ON THE CONVERGENCE OF THE FINITE DIFFERENCE METHOD FOR NONLINEAR ORDINARY BOUNDARY VALUE PROBLEMS

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There are well known conditions which finite difference equations approximating a linear ordinary boundary value problem have to satisfy in order to guarantee consistency and stability of the method and hence convergence of the finite difference solutions. Furthermore, under analogous assumptions a local convergence theorem holds in the non-linear case. In this paper we give two global versions of this local result, one which yields a global stability inequality for the finite difference equations and another one which shows that the number of solutions is the same for the difference equations as for the boundary value problem. Our results are illustrated by two examples.

1. Introduction

Let us consider a nonlinear boundary value problem of order k

(1) $D^k u + f(\cdot,u,\dots,D^{k-1}u) = 0$ in $[0,1]$, $Ru = d \in \mathbb{R}^k$

where f and its partial derivatives $D_{j+1}f(j=1,\dots,k)$ are continuous in $[0,1] \times \mathbb{R}^k$, $u \in C^k[0,1]$ and Ru is of the form

$$Ru = \left\{ \sum_{j=1}^{k} (a_{ij}u^{(j-1)}(0) + b_{ij}u^{(j-1)}(1)): \quad i=1,\dots,k \right\} \in \mathbb{R}^k.$$

We will treat a general class of finte difference equations for (1) which can be written as follows

(2) $T_h u_h := p_k^h E^{-k_1} \Delta^k u_h + M_h f(\cdot, D_h^0 u_h, \ldots, D_h^{k-1}) = 0,\ R_h u_h = d,$

where $u_h \in U_h$ is the unknown grid function and U_h is the space of real valued functions on the grid $J_h = \{0, h, 2h, \ldots, 1-h, 1\}$ ($h = n^{-1}, n \in \mathbb{N}$). $T_h u_h$ is an element of U_h', the space of grid functions on $J_h' = \{k_1 h, (k_1+1)h, \ldots, 1-k_2 h\}$ where $k_1, k_2 \in \mathbb{N}$ and $k_1 + k_2 = k$. Furthermore, we have used in (2) the translation operators E^m (defined by $E^m u_h(x) = u_h(x+mh)$, $x + mh \in \text{domain}(u_h), m \in \mathbb{Z}$) and the divided differences Δ^m (defined by $\Delta^m u_h(x) = (E-I)^m u_h(x)$, $x + jh \in \text{domain}(u_h)$ for $j=0,\ldots,m$, $m \in \mathbb{N}$).

The operators $p_k^h : U_h' \to U_h'$, $D_h^j : U_h \to U_h$ $(j=0,\ldots,k-1)$, $M_h : U_h \to U_h'$ and $R_h : U_h \to \mathbb{R}^k$ in (2) are assumed to be linear. By their use we are able to consider difference formulas of higher order as well as Hermitian expressions ([5] III, §2).

For linear boundary value problems (1) a theory of convergence of the finite difference solutions with respect to different norms has been developed in [7 , 9] (see also [2 , 16]). Following the simplified approach of [2] we will call (2) a linear scheme for (1) if some conditions of consistency and stability are satisfied for p_k^h, D_h^j, M_h and R_h. In the linear case these conditions imply convergence of the finite difference solutions. In the nonlinear case a local theorem of convergence follows for isolated solutions of (1). This local result appears in various abstract versions in the literature [1 , 8 , 13, 14, 15]. The main drawback of this theorem is that it yields existence and uniqueness of a solution of (2) in a small neighbourhood of the exact solution of (1) which is just to be computed. By using some global information on the equation (1) we give two theorems in section 3 which determine the number of solutions and show the stability behaviour of the difference equations (2) in the whole space U_h. These results will be applied to two examples. In particular, we will show that some "parasitic" solutions, which have been found in [3] for a certain finite difference equation at a fixed mesh parameter h, have to disappear as h tends to zero.

2. Linear Schemes

We introduce some further notation. Let $[u]_h \in U_h$ and $[u]'_h \in U'_h$ denote the restrictions of a function $u \in C[0,1]$ to the meshes J_h and J'_h resp.. Furthermore, we will use the norms

$$\|u_h\|_j = \sum_{i=0}^{j} \mathrm{Max}\{|\Delta^i u_h(x)| : x=0,h,\dots,1-ih\},\ u_h \in U_h.$$

$\| \ \|_0$ will also denote the maximum norm in U'_h and $\mathbb{R}^k$.

DEFINITION

The difference equations (2) are called a linear scheme for (1) if the following assumptions hold:

(i) D_h^j is uniformly $\| \ \|_j$-bounded, i.e. for some constant $C > 0$, $\|D_h^j u_h\|_0 \le C\|u_h\|_j$ for all $u_h \in U_h$ and for all h $(j=0,\dots,k-1)$, and D_h^j is consistent with D^j, i.e.

$$\|D_h^j[u]_h - [D^j u]_h\|_0 \to 0 \text{ as } h \to 0, \text{ for all } u \in C^j[0,1]\ (j=0,\dots,k-1).$$

(ii) M_h is uniformly $\| \|_0$-bounded (cf. (i)) and consistent with I, i.e.

$$\|M_h[u]_h - [u]_h'\|_0 \to 0 \text{ as } h \to 0, \text{ for all } u \in C[0,1].$$

(iii) R_h is uniformly $\| \|_{k-1}$-bounded and consistent with R, i.e.

$$\|R_h[u]_h - Ru\|_0 \to 0 \text{ as } h \to 0, \text{ for all } u \in C^{k-1}[0,1].$$

(iv) p_k^h is consistent with I, i.e.

$$\|p_k^h[u]_h' - [u]_h'\|_0 \to 0 \text{ as } h \to 0, \text{ for all } u \in C[0,1],$$

and p_k^h satisfies the root conditions (see [2]).

Conditions (i)-(iv) are satisfied for nearly every reasonable difference approximation to (1), the only nontrivial assumptions being the root conditions in (iv). These have been verified in [2] for a large class of difference methods including those which can be composed of formulas from [5 , Appendix]. Now the following local result is well known, it can easily be derived from the abstract theorem [15, §3(14)] (see [1 , 8 , 13, 14, 16] for related results) and from the linear theory [2].

THEOREM 1

Let $\bar{u} \in C^k[0,1]$ be an isolated solution of (1), which means that the linearization at $\bar{u}$

$$\left(D^k + \sum_{j=0}^{k-1} D_{j+2}f(\cdot,\bar{u},\dots,\bar{u}^{(k-1)})D^j, R\right) : C^k[0,1] \to C[0,1] \times \mathbb{R}^k$$

is invertible.

Then for each linear scheme (2) there exists an $h_0 > 0$ and a $\rho > 0$ such that (2) has a unique solution $\bar{u}_h$ in the ball $K_\rho = \{u_h \in U_h : \|[\bar{u}]_h - u_h\|_k \leq \rho\}$ for all $h \leq h_0$. Moreover, for all $u_h, v_h \in K_\rho$ and $h \leq h_0$ the stability inequality

(3) $\|u_h - v_h\|_k \leq C(\|T_h u_h - T_h v_h\|_0 + \|R_h(u_h - v_h)\|_0)$

holds and $\|[\bar{u}]_h - \bar{u}_h\|_k \to 0$ as $h \to 0$.

We note that the assumption on the linearization at $\bar{u}$ can be weakened by a condition on the Leray Schauder index at $\bar{u}$ [15, §3(43)]. Since the center $[\bar{u}]_h$ and the radius ρ of K_ρ are unknown a priori, theorem 1 doesn't give much information on the numerical solution of (2). This, of course, is natural under the weak assumption of an isolated solution.

3. Two Global Results

In our first global theorem the assumption on the linearization at the solution $\bar{u}$ is extended to the whole space $C^k[0,1]$.

THEOREM 2

Assume that for some $K_j, K^j \in C[0,1]$ $(j=0,\ldots,k-1)$ we have

(4) $K_j(x) \leq D_{j+2} f(x,y) \leq K^j(x)$ for all $x \in [0,1]$ and $y \in \mathbb{R}^k$.

Let there exist an $\varepsilon > 0$ such that the linear pairs $\left(D^k + \sum_{j=0}^{k-1} p_j D^j, R\right)$ are invertible on $C[0,1] \times \mathbb{R}^k$ for all coefficients $p_j \in C[0,1]$ satisfying $K_j(x) - \varepsilon \leq p_j(x) \leq K^j(x) + \varepsilon$ $(x \in [0,1],\ j=0,\ldots,k-1)$. Then the boundary value problem (1) has a unique solution $\bar{u} \in C^k[0,1]$. Moreover, for each linear scheme, for which M_h is represented by a nonnegative band matrix of a band width independent of h, there exists an $h_0 > 0$ such that (2) has a unique solution $\bar{u}_h$ for all $h \leq h_0$. Finally, $\| [\bar{u}]_h - \bar{u}_h \|_k \to 0$ as $h \to 0$, and the stability inequality (3) is valid for all $u_h, v_h \in U_h$ and $h \leq h_0$.

Instead of going into the rather lengthy proof, which will be given elsewhere, we consider two examples.

Example 1

(5) $u'' + \lambda e^u = 0$ in $[0,1]$, $u(0) = u(1) = 0$.

This problem has a unique solution if $\lambda \leq 0$, two solutions if $0 < \lambda < \lambda^*$ (λ^* a certain critical parameter), a unique solution if $\lambda = \lambda^*$ and no solutions if $\lambda > \lambda^*$ (see [6 , 10] and the references given therein). In case $\lambda \leq 0$ the problem (5) has only nonpositive solutions and it is a standard procedure (cf. [12]) to replace e^u in this case by

$$f(u) = \begin{cases} e^u, & \text{if } u \leq 0 \\ 1 + u, & \text{if } u > 0. \end{cases}$$

Assumption (4) is then satisfied for f with $K^0(x) = K^1(x) = K_1(x) = 0$, $K_0(x) = \lambda$ and it is easily seen by monotonicity arguments (cf. [11]) that for some $\varepsilon > 0$ the equation $u'' + p_1 u' + p_0 u = 0$ in $[0,1]$, $u(0) = u(1) = 0$, has only the trivial solution provided $|p_1(x)| \leq \varepsilon$ and $\lambda - \varepsilon \leq p_0(x) \leq \varepsilon$ ($x \in [0,1]$). Hence any linear scheme (2) applied to (5) with the modified nonlinearity has a unique solution for sufficiently small h which converges to the unique solution of (5). We refer to [4] for a totality of linear schemes. Note that the monotonicity methods of [4] also provide results on the difference equations at definite values of h.

Example 2 (cf. [3] and the references therein)

(6) $u'' + \lambda \sin u = 0$ in $[0,1]$, $u(0) = u(1) = 0$ $(\lambda \geq 0)$.

In case $\lambda < \pi^2$, theorem 2 can be applied with $K^1(x) = K_1(x) = 0$, $K_0(x) = -\lambda$, $K^0(x) = \lambda$ $(x \in [0,1])$, so that every linear scheme (2) applied to (6) has only the trivial solution if h is small enough.

But again, as in example 1, we cannot deal with thé case of several solutions which in example 2 occurs as λ exceeds π^2. This problem is covered by the following theorem.

THEOREM 3

Suppose that (1) has exactly N solutions $\bar{u}_i$ $(i=1,\dots,N)$ in $C^k[0,1]$ which are isolated in the sense of theorem 1. Assume further that for some $p_j \in C[0,1]$ $(j=0,\dots,k-1)$ we have $f(x,y) = \sum_{j=0}^{k-1} p_j(x) y_{j+1} + g(x,y)$ $(x \in [0,1], y \in \mathbb{R}^k)$, *where* $\left(D^k + \sum_j p_j D^j, R\right)$ *is invertible and g is sublinear*, i.e.

$$(7) \quad |g(x,y)| \left(\sum_{j=1}^{k} |y_j|\right)^{-1} \to 0 \text{ uniformly in } x \in [0,1]$$

$$\text{as } \sum_{j=1}^{k} |y_j| \to \infty.$$

Then each linear scheme (2) has exactly N solutions $\bar{u}_{ih}$ $(i=1,\dots,N)$ in U_h for h sufficiently small and these satisfy

(8) $\|\bar{u}_{ih} - [\bar{u}_i]_h\|_k \to 0$ as $h \to 0$ $(i=1,\ldots,N)$.

The existence of the solutions $\bar{u}_{ih}$ is guaranteed by theorem 1 whereas the nonexistence of further solutions follows from a compactness argument. It shows that any sequence of possible solutions u_h of (2) has a subsequence which converges to a solution of (1) in the sense of (8) so that the local uniqueness result of theorem 1 applies.

In case $N = 1$ this underlying idea is already contained in an abstract theorem of Vainikko [15, §3(27)]. Note, however, that this argument is valid for arbitrary $N \in \mathbb{N}$ and even in the case $N = 0$, if "N solutions" are interpreted as "no solutions".

Let us reconsider example 2. If $n^2\pi^2 < \lambda < (n+1)^2\pi^2$ for some $n \in \mathbb{N}$, then (6) has $2n + 1$ distinct solutions which we assume to be isolated (we still have no complete proof of this). Since $\sin u$ is sublinear theorem 3 shows that any linear scheme (2) applied to (6) also has $2n + 1$ solutions for sufficiently small h. In [3] some additional solutions to these have been discovered for a certain linear scheme applied to (6) at a fixed value of h. Our theorem then shows that these solutions have to disappear as h tends to zero.

Due to the strong nonlinearity e^u, theorem 3 does obviously not apply to our example 1 in case $\lambda > 0$. However, in some cases it is possible to derive a priori estimates for all solutions of a superlinear boundary value problem (cf. [6] section 4).

For example, in the case of (5) we can proceed as follows. For any solution u of (5) we have $u(x) > 0$, $u''(x) < 0$ $(0 < x < 1)$. Hence u has a unique maximum $M = u(x_0)$. Since

$w(x) = u(2x_0 - x)$ satisfies $w(x_0) = M, w'(x_0) = 0$ and the same differential equation as u we obtain $x_0 = 1/2$. Consequently, $-u''(x) > \lambda e^{r(x)}$ $(0 < x < 1)$ where

$$r(x) = 2M \begin{cases} x, & 0 \leq x \leq 1/2 \\ 1 - x, & 1/2 \leq x \leq 1. \end{cases}$$

Now we define v by $-v'' = e^r$ in $[0,1]$, $v(0) = v(1) = 0$, and by the maximum principle, we have

$$(9) \quad M = u(1/2) > v(1/2) = \lambda(4M^2)^{-1}((M-1)e^M + 1)$$

Hence an upper bound for u is given by the largest positive root M_0 of the equation $4M^3 = \lambda((M-1)e^M + 1)$. If λ is large enough the inequality (9) is false for all $M > 0$ and (5) has no solution. A rough estimate shows $M_0 \leq 24\lambda^{-1} + 1$. Now the problem (5) has no solutions (in case $\lambda > \lambda^*$) or two solutions (in case $0 < \lambda < \lambda^*$) which we assume to be isolated. All solutions belong to $\{u \in C^k[0,1] : u(x) < M_0 \text{ for } x \in [0,1]\}$ and $g(x,y) = \lambda e^y$ is sublinear on $[0,1] \times (-\infty, M_0)$. By a slight modification of theorem 3 we then obtain that every linear scheme (2) applied to (5) also has no solution u_h satisfying $D_h^0 u_h(x) < M_0 (x \in J_h)$ in case $\lambda > \lambda^*$ and two solutions satisfying $D_h^0 u_h(x) < M_0 (x \in J_h)$ in case $0 < \lambda < \lambda^*$. Both statements are true if $h \leq h_0$ where h_0, in general, depends on λ. Nothing can be said about the case $\lambda = \lambda^*$ since the unique solution of (5) in this case is not isolated in the sense of theorem 1.

References

[1] Atkinson, K.E.: The numerical evaluation of fixed points for completely continuous operators. SIAM J. Numer. Anal. (1973), 799-807.

[2] Beyn, W.-J.: Zur Stabilität von Differenzenverfahren für Systeme linearer gewöhnlicher Randwertaufgaben. Numer. Math. 29 (1978), 209-226.

[3] Bohl, E.: On the numerical treatment of a class of discrete bifurcation problems. To appear in IAC, Istituto per le Applicazioni del Calcolo, "Mauro Picone", Pubblicazioni Serie III.

[4] Bohl, E., Lorenz, J.: Inverse monotonicity and difference schemes of higher order. A summary for two-point boundary value problems. To appear in Aequ. Math.

[5] Collatz, L.: The numerical treatment of differential equations, 3rd ed. Berlin-Göttingen-Heidelberg, Springer 1966.

[6] Crandall, M.G., Rabinowitz, P.H.: Bifurcation, perturbation of simple eigenvalues and linearized stability. Arch. Rat. Mech. Anal. 52 (1973), 161-180.

[7] Grigorieff, R.D.: Die Konvergenz des Rand- und Eigenwertproblems linearer gewöhnlicher Differentialgleichungen. Numer. Math. 15 (1970), 15-48.

[8] Keller, H.B.: Approximation methods for nonlinear problems with application to two point boundary value problems. Math. Comput. 29 (1975), 464-474.

[9] Kreiss, H.-O.: Difference approximations for boundary and eigenvalue problems for ordinary differential equations. Math. Comput. 26 (1972), 605-624.

[10] Meyer-Spasche, R.: Numerische Behandlung von elliptischen Randwertproblemen mit mehreren Lösungen und von MHD Gleichgewichtsproblemen. Max Planck Institut für Plasmaphysik, Garching bei München, 1975.

[11] Protter, M.H., Weinberger, H.F.: Maximum principles in differential equations. Prentice Hall, Englewood Cliffs, N.J., 1967.

[12] Shampine, L.F.: Boundary value problems for ordinary differential equations I. SIAM J. Numer. Anal. 5 (1968), 219-242.

[13] Stetter, H.J.: Analysis of discretization methods for ordinary differential equations. Springer Tracts in Natural Philosophy, vol. 23. Berlin-Heidelberg-New York, Springer, 1973.

[14] Stummel, F.: Stability and discrete convergence of differentiable mappings. Rev. Roum. Math. Pures et Appl. Tome XXI, No 1 (1976), 63-96.

[15] Vainikko, G.: Funktionalanalysis der Diskretisierungsmethoden. Teubner Texte zur Mathematik, Leipzig, Teubner Verlag 1976.

[16] Vainikko, G.: Approximate methods for nonlinear equations. Nonlinear Analysis 2, (1978), 647-687.

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