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Spurious Solutions for Discrete Superlinear Boundary Value Problems

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Abstract — Zusammenfassung

Spurious Solutions for Discrete Superlinear Boundary Value Problems. We consider finite dimensional nonlinear eigenvalue problems of the type $Au=\lambda Fu$ where A is a matrix and $(Fu)_i=f(u_i)$, $i=1,\ldots,m$. These may be thought of as discretizations of a corresponding boundary value problem. We show that positive, spurious solution branches of the discrete equations (which have been observed in some cases in [1, 7]) typically arise if f increases sufficiently strong and if A^{-1} has at least two positive columns of a certain type. We treat in more detail the cases $f(u)=e^u$ and $f(u)=u^\alpha$ where also discrete bifurcation diagrams are given.

Key words: spurious solutions, discretizations, nonlinear eigenvalue problems, superlinear functions.

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Zusätzliche Lösungen von Diskretisierungen superlinearer Randwertaufgaben. Es werden endlich dimensionale, nichtlineare Eigenwertprobleme der Form $Au=\lambda Fu$ mit einer Matrix A und einem Feld $(Fu)_i=f(u_i)$, $i=1,\ldots,m$ betrachtet. Diese können als Diskretisierung eines entsprechenden Randwertproblems angesehen werden. Wir zeigen, daß diese diskreten Gleichungen dann zusätzliche, positive Lösungszweige (welche in [1, 7] beobachtet wurden) aufweisen, wenn f hinreichend stark wächst und A^{-1} mindestens zwei positive Spalten von einem bestimmten Typ besitzt. Ausführlicher werden die Fälle $f(u)=e^u$ und $f(u)=u^\alpha$ behandelt, für die auch diskrete Verzweigungsdiagramme angegeben werden.

1. Introduction

Recently, various authors have observed that discrete analogues of nonlinear boundary value problems may have spurious solutions that do not converge to any of the continuous solutions (cf. Gaines [4], Allgower [1], Bohl [2], Peitgen, Saupe, Schmitt [7], Doedel, Beyn [3]).

Let us consider, for example, the nonlinear eigenvalue problem

$$-u''=\lambda f(u) \text{ in } [0,1],\ \lambda\geq 0,\ u(0)=u(1)=0$$

and a discrete analogue of the type

$$Au=\lambda Fu,\ u\in\mathbb{R}^m,\ \lambda\geq 0 \tag{1}$$

where $A\in L[\mathbb{R}^m]$ is an $m\times m$-matrix and F is the diagonal field

$$(F(u))_i=f(u_i), i=1,\ldots,m.$$

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Then there are essentially two types of spurious solution branches for the discrete equation (1):

Type I:
Solution branches which occur for large λ and which tend to solutions of $Fu=0$ as $\lambda \to \infty$.

Type II:
Solution branches which occur for small λ and which tend to infinity (in the maximum norm) as $\lambda \to 0$.

Type-I-solutions are well understood and can be analyzed by means of the reduced equation

$$Fu=0,$$

which is obtained from (1) if we divide by λ and let $\lambda \to \infty$ (cf. Bohl [2], Peitgen, Saupe, Schmitt [7]).

Type-II-solutions are typical for superlinear functions f, but – apart from numerical results – their existence has only been proved in special cases (see Allgower [1] for the case $f(u)=u^4$, $m=3$ and note that the unbounded continua in Peitgen, Saupe, Schmitt [7, section 4] may all consist of numerically relevant solutions).

The purpose of this paper is to clarify the situation for spurious solutions of type II.

In section 2 we show under a certain growth condition on f that any positive column of A^{-1} which attains its maximum on the diagonal leads to a positive branch of type-II-solutions. The shape of the solutions on this branch is given by the respective column of A^{-1}.

In particular, the discrete Gel'fand problem (Gel'fand [5]) where

$$A=(m+1)^2 \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \cdot & \vdots & \vdots & \vdots & \cdot \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \quad f(u)=e^u, \tag{2}$$

has at least m distinct branches of this type.

If m is odd then only one of these contains the symmetric solutions which correspond to solutions of the boundary value problem. All the other branches are spurious. Moreover – as the numerical and theoretical results of section 3 show – there are still further positive spurious solution branches for this example.

Our growth condition on f excludes the important case

$$f(u)=u^\alpha, \quad \alpha \geq 1,$$

which will be treated in some detail in section 4.

It is shown that for α sufficiently large, say $\alpha > \alpha_1 > 1$, the same situation as in section 2 prevails.

Note however that not any $f(u)=u^{\alpha}$, $\alpha>1$ produces spurious solutions. It follows from the results of Lorenz [6] that if A^{-1} has only positive entries there can be computed a number $\alpha_0=\alpha_0(A^{-1})>1$ for which the system (1) with $f(u)=u^{\alpha}$, $|\alpha|<\alpha_0$ has exactly one positive solution branch. Hence, for any matrix A of this kind there is a critical value α^*, $1<\alpha_0\leq\alpha^*\leq\alpha_1<\infty$, at which spurious solutions for the system (1) with $f(u)=u^{\alpha^*}$ begin to exist (see Fig. 2 in section 4).

2. Spurious Solutions in the General Case

Let $\|\ \|$ denote the maximum norm in $\mathbb{R}^m$.

It will be convenient to parametrize solution branches of equation (1) by this norm, i.e. we look for solutions

$$(u,\lambda)=(u(r),\lambda(r))\in\mathbb{R}^{m+1}$$

of the system

$$Au=\lambda F u,\quad \|u\|=r\geq 0. \tag{3}$$

Let us assume that $B=A^{-1}$ exists. Then our analysis is based on the following reformulation of (3)

$$v=\mu f(r)^{-1}F(rBv),\ r\geq 0,\quad \|Bv\|=1. \tag{4}$$

It is easily verified that the solutions $(u,\lambda)\in\mathbb{R}^{m+1}$ of (3) and (v,μ) of (4) are related by

$$(u,\lambda)=(rBv,\ rf(r)^{-1}\mu). \tag{5}$$

As we will show, equation (4) leads to a reasonable reduced problem if we let $r\to\infty$.

The assumptions in the following theorem are easily seen to be fulfilled by example (2) (cf. section 3).

Theorem 1:

(*i*) *Let* $f:(0,\infty)\to(0,\infty)$ *be a* C^1*-function such that*

$$\frac{1}{f(r)}\sup_{0<\tau\leq t} f(\tau r)\to 0,\quad \frac{r}{f(r)}\sup_{0<\tau\leq t} f'(\tau r)\to 0$$

as $r\to\infty$ *holds for every* $t\in(0,1)$.

(*ii*) *Let* $B=A^{-1}$ *exist and let* $B_j=(B_{1j},\ldots,B_{mj})^T$ *be a column of* B *satisfying*

$$B_{jj}>B_{ij}>0\quad \forall i\neq j.$$

Then there exists a continously differentiable branch of positive solutions $(u(r),\lambda(r))$, $r\geq r_0$ *of the system* (3) *which satisfies*

$$r^{-1}u(r)\to B_{jj}^{-1}B_j,\ \lambda(r)r^{-1}f(r)\to B_{jj}^{-1}\quad as\ r\to\infty. \tag{6}$$

Proof:

Let $v^0=B_{jj}^{-1}e^j$, where e^j is the j-th unit vector in $\mathbb{R}^m$. Then from assumption (ii) we have an $\varepsilon>0$ such that

$$1>(Bv)_i>0\ \forall i\neq j,\ v\in V=\{v\in\mathbb{R}^m:\|v-v^0\|\leq\varepsilon\}. \tag{7}$$

We will now apply the implicit function theorem to the equation

$$T(v,s)=0,\ (v,s)\in V\times\mathbb{R}$$

where $T: V\times\mathbb{R}\to\mathbb{R}^m$ is defined by

$$T_i(v,s)=\begin{cases} v_j f\left((Bv)_i|s|^{-1}\right)f(|s|^{-1})^{-1}-v_i, & i\neq j,\ s\neq 0\\ -v_i, & i\neq j,\ s=0\\ (Bv)_j-1, & i=j.\end{cases}$$

By an elementary discussion using (7) and assumption (i) we obtain

$$T,\ \frac{\partial T}{\partial v_k}\in C(V\times\mathbb{R},\mathbb{R}^m),\ k=1,\dots,m$$

as well as

$$\frac{\partial \mathrm{T}_i}{\partial \mathrm{v}_k}(\mathrm{v}^0,0)=\begin{cases}-\delta_{ik}, & i\neq j\\ B_{jk}, & i=j\end{cases}\quad (k=1,\dots,m).$$

Hence

$$\det\left(\frac{\partial T}{\partial v}(v^0,0)\right)=(-1)^{m-1}B_{jj}\neq 0 \tag{8}$$

and we find a continuous solution branch $\bar v(s)\in V$, $|s|<\delta$ such that $\bar v(0)=v^0$. Since

$$\frac{\partial T}{\partial s}\in C\left(V\times(\mathbb{R}\setminus\{0\}),\mathbb{R}^m\right)$$

this branch is also continuously differentiable for $s\neq 0$.

Now let us define

$$v(r)=\bar v(r^{-1}),\ \mu(r)=v_j(r)\ \text{ if }\ r>\delta^{-1},$$

then $(v(r),\mu(r))$ is a solution of (4) – note that

$$(B\,v(r))_j=1>(B\,v(r))_i>0\ \ \forall i\neq j$$

and

$$v_j(r)=\mu(r)=\mu(r)f(r)^{-1}f\left(r\,(B\,v(r))_j\right).$$

Moreover

$$v(r)\to v^0,\ \mu(r)\to B_{jj}^{-1}\ \text{ as }\ r\to\infty$$

which proves (6) if $(u(r),\lambda(r))$ are defined via the relation (5).

3. The Case $f(u)=e^u$

As an application of theorem 1 let us consider the discrete Gel'fand problem (3) where A and f are defined by (2).

In this case $B=A^{-1}$ is given by

$$B_{ij}=(m+1)^{-3}\begin{cases} i(m+1-j), & i\le j\\ j(m+1-i), & i\ge j.\end{cases} \tag{9}$$

Assumption (i) of theorem 1 is obviously satisfied and assumption (ii) holds for each column of B.

Hence there are at least m distinct positive solution branches

$$\left(u^j(r), \lambda^j(r)\right), \; r \geq r_0, j=1, \ldots, m$$

of the discrete Gel'fand problem (3), (2) and these satisfy the asymptotic relations

$$u_i^j(r) \sim r \begin{cases} i j^{-1}, & i \leq j \\ (m+1-i)(m+1-j)^{-1}, & i \geq j \end{cases}, \quad \lambda^j(r) \sim r e^{-r} \frac{(m+1)^3}{j(m+1-j)}.$$

In the case $m=9$ we have computed these branches numerically and obtained the picture as given in Fig. 1.

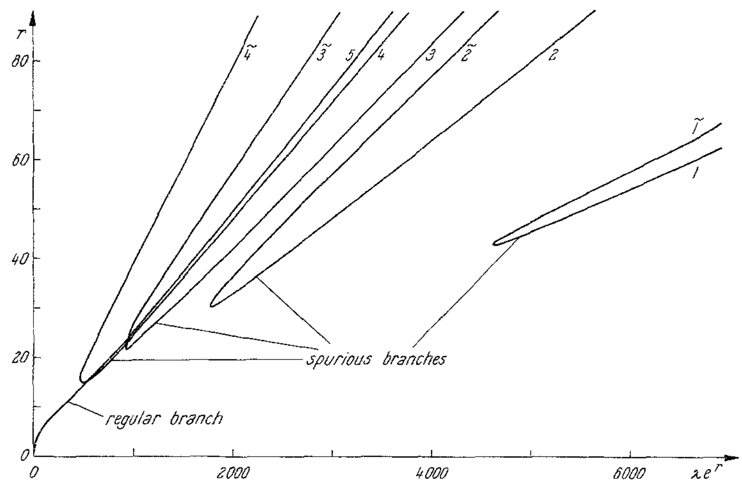

Fig. 1

The numbers $j=1, \ldots, 5$ indicate the above branches. The branch 5 contains the symmetric solutions (i.e. $u_i = u_{10-i}$, $i=1, \ldots, 9$) which correspond to the solutions of the boundary value problem. The spurious branches with indices $j=6, \ldots, 9$ are related to those with $j=1, \ldots, 4$ by

$$u_i^j(r) = u_{10-i}^{10-j}(r), \quad i=1, \ldots, 9, \; j=6, \ldots, 9.$$

Therefore the branches j and $10-j$ coalesce in Fig. 1.

Note that the spurious branches $j=1, \ldots, 4$ are not connected with each other but rather "bend back to infinity" creating another type of spurious solutions (denoted by $\tilde{j}$).

The spurious solutions on the $\tilde{j}$-branches still attain their maximum in the j-th component. Their shape suggests that they can be represented asymptotically as a linear combination of two columns of $B = A^{-1}$. This observation is made precise in the following theorem.

Theorem 2:

Consider the system (3) *where* $f(u)=e^u$. *Suppose that the matrix* $B=A^{-1}$ *exists and has two columns*

$$B_j=(B_{1j},\dots,B_{mj})^T,\ B_k=(B_{1k},\dots,B_{mk})^T$$

with the following properties

$$0<B_{ik}<B_{kk}\ \forall\, i\neq k,\ 0<B_{ij}<B_{jj}\ \forall\, i\neq j, \tag{10 a}$$

$$B_{ik}\leq B_{jk}\ \forall\, i\neq k\ \text{ or }\ B_{ij}\leq B_{kj}\ \forall\, i\neq j, \tag{10 b}$$

$$B_{jj}-B_{kj}<B_{kk}-B_{jk}. \tag{10 c}$$

Then there exists a continuously differentiable branch of positive solutions $(u(r),\lambda(r))$, $r\geq r_0$ *of the equations* (3) *which satisfies*

$$r^{-1}u(r)\to\alpha B_j+\beta B_k,\ \lambda(r)\,r^{-1}e^r\to\alpha\ \text{ as }\ r\to\infty$$

where

$$\alpha=\Delta^{-1}(B_{kk}-B_{jk}),\ \beta=\Delta^{-1}(B_{jj}-B_{kj}),\ \Delta=B_{jj}B_{kk}-B_{jk}B_{kj}. \tag{11}$$

Proof:

We will only give the main steps in the proof and leave the details to the reader.

The system (4) may now be written as

$$v_i=\mu\exp\left(r\left((Bv)_i-1\right)\right),\ i=1,\dots,m,\ \|Bv\|=1. \tag{12}$$

Let us introduce the operator $P:\mathbb{R}^{m+1}\times\mathbb{R}\to\mathbb{R}^m$ by

$$P_i(w,s)=\begin{cases} w_i, & i\neq k\\ w_k-w_{m+1}\,s, & i=k\end{cases},\quad w\in\mathbb{R}^{m+1},\ s\in\mathbb{R}$$

and the operator $T:\mathbb{R}^{m+1}\times\mathbb{R}\to\mathbb{R}^{m+1}$ by

$$T_i(w,s)=\begin{cases} w_i-w_j\exp\left(|s|^{-1}\left[(BP(w,s))_i-1\right]\right), & i\in\{1,\dots,m\}\setminus\{j,k\}\\ (BP(w,s))_j-1, & i=j\\ (BP(w,s))_k+w_{m+1}\,s-1, & i=k\\ w_k-w_{m+1}\,s-w_j\,e^{-w_{m+1}}, & i=m+1.\end{cases}$$

The exponential term in this definition is set to zero if $s=0$. Instead of (12) we consider the equation

$$T(w,s)=0 \tag{13}$$

in a neighbourhood of

$$s=0,\ w=w^0:=\alpha e^j+\beta e^k+\gamma e^{m+1}\in\mathbb{R}^{m+1}$$

where $\alpha,\beta>0$ are given by (11) and $\gamma>0$ is defined by

$$\beta=\alpha e^{-\gamma}. \tag{14}$$

From (11), (14) we have $T(w^0,0)=0$ and using (10 a, b) we find a neighbourhood

$$W=\{(w,s)\in\mathbb{R}^{m+1}\times\mathbb{R}:\ \|w-w^0\|+|s|<\varepsilon\}$$

such that $T \in C^1(W, \mathbb{R}^{m+1})$ and

$$\varepsilon w_{m+1}, (BP(w,s))_i \in (0,1) \ \forall i \neq j, k, (w,s) \in W.$$

Finally,

$$\det\left(\frac{\partial T}{\partial w}(w^0, 0)\right) = (-1)^m \Delta\beta \neq 0$$

and the implicit function theorem yields a continuously differentiable branch of solutions $(w(s), s)$, $|s| < \delta$ of equation (13) which satisfies $w(0) = w^0$.

Now, all our assertions follow from the relation (5), because

$$v(r) = P(w(r^{-1}), r^{-1}),\ \mu(r) = w_j(r^{-1}),\ r > \mathrm{Max}(\varepsilon, \delta^{-1})$$

are solutions of the system (12). □

In the special case where A and B are given by (2), (9) it is readily verified that our assumptions (10 a – c) are satisfied with

$$j < \frac{m}{2},\ k = j+1 \ \text{ and } \ j > \frac{m}{2} + 1,\ k = j-1.$$

Therefore, theorem 2 yields the existence of at least $m-1$ (m odd), $m-2$ (m even) spurious solution branches distinct from each other and distinct from the branches established by theorem 1. The corresponding solutions are unsymmetric.

4. The Case $f(u) = u^\alpha$, $\alpha \geq 1$

Assumption (i) of theorem 1 is obviously not satisfied if

$$f(u) = u^\alpha,\ \alpha \geq 1.$$

However, in this case the system (4) takes the special form

$$v_i = \mu (Bv)_i^\alpha,\ i = 1, \ldots, m,\ \|Bv\| = 1 \tag{15}$$

which is independent of r. The system (15) can now be solved for (v, μ) where α is considered as a parameter.

Theorem 3:

For $\alpha \geq 1$ consider the equations

$$(Au)_i = \lambda u_i^\alpha \ (i = 1, \ldots, m),\ \|u\| = r \tag{16}$$

and let assumption (ii) of theorem 1 hold.

Then there exists $\alpha_1 > 1$ and for every $\alpha > \alpha_1$ a continuously differentiable positive solution branch for (16):

$$u(r) = r u_\alpha,\ \lambda(r) = r^{1-\alpha} \lambda_\alpha,\ r > 0.$$

Here $u_\alpha \in \mathbb{R}^m$, $\lambda_\alpha \in \mathbb{R}$ are independent of r and satisfy

$$u_\alpha \to B_{jj}^{-1} B_j,\ \lambda_\alpha \to B_{jj}^{-1} \ \text{ as } \alpha \to \infty. \tag{17}$$

Proof: As in the proof of theorem 1 let $v^0 = B_{jj}^{-1} e^j$ and choose $\varepsilon > 0$ such that (7) holds.

We consider the equation

$$T(v,s)=0,\ (v,s)\in V\times\mathbb{R}, \tag{18}$$

where T is now defined by

$$T_i(v,s)=\begin{cases} v_i-v_j(Bv)_i^{1/|s|}, & i\neq j,\ s\neq 0\\ v_i, & i\neq j,\ s=0\\ (Bv)_j-1, & i=j.\end{cases}$$

We have $T\in C^1(V\times\mathbb{R},\mathbb{R}^m)$ and as in (8) we find

$$\det\left(\frac{\partial T}{\partial v}(v^0,0)\right)=(-1)^{m-1}B_{jj}\neq 0.$$

Hence, equation (18) can be solved by a C^1-function

$$v(s)\in V,\ |s|<\delta,\ v(0)=v^0.$$

We then obtain $(v(\alpha^{-1}),\mu=v_j(\alpha^{-1}))$, $\alpha>\delta^{-1}$ as solutions of (15) and our assertions hold with

$$u_\alpha=B\,v(\alpha^{-1}),\ \lambda_\alpha=v_j(\alpha^{-1}),\ \alpha>\alpha_1:=\delta^{-1}. \qquad \square$$

The vector $(u_\alpha,\lambda_\alpha)$ of theorem 3 is a solution of

$$(Au)_i=\lambda u_i^\alpha,\ i=1,\dots,m,\ \|u\|=1.$$

Some numerical branches for this equation in the case

$$m=9,\ \text{matrix } A \text{ as in (2)}$$

are given in Fig. 2. Note that any solution in this diagram gives a complete branch for problem (16).

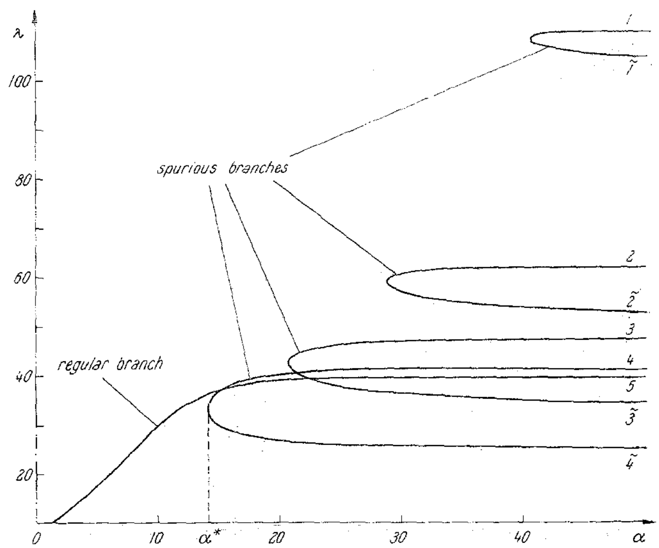

Fig. 2

The numbers $j = 1, \ldots, 5$ denote the branches with the asymptotic behaviour (17) and the branch 5 contains the symmetric solutions. Finally, the additional solutions on the $\tilde{j}$ branches are quite similar to those for the discrete Gel'fand problem as established by theorem 2.

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