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A MONTE CARLO STUDY OF THE β-FUNCTION OF THE SU(3) WILSON ACTION

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We discuss the behaviour of the β-function of the standard SU(3) Wilson action at intermediate couplings. Results obtained from measurements of the deconfinement temperature and string tension on large lattices are compared with those obtained from a systematically optimized Monte Carlo Renormalization Group method.

1. INTRODUCTION

In the large cut-off limit of renormalizable theories it is possible to tune the cut-off and coupling(s) in such a way that the physical content of the theory remains unchanged. The functional relation between the coupling(s) and the cut-off is given by the β-function(s) of the theory. In an SU(N) lattice gauge theory the β-function describes the way the bare coupling g(a) has to be changed when the lattice spacing a is varried in order to leave all physical predictions unchanged: $\beta(g)=-a dg(a)/da$. However, for a generic value of the cut-off the function g(a) depends on the specific quantity which is kept fixed - there is no way to keep all physical predictions unchanged. It is only in the large cut-off limit that a unique β-function can be defined. But also in this case it still depends on the renormalization scheme choosen. In particular, the β-function depends on the lattice action choosen, only the two leading terms in its perturbative expansion are universal:

$$\beta(g) = -b_o\, g^3 - b_1\, g^5 + O(g^7)$$

with (1)

$$b_o = 11N/(48\pi^2);\ b_1 = \frac{34}{3}\,(N/16\pi^2)^2.$$

Early exploratory studies of both SU(2) and SU(3) gauge theories indicated that already at moderate correlation length (i.e. at intermediate coupling constant values) physical quantities seem to scale according to the above perturbative SU(N) β-function [1]. However, recent detailed studies on large lattices (which allow to study the theories at smaller couplings)

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showed that substantial deviations from this universal "asymptotic scaling" behaviour are still present at intermediate couplings [2-5]. This led to sometimes confusing interpretations of quantitative results for physical observables. For a correct interpretation of results obtained at intermediate couplings and their extrapolation to the continuum limit it is thus basically important to reveal and understand the quantitative structure of the β-function.

In the following we will discuss the determination of the SU(3) β-function by using the deconfinement temperature as a physical observable which is held fixed under changes of the cut-off. In section 3 we will discuss a MCRG approach to determine the β-function [6] and compare the results with those of section 2 and recent measurements of the string tension [4,7]. Section 4 contains our conclusions.

2. EXTRACTING A β-FUNCTION FROM THE SU(3) DECONFINEMENT TEMPERATURE

At finite temperature the SU(3) gauge theory exhibits a first order deconfining phase transition [8,9]. On a lattice of size $N_\beta \times N_\sigma^3$ ($N_\sigma \gg N_\beta$) the order parameter for this transition, the thermal Wilson line, is discontinous at the critical coupling $\beta_C\,(N_\beta) = 6/g_c^2\,(a)$. This jump in the order parameter provides a very clear signal for the critical temperature

$$T_c^{-1} = N_\beta g(g_c^2). \tag{2}$$

Demanding that T_c remains unchanged when the cut-off a is varried allows to determine a discretized version of the β-function: Changing the temporal extend of the lattice by a factor n corresponds to a decrease of the lattice spacing by the same factor in order to keep T_c unchanged

$$T_c^{-1} = N_{\beta,1}a\;(g_{c1}^2) = N_{\beta,2}a\;(g_{c2}^2)$$

with (3)

$$\frac{a\;(g_{c2}^2)}{a(g_{c1}^2)} = \frac{N_{\beta,1}}{N_{\beta,2}}$$

In the following we will restrict ourselves to scale changes by a factor n=2. The change Δβ in the critical couplings $\beta_c\;(N_\beta) = 6/g_c^2$,

$$\Delta\beta = \beta_c\;(2N_\beta) - \beta_c(N_\beta) \tag{4}$$

is related to the β-function through

$$\int_{\beta-\Delta\beta}^{\beta} \frac{dx}{x^{3/2}\beta_{funct}\left((6/x)^{1/2}\right)} = -\frac{2\ln 2}{\sqrt{6}} \qquad (5)$$

From the perturbative β-function, eq.(1), one finds that in the limit $g^2 \to 0$ $\Delta\beta$ approaches a constant value:

$$(\Delta\beta)_{g^2\to o} = 132\ \ln 2/(16\pi^2) \approx 0.579 \qquad (6)$$

At finite g^2 the inclusion of the two-loop perturbative corrections leads to somewhat larger values for $\Delta\beta$. For instance in the coupling region $\beta \approx 6.0$ one would expect to find $\Delta\beta \approx 0.61$, if the perturbative β-function is still valid in this intermediate coupling regime. The critical temperature T_c has by now been determined on lattices of various temporal extend N_β [2,5,8,9]. The available results are summarized in table 1. As can be seen the critical temperature does not stay constant when one assumes the validity of the two-loop perturbative result for the β-function.

N_β	N_σ	$6/g^2$	T_c/Λ_L	$\Delta\beta$
2	12	5.11 ±0.01	78±1	
	∞	5.097±0.001		
3	10	5.55±0.01	86±1	
4	10	5.70±0.01	76±1	0.59±0.02
	∞	5.696±0.004		
5	12	5.79-5.82	68.5±1	
6	16	5.92-5.94	65.5±1	0.38±0.02
	∞	5.877±0.006	62±.3	0.33±0.01
8	∞	6.00±0.02	53±1.5	0.30±0.02
10	∞	6.22±0.07	55±4	0.41±0.09

Table 1: Summary of critical couplings of the SU(3) deconfinement transition on lattices of size $N_\sigma^3 \times N_\beta$. The rows with $N_\sigma = \infty$ refer to the extrapolated data of ref. [5]. The other data are taken from ref. [2, 8]. Also given are the critical temperatures T_c/Λ obtained by assuming the validity of eq. (1) and the $\Delta\beta$ obtained from eq. 4.

The last column of table 1 shows the resulting values for $\Delta\beta$ when T_c is kept fixed. They clearly show that the asymptotic scaling regime has not been reached below $\beta = 6.0$. Indeed they are still decreasing between $\beta \approx 5.5$ and 6.0.

To extract $\Delta\beta$ from measurements of T_c for larger values of β much larger lattices are required. In addition T_c is just one physical observable and more observables have to be analysed in order to check whether the β-function is unique for all these observables in the β range considered. In the following we will discuss a MCRG study of the SU(3) β-function which analyses a large set of different physical observables and allows to make contact with the perturbative regime.

3. THE RATIO METHOD

The basic idea of the ratio method [10] is to extract the β -function from ratios of Wilson loop expectation values which are combined in such a way that the self mass and corner contributions cancel. These ratios satisfy the homogeneous renormalization group equation, thus by comparing ratios of loops calculated at some value of β with corresponding once formed from half as large loops at an appropriate β' the change of the couplings $\Delta\beta=\beta-\beta'$ neccessary to achieve matching between these ratios can be determined.

There are two problems, however. First, ratios composed of small loops are contaminated by lattice artifacts resulting in a systematic error which increases linearly with β [6,11]. Second, the matching prediction is distorted by finite size effects if the correlation length is comparable or larger than the lattice size. While the last problem can easily be handled by measuring Wilson loops at β and β' on lattices of size L^4 and $(L/2)^4$ respectively, the first problem requires a selection of systematically improved observables, which are free of lattice artifacts. These are constructed as follows: First the basic ratios are formed as

$$f(i_1, i_2; i_3, i_4) = \frac{W(i_1, i_2)}{W(i_3, i_4)}, \quad i_1+i_2 = i_3+i_4 \tag{7}$$

$$g(i_1, i_2; i_3, i_4; i_5, i_6; i_7, i_8) = \frac{W(i_1, i_2)\, W(i_3, i_4)}{W(i_5, i_6)\, W(i_7, i_8)},$$

$$i_1+i_2+i_3+i_4 = i_5+i_6+i_7+i_8.$$

and so on. Here $W(i_1,i_2)$ is the expectation value of a planar Wilson loop of size i_1,i_2. Apart from lattice artifacts these loops satisfy the RG-equation

$$f\,(2i_1,\ 2i_2;\ 2i_3,\ 2i_4;\ \beta,L) = f\,(i_1,\ i_2;\ i_3,\ i_4;\ \beta',\ L/2) \tag{8}$$

and similar equations for g,.... Any linear combination of the functions f,g,... defined in eq.(7) satisfy eq.(8) also. In the improved ratio method the mixing coefficients are determined by the requirement of cancelling the lattice artifact corrections to eq.(8) systematically order by order in perturbation theory [6,12]. The improvement procedure is illustrated in table 2.

BASIC RATIOS	WEAK COUPLING $\Delta\beta$
$R_1 = \dfrac{W(3,3)}{W(2,4)}$	-0.158β
$R_2 = \dfrac{W(1,1)W(3,3)}{W(1,2)W(2,3)}$	-0.057β
$R_3 = \dfrac{W(1,2)W(2,3)}{W(1,3)W(1,3)}$	0.046β
TREE LEVEL IMPROVED RATIOS	
$R_{13} = R_1 + 0.688298\ R_3$ $R_{23} = R_2 + 0.523659\ R_3$	0.582 0.492
1 - LOOP IMPROVED RATIOS	
$R_{123} = R_1 + 0.027917\ R_2 + 0.702917\ R_3$	0.579

Table 2: Illustration of the improvement procedure for three basic ratios. Tree level and one-loop improved ratios are formed fromm the three basic ratios R_1, R_2 and R_3. The mixing leads to a systematic improvement of the weak coupling behaviour by canceling the $O(\beta^{-1})$ (tree level) and $O(\beta^{o})$ (one-loop level) lattice artifacts for the observables considered. The last column shows the shift $\Delta\beta$ for the listed ratios obtained in the weak coupling limit.

A large number of systematically improved ratios can be obtained this way. These mixed ratios have been used in a MC analysis to determine $\Delta\beta$ [6]. Ratios of Wilson loops measured on a 16^4 lattice at several values of β [4] have been compared with those on a 8^4 lattice at β'. The results obtained for $\Delta\beta$ are shown in fig. 1 together with those deduced from the deconfinement temperature and measurements of the string tension [4,7].

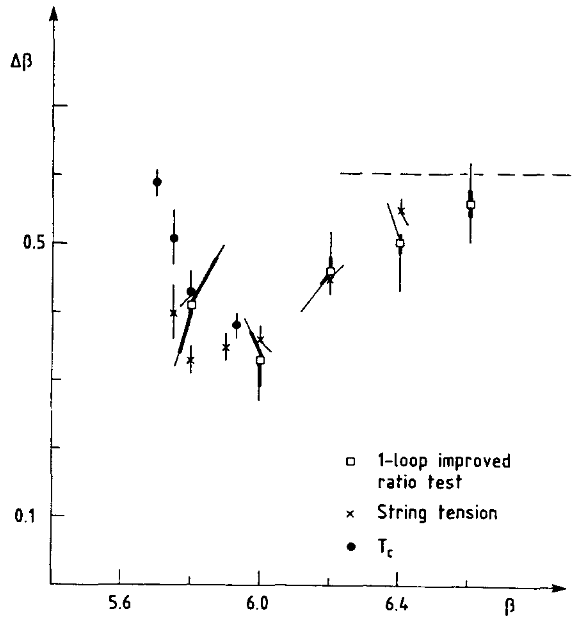

Figure 1: The average shift $\Delta\beta$ as a function of β obtained from the analysis of one-loop improved ratios (squares). (At β = 5.8 the basic ratios are used.) The error bars refer to statistical fluctuations (thin bars) and the average fluctuations (thick bars) of the matching predictions obtained from different ratios. Also shown are the predictions for $\Delta\beta$ obtained from the string tension (crosses) and the critical temperature (full points).

As can be seen the matching predictions start approaching the perturbative result above β = 6.0. In particular the result at β=6.6

$$\Delta\beta \ (\beta = 6.6) = 0.56 \pm 0.06 \tag{9}$$

shows that there are only small deviations from asymptotic scaling for $\beta < 6.0$.

4. CONCLUSIONS

The available information on the SU(3) β-function shows that there are substantial deviations from asymptotic scaling in the intermediate coupling regime $\beta \approx 5.5 - 6.0$. However, above $\beta = 6.0$ the deviations from asymptotic scaling seem to be small. The fact that quite differnt approaches and physical observables lead to compatible results for $\Delta\beta$ seems to indicate that already in the non-asymptotic regime a unique β-function exists. This is supported by recent MCRG calculations [13,14] where block-spin transformations have been used to extract $\Delta\beta$. Thus a consistent quantitative understanding of the rather non-trivial way the standard Wilson action approaches continuum seems to be emerging.

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REFERENCES

1) M. Creutz, Phys. Rev. D21 (1980) 2308;
E. Pietarinen, Nucl. Phys. B190 [FS3] (1981) 349.

2) F. Karsch and R. Petronzio, Phys. Lett. 139b (1984) 403.

3) F. Gutbrod and I. Montvay, Phys. Lett. 136B (1984) 411.

4) D. Barkai, K. J. M. Moriarty and C. Rebbi, Brookhaven preprint, BNL-34462 (1984).

5) A. Kennedy, J. Kuti, S. Meyer and B. Pendelton, Santa Barbara preprint, NSF-ITP-84-61 (1984).

6) A. Hasenfratz, P. Hasenfratz, U. Heller and F. Karsch, Phys. Lett. 143B (1984) 193.

7) F. Gutbrod, P. Hasenfratz, Z. Kunstz and I. Montvay, Phys. Lett. 128B (1983) 415.

8) T. Celik, J. Engels and H. Satz, Phys. Lett. 125B (1983) 411.

9) J. Kogut, M. Stone, H.W. Wyld, W.R. Gibbs, J. Shigemitsu, S.H. Shenker and D.K. Sinclair, Phys. Rev. Lett. 50 (1983) 393.

10) M. Creutz, Phys. Rev. D23 (1981) 1815; R.W.B. Ardill, M. Creutz and K.J.M. Moriarty, Phys. Rev. D27 (1983) 1956.

11) A. Hasenfratz, P. Hasenfratz, U. Heller and F. Karsch, Phys. Lett. 140B (1984) 76.

12) U. Heller and F. Karsch, CERN preprint, Ref. TH.-3879-CERN (1984).

13) K.C. Bowler, R.D. Kenway, G.S. Pawley, D.J. Wallace, A. Hasenfratz, P. Hasenfratz, U. Heller, F. Karsch and I. Montvay, CERN preprint, Ref. TH.-3952-CERN (1984).

14) R. Gupta and A. Patel, CALTECH preprint (1984).