2})\|_0$$

which tends to zero by Theorem 6.1. In addition we have computed numerical estimates of the order of convergence

$$\mathrm{Ord}(\varepsilon_h) = \ln\left(\frac{\varepsilon_h}{\varepsilon_{h/2}}\right)\Big/\ln 2.$$

These are not determined by our theorems but illustrate the speed of convergence.

Example 1 $\quad -x'' - 4x = r$ in $[0,1]$, $x(0) = x(1) = 0$.

Our difference schemes are defined by

$$a_h = 0,\ b_h = 1,\ k_1 = k_2 = 1$$

and by the following formulas (the underlined position corresponds to t)

Scheme I $\qquad x''(t) \sim h^{-2}(1, \underline{-2}, 1),$

Scheme II $x''(t) \sim h^{-2}(1, \underline{-2}, 1)$ for $t = h,\ 1-h$

$$x''(t) \sim h^{-2}(-1, 16, \underline{-30}, 16, -1) \text{ for } t = 2h, \ldots, 1-2h.$$

See [7] for an extensive analysis of these schemes.

Our example clearly shows that difference schemes which are of higher order for smooth solutions may give poor results for non smooth solutions—even worse than those produced by a method which is of low order for smooth solutions.

h	Scheme I($0(h^2)$) ε_h	Scheme I($0(h^2)$) Ord (ε_h)	Scheme II($0(h^4)$) ε_h	Scheme II($0(h^4)$) Ord (ε_h)
$\frac{1}{10}$	0.3406E-2	2.01	0.7102E-2	0.98
$\frac{1}{20}$	0.8455E-3	2.00	0.3597E-2	0.99
$\frac{1}{40}$	0.2110E-3	2.00	0.1803E-2	1.00
$\frac{1}{80}$	0.5273E-4	2.00	0.9020E-3	1.00
$\frac{1}{160}$	0.1318E-4		0.4510E-3	

The situation for the higher order scheme is somewhat better for the next example of a fourth order boundary value problem.

Example 2

$$x^{(4)} - x = r \text{ in } [0, 1],\ x(0) = x''(0) = x(1) = x''(1) = 0.$$

Let $a_h = -h$, $b_h = 1 + h$, $k_1 = k_2 = 2$ and consider

Scheme I $x''(t) \sim h^{-2}(1, \underline{-2}, 1)$ for $t = 0,1$

$$x^{(4)}(t) \sim h^{-4}(1, -4, \underline{6}, -4, 1) \text{ for } t = h, \ldots, 1-h,$$

Scheme II $x''(0) \sim (12h^2)^{-1}(10, \underline{-15}, -4, 14, -6, 1)$,

$x^{(4)}(t)$ as in Scheme I for $t = h,\ 1-h$,

$$x^{(4)}(t) \sim (6h^4)^{-1}(-1, 12, -39, \underline{56}, -39, 12, -1) \text{ for } t = 2h, \ldots, 1-2h,$$

$$x''(1) \sim (12h^2)^{-1}(1, -6, 14, -4, \underline{-15}, 10).$$

Scheme II is $0(h^4)$-convergent for C^8-solutions (cf. [4]).

h	Scheme I ε_h	Scheme I Ord (ε_h)	Scheme II ε_h	Scheme II Ord (ε_h)
$\frac{1}{10}$	0.4238E-3	2.00	0.2011E-4	2.89
$\frac{1}{20}$	0.1059E-3	2.00	0.2699E-5	3.00
$\frac{1}{40}$	0.2648E-4	2.00	0.3367E-6	3.00
$\frac{1}{80}$	0.6620E-5	2.00	0.4206E-7	2.99
$\frac{1}{160}$	0.1655E-5		0.5283E-8	

As a conclusion of our theoretical and numerical results we may say that ordinary boundary value problems with δ-functions on the right hand side (for example mechanical systems with point loads) may be well solved by the simplest compact finite difference method. However, it usually makes no sense to employ difference methods which are of higher order for smooth solutions.

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