

Scaling and asymptotic scaling in the SU(2) gauge theory

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Received 14 August 1992

Accepted for publication 14 October 1992

We determine the critical couplings for the deconfinement phase transition in SU(2) gauge theory on $N_\tau \times N_\sigma^3$ lattices with $N_\tau = 8$ and 16 and N_σ varying between 16 and 48. A comparison with string tension data shows scaling of the ratio $T_c/\sqrt{\sigma}$ in the entire coupling regime $\beta = 2.30$ – 2.75 , while the individual quantities still exhibit large scaling violations. We find $T_c/\sqrt{\sigma} = 0.69(2)$. We also discuss in detail the extrapolation of $T_c/\Lambda_{\overline{\text{MS}}}$ and $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}}$ to the continuum limit. Our result, which is consistent with the above ratio, is $T_c/\Lambda_{\overline{\text{MS}}} = 1.23(11)$ and $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}} = 1.79(12)$. We also comment upon corresponding results for SU(3) gauge theory and four-flavour QCD.

1. Introduction

Ever since the pioneering work of Creutz [1] the approach to asymptotic scaling, and thus the continuum limit, was one of the central issues in studies of gauge theories on the lattice. Although the first results were promising, it soon became clear that simulations on large lattices are needed in order to establish asymptotic scaling for asymptotically free quantum field theories such as QCD. In fact, recent numerical studies of the O(3) σ -model in two-dimensions [2] suggest that the asymptotic scaling regime may not be reached even for quite large correlation length while at the same time ratios of physical observables show scaling behaviour.

The lack of asymptotic scaling as well as the scaling of certain ratios of physical observables has also been observed in SU(N) gauge theories. In particular in the case of the SU(3) gauge theory the deviations from asymptotic scaling are large and have been noticed early as a dip in the discrete β -function [3,4], which led to deviations from asymptotic scaling by more than 50% for certain values of the gauge couplings. Although the dip is not that pronounced for SU(2) gauge theories, there are clear deviations from asymptotic scaling seen even for the

largest values of the coupling, $\beta = 2N/g^2$ for colour group SU(N), studied so far. In particular, the analysis of the heavy quark potential and the string tension has been performed up to couplings as large as $\beta = 2.85$ [5]. Still there is no hint for asymptotic scaling at this large β -value, which already corresponds to lattice spacings as small as ~ 0.05 fm. Furthermore, an analysis of the short-distance part of the heavy quark potential suggests, that the approach to the continuum limit may even be as slow as in the two-dimensional σ -model [6].

One of the best studied quantities in SU(2) gauge theory is the finite-temperature deconfinement phase transition. For lattices of size $N_\tau \times N_\sigma^3$ with $N_\tau \leq 6$ the critical coupling has been determined with high accuracy [7,8] and an extrapolation to spatially infinite volume could be performed. Moreover, an analysis of the critical exponents at the transition point were in perfect agreement with those of the three-dimensional Ising model. Here it turned out that the Binder cumulant of the order parameter is an observable which is well suited to locate the critical coupling for given N_τ as finite spatial size corrections are only due to the presence of irrelevant operators.

So far the analysis of the scaling of the ratio $T_c/\sqrt{\sigma}$ was limited to a rather small coupling regime in the case of SU(2) as the critical couplings for the deconfinement transition have been determined only for $N_\tau \leq 6$. It is the purpose of this paper to further investigate the scaling properties of the SU(2) gauge theory. We will extend earlier studies of the deconfinement transition to lattices up to a temporal size $N_\tau = 16$. This will enable us to perform a quantitative test of scaling in SU(2). Furthermore, we can follow the apparent scaling violations over a large range of couplings, which allows us to analyze various extrapolation schemes to extract $T_c/\Lambda_{\overline{\text{MS}}}$ in the continuum limit.

This paper is organized as follows: In sect. 2 we discuss our strategy of calculating the critical couplings for the deconfinement transition on lattices with large temporal extent. In particular we discuss the finite-size scaling of cumulants of the Polyakov loop expectation value. In sect. 3 we present our numerical results. Sect. 4 is devoted to a discussion of scaling and asymptotic scaling and a detailed discussion of the extrapolation of these results to the continuum limit. Finally sect. 5 contains our conclusions.

2. Finite-size scaling and the continuum limit

Usually finite-size scaling (FSS) in the vicinity of a finite-temperature phase transition is discussed for lattice SU(N) gauge models, without trying to make contact with the continuum limit, i.e. the scaling properties are studied on lattices of size $N_\tau \times N_\sigma^d$ with fixed N_τ and varying N_σ , where d denotes the spatial dimension and the model is viewed as a d -dimensional spin system. In the continuum limit the FSS properties of these non-abelian models should, of course,

be discussed in terms of the physical volume, $V = L^d$, and the temperature, T , in the vicinity of the deconfinement transition temperature T_c . We will study here how the scaling behaviour of the continuum theory emerges from the lattice free energy on arbitrary lattices, i.e. when varying N_τ and N_σ .

For a continuum theory having a simple critical point and a characteristic length $L = V^{1/d}$ the singular part of the free energy density,

$$f_s = \frac{F_s}{TV} = - \frac{\ln Z_s}{TV}, \tag{1}$$

is described by a universal finite-size scaling form [9,10],

$$f_s(T, H; L) = L^{-d} Q_{f_s}(g_T L^{1/\nu}, g_H L^{(\beta+\gamma)/\nu}). \tag{2}$$

Here we assume that corrections to scaling from irrelevant scaling fields $g_i L^{y_i}$, proportional to negative powers of L , can be neglected; for $N_\tau = 6$ a value of $y_1 = -0.9$ has been found for the SU(2) gauge theory [8], showing that irrelevant contributions disappear rather fast with increasing N_σ .

On a lattice of size $N_\tau \times N_\sigma^d$ the length scale L and the temperature T are given in units of the lattice spacing,

$$L = N_\sigma a, \quad T^{-1} = N_\tau a. \tag{3), (4)}$$

In general the lattice spacing a is a complicated function of the coupling $\beta = 2N/g^2$. The dependence of the lattice spacing on β is known only in the continuum limit in form of the renormalization group equation

$$a \Lambda_L = \left(\frac{\beta}{2N b_0} \right)^{b_1/2b_0^2} \exp\left(- \frac{\beta}{4N b_0} \right). \tag{5}$$

Therefore it is advantageous to replace the length scale L by the dimensionless combination

$$L T = \frac{N_\sigma}{N_\tau}. \tag{6}$$

Using this ratio in the FSS relation for the singular part of the free energy density we get

$$f_s(t, h; N_\sigma; N_\tau) = \left(\frac{N_\sigma}{N_\tau} \right)^{-d} Q_{f_s} \left(g_t \left(\frac{N_\sigma}{N_\tau} \right)^{1/\nu}, g_h \left(\frac{N_\sigma}{N_\tau} \right)^{(\beta+\gamma)/\nu} \right). \tag{7}$$

The scaling function Q_{f_s} depends on the temperature T and the external field strength h through thermal and magnetic scaling fields,

$$g_t = c_t t(1 + b_t t) + O(th, t^3), \tag{8}$$

$$g_h = c_h h(1 + b_h t) + O(th^2, t^2 h, h^2), \tag{9}$$

with non-universal metric coefficients c_t, c_h, b_t and b_h , still carrying a possible N_τ dependence. Here t is the reduced temperature, $t = (T - T_c)/T_c$, which in the neighbourhood of the transition point can be approximated by

$$t = (\beta - \beta_{c,\infty}) \frac{1}{4Nb_0} \left[1 - \frac{2Nb_1}{b_0} \beta_{c,\infty}^{-1} \right]. \tag{10}$$

This approximation reproduces the correct reduced temperature in the continuum limit, which is easily verified by using eq. (5). We note that t has $O(\beta_{c,\infty}^{-1})$ corrections to the leading term, which however, contribute less than 8% in the relevant coupling regime, i.e. $\beta > 2.0$ for SU(2) and $\beta > 5.5$ for SU(3). This non-leading term introduces a logarithmic dependence of t , and thus Q_{f_s} , on N_τ through eqs. (4) and (5). A priori we cannot exclude that this violation of the otherwise universal N_σ/N_τ dependence is enhanced in the non-asymptotic scaling regime. However, as we will show later we do not find any hints for this in our Monte Carlo data.

A non-vanishing magnetic field strength h corresponds to adding a symmetry breaking term of the form $hZ(a, N_\tau)N_\sigma^d P$ to the action. Here P denotes the Polyakov loop, defined as

$$P = N_\sigma^{-d} \sum_x \prod_{x_0=1}^{N_\tau} U_{(x_0, x), 0}, \tag{11}$$

and $Z(a, N_\tau)$ is a renormalization factor necessary to remove divergent self-energy contributions to the Polyakov loop, which represents a static heavy quark source. A physical order parameter, $\langle P_p \rangle$, not vanishing in the continuum limit, a susceptibility χ_p and a normalized fourth cumulant g_4 may then be defined through derivatives of f_s with respect to the external magnetic field strength h at $h = 0$,

$$\langle P_p \rangle = N_\tau^d Z(a, N_\tau) \langle P \rangle = - \left. \frac{\partial f_s}{\partial h} \right|_{h=0}, \tag{12}$$

$$\chi_p = N_\tau^d Z(a, N_\tau)^2 \chi = \left. \frac{\partial^2 f_s}{\partial h^2} \right|_{h=0}, \tag{13}$$

$$g_4 = \left. \frac{\partial^4 f_s}{\partial h^4} \right|_{h=0} / \left(\chi_p^2 \left(\frac{N_\sigma}{N_\tau} \right)^d \right). \tag{14}$$

For the cumulant g_4 the renormalization factors cancel and we end up with the usual expression known for the Binder cumulant of the order parameter [11–13]. The general form of the scaling relations derived from eq. (7) is

$$\partial^n f_s / \partial h^n |_{h=0} = \left(\frac{N_\sigma}{N_\tau} \right)^{n \frac{\beta+\gamma}{\nu} - d} \cdot Q_n \left(g_t \left(\frac{N_\sigma}{N_\tau} \right)^{1/\nu} \right), \tag{15}$$

where the function Q_n is defined as

$$Q_n(x_1) = \frac{\partial^n}{\partial x_2^n} Q_f(x_1, x_2) |_{x_2=0}. \tag{16}$$

The finite-size scaling behaviour of higher-order cumulants which are closely related to the derivatives defined in eq. (15) is discussed in appendix A. We note that for g_4 , as well as the higher cumulants, the prefactors of the scaling functions cancel. Thus they take on unique fixed point values at $g_t = 0$ even on finite lattices, if we ignore corrections from irrelevant operators. The cumulants are thus well suited to determine the critical coupling from simulations on finite lattices.

Let us first consider the case when N_τ is kept fixed. Then N_τ can be absorbed in the non-universal constants in g_t and g_h and we end up with the usual form of the finite-size scaling ansatz for f_s . The critical coupling can be determined from the fixed point of $g_4(\beta, N_\sigma)$ [8,11,13,14].

Next we consider $y = N_\sigma/N_\tau$ fixed, varying N_σ and therefore N_τ accordingly as is needed to reach the continuum limit. Rescaling N_σ and N_τ by a factor b leads to a phenomenological renormalization $\tilde{\beta}(\beta; b; y)$ by the following identity for a scaling function Q :

$$Q \left(g_t(\beta, N_\tau) \left(\frac{N_\sigma}{N_\tau} \right)^{1/\nu} \right) = Q \left(g_t(\tilde{\beta}, bN_\tau) \left(\frac{bN_\sigma}{bN_\tau} \right)^{1/\nu} \right). \tag{17}$$

Here we have kept explicit the dependence of g_t on N_τ that comes in through the non-universal metric coefficients c_t and b_t in eq. (8).

The property that the normalized fourth cumulant g_4 is directly a scaling function can be used to measure the discrete β -function, $\Delta\beta$, by using the above identity for $Q = g_4$. By writing eq. (17) in terms of the scaling function Q and not for the scaling field g_t directly we do not have to determine the metric coefficients in eq. (8). The discrete β -function is given by the shift in the coupling β , which is necessary to get the same value of g_4 for the two different lattice sizes,

$$g_4(\beta - \Delta\beta_y(\beta); N_\sigma; N_\tau) = g_4(\beta; bN_\sigma; bN_\tau). \tag{18}$$

The function $\Delta\beta_y$ defined in eq. (18) may depend on y through contributions from irrelevant scaling fields $g_i(\beta, N_\tau)N_\sigma^{y_i}$ with negative exponents y_i . The discrete β -function can then be obtained by an extrapolation to $y = \infty$,

$$\Delta\beta(\beta) = \lim_{y \rightarrow \infty} \Delta\beta_y(\beta). \quad (19)$$

The knowledge of $\beta_{c,\infty}(N_\tau)$ and $\Delta\beta$ allows us to calculate the critical coupling for the rescaled lattice size,

$$\beta_{c,\infty}(bN_\tau) = \beta_{c,\infty}(N_\tau) + \Delta\beta. \quad (20)$$

In the following we will use this approach to determine $\beta_{c,\infty}$ for $N_\tau = 8$ and 16. In particular we will check the y -independence for $N_\tau \leq 8$ and use this information to justify our calculations for $N_\tau = 16$ with moderate values for y .

3. Critical couplings and the Binder cumulant for $N_\tau = 8$ and $N_\tau = 16$

As discussed in sect. 2 the Binder cumulants are well suited to determine the critical couplings for the deconfinement transition. Previously they have been used to determine the critical couplings for the SU(2) gauge theory on lattices with temporal extent $N_\tau = 4$ and 6 [7,8]. We follow here the same strategy to determine the critical coupling for $N_\tau = 8$. We have performed simulations on lattices of size $8 \times N_\sigma^3$ with $N_\sigma = 16, 24$ and 32 at several values of β in the vicinity of the estimated critical point. We have used an overrelaxed heat-bath algorithm with 14 overrelaxation steps between subsequent ‘‘incomplete’’ heat-bath updates [15]. We used the Kennedy–Pendleton [16] algorithm with one trial per link. The acceptance rate was always larger than 96%. Most runs were done on the massively parallel CM-2 and so the use of a complete heat-bath algorithm would have led to a considerable waste of resources. Measurements were taken after each heat-bath update. Details on our parameter choices and number of iterations are given in table 1, where we also give results for the estimated integrated autocorrelation times for the expectation value of the Polyakov loop.

In fig. 1 we show the Binder cumulant, which on a lattice is measured by

$$g_4 = \frac{\langle P^4 \rangle}{\langle P^2 \rangle^2} - 3, \quad (21)$$

with the Polyakov loop, P , given by eq. (11). Also shown in this figure is an interpolation between results obtained at the various values of β , which is based on the density-of-states method (DSM) [17]. Our implementation of the DSM takes into account each measurement and does not require the usual division of

TABLE 1
Run parameters

N_σ	N_τ	β	N_{meas}	τ_{int}
16	8	2.500	25010	3.4
16	8	2.510	21055	3.7
16	8	2.520	21200	3.6
24	8	2.500	55500	10.6
24	8	2.505	29500	10.0
24	8	2.510	29500	13.5
24	8	2.515	39500	9.8
24	8	2.520	14500	14.9
32	8	2.500	30354	29.6
32	8	2.505	34750	17.1
32	8	2.510	38540	15.3
32	8	2.515	28750	18.0
32	8	2.520	18800	30.0
32	16	2.720	30910	10.9
32	16	2.740	40350	12.6
32	16	2.750	35000	12.0
48	16	2.740	43850	45.0

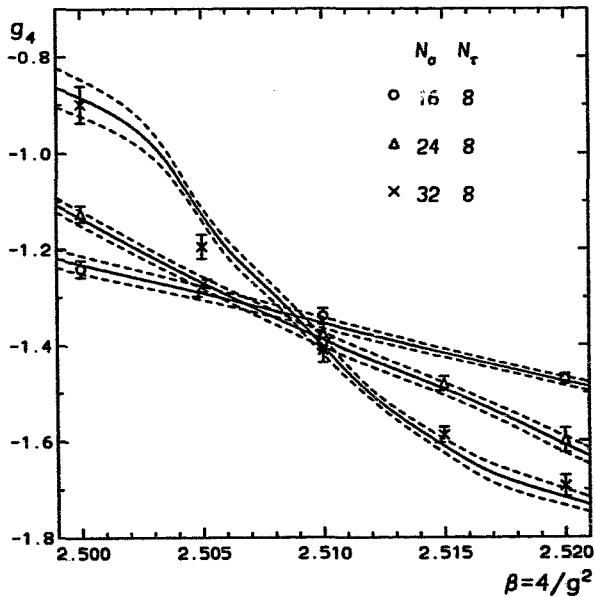


Fig. 1. The cumulant g_4 for $N_\tau = 8$ and various values of the spatial lattice size as a function of the coupling β . Solid curves are interpolation curves based on the density-of-states method. The dashed lines indicate the error on this curves estimated by the jackknife method.

the range of the expectation value of the plaquette operator $U_p = N^{-1} \text{Tr} U_1 U_2 U_3^\dagger U_4^\dagger$ in bins. This corresponds to an infinite number of bins and has the advantage to remove a free parameter, the number of bins, from the method. Because of the finite length of the Monte Carlo runs, the DSM provides reliable results only if histograms of U_p belonging to adjacent couplings overlap. We convinced ourself that at least 2.5% of the data of one sample is contained in each of the overlapping tails of the distribution.

The DSM allows an accurate determination of the intersection points of $g_4(\beta; N_\sigma; N_\tau)$ and $g_4(\beta; bN_\sigma; N_\tau)$, where N_σ is the spatial size of the smaller lattice and b is given by the ratio N'_σ/N_σ . Taking also the largest irrelevant scaling field into account, the coupling $\beta_c(N_\sigma, b)$ at which two cumulants intersect varies with N_σ and b as [8,11,13]

$$\beta_c(N_\sigma, b) = \beta_{c,\infty}(1 - \alpha\epsilon),$$

$$\epsilon = N_\sigma^{y_1-1/\nu} \frac{1 - b^{y_1}}{b^{1/\nu} - 1}. \quad (22)$$

Using the value $\nu = 0.628$ calculated for the three-dimensional Ising model [14] and $y_1 = -1$ [8] we determine the critical coupling for $N_\tau = 8$ by an extrapolation to $\epsilon = 0$ with the result

$$\beta_{c,\infty}(N_\tau = 8) = 2.5115 \pm 0.0040. \quad (23)$$

The error has been obtained from a weighted linear fit to $\beta_c(N_\sigma, b)$, where the error for the intersection points has been calculated using the jackknife method. The extrapolation is shown in fig. 2.

Taking into account the error band on the continuous curves obtained by the density-of-states method shown in fig. 1 as well as the error on $\beta_{c,\infty}$ we obtain for the fixed-point value of g_4 ,

$$\bar{g}_4(N_\tau) = \lim_{N_\sigma \rightarrow \infty} g_4(\beta_{c,\infty}; N_\sigma; N_\tau), \quad (24)$$

$$\bar{g}_4(N_\tau = 8) = -1.48 \pm 0.10. \quad (25)$$

From the FSS theory as discussed in sect. 2 we expect, in fact, that \bar{g}_4 is independent of N_τ , i.e. is universal also in the continuum limit. Indeed, this seems to be supported by our data. In table 2 we give results for $\bar{g}_4(N_\tau)$ obtained from simulations with $N_\tau = 4, 6$ and 8.

All three values agree reasonably well within errors, although there seems to be a tendency to lower values of \bar{g}_4 for larger N_τ . At least partially this may be related to still too low statistics for lattices with large N_τ as g_4 is a non-self-averaging quantity [18]. We note, however, that a common value of \bar{g}_4 taken from the

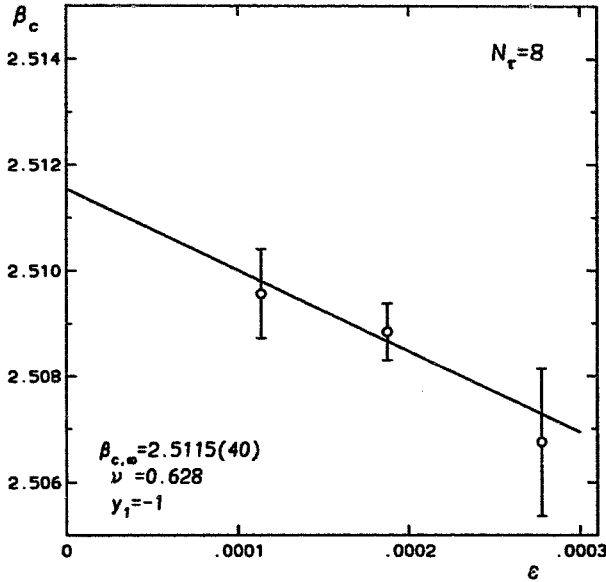


Fig. 2. The coupling value at the intersection points of g_4 for $N_\sigma = 16, 24$ and 32 as a function of ϵ which is defined in eq. (22). The critical coupling $\beta_{c,\infty}$ can be read from the figure as the section of the y-axis at $\epsilon = 0$.

spread of the data for $N_\tau = 4, 6, 8$ and 16 shown in fig. 3 at $t(N_\sigma/N_\tau)^{1/\nu} = 0$ shows good agreement with the value of the three-dimensional Ising model [14,19],

$$\bar{g}_4 = -1.40(10), \quad \bar{g}_{4,\text{Ising}} = -1.41(1). \quad (26), (27)$$

As discussed in sect. 2, g_4 is a scaling function, which in the continuum limit depends only on the reduced temperature, $t = (T - T_c)/T_c$, and the dimensionless quantity $y = LT$. In terms of lattice variables, y is given by

$$y = \frac{N_\sigma}{N_\tau}, \quad (28)$$

TABLE 2
Fixed point value of g_4 for different N_τ

N_τ	\bar{g}_4
4	-1.38(3)
6	-1.43(8)
8	-1.48(10)

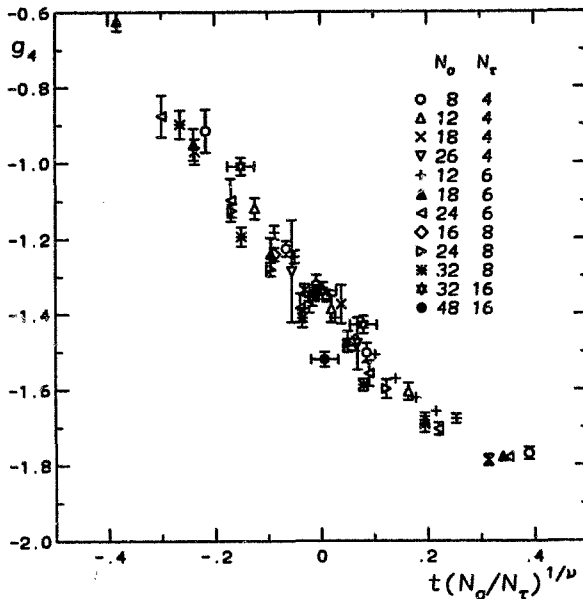


Fig. 3. The Binder cumulant g_4 as a function of $ty^{1/\nu}$ for various lattice sizes as given in the figure. The critical exponent ν has been taken to be the one of the three-dimensional Ising model, $\nu = 0.628$. For $N_\tau = 16$ and $N_\sigma = 32$ and 48 we also mark the error in x -direction caused by the uncertainty in the critical coupling $\beta_{c,\infty}$.

and the reduced temperature in the asymptotic scaling regime is given by eq. (10). For small values of t one expects

$$g_4(t, y) = g_4(ty^{1/\nu}) = \bar{g}_4 + g_{4,1}ty^{1/\nu}. \quad (29)$$

This universal scaling behaviour indeed seems to be fulfilled quite well, as can be seen from fig. 3, where we show $g_4(t, y)$ for various values of N_τ and N_σ . The critical exponent ν has been taken from the three-dimensional Ising model [14]. Otherwise the presentation in fig. 3 is parameter free. We note that indeed the slope, $g_{4,1}$, seems to be universal and within our accuracy does not show any further N_τ or even N_σ dependence.

The universal scaling behaviour of g_4 in the vicinity of the critical point can be explored to determine the critical couplings for larger values of N_τ without an explicit determination of the fixed point \bar{g}_4 from simulations on lattices of varying spatial size. We have calculated g_4 at three β -values on a 16×32^3 lattice and at one β -value on a 16×48^3 lattice. Details for these runs are also given in table 1. In fig. 4 we show the Binder cumulant as a function of β . The critical coupling $\beta_c(N_\tau = 16)$ can now be determined from the shift of β required to overlay the data of fig. 4 with those for $N_\sigma = 16$ and 24 shown in fig. 1. A matching procedure according to eq. (18) gives the shift values $\Delta\beta_{y=2} = 0.232(2)$ and $\Delta\beta_{y=3} = 0.224(2)$.

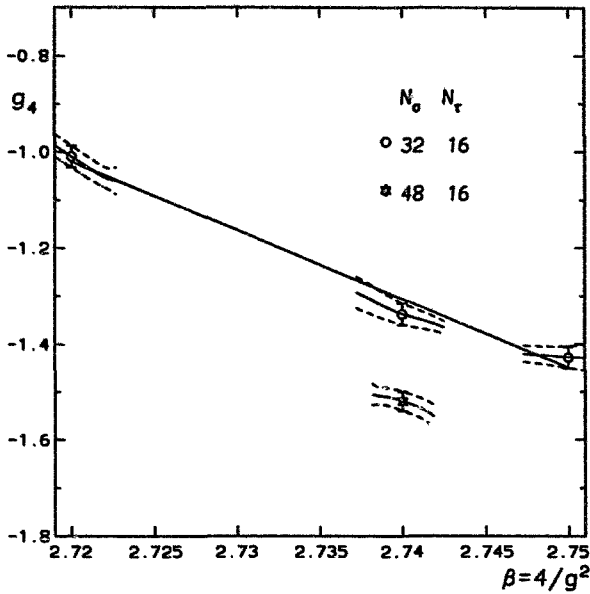


Fig. 4. The cumulant g_4 for N_τ and $N_\sigma = 32$ (dots) and 48 (star) as a function of β . The straight line represents a linear fit to the data points for $N_\sigma = 32$. The four curves through the data points are obtained using the single histogram version of the DSM. The range of validity follows from the condition that at least 2.5% of the measurements in the data sample is contained in each tail of the distribution of U_p . This gives the range of validity in U_p , which is converted to a coupling range using the computed energy coupling relation. The dashed lines indicate the error on these curves.

The two values show a noticeable difference, although the statistical error is relatively large. To investigate whether the observed y -dependence is of significance, we checked if this is present also for $N_\tau = 4, 6$ and 8. We determined the shift $\Delta\beta_y$ for $y = 2, 3$ and 4 using data from refs. [7,8] and found that there is no significant y -dependence of $\Delta\beta_y$. Furthermore $\Delta\beta_y$ agrees with the shift $\Delta\beta = \beta_{c,\infty}(N_{\tau,1}) - \beta_{c,\infty}(N_{\tau,2})$ calculated from the infinite-volume critical coupling. For the largest lattice size $N_\tau = 16$ the critical slowing down is most severe, as is seen from τ_{int} given in table 1. As mentioned before g_4 is a non-self-averaging quantity and like for the susceptibility the expectation value will be too small, if one has not enough independent measurements [18]. This seems to be true for our dataset on the $N_\sigma = 48, N_\tau = 16$ lattice and we can not obtain the same precision as in the previous cases of $N_\tau = 4, 6$ and 8.

Averaging over the two values of $\Delta\beta_y$ at $y = 2$ and 3 and assigning a large error, which includes both numbers we get $\Delta\beta(N_\tau = 16) = 0.228(6)$, which is consistent with results obtained on lattices of size 32^4 [20]. Using eqs. (20) and (23) this corresponds to a critical coupling

$$\beta_{c,\infty}(N_\tau = 16) = 2.7395 \pm 0.0100. \tag{30}$$

We want to point out that the combination of Monte Carlo simulation and the DSM is especially suited to determine the intersection points of g_4 for small volumes. If the system becomes too large, the width of the histograms of U_p becomes very small and one needs a lot of simulations at different β -values to cover a given range of the coupling. This was the case for our results on lattices with $N_\tau = 16$ and for these we could only use the single histogram version of the DSM.

4. Scaling and asymptotic scaling

The critical couplings determined in sect. 3 can be used to test scaling and asymptotic scaling in SU(2) gauge theory over a much wider range of the coupling than was previously possible. Additionally we will compare the SU(2) results to existing data for SU(3) and four-flavour QCD. After that we will try to point out common features of the data.

Recently the heavy quark potential has been studied in the pure SU(2) gauge theory on quite large lattices and several values of the gauge coupling [5,6,21–23] which cover the range of couplings we have studied here. The string tension extracted from the long-distance part of the heavy quark potential can be used to test the scaling of dimensionless ratios of physical quantities. In particular, we will discuss here the scaling of $T_c/\sqrt{\sigma}$. In tables 3 and 4 we have collected results from most recent calculations of the string tension in SU(2) [5,6,21–23] as well as SU(3) [24,25] gauge theory. For four-flavour QCD we used the value $\sqrt{\sigma} a = 0.332(2)$ [26].

We have used a spline interpolation between these values to determine the string tension $\sqrt{\sigma} a(\beta)$ at the critical couplings of the deconfinement transition. These critical couplings are displayed in tables 5 and 6 [27–29]. The resulting values for the ratio,

$$\frac{T_c}{\sqrt{\sigma}} = (\sqrt{\sigma} a(\beta_c^\infty(N_\tau))N_\tau)^{-1}, \quad (31)$$

TABLE 3
SU(2) string tension

N_σ	N_τ	β	$\sqrt{\sigma} a$
8	10	2.20	0.4690(100)
10	10	2.30	0.3690(30)
16	16	2.40	0.2660(20)
32	32	2.50	0.1905(8)
20	20	2.60	0.1360(40)
32	32	2.70	0.1015(10)
48	56	2.85	0.0630(30)

TABLE 4
SU(3) string tension

N_σ	N_τ	β	$\sqrt{\sigma} a$
16	32	5.60	0.5295(9)
16	24	5.70	0.4099(12)
16	24	5.80	0.3302(15)
16	24	5.90	0.2702(19)
32	32	6.00	0.2269(62)
24	32	6.20	0.1619(19)
32	32	6.40	0.1214(12)
32	32	6.80	0.0738(20)

TABLE 5
Critical couplings for SU(2)

N_τ	β_c	T_c/A_L	$T_c/A_{\overline{MS}}$	$T_c/A_{\overline{MS}} _E$
2	1.8800(30)	29.7(2)	1.499(11)	0.852(6)
3	2.1768(30)	41.4(3)	2.089(16)	1.213(11)
4	2.2986(6)	42.1(1)	2.125(3)	1.313(3)
5	2.3726(45)	40.6(5)	2.047(23)	1.360(21)
6	2.4265(30)	38.7(3)	1.954(15)	1.354(13)
8	2.5115(40)	36.0(4)	1.815(18)	1.325(16)
16	2.7395(100)	32.0(8)	1.616(41)	1.271(37)

are shown in fig. 5 as a function of $aT_c \equiv 1/N_\tau$. Also shown in this figure is a result for four-flavour QCD, which combines the measured critical coupling for $N_\tau = 8$ ($\beta_c = 5.15 \pm 0.05$) [30] and the result for the string tension measured at $\beta = 5.15$ on a 24×16^3 lattice [26].

It is rather remarkable that scaling of $T_c/\sqrt{\sigma}$ is valid in the case of SU(2) to a high degree over the entire regime of lattice spacings from $aT_c \leq 0.25$ ($N_\tau \geq 4$) onwards. For SU(3) this seems to hold for $aT_c \leq 0.125$ ($N_\tau \geq 8$), although for larger values of the lattice spacing deviations are also only on the few percent level.

TABLE 6
Critical couplings for SU(3)

N_σ	N_τ	β_c	T_c/A_L	$T_c/A_{\overline{MS}}$	$T_c/A_{\overline{MS}} _E$
12	3	5.5500(100)	85.70 ± 0.96	2.975 ± 0.033	1.217 ± 0.038
∞	4	5.6925(2)	75.41 ± 0.02	2.618 ± 0.001	1.318 ± 0.012
∞	6	5.8941(5)	63.05 ± 0.04	2.189 ± 0.001	1.299 ± 0.002
16	8	6.0010(250)	53.34 ± 1.50	1.851 ± 0.052	1.152 ± 0.043
16	10	6.1600(70)	51.05 ± 0.40	1.772 ± 0.014	1.155 ± 0.012
16	12	6.2680(120)	48.05 ± 0.65	1.668 ± 0.023	1.110 ± 0.017
16	14	6.3830(100)	46.90 ± 0.53	1.628 ± 0.018	1.111 ± 0.016
24	16	6.4500(500)	46.27 ± 2.50	1.537 ± 0.087	1.064 ± 0.069

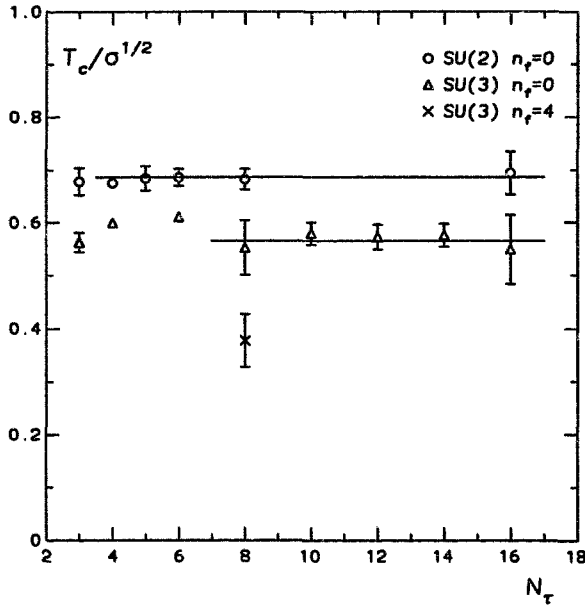


Fig. 5. The critical temperature in units of the square root of the string tension versus the $aT_c \equiv 1/N_f$ for the case of SU(2) (circles) and SU(3) (triangles) pure gauge theory as well as QCD with four flavours of the dynamical fermions of mass $ma = 0.01$ (cross). The straight lines correspond to one-parameter fits to the data.

Averaging over the region $0.0625 \leq aT_c \leq 0.25$ ($4 \leq N_f \leq 16$) for the SU(2) data and over $0.0714 \leq aT_c \leq 0.125$ ($8 \leq N_f \leq 14$) for SU(3) we obtain

$$\frac{T_c}{\sqrt{\sigma}} = \begin{cases} 0.69 \pm 0.02, & \text{SU(2),} \\ 0.56 \pm 0.03, & \text{SU(3)} \\ 0.38 \pm 0.05, & \text{QCD, } n_f = 4. \end{cases} \quad (32)$$

A strong decrease of this ratio with increasing number of degrees of freedom (partons) is obvious. Such a behaviour is expected, as a certain critical energy density can be reached already at a lower temperature when the number of partons is larger. Moreover, it is natural to assume that the string tension increases when the number of gluonic degrees of freedom increases. Both effects tend to lower the value of $T_c / \sqrt{\sigma}$ with increasing N and/or n_f .

A more natural scale to compare the critical temperature in pure SU(N) gauge theories is given by the glueball mass. The energy density in units of T^4 of a massive, free glueball gas is a function of $x = m_G/T$ only,

$$\epsilon/T^4 = d \frac{x^3}{2\pi^2} \sum_{n=1}^{\infty} \left[K_1(nx) + \frac{3}{nx} K_2(nx) \right], \quad (33)$$

where d is a degeneracy factor, K_1 and K_2 are modified Bessel functions. One thus might expect that the critical behaviour in the SU(2) and SU(3) theory is controlled by the ratio T_c/m_G , with m_G denoting the mass of the lightest glueball. In fact, using recent glueball data [21,31,32], we find that this ratio varies only little between SU(2) and SU(3),

$$\frac{T_c}{m_G} = \begin{cases} 0.180 \pm 0.016, & \text{SU(2),} \\ 0.176 \pm 0.020, & \text{SU(3).} \end{cases} \quad (34)$$

It is well known that asymptotic scaling does not hold in the coupling regime considered by us, despite the fact that scaling works remarkably well. This suggests that there are universal scaling violating terms, which are common to the different physical observables and thus may partially be absorbed in a redefinition of the coupling constant. The need for such a resummation has been noticed quite early and various schemes have been suggested [33–37]. In the case of the SU(3) deconfinement transition a variant of the scheme originally introduced by Parisi [33] has been found to yield quite good results already for rather large lattice spacings [38]. Here the bare coupling, β , is replaced by an effective coupling, β_E , which is related to the plaquette expectation value and thus takes into account rapid fluctuations in the pure gauge action at intermediate values of the coupling. Further details on the definition of β_E and a collection of plaquette expectation values needed to calculate β_E are given in appendix B.

In fig. 6 we show $T_c/\Lambda_{\overline{\text{MS}}}$, using a conversion factor $\Lambda_{\overline{\text{MS}}}/\Lambda_L$ [39] given by

$$\frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_L} = 38.852704 \times \exp\left(-\frac{3\pi^2}{11N^2}\right). \quad (35)$$

In the case of full QCD with four flavours we used the value $\Lambda_{\overline{\text{MS}}}/\Lambda_L = 76.45$ for the conversion factor.

It is evident, that the asymptotic scaling violations are much reduced when β_E is used as a coupling constant. This is particularly true for the case of SU(3) shown in fig. 6b. The $O(g^7)$ term in the SU(N) β -function would lead to violations of asymptotic scaling, which are of $O(1/\ln a)$. Our data for $T_c/\Lambda_{\overline{\text{MS}}}$ at finite values of the cut-off suggest, however, that the deviations from asymptotic scaling are well approximated by a correction term which is $O(a)$. This holds for both coupling schemes, with the bare coupling β and with the effective coupling β_E . In the latter case, though, the coefficient of the correction term is much smaller. Performing a linear extrapolation of $T_c/\Lambda_{\overline{\text{MS}}}$ to $a = 0$ in the effective coupling scheme we find

$$\frac{T_c}{\Lambda_{\overline{\text{MS}}}} = \begin{cases} 1.23 \pm 0.11, & \text{SU(2),} \\ 1.03 \pm 0.19, & \text{SU(3),} \end{cases} \quad (36)$$

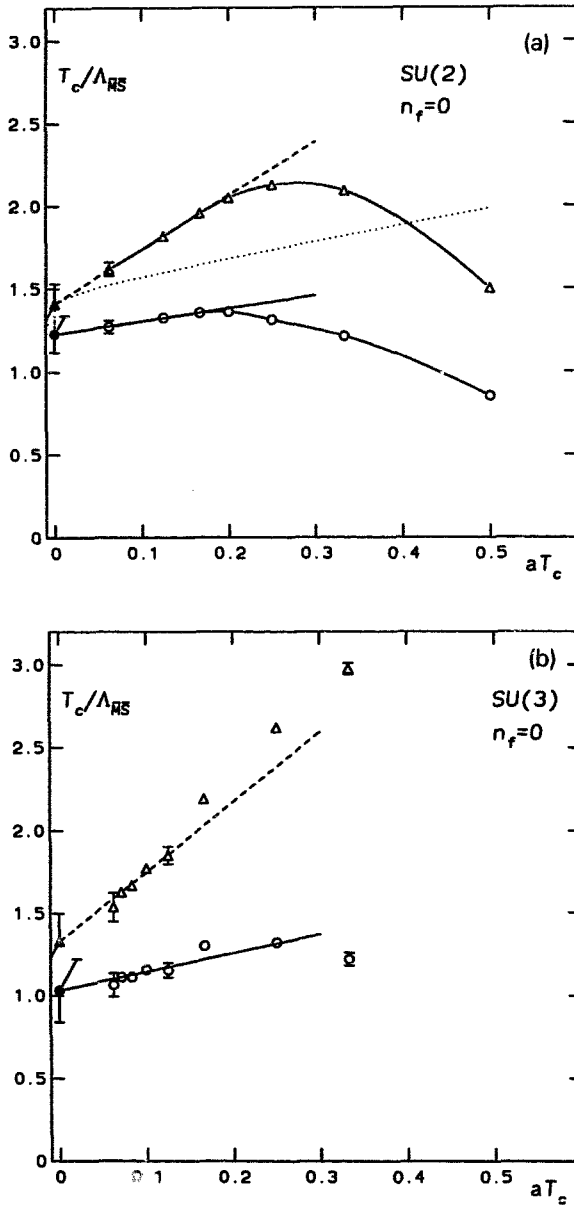


Fig. 6. The critical temperature in units of $\Lambda_{\overline{MS}}$ for the $SU(2)$ (a) and $SU(3)$ (b) gauge theory. Shown are data obtained by using the bare coupling constant (triangles) and the effective coupling β_E (circles) as input in the asymptotic renormalization group equation. The solid lines give linear extrapolations of these data sets to the continuum limit, $a=0$, in the effective coupling scheme. The broken lines indicate a corresponding linear extrapolation using the bare coupling. The position of the filled symbols marks the result of the extrapolation. The dotted line in (a) marks how both coupling schemes approach in the continuum limit, where we assumed a linear form of $T_c/\Lambda_{\overline{MS}}$ as a function of a given by the fit for the effective coupling scheme and used the third-order expansion of the internal energy given by eq. (B.1).

where we used the region $0.0625 \leq aT_c \leq 0.1667$ ($6 \leq N_\tau \leq 16$) for the SU(2) data and $0.0714 \leq aT_c \leq 0.1250$ ($8 \leq N_\tau \leq 14$) for SU(3). We want to note that this result does not disagree with a linear extrapolation for the bare coupling scheme within errors.

It is remarkable that T_c in units of $\Lambda_{\overline{\text{MS}}}$ differs only by less than 20% between SU(2) and SU(3) and also the corresponding value for QCD with four flavours, $T_c/\Lambda_{\overline{\text{MS}}} = 1.05(20)$ [30], is surprisingly close to the SU(3) value.

A corresponding linear extrapolation of the string tension $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}}$ in the region $0.0625 \leq aT_c \leq 0.1667$ ($6 \leq N_\tau \leq 16$) for SU(2) and $0.0714 \leq aT_c \leq 0.1250$ ($8 \leq N_\tau \leq 14$) for SU(3) to the continuum limit $a = 0$ gives the result

$$\frac{\sqrt{\sigma}}{\Lambda_{\overline{\text{MS}}}} = \begin{cases} 1.79 \pm 0.12, & \text{SU(2),} \\ 1.75 \pm 0.20, & \text{SU(3).} \end{cases} \quad (37)$$

The data from table 4 together with the extrapolated values are shown in fig. 7. The corresponding value for four-flavour QCD is $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}} = 2.78 \pm 0.50$. This result is, of course, consistent with the scaling value of $T_c/\sqrt{\sigma}$ given in eq. (32).

We note that our value of the SU(2) string tension is only slightly larger than a recent estimate from the heavy quark potential [6], which yielded $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}} = 1.61(9)$. The agreement between these entirely different approaches gives additional support for our $a = 0$ extrapolation in the β_E -scheme.

5. Conclusions

We determined the critical coupling for the SU(2) deconfinement transition on lattices with large temporal extent, $N_\tau = 8$ and 16, using a FSS analysis of the Binder cumulant g_4 . We find

$$\beta_{c,\infty}(N_\tau = 8) = 2.5115(40), \quad \beta_{c,\infty}(N_\tau = 16) = 2.7395(100).$$

A comparison with existing string tension data showed scaling in the entire coupling range $2.30 \leq \beta \leq 2.74$ under consideration. For the SU(3) gauge group scaling sets in at a value of $\beta \approx 6$. We observe a 20% increase in $T_c/\sqrt{\sigma}$ when going from SU(3) to SU(2), which can be understood in terms of the larger number of gluons in the SU(3) gauge theory, which tend to decrease T_c and increase $\sqrt{\sigma}$ at the same time.

In the bare coupling $\beta = 2N/g^2$ we observe no sign of asymptotic scaling up to $\beta = 2.74$. The alternative coupling scheme β_E , derived from the measured pla-

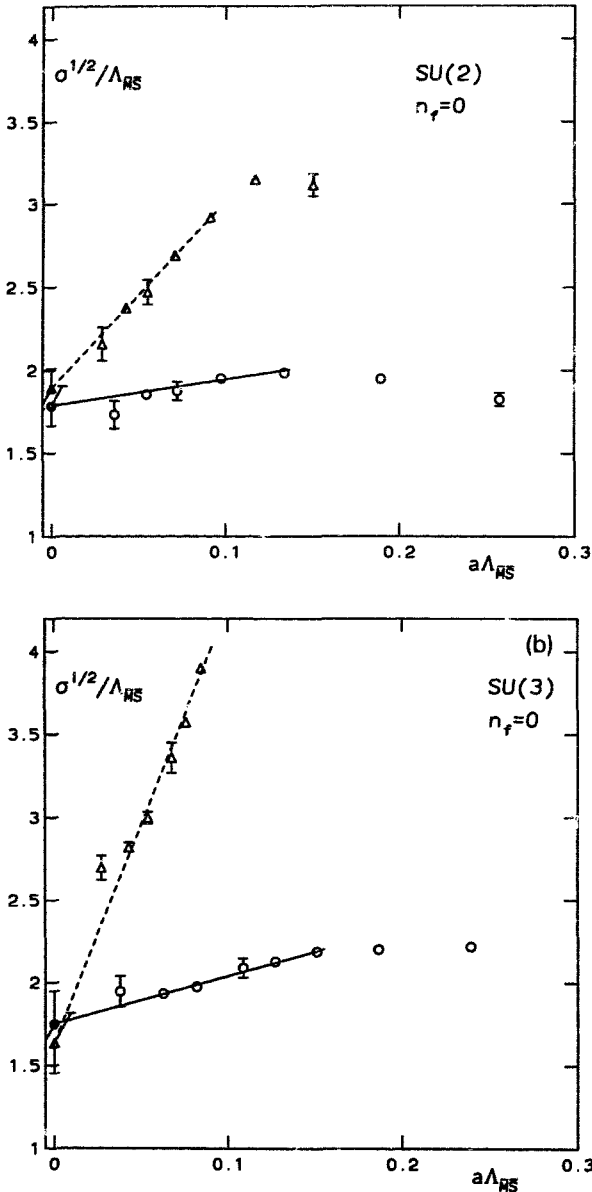


Fig. 7. The string tension $\sqrt{\sigma}$ in units of $\Lambda_{\overline{MS}}$ as a function of $a\Lambda_{\overline{MS}}$ obtained from the two-loop renormalization group equation (5) for the SU(2) (a) and SU(3) (b) gauge theory. The convention for symbols and lines is the same as in fig. 6.

quette expectation value, shows much less deviations from asymptotic scaling. Our data indicate that these deviations are well described by $O(a)$ corrections and a linear extrapolation of T_c/Λ_L and $\sqrt{\sigma}/\Lambda_L$ to $a = 0$ seems to be justified.

We note, however, that our analysis clearly is not sensitive to $O(1/\ln a)$ correction terms, which would be nearly constant in the rather small interval of couplings considered by us.

Our Monte Carlo simulations have been performed on the Connection Machines CM-2 of the HLRZ-Jülich at GMD, Birlingshoven, of SCRI, Tallahassee, on the Intel iPSC/860 at the ZAM, Forschungszentrum Jülich, and on the NEC SX-3 computer of the RRZ Köln.

We thank these computer centers for providing us with the necessary computational resources and the staff for their continuous support. In particular we thank Dr. Völpel, GMD, for making the CM-2 available to us during the Christmas break. Financial support from DFG under contact Pe 340/1-3 and from the Ministerium für Wissenschaft und Forschung NRW under contract IVA5-10600990 is gratefully acknowledged (J.F. and F.K.). The work of U.M.H. was supported in part by the DOE under grant DE-FG05ER250000. He also acknowledges partial support by the NSF under grant INT-8922411 and the kind hospitality at the Fakultät für Physik at the University of Bielefeld while this research was begun.

Note added in proof

The perturbative $O(\beta^{-3})$ term, c_3 , in the weak coupling expansion of the internal energy quoted in eq. (B.1), which had been taken from ref. [41], has changed slightly in the final publication [47]. The corrected value is

$$c_3 = (N^2 - 1)N^4(4/3)(0.0546 - 0.0205125/N^2 + 0.0346125/N^4).$$

Appendix A

CUMULANTS OF THE ORDER PARAMETER

We define cumulants of the Polyakov loop expectation value in the following way:

$$K_n = \left(\frac{N_\sigma}{N_\tau} \right)^{d(1-n)} \frac{\partial^n f_s}{\partial h^n} \Big|_{h=0}. \tag{A.1}$$

On a finite lattice all odd cumulants are zero due to the $Z(N)$ center symmetry.

$$K_i = 0, \quad i \text{ odd}, \tag{A.2}$$

$$K_2 = \langle P^2 \rangle, \tag{A.3}$$

$$K_4 = \langle P^4 \rangle - 3\langle P^2 \rangle^2, \quad (\text{A.4})$$

$$K_6 = \langle P^6 \rangle - 15\langle P^2 \rangle \langle P^4 \rangle + 30\langle P^2 \rangle^3 \quad (\text{A.5})$$

$$K_8 = \langle P^8 \rangle - 35\langle P^4 \rangle^2 - 28\langle P^2 \rangle \langle P^6 \rangle + 420\langle P^2 \rangle^2 \langle P^4 \rangle - 630\langle P^2 \rangle^3. \quad (\text{A.6})$$

Then moments of the order parameter can be expressed by cumulants:

$$\langle P^2 \rangle = K_2, \quad (\text{A.7})$$

$$\langle P^4 \rangle = K_4 + 3K_2^2, \quad (\text{A.8})$$

$$\langle P^6 \rangle = K_6 + 15 K_2 K_4 + 75 K_2^3, \quad (\text{A.9})$$

$$\langle P^8 \rangle = K_8 - 28 K_6 K_2 - 35 K_4^2 - 210 K_4 K_2^2 - 1785 K_2^4. \quad (\text{A.10})$$

Now we consider three possible cumulant ratios given by

$$g_4 = \frac{K_4}{K_2^2} = \frac{\langle P^4 \rangle}{\langle P^2 \rangle^2} - 3, \quad (\text{A.11})$$

$$g_6 = \frac{K_6}{K_2^3} = \frac{\langle P^6 \rangle}{\langle P^2 \rangle^3} - 15 g_4 - 75, \quad (\text{A.12})$$

$$g_8 = \frac{K_8}{K_2^4} = \frac{\langle P^8 \rangle}{\langle P^2 \rangle^4} + 28 g_6 + 35 g_4^2 - 210 g_4 - 315. \quad (\text{A.13})$$

The general form of these cumulant ratios obtained from eq. (A.1) is

$$g_{nm} = \frac{K_{nm}}{K_n^m} = \frac{(N_\sigma/N_\tau)^{d(1-nm)}}{(N_\sigma/N_\tau)^{dm(1-n)}} \frac{\partial^{nm} f_s / \partial h^{nm} |_{h=0}}{(\partial^n f_s / \partial h^n |_{h=0})^m} = \frac{Q_{nm}}{Q_n^m}. \quad (\text{A.14})$$

The ratio K_{nm}/K_n^m is directly a scaling function. Therefore it can be expressed as a function of g_4 since both are derived from f_s and depend on the same argument $g_4(N_\sigma/N_\tau)^{1/\nu}$. Then we expect g_{nm} to be constant for a fixed value of g_4 .

In order to test the consistency of the FSS form for f_s , eq. (7), we measured the normalized moments $M_6 = \langle P^6 \rangle / \langle P^2 \rangle^3$ and $M_8 = \langle P^8 \rangle / \langle P^2 \rangle^4$. From eqs. (A.12) and (A.13) we see, that if g_6 and g_8 are constant then M_6 and M_8 should also be constant. The measured values of these moments together with the size of a typical error are given in table A.1. We see, that M_6 and also M_8 are very stable for the different lattice sizes, thus supporting our FSS ansatz for the singular part of the free energy density.

TABLE A.1
Normalized moments at $g_4 = -1.4$

N_σ	N_τ	M_6	M_8
8	4	3.13	6.94
12	4	3.11	6.82
16	8	3.16	7.17
24	8	3.13	6.97
32	16	3.19(10)	7.39(30)

Appendix B

THE EFFECTIVE COUPLING SCHEME

In the β_E scheme the bare coupling, β , is replaced by an effective coupling, β_E , which is related to the internal energy. The starting point is a perturbative weak coupling expansion of the internal energy, which is known up to $O(\beta^{-3})$ [40,41] *,

$$\langle U_p \rangle = 1 - c_1 \beta^{-1} - c_2 \beta^{-2} - c_3 \beta^{-3} + O(\beta^{-4})$$

$$c_1 = (N^2 - 1)(1/4)$$

$$c_2 = (N^2 - 1)N^2(0.0204277 - 1/(32N^2))$$

$$c_3 = (N^2 - 1)N^4(4/3)(0.0066599 - 0.020411/N^2 + 0.0343399/N^4). \quad (B.1)$$

An alternative coupling β_E can now be defined from the first two terms of $\langle U_p \rangle$,

$$\beta_E = \frac{c}{1 - \langle U_p \rangle}, \quad (B.2)$$

with U_p denoting the plaquette operator as defined in sect. 3. The value of $\langle U_p \rangle$ has to be determined by numerical simulations. Our data for the average action are listed in table B.1. For SU(3) the data of the average action were taken from large symmetric lattices [24,42,43]. They are summarized in table B.2.

The normalization in eq. (B.2) is chosen such that the leading term in the weak-coupling expansion of $\beta_E = 2N/g_E^2$ is identical to the bare coupling,

$$\beta_E = \beta - c_2/c_1 + O(\beta^{-1}). \quad (B.3)$$

* We thank H. Panagopoulos for informing us about the result of his $O(\beta^{-3})$ calculation prior to publication.

TABLE B.1
Expectation value of the average action for SU(2)

β	$1 - \langle U_p \rangle$
1.8800	0.52637(5)
2.1768	0.43158(3)
2.2986	0.39746(1)
2.3726	0.37661(2)
2.4265	0.36352(1)
2.5115	0.34564(1)
2.7395	0.30869(2)

Both couplings g and g_E agree in the limit when g goes to zero,

$$g_E^2 = g^2 + (c_2/c_1)g^4/(2N) + (c_3/c_1)g^6/(2N)^2 + O(g^8). \quad (\text{B.4})$$

Additionally in both schemes the β -function dg/da starts with the two universal coefficients b_0 and b_1 , which can be seen from an expansion in the bare coupling g ,

$$\begin{aligned} a \frac{dg_E}{da} &= -b_0 g^3 - b_1 g^5 + \frac{3}{2}(c_2/c_1)b_0 g^5 + O(g^7) \\ &= -b_0 g_E^3 - b_1 g_E^5. \end{aligned} \quad (\text{B.5})$$

The constant c_2/c_1 , however, redefines the lattice Λ -parameter in the β_E -scheme relative to the β -scheme,

$$\frac{\Lambda_E}{\Lambda_L} = \exp\left(\frac{c_2/c_1}{4Nb_0}\right). \quad (\text{B.6})$$

TABLE B.2
Expectation value of the average action for SU(3)

β	$1 - \langle U_p \rangle$
5.40	0.52823(30)
5.51	0.50120(100)
5.60	0.47520(2)
5.70	0.45100(80)
5.75	0.44105(9)
5.80	0.43236(5)
5.90	0.41825(6)
6.00	0.40626(2)
6.10	0.39592(3)
6.20	0.38635(1)
6.30	0.37788(1)
6.40	0.36935(1)
6.60	0.35438(4)
6.80	0.34078(1)
8.00	0.27935(2)

This yields a much larger Λ -parameter for the alternative coupling scheme,

$$\frac{\Lambda_E}{\Lambda_L} = \begin{cases} 1.7217, & \text{SU}(2), \\ 2.0756, & \text{SU}(3). \end{cases} \quad (\text{B.7})$$

The β -function in the β_E scheme can be separated into two terms,

$$a \frac{dg_E}{da} = a \frac{dg}{da} \frac{dg_E}{dg}, \quad (\text{B.8})$$

$$\frac{dg_E}{dg} = \frac{1}{g_E g^3} \frac{16N^2}{N^2 - 1} \frac{dE}{d\beta}. \quad (\text{B.9})$$

The first term is the usual β -function while the second term $dE/d\beta$ is proportional to the “specific heat” c_v *. The entire function dg_E/dg asymptotically approaches unity as the coupling g goes to zero.

For the SU(2) gauge theory a peak of the “specific heat” has been observed at $\beta \approx 2.2$ [44] which can be related to a nearby singularity in a generalized two-parameter coupling space [45]. For SU(3) the same structure was found in form of a huge bump in the region $5.2 \leq \beta \leq 5.8$ [46].

As a consequence the dip in the discrete β -function of the bare coupling seems to be compensated by the peak of the “specific heat”, which motivates the choice of g_E as a better behaved coupling.

References

- [1] M. Creutz, Phys. Rev. D21 (1980) 2308
- [2] U. Wolff, Phys. Lett. B222 (1989) 473; Nucl. Phys. B334 (1990) 581
- [3] A. Hasenfratz, P. Hasenfratz, U. Heller and F. Karsch, Phys. Lett. B140 (1984) 76
- [4] R. Gupta, G.W. Kilcup, A Patel and S.R. Sharpe, Phys. Lett. B211 (1988) 132
- [5] UKQCD Collaboration, S.P. Booth et al., Phys. Lett. B275 (1992) 424
- [6] C. Michael, Phys. Lett. B283 (1992) 103
- [7] J. Engels, J. Fingberg, and M. Weber, Nucl. Phys. B332 (1990) 737
- [8] J. Engels, J. Fingberg and D. Miller, Nucl. Phys. B, to be published
- [9] M.N. Barber, *in* Phase transitions and critical phenomena, Vol. 8, ed. C. Domb and J. Lebowitz (Academic Press, New York, 1981)
- [10] V. Privman, *in* Finite-size scaling and numerical simulations of statistical systems (World Scientific, Singapore, 1990)
- [11] K. Binder, Z. Phys. B43 (1981) 119;
K. Binder, Phys. Rev. Lett. 47 (1981) 693
- [12] G.A. Baker and J. Kincaid, J. Stat. Phys. 24 (1981) 469;
G.A. Baker and B.A. Freedman, J. Phys. A15 (1982) L715

* Here we consider the system as a $(d + 1)$ -dimensional model at “inverse temperature” β , which is not to be confused with the physical temperature T .

- [13] M.N. Barber, R.B. Pearson, D. Toussaint and J.L. Richardson, Phys. Rev. B32 (1985) 1720
- [14] A.M. Ferrenberg and D.P. Landau, Phys. Rev. B44 (1991) 5081
- [15] K. Fredenhagen and M. Marcu, Phys. Lett. B193 (1987) 486
- [16] A. Kennedy and B. Pendleton, Phys. Lett. B156 (1985) 393
- [17] M. Falconi, E. Marinari, M.L. Paciello, G. Parisi and B. Taglienti, Phys. Lett. B108 (1982) 331;
E. Marinari, Nucl. Phys. B235 (1984) 123;
G. Bhanot, S. Black, P. Carter and R. Salvador, Phys. Lett. B183 (1986) 331;
G. Bhanot, K. Bitar, S. Black, P. Carter and R. Salvador, Phys. Lett. B187 (1987) 381;
G. Bhanot, K. Bitar and R. Salvador, Phys. Lett. B188 (1987) 246;
A.M. Ferrenberg and R.H. Swendsen, Phys. Rev. Lett. 61 (1988) 2635; 63 (1989) 1195
- [18] A.M. Ferrenberg, D.P. Landau and K. Binder, J. Stat. Phys. 63 (1991) 867
- [19] V. Privman, P. Hohenberg and A. Aharony, in Phase transitions and critical phenomena, Vol. 14, ed. C. Domb and J. Lebowitz (Academic Press, New York 1991)
- [20] K.M. Decker and Ph. de Forcrand, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 567
- [21] C. Michael and M. Teper, Phys. Lett. B199 (1987) 95
- [22] S. Perantonis, A. Huntley and C. Michael, Nucl. Phys. B326 (1989) 544;
S. Perantonis and C. Michael, Nucl. Phys. B347 (1990) 854; C. Michael and S. Perantonis, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 177
- [23] UKQCD Collaboration, S.P. Booth et al., Liverpool preprint LTH271 (October 1991) and LTH284 (August 1992)
- [24] D. Barkai, K.J.M. Moriarty and C. Rebbi, Phys. Rev. D30 (1984) 1293;
K.C. Bowler, F. Gutbrod, P. Hasenfratz, U. Heller, F. Karsch, R.D. Kenway, I. Montvay, G.S. Pawley, J. Smith and D.J. Wallace, Phys. Lett. B163 (1985) 367
- [25] G.K. Bali and K. Schilling, Wuppertal preprint WUB 92-02 and WUB92-29
- [26] MTc Collaboration, K.D. Born, R. Altmeyer, W. Ibes, E. Laermann, R. Sommer, T.F. Walsh and P. Zerwas, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 394
- [27] N.H. Christ, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 267;
H. Ding and N. Christ, Phys. Rev. Lett. 60 (1988) 1367;
F. Brown et al., Phys. Rev. Lett. 61 (1988) 2058
- [28] Y. Iwasaki, K. Kanaya, T. Yoshié, T. Hoshino, T. Shirakawa, Y. Oyanagi, S. Ichii and T. Kawai, Phys. Rev. Lett. 67 (1991) 3343;
Y. Iwasaki et al., University of Tsubuka preprint UTHE-237 (1992)
- [29] A.D. Kennedy, J. Kuti, S. Meyer and B.J. Pendleton, Phys. Rev. Lett. 54 (1985) 87;
S.A. Gottlieb, J. Kuti, D. Toussaint, A.D. Kennedy, S. Meyer, B.J. Pendleton and R.L. Sugar, Phys. Rev. Lett. 55 (1985) 1958
- [30] MTc Collaboration, R.V. Gavai, S. Gupta, A. Irbäck, F. Karsch, S. Meyer, B. Petersson, H. Satz and H.W. Wyld, Phys. Lett. B241 (1990) 567
- [31] P. van Baal and A.S. Kronfeld, Nucl. Phys. B (Proc. Suppl.) 9 (1990) 227
- [32] C. Michael and M. Teper, Nucl. Phys. B314 (1989) 347;
C. Michael, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 59
- [33] G. Parisi, Proc. XXth Conf. on High energy Physics, Madison 1980
- [34] G.P. Lepage and P.B. Mackenzie, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 173;
A.X. El-Khadra, G. Hockney, A. Kronfeld and P.B. Mackenzie, FERMILAB-PUB-91/354-T (1991)
- [35] F. Green and S. Samuel, Nucl. Phys. B194 (1982) 107;
S. Samuel, O. Martin and K. Moriarty, Phys. Lett. B152 (1984) 87
- [36] Y.M. Makeenko and M.I. Polykarpov, Nucl. Phys. B205 (1982) 386
- [37] M. Lüscher, R. Sommer and U. Wolff, CERN preprint CERN-TH 6566/92 (1992)
- [38] F. Karsch and R. Petronzio, Phys. Lett. B153 (1985) 87
- [39] R. Dashen and D.J. Gross, Phys. Rev. D23 (1981) 2340
- [40] U. Heller and F. Karsch, Nucl. Phys. B258 (1985) 29
- [41] H. Panagopoulos, private communication
- [42] F. Gutbrod, P. Hasenfratz, Z. Kunszt and I. Montvay, Phys. Lett. B128 (1983) 415;
A. Hasenfratz, P. Hasenfratz, U. Heller and F. Karsch, Phys. Lett. B143 (1984) 193;

- K.C. Bowler, A. Hasenfratz, P. Hasenfratz, U. Heller, F. Karsch, R.D. Kenway, H. Meyer-Ortmanns, I. Montvay, G.S. Pawley and D.J. Wallace, Nucl. Phys. B257 (1985) 155
- [43] N. Attig, PhD thesis, University of Bielefeld (1988)
- [44] M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Lett. 42 (1979) 1390; Phys. Rev. D20 (1979) 1915; M. Creutz, Phys. Rev. Lett. 43 (1979) 553; B. Lautrup and M. Nauenberg, Phys. Rev. Lett. 45 (1980) 1755
- [45] G. Bhanot and M. Creutz, Phys. Rev. D24 (1981) 3212
- [46] N. Christ and A. Terrano, Columbia preprint CU-TP-266
- [47] B. Allés, M. Campostrini, A. Feo and H. Panagopoulos, Three-loop results on the lattice, UG-FT-26/92