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THE SPECTRUM OF A CLASS OF SUPERSYMMETRIC THEORIES WITH FALSE VACUA

F. KARSCH

CERN, Geneva, Switzerland

E. RABINOVICI*

*Racah Institute of Physics and Hebrew University, Jerusalem, Israel***

G. SHORE and G. VENEZIANO

CERN, Geneva, Switzerland

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The spectrum of a supersymmetric quantum mechanics model, whose potential has a steep supersymmetric minimum and a broad non-supersymmetric minimum, is analyzed. With the exception of the supersymmetric ground state, the low-energy spectrum is found to be determined entirely by the non-supersymmetric well. The model is motivated by effective lagrangians proposed for supersymmetric QCD. It is speculated that in an equivalent field theory exhibiting a supersymmetric true vacuum and a non-supersymmetric false vacuum, the false vacuum can play an important rôle in the physics, and that the lowest energy excitations are extended field configurations involving a new mass scale.

1. Introduction

Recently, Peskin [1] proposed an effective lagrangian for supersymmetric QCD with a very curious potential. His potential is very similar to that shown in fig. 1. It depends on a parameter m, the quark mass. It comprises, for small m, a double well, the left-hand well being broad and with $V > 0$ at the minimum, and the right-hand well deep and narrow with $V = 0$ at the minimum. The wells are separated by a large potential barrier.

A potential of this form is very unusual in field theory, and it is clearly an interesting question to ask what is the spectrum of such a model. For reasons that will be clearer at the end of this paper, this is unlikely to be a simple task. We have

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** Permanent address.

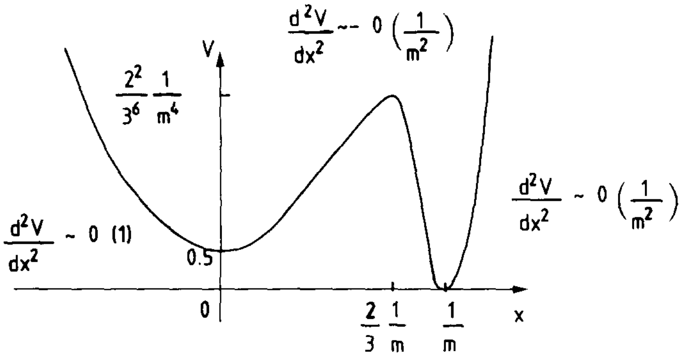

Fig. 1. Sketch of the potential $V(x)$ for the model of sect. 3 for small m.

therefore addressed the much simpler problem of determining the spectrum of a supersymmetric quantum mechanical model with the same features.

The results are quite interesting. Essentially, we find that the model has a supersymmetric ground state localized in the steep well, but that for small m, the lowest excited states are localized in, and take their characteristics from, the broad well. The low-energy spectrum of this model, which has a supersymmetric minimum, is therefore determined entirely by the non-supersymmetric minimum, with the important exception that it includes also a supersymmetric ground state.

After presenting first the necessary formalism of supersymmetric quantum mechanics, we give general arguments for this phenomenon, and then support these by explicit numerical solutions of the Schrödinger equation for the energy levels and wave functions of the model.

Finally, we present some speculations about the spectrum of the equivalent field theory, based on these results. It is suggested that the lowest excitations in the field theory may be unusual "lump" type configurations characterised by a mass scale involving the height of the false vacuum as well as the shape of the barrier. Excitations of the lowest energy lump would then depend upon the curvature of the potential around the false minimum.

2. Supersymmetric quantum mechanics

The formalism for describing the quantum mechanics of particles with fermionic as well as bosonic degrees of freedom was established by Witten [2] and has been further developed by several authors [3]. The wave function is promoted to be a two-component Pauli spinor

$$\psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \tag{2.1}$$

and the hamiltonian is the 2×2 matrix

$$H = \tfrac{1}{2}p^2 + V(x) + \tfrac{1}{2}\hbar\sigma_3 W(x), \tag{2.2}$$

where $p = -i\hbar(\mathrm{d}/\mathrm{d}x)$ and σ_i are the Pauli matrices.

The operator σ_3, which commutes with the hamiltonian, plays the role of a fermionic quantum number. We define the fermion number operator $N = \frac{1}{2}(1 - \sigma_3)$. Wave functions of the form $\begin{pmatrix}\phi_1\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ \phi_2\end{pmatrix}$ then correspond to eigenstates of the hamiltonian with fermion number zero and one respectively.

A supersymmetric quantum mechanical model is one in which the functions $V(x)$ and $W(x)$ are related in a particular way, viz. there exists a superpotential $v(x)$ such that

$$V(x) = \tfrac{1}{2}v^2(x),$$

$$W(x) = v'(x). \tag{2.3}$$

In this case, one can construct supersymmetry generators Q_i, $i = 1, 2$, which commute with the hamiltonian

$$[Q_i, H] = 0, \tag{2.4}$$

and satisfy the algebra

$$\{Q_i, Q_j\} = \delta_{ij}H. \tag{2.5}$$

Explicitly

$$Q_i = \tfrac{1}{2}\sigma_i(p + i\sigma_3 v). \tag{2.6}$$

A supersymmetric ground state is a zero-energy eigenstate of H. At the tree level ($\hbar = 0$) such states are found by looking for zeros of the superpotential v, or equivalently of the (positive definite) classical potential V.

It is interesting to analyze the spectrum in weak coupling perturbation theory, approximating the first two terms of H as a simple harmonic oscillator, in the case when such a state exists. Suppose x_0 is a minimum of V with $V(x_0) = 0$, $W(x_0) < 0$. To $O(\hbar)$, there are contributions to the energy from quantum fluctuations and directly from the fermionic term in H. Thus

$$E_n = (n + \tfrac{1}{2})\hbar\omega + \tfrac{1}{2}\hbar\sigma_3 W(x_0), \tag{2.7}$$

where

$$\omega^2 = \left.\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right|_{x=x_0}.$$

But now

$$\omega^2 = vv'' + v'^2|_{x=x_0} = v'^2(x_0) \tag{2.8}$$

and we observe $\omega = |W(x_0)|$. The spectrum, which is labelled by the quantum number n and fermion number N, is thus

$$E_{n,N} = (n+N)\hbar\omega. \tag{2.9}$$

The ground state ($n = N = 0$) is a non-degenerate Bose state and has zero energy. This illustrates the familiar cancellation of bosonic and fermionic contributions to the vacuum energy. The excited states are all degenerate, with Bose and Fermi states being paired. For example, the first excited states are a Bose state with $n=1$, $N=0$ and a degenerate Fermi state with $n=0$, $N=1$.

Of course, perturbation theory is not adequate to expose the phenomena we are interested in here. For this, we must consider the exact spectrum of the theory. As remarked above, the eigenstates of H are labelled by the fermion number N, or equivalently by the eigenvalues ± 1 of σ_3. Finding the spectrum, therefore, reduces to solving a standard Schrödinger equation for a one-component wave function in each of these sectors separately, the appropriate potentials being $U_{\pm} = \frac{1}{2}(v^2 \pm \hbar v')$ for $N = 0, 1$ respectively.

The non-zero energy states again appear in Bose-Fermi pairs, as a direct consequence of the supersymmetry algebra. Noting that

$$\tfrac{1}{2}H = Q_1^2 = Q_2^2, \qquad Q_2 = -i\sigma_3 Q_1, \tag{2.10}$$

we may define, given a Bose state $|b\rangle$ with non-zero energy E, a degenerate Fermi state $|f\rangle$ by

$$|f\rangle = \sqrt{2/E}\, Q_1|b\rangle = i\sqrt{2/E}\, Q_2|b\rangle. \tag{2.11}$$

That this is indeed a Fermi state can be checked by noting that $N|f\rangle = |f\rangle$ provided $N|b\rangle = 0$, which follows from the commutation relations

$$[N, Q_i] = i\varepsilon_{ij}Q_j. \tag{2.12}$$

Finally, the question of whether a supersymmetric ground state exists in the exact theory is more subtle. From eq. (2.10), it is clear that any supersymmetric ground state must be a solution of

$$Q_1\psi(x) = 0. \tag{2.13}$$

Multiplying by σ_1 and using the explicit form of Q_1, this reduces to

$$\frac{d\psi}{dx} = \frac{1}{\hbar}v(x)\sigma_3\psi(x), \tag{2.14}$$

the solution being

$$\psi(x)=\exp\left\{\frac{1}{\hbar}\int_0^x \mathrm{d}y\, v(y)\sigma_3\right\}\psi(0). \tag{2.15}$$

With the major proviso that this wave function is *normalizable*, it defines a supersymmetric ground state.

Under the assumption that $|v(x)|\to\infty$ as $x\to\infty$, the condition of normalizability of $\psi(x)$ is equivalent to requiring that $v(x)$ has an odd number of zeros, i.e. the sign of $v(x)$ for large positive and negative x should be opposite. So while at the tree level, the number of supersymmetric ground states is equal to the number of zeros of v, in the exact spectrum, the number of supersymmetric ground states is equal to one or zero, depending on whether v has an odd or even number of zeros. In other words, the Witten index [4] $\Delta=1(0)$ if v has an odd (even) number of zeros.

3. The model

The specific model we shall consider has the following form for the super-potential $v(x)$:

$$v(x)=(1+x^2)(1-mx). \tag{3.1}$$

The bosonic potential is thus

$$V(x)=\tfrac{1}{2}(1+x^2)^2(1-mx)^2, \tag{3.2}$$

and depends on a single parameter m. From now on, we set $\hbar=1$.

The main features of $V(x)$ in the interesting case of small m are shown in fig. 1. There are two minima, one at $x_1\simeq 0$ and one at $x_2=1/m$. The local minimum at $x_1\simeq 0$ is a broad well with curvature of O(1). The absolute minimum is a steep, narrow well with curvature of $O(1/m^2)$. The two wells are separated by a large potential barrier with height of $O(1/m^4)$, and the energy difference between the minima is O(1).

As m is taken to zero, the potential barrier grows indefinitely and the absolute minimum slides off to infinity, disappearing altogether at $m=0$ where the potential reduces to a quartic, $V(x)|_{m=0}=\frac{1}{2}(1+x^2)^2$.

We can analyze the question of whether there exists a supersymmetric ground state using the general arguments presented in the last section. For $m\neq 0$, the superpotential $v(x)\sim -mx^3$ for large x, and therefore has an odd number of zeros. So there does exist a supersymmetric ground state in the exact spectrum. We expect this to be a state localized around the absolute minimum, where $V(x_2)=0$. Notice though that the existence of this state cannot be concluded simply from the existence of such a minimum – in the presence of fermionic degrees of freedom, the superpotential $v(x)$ must be inspected to check that the candidate ground state is normalizable.

For $m = 0$ exactly, the situation changes. Now $v(x) \sim x^2$ for large x and has an even number of zeros. Therefore, a supersymmetric ground state does not exist as we expect from the disappearance of the minimum with $V = 0$.

Our explicit choice for the shape of V is motivated by the study of effective lagrangians for supersymmetric QCD, where the basic field is taken to be the chiral symmetry breaking order parameter superfield

$$T = s_+ s_- + \theta^\alpha (s_+ q_{-\alpha} + s_- q_{+\alpha}) + \theta^2 (s_+ F_- + s_- F_+ - \tfrac{1}{2} q_+^\alpha q_{-\alpha}), \qquad (3.3)$$

where $s_\pm, q_\pm$ are the squark and quark fields, and $F_\pm$ are auxiliary fields. The parameter m above plays the role of the quark mass. The phenomenon of the VEV of T in a supersymmetric vacuum running off to infinity as $m \to 0$ was first noted by Taylor et al. [5]. The specific form of $v(x)$ has been tailored to match as closely as possible the potential suggested by Peskin [1]. In fact, writing Peskin's effective lagrangian in terms of T, reducing to one flavour and taking T real, leads to an equivalent quantum mechanical model with $V(x) = \frac{1}{2}(1 + x^2)^2 (1/x - m)^2$. However, Peskin's lagrangian uses non-canonical kinetic terms, and these substantially affect the physics derived from $V(x)$. All the crucial physical features of his lagrangian (supersymmetric unbroken for $m \neq 0$, broken for $m = 0$, $m_\pi^2 \to \infty$ as $m \to 0$ in the supersymmetric vacuum) are preserved most simply by the quantum mechanical model described above.

Before turning to the exact numerical solutions for the lowest energy eigenstates of this model, we shall make a few comments on the general features we expect to find for small m.

We have already seen that a normalizable, supersymmetric ground state exists, for which, of course, $E = 0$, and we expect this state to have a wave function confined to the steep well. Excited states centred around the steep well would, however, have very high energies, very roughly (see the last section) of order $\hbar\omega$, where $\omega^2 \sim O(1/m^2)$ is the curvature in the steep well. However, we can also build states centred around the broad well. Although penalized by having $V(x_1) \neq 0$, the curvature in this well is sufficiently small so that such states will have energies of O(1), lower than the excited states built in the steep well.

Indeed, in an ordinary quantum mechanical model where the cancellation of bosonic and fermionic contributions to the ground state energy is not enforced, even the lowest state built in the steep well would have an energy of $O(1/m)$, greater than that of the broad well states.

The above general reasoning is based entirely on inspection of the bosonic potential V and neglects the effect of the fermionic degrees of freedom. In this case, however, it is not misleading, as we can see by the following much sharper argument.

As pointed out in the last section, the spectrum is determined in the $N=0$ and $N=1$ sectors by solving a conventional Schrödinger equation with the effective potentials $U_{\pm}=\frac{1}{2}(v^2 \pm v')$ respectively. Since at the steep minimum of $V=\frac{1}{2}v^2$, v' is negative, we expect that the potential U_+ shows an exaggerated steep well relative to V, whereas in U_- the right-hand well is lifted much higher than the left-hand well for small enough m.

The ground state is expected from the general analysis of sect. 2 to be a single Bose ($N=0$) state. The first excited states will be a degenerate Bose ($N=0$) and Fermi ($N=1$) pair. Consider the $N=1$ state. Since it is derived from the potential U_-, which does not have a deeper right-hand minimum, its wave function will certainly be localized in the broad, left-hand well. As m is made progressively smaller, this will become true of all the excited states.

Having thus established that for small m the wave functions of the fermionic states are localized in the left-hand well, it remains only to assert that the wave functions of the degenerate bosonic states must be similarly localized. The only exception to this argument is the non-degenerate bosonic ground state.

The picture we expect for small m from these qualitative arguments is therefore as follows. The ground state is enforced by supersymmetry to be a zero-energy state centred in the steep well. However, the low-lying excited states are effectively independent of the existence of the steep well and its supersymmetric minimum, being centred around, and taking their characteristics from, the non-supersymmetric broad well.

In the next section, we shall present the exact low-lying spectrum of this model for different values of m, and shall see to what extent these qualitative expectations are realized.

4. Energy levels and wave functions

We have solved the Schrödinger equation numerically for the quantum mechanical model described by the superpotential of eq. (3.1) in the sectors with fermion number $N=0$, 1 separately. As already remarked, the problem is then reduced to a standard one-component wave function Schrödinger equation and is soluble by standard numerical methods. Our results for the energy levels of the ground state

TABLE 1
Energies of the ground state and first two excited states for various values of m

m	E_0	E_1	E_2
0.5	0	1.02	2.40
0.4	0	0.385	2.12
0.3	0	1.07	2.29
0.2	0	1.15	2.91
0.1	0	1.19	3.09

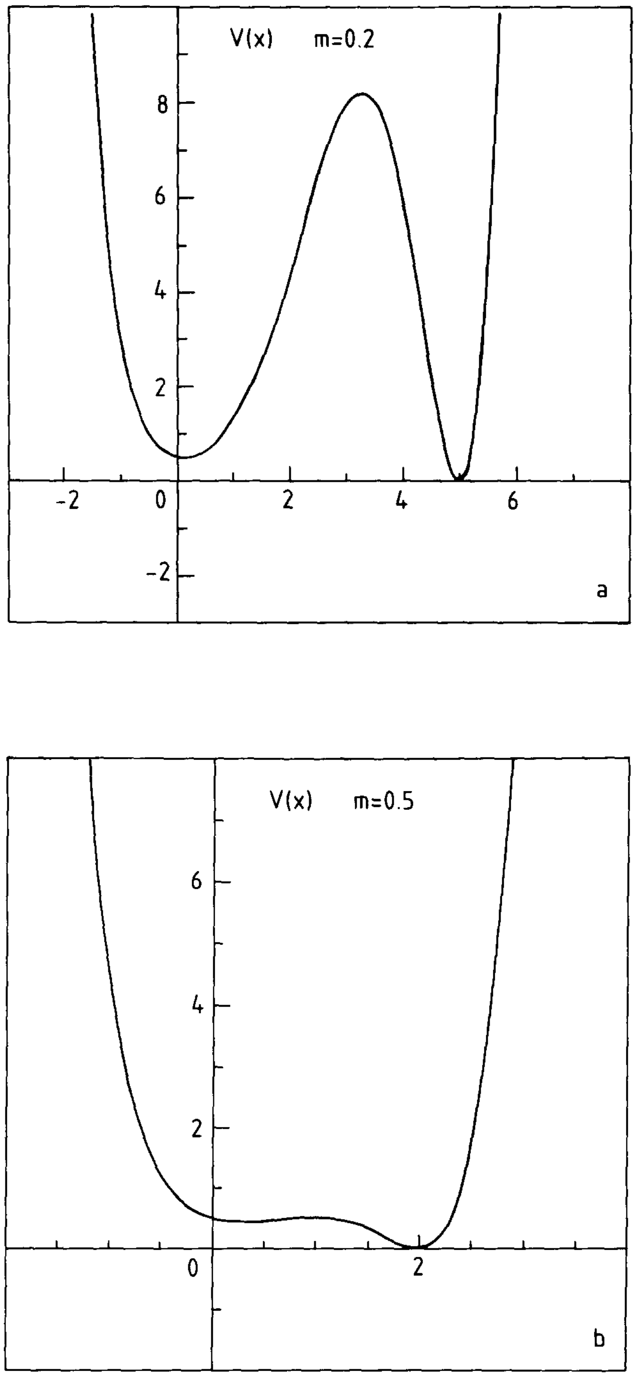

Fig. 2. Plot of the bosonic potential $V(x)$ for (a) $m = 0.2$ and (b) $m = 0.5$.

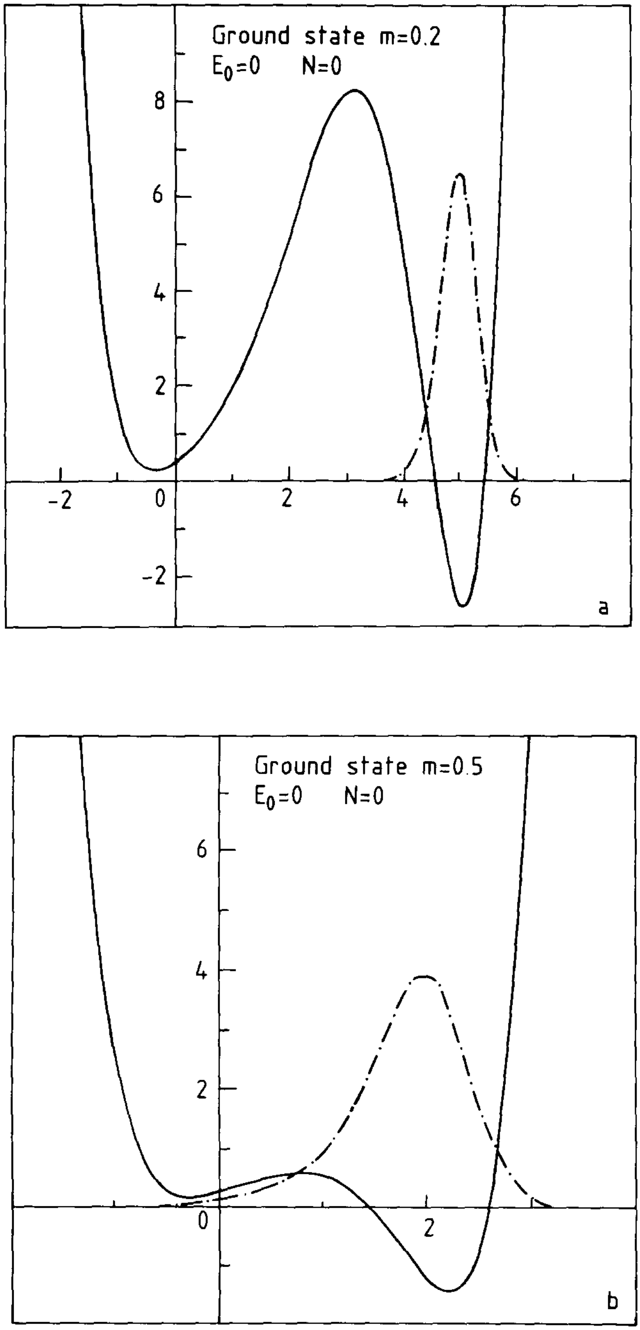

Fig. 3. Plots showing the modulus squared of the wave functions (dotted curves) for the ground and first two excited states superimposed on the relevant effective potentials U_+ or U_- (solid curves) for (a) $m = 0.2$ and (b) $m = 0.5$. The scale of the wave function plot is $\times 10$.

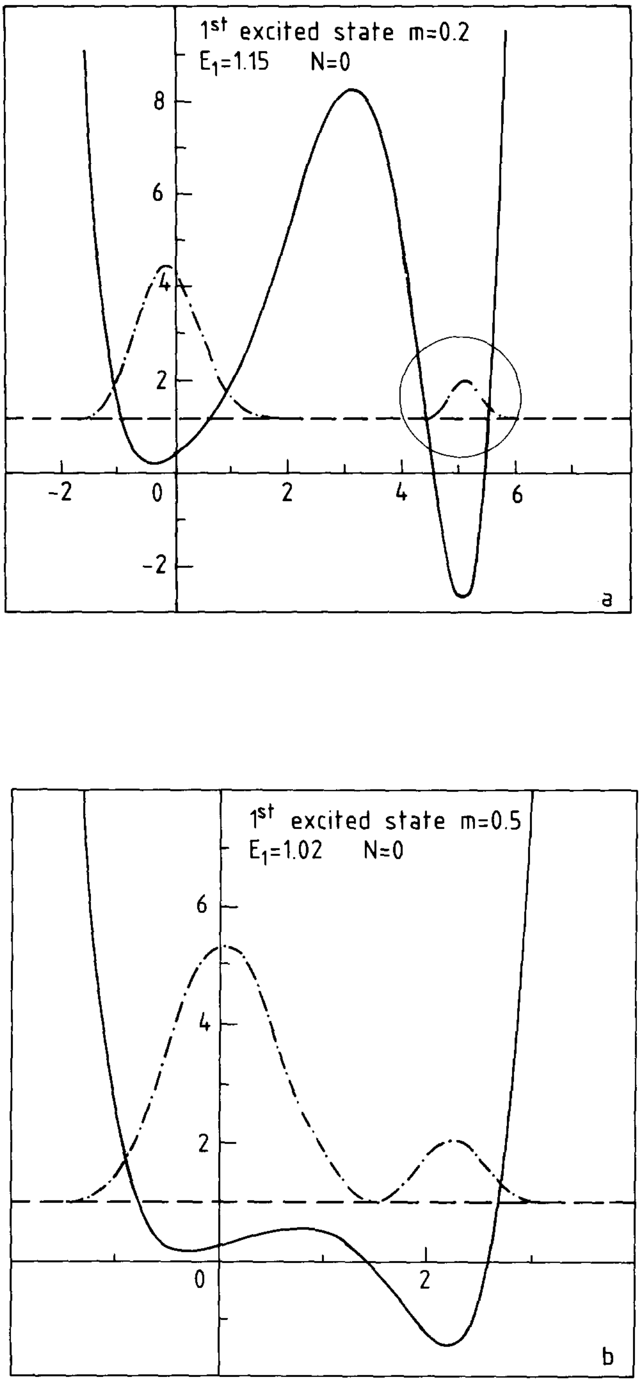

Fig. 4. See fig. 3 caption. In fig.4a the scale of the encircled wave function is $\times 10^4$.

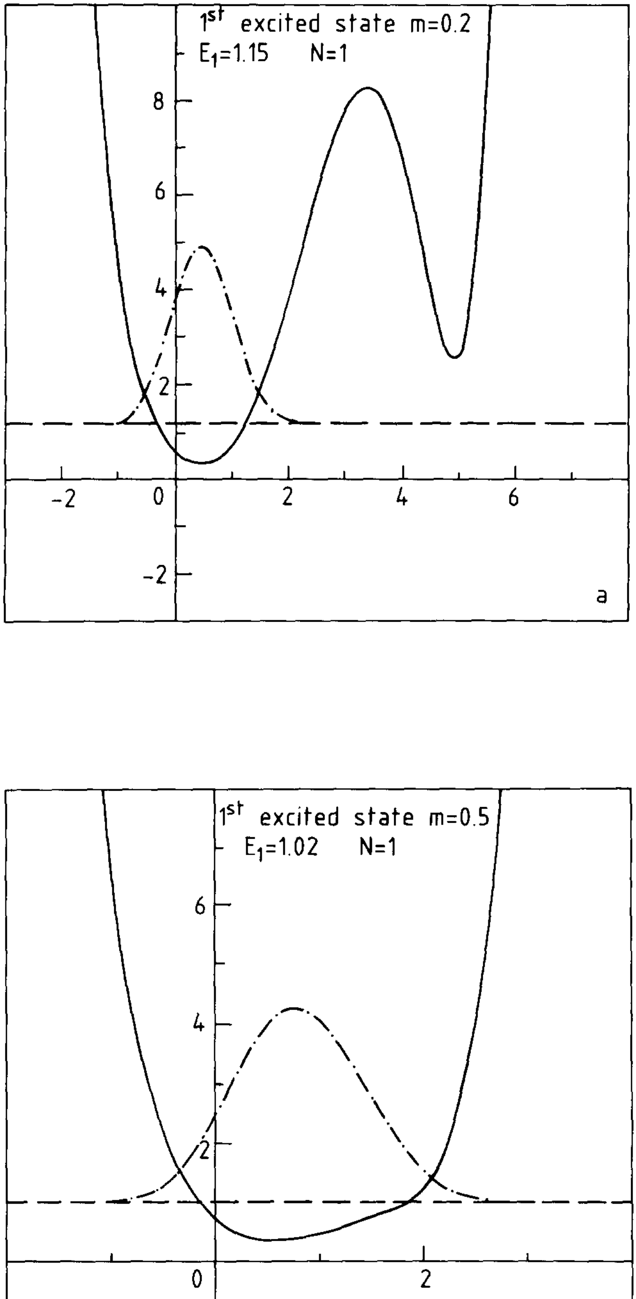

Fig. 5. See fig. 3 caption.

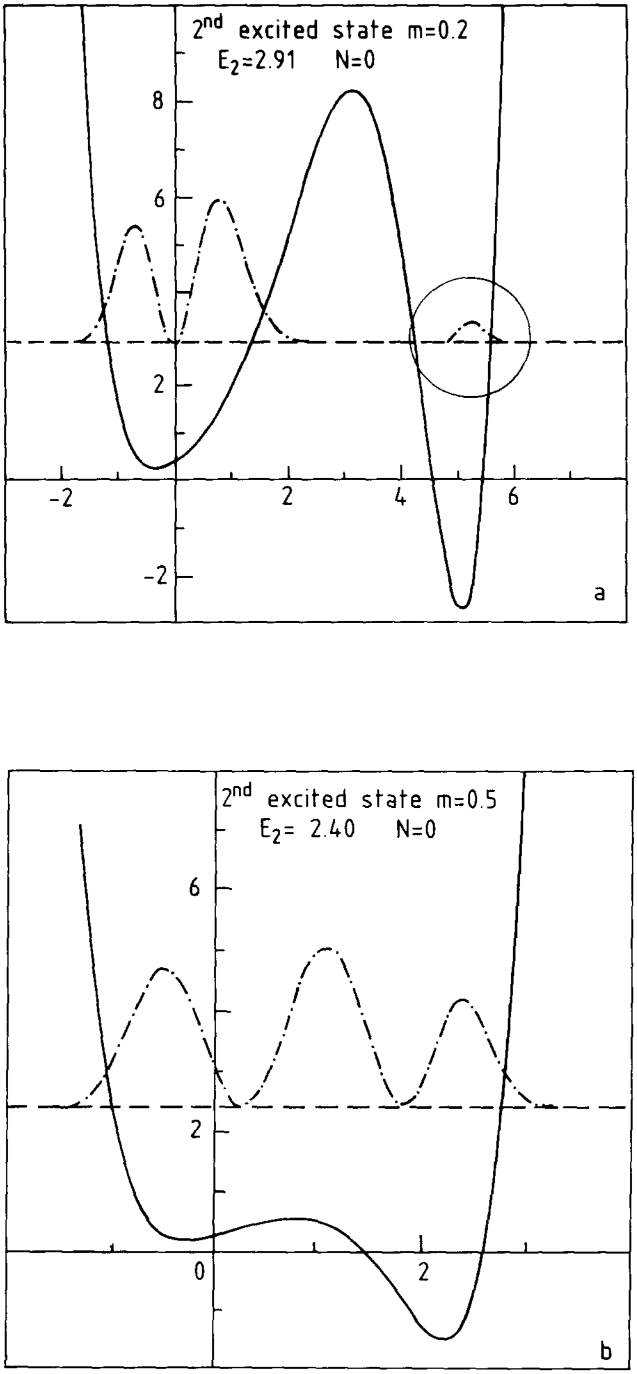

Fig. 6. See fig. 3 caption. In fig. 6a the scale of the encircled wave function is $\times 10^3$.

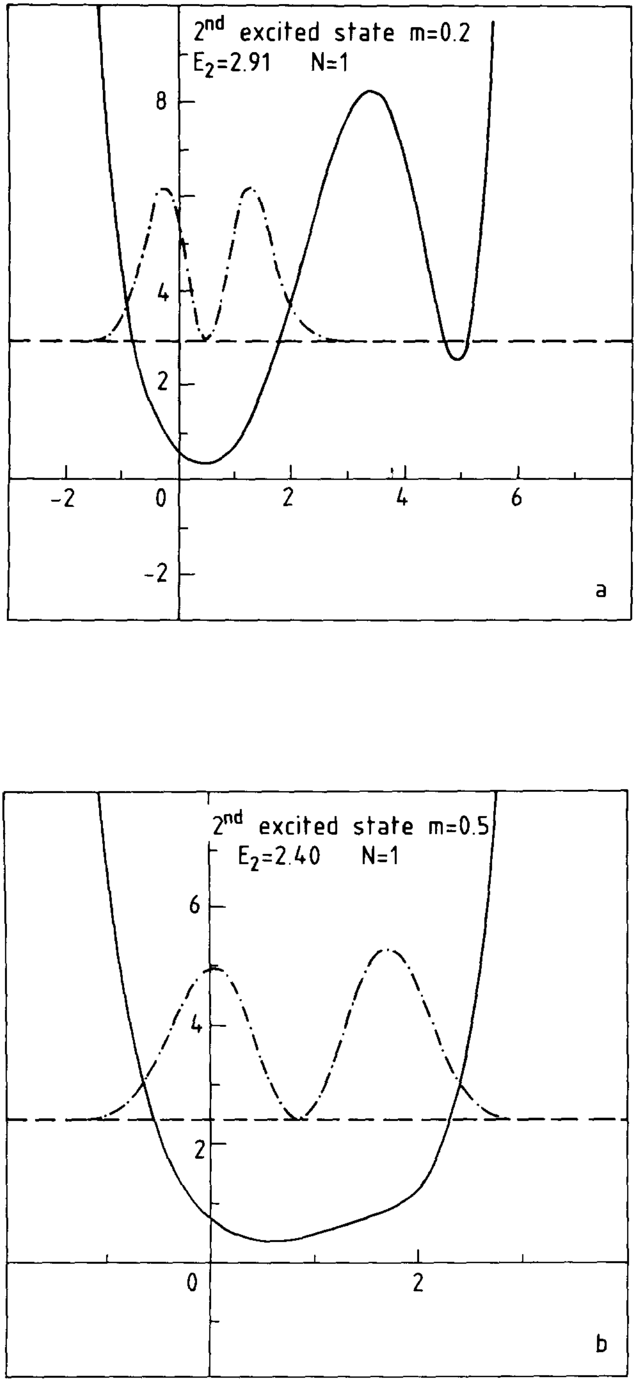

Fig. 7. See fig. 3 caption.

and first two excited states, and their wave functions, are shown for various values of m in table 1 and in figs. 2 to 7.

To discuss these results, it is best to make a systematic comparison for two different values of m, chosen here to be $m = 0.5$ and $m = 0.2$, to see what special phenomena occur and what features may be expected as $m \to 0$.

The bosonic potential $V(x)$ is shown in fig. 2. The effective potentials $U_{\pm} = \frac{1}{2}(v^2 \pm v')$, needed to solve the one-component Schrödinger problem in the sectors $N = 0$, 1 respectively, are depicted by the solid lines in figs. 3 to 7. Observe that the potential U_+ for $N = 0$ has its deep minimum depressed below zero – the effect of zero-point quantum fluctuations restores the energy level of the ground state to precisely zero.

The ground state is, as expected by the general supersymmetry arguments, a zero-energy supersymmetric state with $N = 0$ and is not degenerate. The (modulus squared of the) wave functions are shown in fig. 3, superimposed on the effective potentials U_+. In both cases, the states are well localized in the right-hand well, particularly so for the smaller value $m = 0.2$ where the well is deep and narrow.

The first excited state (figs. 4 and 5), again in accordance with general arguments, is doubly degenerate, comprising a Bose-Fermi pair of states with $N = 0$ and 1. The Bose ($N = 0$) state is the first excited level of the effective potential U_+, while the degenerate Fermi ($N = 1$) state is the lowest level of the potential U_-. This difference is reflected in the number of nodes in the wave functions, one for the Bose state and none for the Fermi state.

The crucial feature, however, is how already for $m = 0.2$ this excited state is almost completely localized in the broad well. Only a tiny tail remains in the steep well. Our expectations for the small m behaviour of this model seem therefore to be justified. As m becomes small and the right-hand well becomes steep, the lowest excited states lie entirely in the broad well, the supersymmetric ground state only remaining localized in the steep well.

The situation is similar for the second excited state (figs. 6 and 7), which is again degenerate. For $m = 0.5$, there is considerable mixing between the wells, as expected, since the energy of this state is much larger than the barrier height. However, for $m = 0.2$, the wave function is clearly localized in the broad well.

In conclusion, we see that the results of this numerical study confirm very well our expectations for the small m behaviour of the model.

5. Conclusions and speculations

In this final section, we shall discuss what conclusions are to be drawn from the analysis of this quantum mechanical model, and offer some speculations as to what might occur in a corresponding field theory.

The first conclusion is that *fermions are important*. The fact that our model is supersymmetric highlights the role played by the fermionic degrees of freedom, e.g.

compare the potentials V, U_+ and U_-. But, of course, this is not peculiar to supersymmetry. In general, it is clear that fermions can play an important role in such models, and that arguments based simply on inspection of the bosonic potential, especially where tunnelling is involved, must be treated with caution.

The most important observation, however, is that in models of this type supersymmetry influences the spectrum in an unusual way. To highlight this effect, consider the following slight idealizations of the model considered here: (i) a supersymmetric model whose potential has a broad well with $V(x)=\Delta \neq 0$, and an additional steep well with $V(x)=0$ separated by a large barrier; and (ii) the same model with the steep well removed (see fig. 8). We make the assumption that for states confined to the broad well, the energy levels are approximately equally spaced, as for the simple harmonic oscillator. (In a field theory, this is equivalent to assuming the validity of weak coupling perturbation theory in this well.) This situation is essentially realized by the present model for (i) small m and (ii) $m=0$.

The spectrum is the two cases is shown schematically in fig. 9. For model (ii), the energy levels are evenly spaced, with a separation m_b proportional to the curvature of the broad well, and are degenerate, forming Bose-Fermi pairs. The situation is identical in model (i), with degenerate states localized in the broad well, *except* that there is an extra state at $E=0$ localized in the steep well. This is separated from the others by an energy gap M, where in this quantum mechanics model $M=\Delta+\frac{1}{2}m_b$.

The spectrum of the model (i), with a *supersymmetric* minimum of the potential, is therefore determined entirely by the *non-supersymmetric* minimum, except for the presence of the ground state.

What can we deduce for the spectrum of a supersymmetric field theory with a similar potential? We must be careful not to draw too strong conclusions here in the absence of any proper analysis, but we can make some interesting speculations.

In field theory language, for a model of type (ii), supersymmetry is spontaneously broken. The spectrum suggested by fig. 9 is that bosons exist with mass m_b, and also

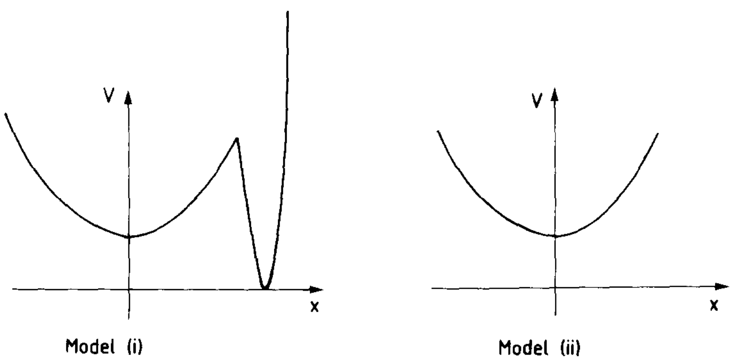

Fig. 8. Sketch of the potentials $V(x)$ for the idealized models discussed in sect. 5.

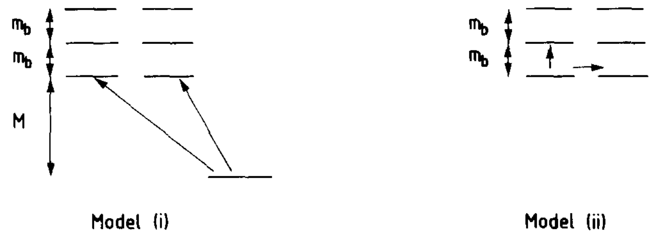

Fig. 9. Spectrum for the potentials of fig. 8. The arrows indicate that in an equivalent field theory, there is an equal mass gap M to create the lowest fermion of boson states in model (i), while in model (ii) there is a massless fermion and a boson of mass m_b.

zero-mass fermions. This generalizes to field theory the degeneracy of states in the quantum mechanical model. Such zero-mass fermions are expected, as the Goldstone fermions of spontaneously broken supersymmetry.

What about model (i)? The presence of the supersymmetric vacuum state changes the above picture drastically. Generalizing from the quantum mechanics spectrum of fig. 9, we expect the spectrum to comprise here an equal mass boson and fermion with some large mass M, each with further light excitations of mass m_b. Supersymmetry is unbroken, and ensures the equality of the masses of the heavy boson and fermion. This is indeed a strange spectrum but it is reminiscent of two-dimensional models with solitons, where it is possible to have a heavy object (the soliton) with a spectrum of light excitations ("mesons") corresponding to its excited states. Furthermore, soliton-antisoliton excitations are known to occur in the massive Schwinger model [6].

We would like to emphasize that the appearance of the mass scale M into the excitation spectrum as a consequence of the supersymmetric vacuum state is non-trivial. A naive inspection of a potential of type (i) would not indicate that any particle mass should depend on this scale.

We are thus led to believe that the lowest mass particle in a model of type (i) is a very peculiar object. Let ϕ_1, ϕ_2 be the positions of the minima in the broad and steep wells, respectively. What we have in mind is a field configuration in which the field asymptotically takes the value ϕ_2, but has the value ϕ_1 over some finite region of space. The low-energy spectrum comprises a particle of this type (and its superpartner) and the meson-like small oscillations of it.

We have described this spectrum as corresponding to unbroken supersymmetry. Consequently the bosonic lump would be accompanied by a degenerate fermionic state resulting in general from the existence of fermionic zero modes in the bosonic lump configuration. An amusing feature of our particular case is that, in the region where $\phi = \phi_1$ (false vacuum), supersymmetry would appear to be broken and an *effective* massless goldstino could exist inside the lump itself.

Can such a picture be correct? Developing the mathematical techniques to describe such a lump configuration is not an easy task, and we leave it for further work. What seems clear, however, is that the field theory of potentials of this type is interesting and unconventional, and may well hold further surprises.

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