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MONTE CARLO RENORMALISATION GROUP STUDIES OF SU(3) LATTICE GAUGE THEORY*

CERN-DESY-Edinburgh Collaboration

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Results are reported of Monte Carlo renormalisation group studies of the approach to asymptotic scaling in SU(3) lattice gauge theory. By comparing measurements on 8^4 and 16^4 lattices, estimates are obtained for the shift, $\Delta\beta$, in the fundamental plaquette coupling, β, corresponding to a change of length scale by a factor of 2. The definitions of block link variables contain a free parameter whose value can be optimised to minimise the transient flow to a renormalised trajectory. Our results, at $\beta = 6.0$, 6.3 and 6.6, are consistent with those obtained previously with the improved ratio method, which is also briefly discussed. In both methods simulation is performed only with the standard Wilson action. An important feature of the results is the appearance of a pronounced dip in $\Delta\beta$ which implies that in the presently accessible range of β the asymptotic value is approached from below, and its onset is delayed.

1. Introduction

In the large cut-off limit of renormalisable theories it is possible to tune the cut-off and the coupling(s) together in such a way that the physical content of the theory remains unchanged. The functional relation between the coupling(s) and the cut-off is given by the β-function(s) of the theory. Perturbation theory suggests that in an SU(N) gauge theory the leading cut-off dependent corrections are exponentially small in the inverse of the bare coupling constant g^2. Only the two leading terms of the β-function

$$\beta(g) = -b_0 g^3 - b_1 g^5 + \cdots \tag{1}$$

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are universal: the higher-order corrections are power-like and not necessarily small in the region where the cut-off-dependent corrections are already negligible. There might also be sizeable contributions to the β-function from non-perturbative phenomena.

In the lattice formulation of Yang-Mills theories, where numerical studies are performed at moderate correlation lengths (i.e. at intermediate coupling constant values), the quantitative knowledge of the β-function is of fundamental importance for the correct interpretation of the results. One should confirm also that the β-function approaches the asymptotic form of eq. (1) without passing through a phase transition, assuring a continuum limit with the expected properties of asymptotic freedom and confinement.

In this paper first results obtained by an extended collaboration for a Monte Carlo renormalisation group (MCRG) study of SU(3) lattice gauge theory are reported. In this study block loop expectation values on 16^4 and 8^4 lattices are matched in order to determine the shift $\Delta\beta = \Delta\beta(\beta)$ in the fundamental plaquette coupling β $(=6/g^2)$ of the standard Wilson action, corresponding to a change of scale by a factor of 2. The function $\Delta\beta(\beta)$ is directly related to the integral of the inverse of the β-function and contains the same information:

$$\int_{\beta-\Delta\beta}^{\beta} \frac{dx}{x^{3/2}\beta_{\text{funct.}}(\sqrt{6/x})} = -\sqrt{\tfrac{1}{6}}\,2\ln 2\,. \tag{2}$$

The anticipated renormalisation group (RG) flow [1] in a reduced (3-dimensional) coupling constant space is indicated schematically in fig. 1. The axes refer to a parametrisation of the action in terms of a bare coupling, g^2, and dimensionless couplings, c_i, characterising the relative strengths of fundamental, adjoint, 6-link etc. couplings. The continuum fixed point (FP) lies in the $g^2 = 0$ hyperplane and is stable to all perturbations in this hyperplane. The renormalised trajectory (RT) is the one-dimensional unstable manifold emerging from the fixed point. The transient

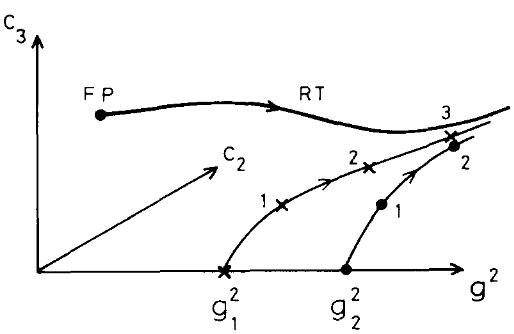


Fig. 1. Schematic diagram of the anticipated RG flow in a three-dimensional coupling constant space. FP denotes the fixed point in the critical hypersurface, whilst RT denotes the one-dimensional unstable manifold.

flow from two points on the g^2 axis (which we conventionally choose to correspond to the standard Wilson action for the pure gauge theory) into the renormalised trajectory is indicated. The positions of the fixed point and renormalised trajectory are not universal - they depend on the choice of RG transformation - but flow transverse to the RT is contractive so that all lattice actions describe the same long-distance physics.

For fixed g_1^2, provided that there exists a universal β-function so that all physical quantities scale in the same way, we are guaranteed to be able to find a value of g_2^2 such that matching is achieved after a suitably large number of blockings, due to the contractive nature of the RG flow. The resulting $\Delta\beta$ is universal. However, there is a finite-size limitation arising from the fact that our starting configurations are on 16^4 and 8^4 lattices, so that we can carry out at most three blockings of the smaller lattice. It is consequently important to take advantage of our freedom in the choice of the RG transformation to extend the definition of the block link variables to include a free parameter which can be used to maximise the rate of convergence of the trajectories through g_1^2 and g_2^2 [2]*. If, by varying this parameter, we could get perfect matching, after one blocking of the larger lattice, of all physical quantities which can be fitted on to the finite lattice, i.e. the points g_1^2 and g_2^2 lie on the same RG trajectory as shown in fig. 2, then we would have determined $\Delta\beta$ *exactly* from the first blocking. All subsequent blockings would yield the same $\Delta\beta$, as the two lattices "follow each other" along the same trajectory. In practice of course perfect matching is never achieved but optimisation speeds up convergence of the sequence of estimates $\{\Delta\beta^{(n)}\}$ by making the trajectory through g_1^2 initially flow close to the g^2 axis (and hence to g_2^2). Furthermore, knowledge of the sequences $\{\Delta\beta^{(n)}\}$ for a range of block transformations around the optimum can be used to obtain monotonically increasing (decreasing) lower (upper) bounds on $\Delta\beta = \Delta\beta^{(\infty)}$, which are important for estimating the systematic error induced by this finite-size effect.

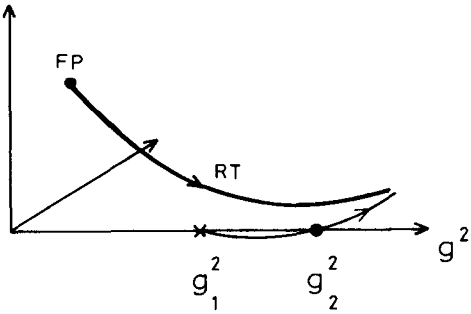


Fig. 2. Schematic diagram of the effect of optimising the RG transformation as discussed in the text.

* In [2] a similar idea to that of Hasenfratz et al. has been put forward and applied to the 3d Ising model recently by Swendsen. The basic idea of a MCRG analysis is suggested by Ma and by Swendsen in the second reference by him.

Our main results are the following matching values at three β-values:

$$\Delta\beta\ (\beta=6.0)=\begin{cases}0.35\pm0.02\,, & \text{scheme 1}\,,\\ 0.34\pm0.02\,, & \text{scheme 2}\,,\end{cases}$$

$$\Delta\beta\ (\beta=6.3)=0.43\pm0.03\,,\quad \text{scheme 1}\,,$$

$$\Delta\beta\ (\beta=6.6)=0.56\pm0.06\,,\quad \text{scheme 1}\,, \tag{3}$$

where the errors include our estimate of both statistical and systematic uncertainties. Results at $\beta=6.9$, 7.2, and of other calculations will be presented elsewhere [3].

The results quoted above are consistent with those obtained earlier by a different MCRG method, the improved ratio method (fig. 3) [4]. The starting point of this latter procedure is the observation [5] that those ratios of Wilson loop expectation values from which the self-mass and corner contributions cancel satisfy the homogeneous renormalisation group (RG) equation, at least if the loops involved are large compared to the lattice spacing. Since the homogeneous RG equation is

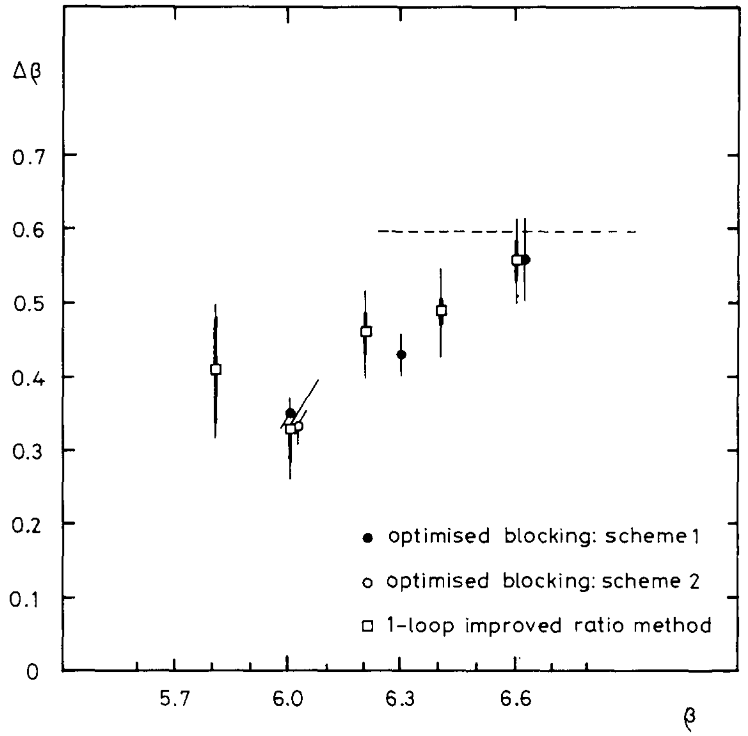


Fig. 3. The shift $\Delta\beta$ as a function of β obtained from 1-loop improved ratios (□) [4] and from optimised blocking scheme 1 (●) and scheme 2 (○) in this work. In the case of the ratio results the thin error bars refer to the statistical error while the thick error bars refer to the average fluctuation of the large number of different ratios included in the analysis. The dashed line is the asymptotic prediction.

linear in these ratios, arbitrary linear combinations of the basic ratios satisfy it also. The method can be optimised by taking particular combinations of the basic ratios to cancel lattice artifacts order by order in perturbation theory. An advantage of this method is its simplicity. Any high statistics measurement of the potential or of the string tension can easily be extended to perform this analysis, and it is expected to work even at large correlation lengths. Its disadvantage is that it is difficult to see how to do the optimisation non-perturbatively and the method requires very good statistics. The statistical error of the block loop matching results at $\beta = 6.0$ quoted in eq. (3) is about 4 times smaller than the corresponding error quoted in ref. [4].

2. The block transformations

The block variable V_{AB} associated with the block link A–B in fig. 4a is chosen with the probability

$$\operatorname{prob}(V_{AB}) \sim \exp \tfrac{1}{6} P \operatorname{Tr}(V_{AB}^{\dagger} X + \text{h.c.}) . \tag{4}$$

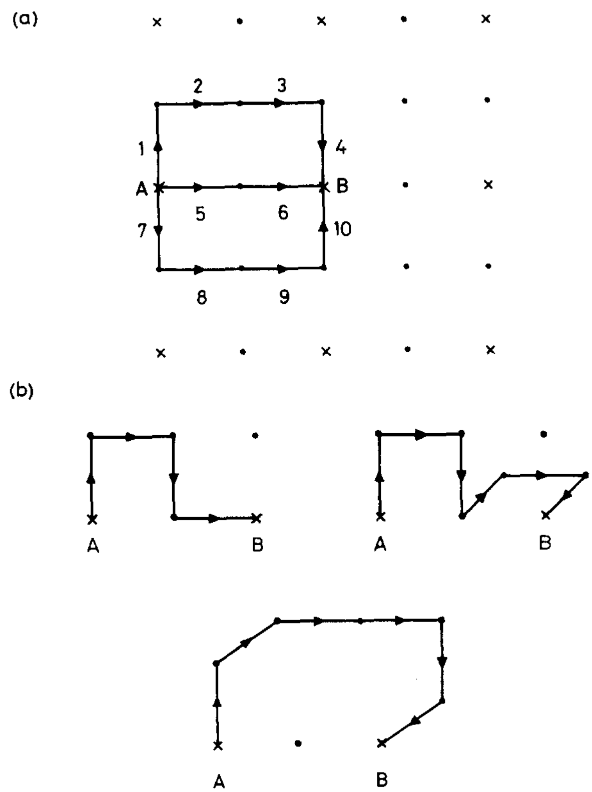


Fig. 4. (a) Construction of the block link in scheme 1, as described in the text. (b) The additional classes of paths used in the definition of the block link in scheme 2.

In blocking scheme 1, X is taken to be the sum of the matrix products along 7 different paths connecting the sites A and B: U_5U_6, $U_1U_2U_3U_4$, $U_7U_8U_9U_{10}$ and the corresponding paths in the orthogonal planes [6]. To test for possible systematic effects arising from this choice of blocking scheme, a second scheme, 2, was tried at $\beta = 6.0$, in which the set of paths connecting A and B was extended to include a further 36 paths of the classes indicated in fig. 4b. $\Delta\beta$ should of course be independent of the blocking scheme. The parameter P is used for optimisation as described in sect. 1.

3. Configurations and statistics

The 16^4 SU(3) configurations at $\beta = 6.0$, 6.3 and 6.6 were created on the DAPs at Edinburgh. After every 112 pseudo-heatbath [7] sweeps the configuration was stored for later blocking and other measurements [3]. The first 1500 sweeps were discarded. The limited memory of the Edinburgh DAPs forced us to store the link variables as 16 bit integers after multiplication by a scale factor $N = 32\,000$, while the matrix multiplications in the updating were done in 3 byte real arithmetic. The corresponding rounding errors introduce additional randomness into the system resulting in a slight systematic error: the configuration looks somewhat "hotter" than the nominal β-value would require. However this effect is very small. Theoretical considerations and test runs with an artificially decreased scale factor, N, suggest that the corresponding error in the plaquette expectation value is $O(10^{-8})$ - well below our statistical accuracy. Details are given in table 1.

The 8^4 configurations at $\beta = 5.4$, 5.6, 5.7, 5.8, 5.9, 6.0 and 6.1 were created at CERN and DESY starting from the last, well-equilibrated configurations of earlier studies. These configurations were separated by 10 pseudo-heatbath sweeps.

Using scheme 1 the blocking was done on the CERN IBM machines at $\beta = 6.0$ at four values of the free parameter, P: 20, 30, 35 and 40. These values were picked after a few trial runs. Similarly at $\beta = 6.6$, the values 22, 25, 30 and 40 were chosen but a preliminary analysis clearly indicated that all the matching results behaved linearly in $1/P$, as observed at $\beta = 6.0$. Thus the complete analysis at $\beta = 6.6$ and at 6.3 was done at just two P values: 25 and 40.

TABLE 1

Effect of various scale factors, N, used for integer storage as described in the text, on the average plaquette for a single 16^4 configuration at $\beta = 6.0$

N	32 000	284	248	124	62	31	15	11	9
$\langle\square\rangle$	1.781	1.781	1.780	1.779	1.781	1.773	1.768	1.755	1.744
ΔE	0	0.000	0.001	0.002	0.000	0.008	0.013	0.026	0.037

ΔE is the shift in the average plaquette relative to the value obtained with the maximal scale factor of 32 000.

The results quoted in eq. (3) for scheme 1 are based on the following statistics:

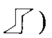
16^4:	$\beta=6.0$,	50 configurations,
	$\beta=6.3$,	59 configurations,
	$\beta=6.6$,	99 configurations,
8^4:	$\beta=5.4$,	32 configurations,
	$\beta=5.6$,	96 configurations,
	$\beta=5.7$,	64 configurations,
	$\beta=5.8$,	96 configurations,
	$\beta=5.9$,	288 configurations,
	$\beta=6.0$,	96 configurations,
	$\beta=6.1$,	96 configurations.

The result at $\beta=6.0$ for the second blocking scheme is based on an analysis of 36 16^4 configurations using three P-values: 26, 21 and 17. Linear interpolation was used throughout to get intermediate β-values.

The statistical errors were estimated by measuring time correlations and also by the usual binning.

4. Results

If for some block transformation the renormalised trajectory runs along the line of the standard Wilson action in the multi-parameter coupling constant space*, then after the first blocking step the effective action is again a standard action at some coupling $\beta'=\beta-\Delta\beta$. In this case the block loop expectation values are equal to the corresponding Wilson loop expectation values of the standard action at coupling β' on a lattice of half the size. The parameter P is fixed by requiring that one gets as close as possible to this situation. To say it in another way: an optimal value of P at each β is determined by requiring the best possible consistent matching for many observables after the first blocking step.

Fig. 5 illustrates the matching values of 12 different loops (1×1, $1\times2,\ldots,4\times4$, , ) after the first blocking step at $\beta=6.0$, using scheme 1 while fig. 6 shows the effect of subsequent blockings on the matching of the 1×1 Wilson loop. Decreasing P-dependence at large length scales is manifested in two ways in these figures. Firstly, at a given level of blocking, matching for the larger loops is in general less P-dependent than for smaller loops. Secondly, matching becomes less P-dependent as the level of blocking increases. The results suggest

$$P^{\rm opt}=35^{+10}_{-5}\,. \qquad (5)$$

It should be emphasised that in principle any value of P is appropriate; the different block transformations should give the same final prediction for $\Delta\beta$.

* Even in principle this is possible only up to the exponentially small corrections discussed in the introduction. We are indebted to J. Kripfganz for a discussion of this point.

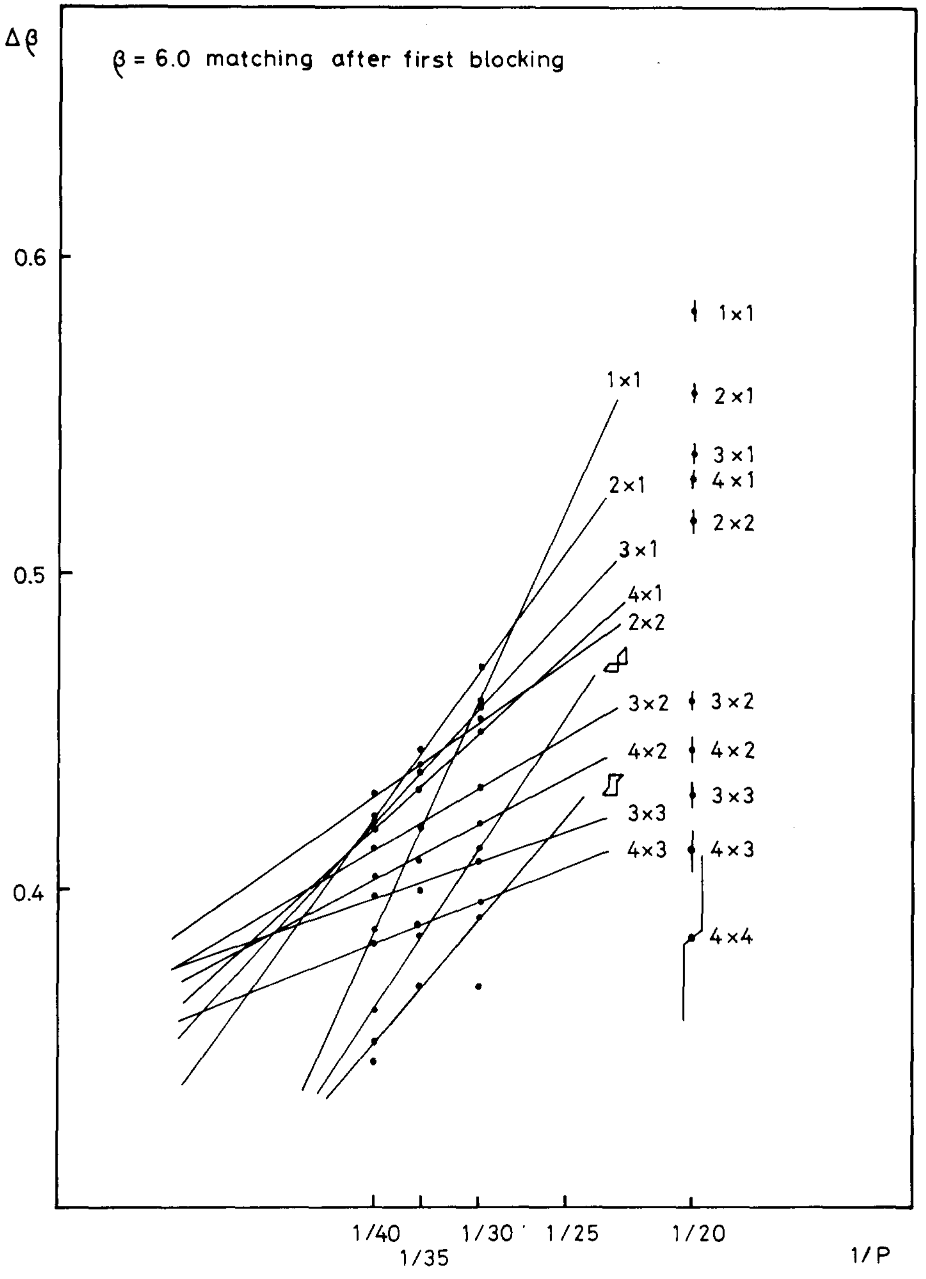


Fig. 5. The matching predictions obtained in scheme 1 from 12 different block loops after the first blocking step at $\beta = 6.0$. P is the free parameter in the block transformation. For $P \gtrsim 30$ the predictions obtained from a given loop are linear in $1/P$. The mean deviation of the matching predictions has a broad minimum in the region $P = 35^{+10}_{-5}$.

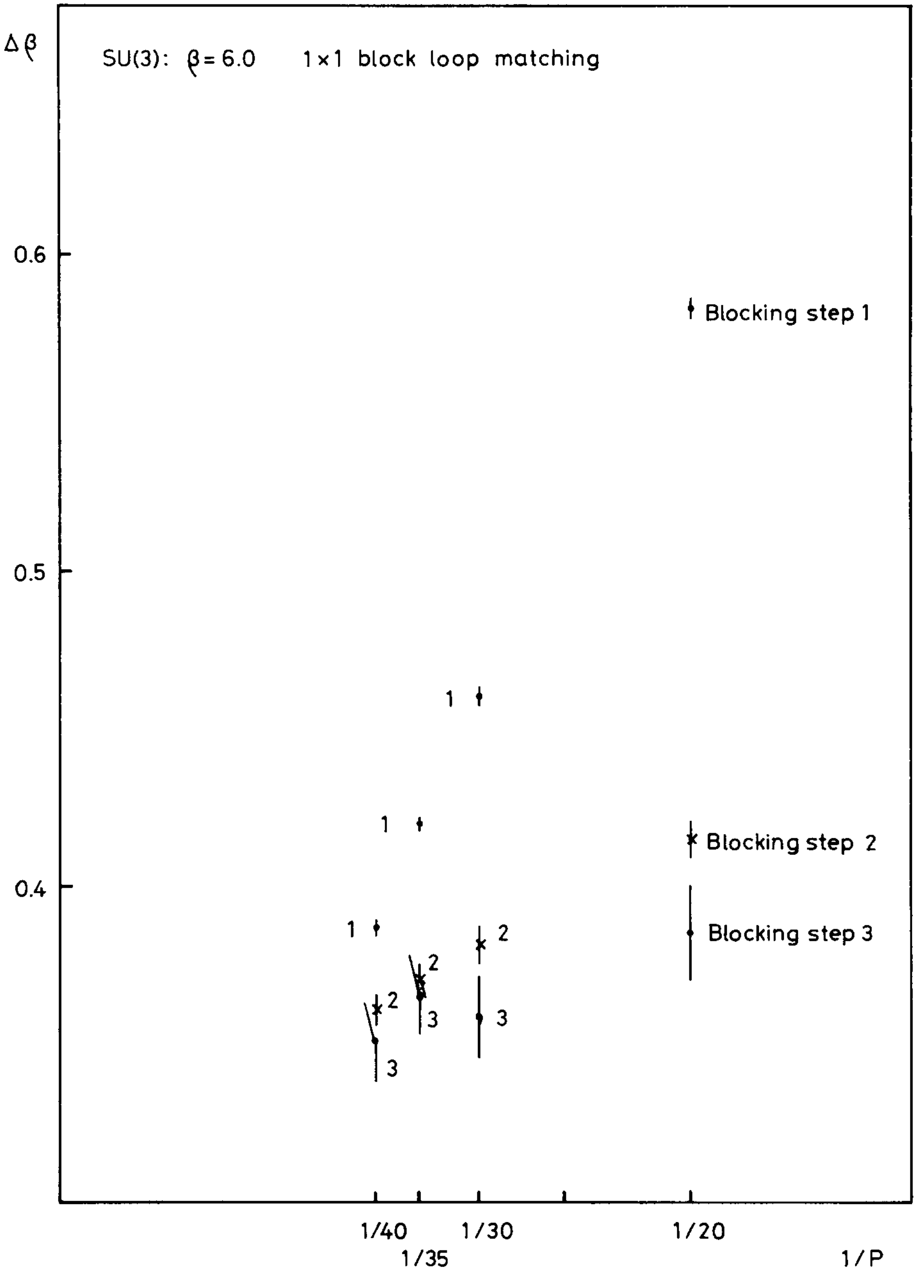


Fig. 6. The matching predictions obtained for the 1 × 1 block loop in scheme 1 at subsequent blocking levels as a function of $1/P$ for $\beta = 6.0$.

A similar analysis at $\beta = 6.3$ yields, after the first blocking step,

$$P_1^{\text{opt}} = 27^{+5}_{-3}\,, \tag{6}$$

whilst after the second blocking step

$$P_2^{\text{opt}} = 32^{+8}_{-6}\,. \tag{7}$$

At $\beta = 6.6$ the corresponding results are

$$P_1^{\text{opt}} = 23.5^{+4}_{-2}\,, \tag{8}$$

$$P_2^{\text{opt}} = 28^{+8}_{-5}\,. \tag{9}$$

Figs. 7–10 show the raw data on which these estimates are based. They illustrate that as β increases, so also does the P-dependence of $\Delta\beta$ (for particular Wilson loops at a given blocking level); hence it is increasingly important to optimise the blocking prescription at the higher β-values.

The predictions for $\Delta\beta$ ($\beta = 6.0$) obtained from matching four different block loops (plaquette, 6_1(= ), 6_2(= ) and 6_3(= )) are given for $P = 30$, 35 and 40 at subsequent blocking levels in table 2. As the statistical errors of the block loops 6_1 and 6_3 after the third blocking step are very large no matching value is quoted there. The P-dependence is linear in $1/P$ (as suggested by perturbation theory) for $P \geqslant 30$, which makes it possible to follow the predictions for large P even without actually measuring them. For the 1×1 block loop matchings $P^{\text{opt}} \sim 50$. This is the value where the predicted $\Delta\beta$ is the same after the first and second blocking step: $\Delta\beta = 0.35 \pm 0.01$ and is consistent with the extrapolated third blocking result. A simple averaging of the third blocking step predictions at $P = 30$, 35 and 40 gives $\Delta\beta = 0.359 \pm 0.013$. It is also encouraging that using the alternative blocking scheme yields results for $\Delta\beta$ ($\beta = 6.0$) which are completely consistent with those of scheme 1 (see eq. (3)).

TABLE 2

The matching predictions $\Delta\beta$ ($\beta = 6.0$) are summarised for 4 different block loops at different blocking levels and different values of P in scheme 1

P	Blocking step				
30	1	0.461 (3)	0.471 (3)	0.414 (3)	0.392 (3)
	2	0.382 (6)	0.375 (7)	0.372 (5)	0.368 (5)
	3	0.359 (13)		0.383 (25)	
35	1	0.420 (2)	0.444 (2)	0.386 (3)	0.370 (3)
	2	0.371 (6)	0.369 (5)	0.365 (5)	0.362 (5)
	3	0.368 (13)		0.348 (21)	
40	1	0.388 (2)	0.424 (2)	0.363 (3)	0.352 (3)
	2	0.361 (5)	0.360 (6)	0.367 (5)	0.355 (5)
	3	0.351 (13)		0.342 (20)	

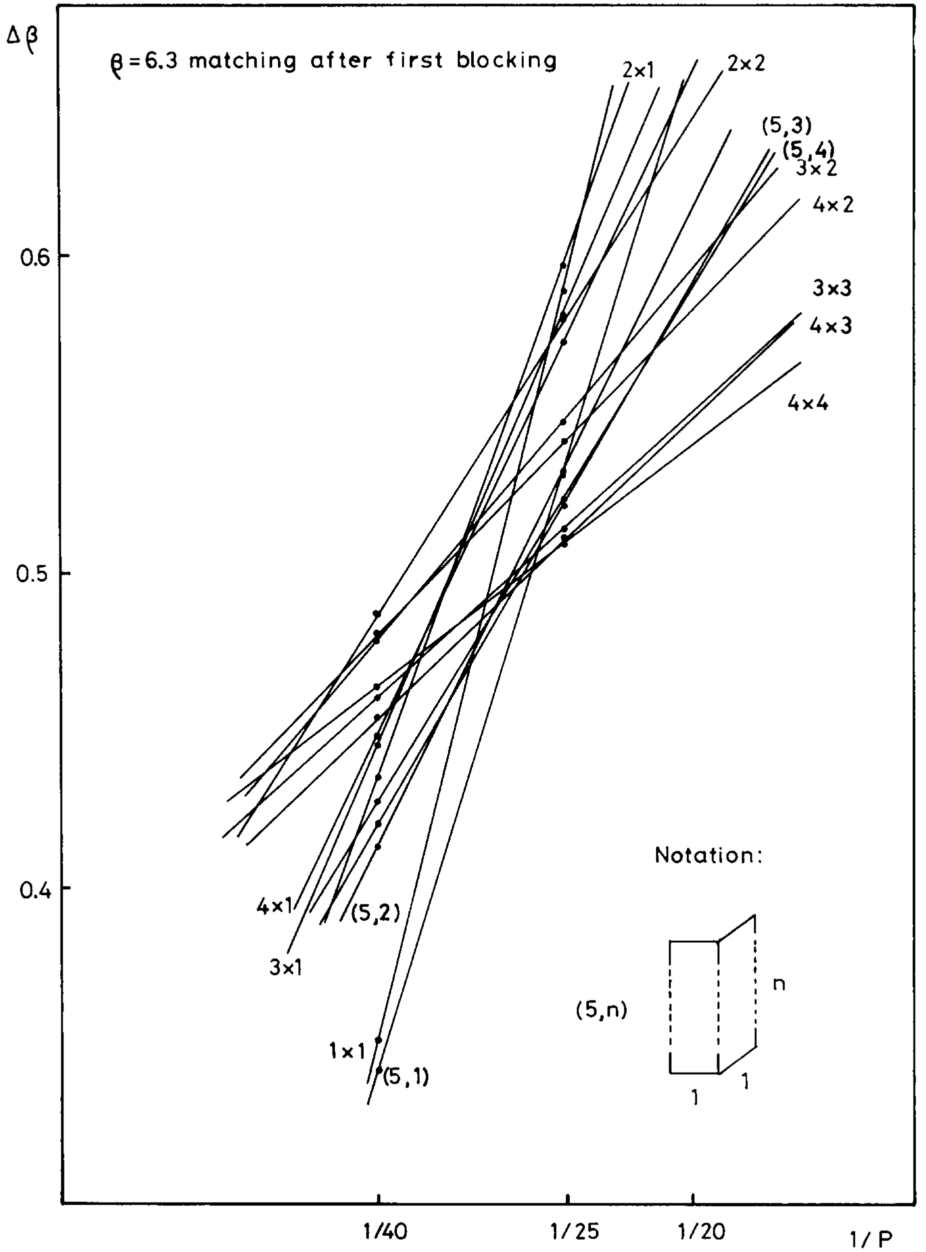


Fig. 7. The matching predictions obtained from various block loops after the first blocking step at $\beta = 6.3$. Error bars have been omitted for clarity but are broadly comparable with those shown in fig. 5.

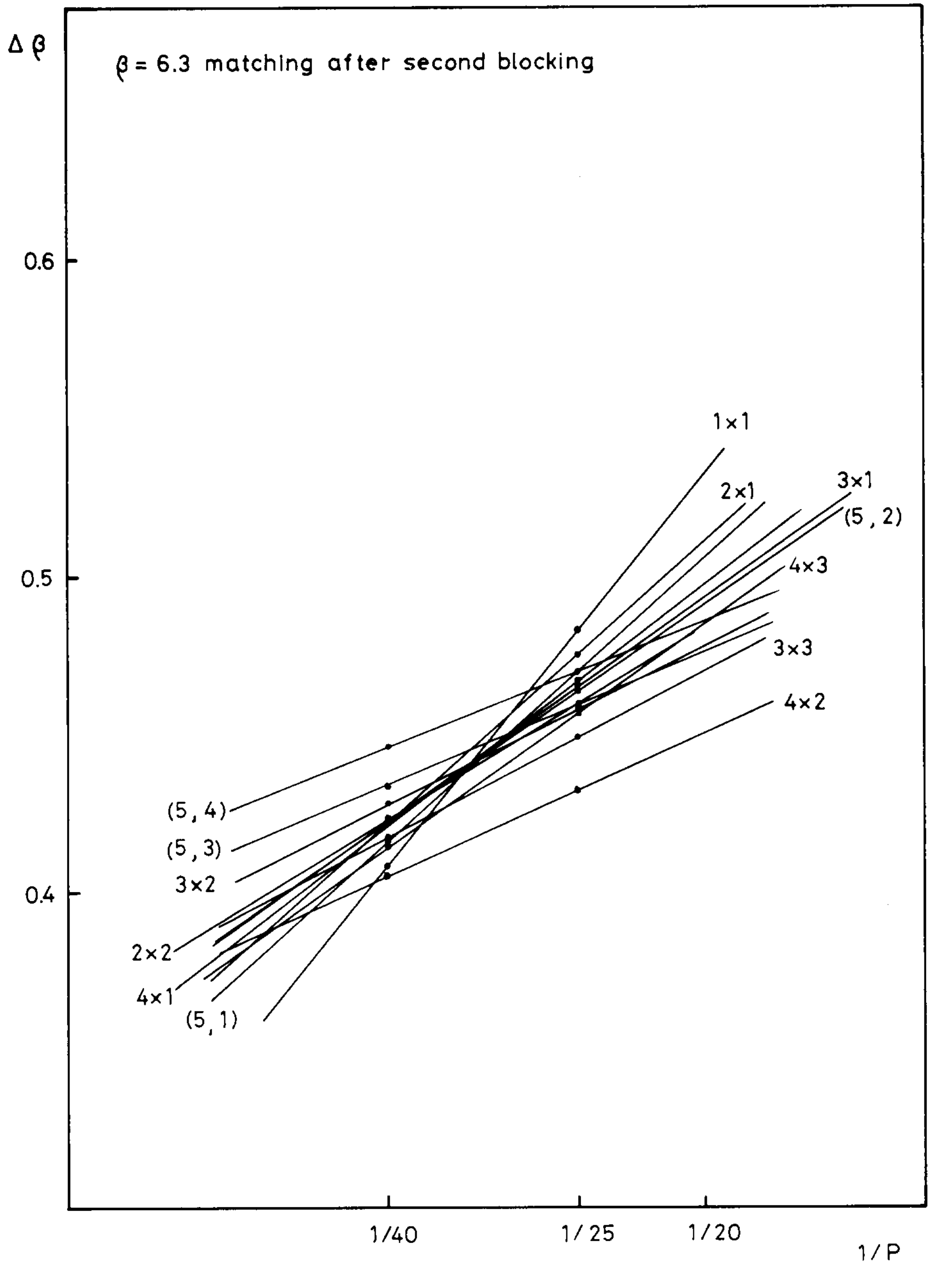


Fig. 8. The matching predictions obtained from various block loops after the second blocking step at $\beta = 6.3$.

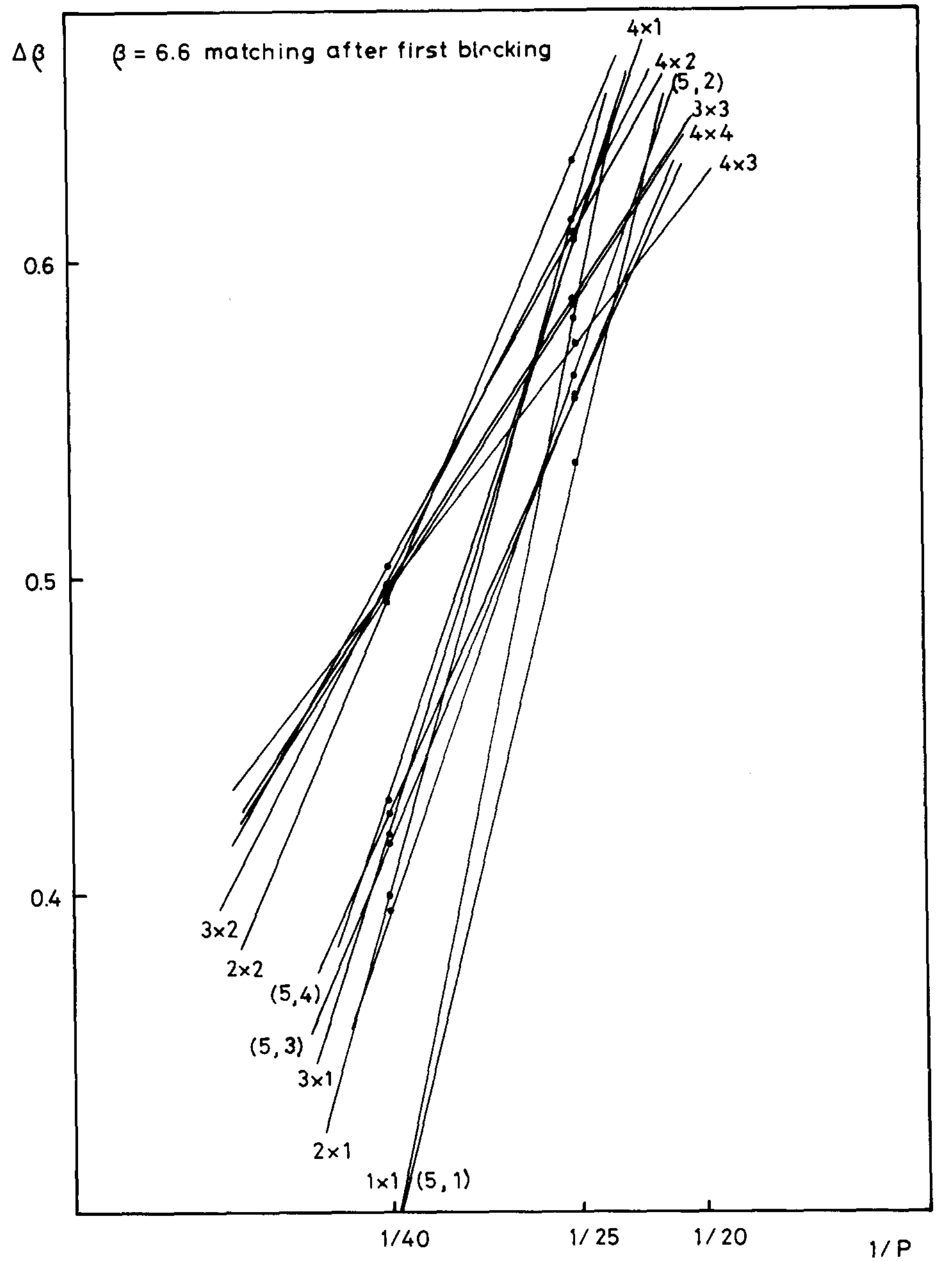


Fig. 9. As fig. 7 but for $\beta = 6.6$.

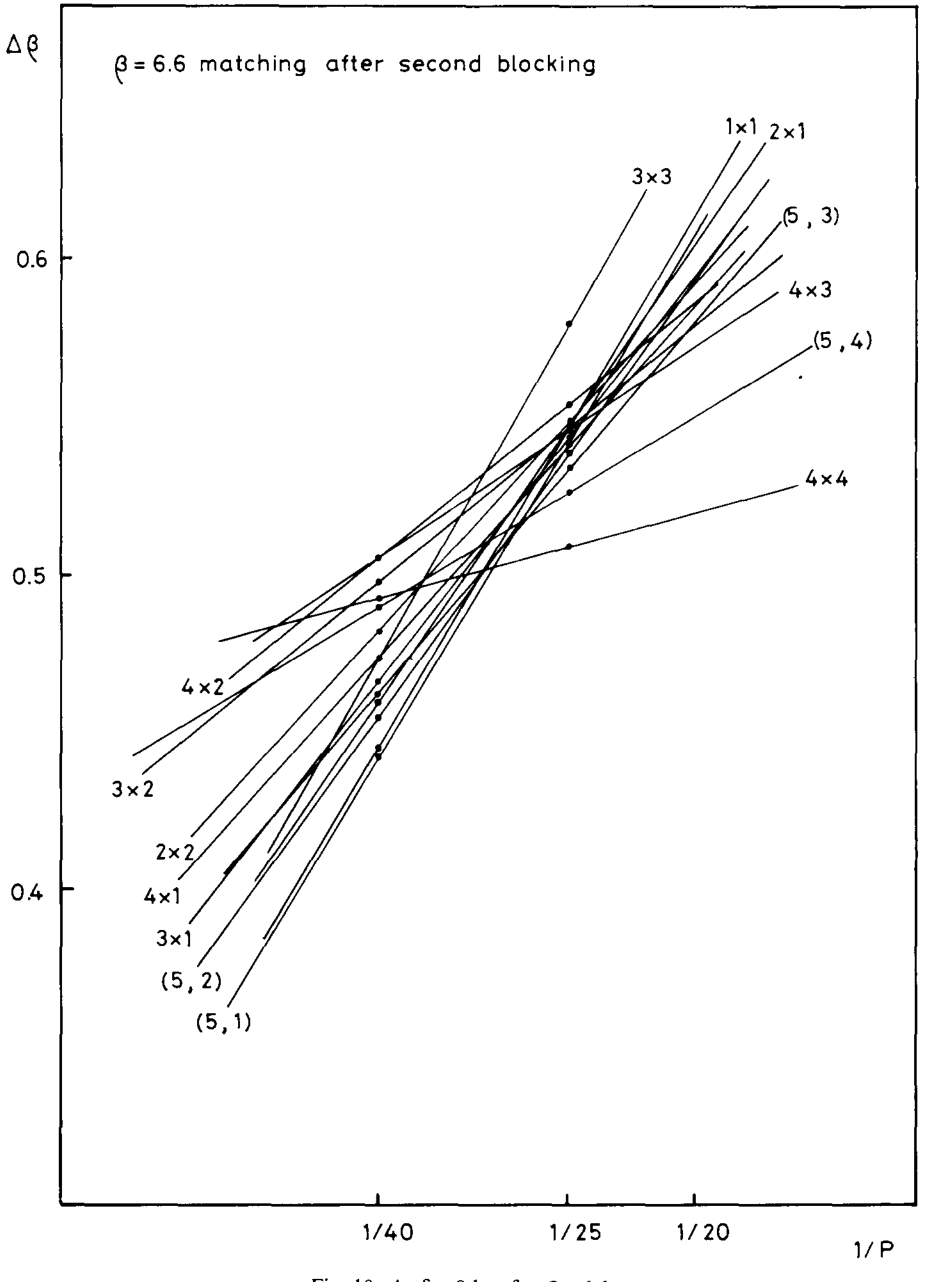


Fig. 10. As fig. 8 but for $\beta = 6.6$.

TABLE 3

$\Delta\beta$ ($n=\infty$) is given as obtained by assuming that the subleading eigenvalue of the linearised RG transformation is $\frac{1}{4}$

P				
30	0.358 (8)	0.346 (9)	0.358 (7)	0.360 (7)
35	0.356 (8)	0.346 (7)	0.358 (7)	0.359 (7)
40	0.353 (7)	0.340 (8)	0.355 (7)	0.356 (7)

The following simple consideration helps to give us confidence that the systematic errors are really under control. It is expected that in the continuum limit the subleading eigenoperator is of dimension 6 with an eigenvalue of $\frac{1}{4}$ (up to negligible logarithmic corrections). This implies the behaviour

$$\Delta\beta^{(n)} = \Delta\beta^{(n=\infty)} + \alpha(P)(\tfrac{1}{4})^n , \tag{10}$$

where n is the number of blocking steps and $\Delta\beta^{(n=\infty)}$ is the P-independent result after $n=\infty$ blocking steps. Eq. (10) gives

$$\Delta\beta^{(n=\infty)} = \tfrac{1}{3}[4\Delta\beta^{(n=2)} - \Delta\beta^{(n=1)}] . \tag{11}$$

If this procedure is consistent the predicted $\Delta\beta^{(n=\infty)}$ should be independent of P and the block loop considered. The numbers are summarised in table 3, where the errors quoted are statistical. On the basis of this table and of the previous considerations we feel that the error estimates in eq. (3) are rather conservative. The matching results for the same four block loops at $\beta=6.3$ and at $\beta=6.6$ are summarised in tables 4 and 5 respectively. The estimate of $\Delta\beta^{(n=\infty)}$ in eq. (11) does not work quite so well at these larger values of β. (The previous argument using the dimension of the subleading eigenoperator is valid only if finite-size effects are negligible in correlation functions, which is certainly not the case when β is as large

TABLE 4

The matching predictions $\Delta\beta$ ($\beta=6.3$) together with estimates for $\Delta\beta^{(n=\infty)}$ as discussed in the text

P	Blocking step				
25	1	0.592 (6)	0.600 (6)	0.534 (5)	0.503 (5)
	2	0.487 (9)	0.475 (10)	0.474 (9)	0.467 (10)
	3	0.44 (2)	$0.45\,(^{+2}_{-3})$	$0.44\,(^{+2}_{-3})$	$0.45\,(^{+2}_{-3})$
	$\frac{1}{3}(4\Delta\beta^{(n=2)} - \Delta\beta^{(n=1)})$	0.451 (14)	0.433 (15)	0.454 (13)	0.456 (14)
	$\frac{1}{3}(4\Delta\beta^{(n=3)} - \Delta\beta^{(n=2)})$	0.43 (3)	$0.44\,(^{+3}_{-4})$	$0.43\,(^{+3}_{-4})$	$0.44\,(^{+3}_{-4})$
40	1	0.356 (3)	0.439 (5)	0.347 (3)	0.348 (4)
	2	0.421 (9)	0.426 (11)	0.419 (10)	0.419 (12)
	3	$0.428\,(^{+14}_{-15})$	0.43 (2)	0.43 (2)	$0.43\,(^{+2}_{-3})$
	$\frac{1}{3}(4\Delta\beta^{(n=2)} - \Delta\beta^{(n=1)})$	0.442 (13)	0.422 (17)	0.443 (14)	0.443 (17)
	$\frac{1}{3}(4\Delta\beta^{(n=3)} - \Delta\beta^{(n=2)})$	0.43 (2)	0.43 (3)	0.43 (3)	$0.44\,(^{+3}_{-4})$

TABLE 5

As table 4 but for $\beta = 6.6$

P	Blocking step				
25	1	0.592 (5)	0.625 (7)	0.544 (5)	0.520 (5)
	2	0.551 (10)	0.554 (21)	0.547 (16)	0.545 (19)
	3	0.55 (5)	0.56 (6)	0.56 (6)	0.56 (6)
	$\frac{1}{3}(4\Delta\beta^{(n=2)} - \Delta\beta^{(n=1)})$	0.538 (15)	0.53 (3)	0.548 (23)	0.553 (28)
	$\frac{1}{3}(4\Delta\beta^{(n=3)} - \Delta\beta^{(n=2)})$	0.55 (7)	0.56 (9)	0.56 (8)	0.57 (9)
40	1	0.285 (3)	0.401 (5)	0.289 (3)	0.303 (4)
	2	0.453 (17)	0.474 (26)	0.457 (20)	0.459 (22)
	3	0.52 (5)	0.53 (6)	0.53 (6)	0.53 (7)
	$\frac{1}{3}(4\Delta\beta^{(n=2)} - \Delta\beta^{(n=1)})$	0.508 (23)	0.499 (36)	0.512 (28)	0.511 (30)
	$\frac{1}{3}(4\Delta\beta^{(n=3)} - \Delta\beta^{(n=2)})$	0.57 (7)	0.55 (9)	0.55 (9)	0.55 (9)

as 6.6.) A more reliable estimate of $\Delta\beta^{(n=\infty)}$ based upon $\Delta\beta^{(n=2)}$ and $\Delta\beta^{(n=3)}$ is also given. An alternative approach is to note that even without any extrapolation $\Delta\beta^{(n)}$ is decreasing as n increases for $P = 25$, whereas $\Delta\beta^{(n)}$ is increasing as n increases for $P = 40$, in agreement with the expectation that P^{opt} lies somewhere between 25 and 40.

There is one trivial type of systematic error which we did not check, however: the error coming from the linear interpolation between β-values on the 8^4 lattice. This error is very easy to avoid completely (by blocking 8^4 configurations at the estimated value of β') and we intend to do so in the future. The effect is expected to be small.

5. Discussion

Fig. 3 shows a pronounced dip in $\Delta\beta(\beta)$ around $\beta = 6.0$ which implies that in the accessible range of β the asymptotic value is approached from below and that its onset is delayed. For $\beta = 6.6$ this deviation is rather small, which is supported by recent MCRG measurements by Gupta and Patel [8] using a special "$\sqrt{3}$" block transformation [9] at $\beta = 6.5$ and 7.0. However, whilst for us optimisation of the block transformation appears to be an essential ingredient, this is not the case in ref. [8]; this aspect requires clarification e.g. by perturbative analysis. Recent precise string tension [10] and critical temperature [11] (in [11] deviations from asymptotic scaling for the deconfinement temperature have also been observed by Montvay and Pietarinen) measurements show the same qualitative behaviour for $\Delta\beta(\beta)$. A similar structure seems to be emerging in SU(2) according to string tension [12] and preliminary MCRG results [13]. (However, it appears [14] that the SU(3) mass gap has a qualitatively different behaviour and is consistent with asymptotic scaling in the range $5.5 < \beta < 5.9$.) It is an interesting theoretical problem to understand the origin of this "unnatural" behaviour, which is unlike that found in the $d = 2$ standard non-linear σ-model, where the asymptotic value is approached smoothly from above [2, 15]. It is even more important to determine β_{min} above which the different methods give quantitatively the same $\Delta\beta(\beta)$. For $\beta > \beta_{min} - \Delta\beta(\beta_{min})$ a unique β-function can be defined and the theory reflects the continuum properties.

Concerning the first question it is a natural assumption that the dip in $\Delta\beta$ is related to the critical point at the end of the first-order transition line in the fundamental-adjoint coupling constant plane. The flow away from the spurious critical point is expected to slow down the flow from $\beta_f = \infty$ until the flow has passed the neighbourhood of the spurious critical point, when the two flows reinforce and speed up the flow towards the fixed point at $\beta_f = \beta_a = 0$. The slowing down implies that $\Delta\beta_f$ approaches its asymptotic value from below. A related explanation was suggested by Makeenko and Polikarpov recently [16]. According to these ideas smoother behaviour and earlier onset of asymptotic scaling is expected along the lines $\beta_f/\beta_a = -c$ $(c > 0)$. It is known that the peak in the specific heat is strongly reduced in this region [17].

The explanation of the dip in $\Delta\beta$ in terms of higher-order perturbative terms of the β-function is very improbable [18].

The answer to the second question requires precision data. It is exciting and reassuring that the kind of precision quoted here and in related works might pin down β_{min} and predict the β-function with a reasonable accuracy.

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