MEAN FIELD ANALYSIS OF SU(N) DECONFINING TRANSITIONS IN THE PRESENCE OF DYNAMICAL QUARKS

F. GREEN[1]

Department of Physics, Northeastern University, Boston, USA

F. KARSCH

CERN, Geneva, Switzerland

The deconfining transitions of SU(N) lattice gauge theories, both with and without quarks, are studied using strong coupling techniques combined with a mean field analysis. In the pure gauge sector our analysis suggests first-order transitions for $N \geqslant 3$ and a second-order transition only for $N = 2$. Quarks are incorporated via an effective external field h related to the Wilson hopping parameter. For $N = 2$ the transition disappears for arbitrarily small h, whereas for finite $N \geqslant 3$ it disappears above a non-zero critical field h_c. h_c approaches zero as $N \to \infty$ even though the pure gauge sector transition remains first order. Our results for h_c in the SU(3) case agree well with recent Monte Carlo simulations.

1. Introduction

Deconfinement in finite temperature SU(N) gauge theories is known to occur in the pure gauge sector [1–5]. However, it is profoundly influenced by the presence of dynamical quarks. Recent Monte Carlo results [6, 7] and theoretical considerations [8] indicate that the transition actually disappears in the real world of interacting quarks and gluons. This is because the quark fields break the global Z(N) symmetry which signals deconfinement for all temperatures. Such a breaking smooths out the phase transition if it is second order [8], just as an external magnetic field smooths out the phase transition of an Ising model. In the case of a first-order transition, the jump in the order parameter disappears only if the breaking is strong enough. Thus the quarks must have a sufficiently small mass. The numerical results for SU(3) [6] and Z(3) [7] indicate that this critical mass, below which there is no phase transition, is quite large. Certainly it is desirable to check these findings with analytic calculations. That is, in part, the purpose of this paper.

In addition, there are some questions as to the nature of the $N \geqslant 4$ transitions. It has been suggested that these theories may have second-order transitions in the same universality class as certain Z(N) spin systems [8–10]. It should be emphasized that this is a weak statement for $N \geqslant 4$ since the order of the transitions depends

---

on the exact structure of the $Z(N)$ invariant model under consideration [10]. However, ref. [11] showed a relation, for ultra-strong coupling, connecting pure $SU(N)$ gauge theories with N state clock models. This picture favours second-order transitions for all $N \neq 3$. Then the critical quark mass would be finite only for the $N = 3$ theory. This has been noted in ref. [12]. It is nonetheless important to do explicit calculations since the order of the $Z(N)$ transitions is very sensitive to the type of model [13].

In this paper we present a mean field analysis of the $SU(N)$ lattice gauge theories at finite temperature and strong coupling. We find, as expected, second- and first-order transitions for the $N = 2$ and 3 pure gauge theories, respectively. Remarkably, however, the pure gauge sector has *first-order transitions for all $N \geq 4$*. Thus the critical masses are finite for all $N \geq 3$ (with the exception of $N = \infty$!). Utilizing a hopping parameter expansion to incorporate quarks, we calculate critical hopping parameters $K_c(N)$ (i.e., critical masses) for the various theories. The K_c's are small, thus justifying the expansion, and in agreement with Monte Carlo. In fact for $N = 3$ the numbers themselves are in excellent agreement, both for K_c as well as the critical coupling. Our values for these critical parameters for $N \geq 4$ are offered as rough estimates for future more detailed investigations. In any case, the $N \geq 4$ results are significant and are in agreement with a qualitative argument [14] that they are first order. The approximation of ref. [11] which leads to clock models seems to be too drastic.

2. Generalities

The theories are defined on an euclidean lattice with N_τ links in the time direction and $N_\sigma \to \infty$ links in the three spatial directions. Thus for lattice spacing a, the temperature $\hat{T} = (N_\tau a)^{-1}$. Wilson fermions are incorporated as usual by first integrating over the fermionic degrees of freedom. The partition function is, for n_f flavours,

$$Z = \int [\mathrm{d}U] \prod_{f=1}^{n_f} \det Q_f \exp\left[\beta N \sum_p \mathrm{tr}\, U_p + \text{c.c.} \right], \tag{2.1}$$

where

$$Q_f = 1 - K_f M, \tag{2.2}$$

$$M_{\mu,xy} = (1 - \gamma_\mu)\tilde{U}_{x,\mu}\delta_{x,y-\mu} + (1 + \gamma_\mu)\tilde{U}^+_{y,\mu}\delta_{y,x+\mu}. \tag{2.3}$$

Due to the antiperiodic boundary conditions for fermions in the temperature direction, the $\tilde{U}$'s obey

$$\tilde{U}_{x,\mu} = \tilde{U}_{(x,x_4),\mu} = (1 - 2\delta_{x_4,N_\tau}\delta_{4,\mu})U_{x,\mu}, \tag{2.4}$$

where the $U_{x,\mu} \in SU(N)$ are ordinary link variables. The U_p's are the usual plaquette variables, $\beta = 1/g^2 N$, and K_f is the hopping parameter for quark flavour f.

Consider first the pure gauge sector ($K_f = 0$). For small β spacelike plaquettes can be neglected, since they tend to deconfine [1, 2, 15]. In that case all spacelike links can be integrated [15] and an effective theory obtained in terms of Wilson line variables,

$$W_x = \prod_{x_4=0}^{N_\tau - 1} U_{(x,x_4),4} . \qquad (2.5)$$

The effective partition function, up to an irrelevant constant, can be written in terms of a character expansion,

$$Z_{\text{eff}}(\beta, K = 0) = \int \prod_x dW_x \prod_{x,e} \left(1 + \sum_\tau z_\tau(\beta)^{N_\tau} \chi_\tau(W_x) \chi_\tau(W_{x+e}^+) \right), \qquad (2.6)$$

where the first product runs over the sites and the second over the links of a $3D$ lattice. The χ_τ's and z_τ's in eq. (2.6) are the characters and character coefficients in the rth representation, respectively (see ref. [16] for notation). The z_τ's are of increasingly higher order in β as we increase r. We assume β is sufficiently small that we can keep only the fundamental term in eq. (2.6), proportional to $z_{1;0}$ (($1;0$) denoting the fundamental representation [16]). Thus

$$Z_{\text{eff}}(\beta, K = 0) \cong \int \prod_x dW_x \exp \left(\beta' \sum_e \text{tr } W_x \text{ tr } W_{x+e}^+ + \text{c.c.} \right), \qquad (2.7)$$

where★

$$\beta' = z_{1;0}^{N_\tau} . \qquad (2.8)$$

To leading order, $z_{1;0} = \beta$ so $\beta' = \beta^{N_\tau} + \cdots$. Higher orders in β are included, in part, by expanding $z_{1;0}$.

Eq. (2.7) describes a $Z(N)$ invariant model with nearest-neighbour interactions between the spins tr W_x. However, it is complicated by self-interactions induced by the group measure. At least qualitatively, this can help us determine the nature of the phase transition via an effective potential formalism [9, 11]. However, the effective potential is hard to find for $N \geqslant 4$, and mean field theory is more trustworthy quantitatively.

Now let the quarks have a finite, but still large, mass ($K \ll 1$). In order to avoid dealing with spacelike plaquettes, keep $N_\tau < 4$ and expand ln det Q in K. To leading order,

$$\sum_{f=1}^{n_f} \ln \det Q_f \cong h \sum_x \text{tr } W_x + \text{c.c.} \equiv \sum_x f_1(W_x, h) , \qquad (2.9)$$

where

$$h \equiv 2n_f(2K)^{N_\tau} . \qquad (2.10)$$

★ Eq. (2.7) is true for $N \geqslant 3$. For $N = 2$, β' in eq. (2.7) must be replaced with $\frac{1}{2}\beta'$.

(For $N_\tau > 4$ the leading term arises from spacelike quark paths which, however, do not break the $Z(N)$ symmetry.) It is also possible to sum the K expansion for $N = 3$ for all quark paths wrapping around the four-direction an arbitrary number of times. This results in (for SU(3)),

$$\ln \det Q \cong 2n_f \sum_x \ln \left[(1 + \bar{K}^3 + (\bar{K} + \bar{K}^2) \operatorname{Re} \operatorname{tr} W_x)^2 + ((\bar{K} - \bar{K}^2) \operatorname{Im} \operatorname{tr} W_x)^2 \right]$$

$$\equiv \sum_x f_2(W_x, h) , \qquad (2.11)$$

where $\bar{K} = h/2n_f$. Thus the strong coupling effective partition function becomes

$$Z_{\text{eff}}(\beta, h) = \int \prod_x dW_x \exp \left[\beta' \sum_{x,e} \operatorname{tr} W_x \operatorname{tr} W_{x+e}^\dagger + \text{c.c.} + \sum_x f_i(W_x, h) \right] , \quad (2.12)$$

where $f_i(W, h)$ is one of the functions f_1, f_2 defined above. Clearly both f's break the global $Z(N)$ symmetry and thus h acts as an external field. In general we will use only f_1; for SU(3), the results are changed very little by switching to f_2.

As long as $K \ll K_{m_\pi=0}$, where $K_{m_\pi=0}$ is the critical hopping parameter which corresponds to massless quarks, a suitable way to relate K to a quark mass is via [6, 7]

$$K \simeq \tfrac{1}{2} e^{ma} . \qquad (2.13)$$

Using $N_\tau a = 1/T$, this is equivalent to

$$m/T = \ln (h/2n_f) . \qquad (2.14)$$

The mean field (MF) analysis of eq. (2.12) for $h = 0$ yields the same self-consistent equations encountered in $SU(N) \times SU(N)$ chiral models [17]. In fact the model of eq. (2.7) is intimately related to chiral models in many ways [15]. Since the MF results of ref. [17] were quite reliable in their agreement with Monte Carlo, we expect the same to hold here. More importantly, the qualitative predictions of MF theory (e.g., the order of the transitions) are almost certainly correct here.

Following ref. [17], we define a MF free energy*,

$$\mathscr{F}_{\text{MF}} \equiv 6\beta' M^2 + \mathscr{F}_{\text{ss}} , \qquad (2.15)$$

$$\mathscr{F}_{\text{ss}} = -\ln Z_{\text{ss}} , \qquad (2.16)$$

where the 'single-site' partition function is,

$$Z_{\text{ss}}(M, h) = \int dW \exp \left[6\beta' M \operatorname{tr} W + \text{c.c.} + f_i(W, h) \right] . \qquad (2.17)$$

For simplicity we take M to be real. The value of the mean field M is determined by locating the minima of $\mathscr{F}_{\text{MF}}$:

$$\frac{\partial \mathscr{F}_{\text{MF}}}{\partial M} = 0 . \qquad (2.18)$$

---

* Once again, for SU(2), eq. (2.15) should have $\beta' \to \tfrac{1}{2}\beta'$.

TABLE 1
Summary of mean field results for the critical parameters of the deconfinement transition and the order of the transition in the pure gauge sector for various N

N	Order	h_c	$\beta_c'(0)$	$\beta_c'(h_c)$
2	2	0	0.17	0.17
3	1	0.059	0.13	0.12
4	1	0.047	0.16	0.15
5	1	0.11	0.16	0.14
6	1	0.087	0.16	0.15
8	1	0.039	0.17	0.16
10	1	0.013	0.17	0.17
∞	1	0	0.17	0.17

The solution $\bar{M}$ is the expectation value of the Wilson line, $\langle \text{tr } W \rangle$, in the MF approximation. One must check that these minima are stable, i.e., that $\mathcal{F}_{\text{MF}}$ is a global minimum. The existence of two stable minima indicates the presence of a first-order transition.

Note that eq. (2.12) with $f_i = f_1$ can be written as

$$Z_{\text{ss}}(y) = \int dW \exp\left[y(\text{tr } W + \text{tr } W^\dagger)\right], \tag{2.19}$$

where $y = 6\beta'M + h$. This is a famous integral, and has been done [16–18]. We utilized both the series expansions of refs. [17, 18] as well as numerical integrations to evaluate eq. (2.19).

The procedure is now clear: increase h from zero. At each value of h, check to see if there is a phase transition at some critical β', $\beta_c'(h)$. It should disappear at some h_c (hopefully small in order to justify the approximations made). We now describe the specific examples. A summary of the results appears in table 1.

3. Results

3.1. SU(2)

For SU(2) (see first footnote in sect. 2, keeping $y = 6\beta'M + h$)

$$Z_{\text{ss}}(y) = \int dW \exp\left[y \text{ tr } W\right]. \tag{3.1}$$

This can be written in closed form,

$$Z_{\text{ss}}(y) = I_1(2y)/y. \tag{3.2}$$

The MF equation to be solved (see previous footnote), rewriting eq. (2.18) in terms of y, is

$$\frac{y-h}{6\beta'} = 2\frac{I_2(2y)}{I_1(2y)} .$$

(3.3)

Consider small y. Then this becomes

$$\frac{y-h}{6\beta'} = y + \cdots .$$

(3.4)

For $h = 0$, this has the unique solution $y = 0$ for $\beta' < \frac{1}{6}$. Since the derivative of I_2/I_1 never increases, this is also the unique solution to eq. (3.3). It is stable, so $\bar{M} = 0$ for $\beta' < \frac{1}{6}$. $\bar{M}$ increases continuously from 0 for $\beta' > \frac{1}{6}$, so there is a *second*-order transition at $\beta'_c = \frac{1}{6}$.

For $h \neq 0$, but small, eq. (3.4) yields the following stable solution for $\beta' \ll \frac{1}{6}$,

$$\bar{M} = \frac{h}{1-6\beta'} .$$

(3.5)

Thus $\bar{M}$ is non-zero for any non-zero h even at very small temperatures, and the phase transition disappears at $h_c = 0$.

3.2. SU(3)

For $h = 0$, the first-order nature of the transition is exhibited in fig. 1. $\mathscr{F}_{MF}$ has two degenerate minima at $\beta'_c(0) = 0.134$, which could also be obtained directly from ref. [17]. In arriving at fig. 1, the group integration was performed numerically. A
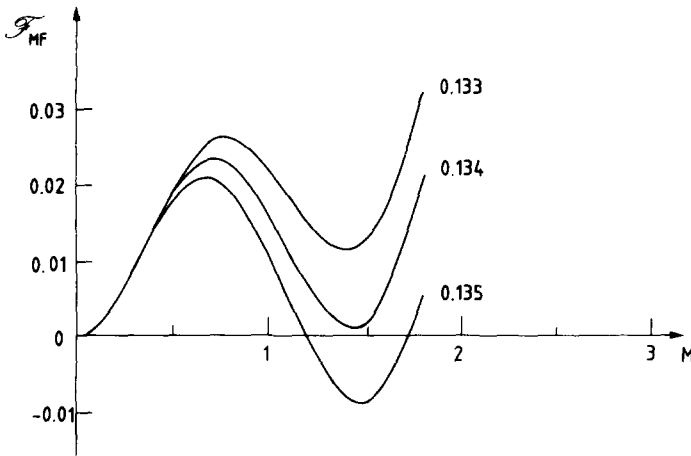

Fig. 1. $\mathscr{F}_{MF}[SU(3)]$ is plotted versus M for $\beta' = 0.133$, 0.134 and 0.135, using numerical integration to evaluate eq. (2.19). The phase transition occurs at $\beta' = 0.134$, where there are two stable minima.
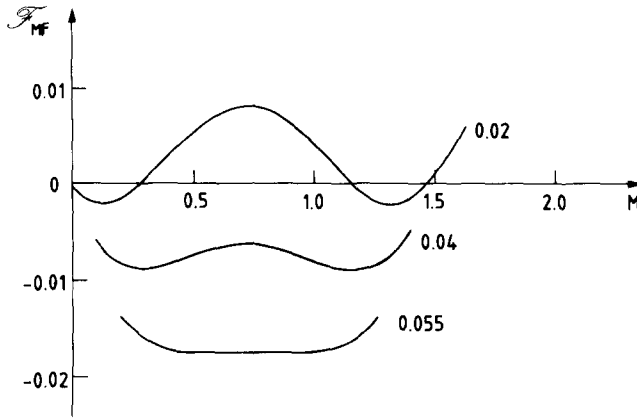
Fig. 2. Behaviour of $\mathscr{F}_{MF}[SU(3)]$ at $\beta_c'(h)$ for three different values of h: 0.02 (top curve), 0.04 (middle curve) and 0.055 (bottom curve). The series expansion of ref. [18] was taken to 30th order to arrive at these curves. The respective values of β_c are 0.130, 0.125 and 0.122.

similar calculation using the series expansion (eq. (2.22) of ref. [17]) yields $\beta_c'(0) = 0.14$. Thus the series results can be trusted to within (roughly) 10%, at least to the orders given in ref. [17].

For f_1 with $h \neq 0$, the two minima of fig. 1 approach each other and $\beta_c'(h)$ decreases (see fig. 2). The behaviour of the order parameter β' for various values of h is displayed in fig. 3. At $h_c = 0.059$ the two minima of $\mathscr{F}_{MF}$ merge (fig. 2) and the jump in $\bar{M}$ disappears (fig. 3). The variation of $\beta_c'(h)$ with h can be determined
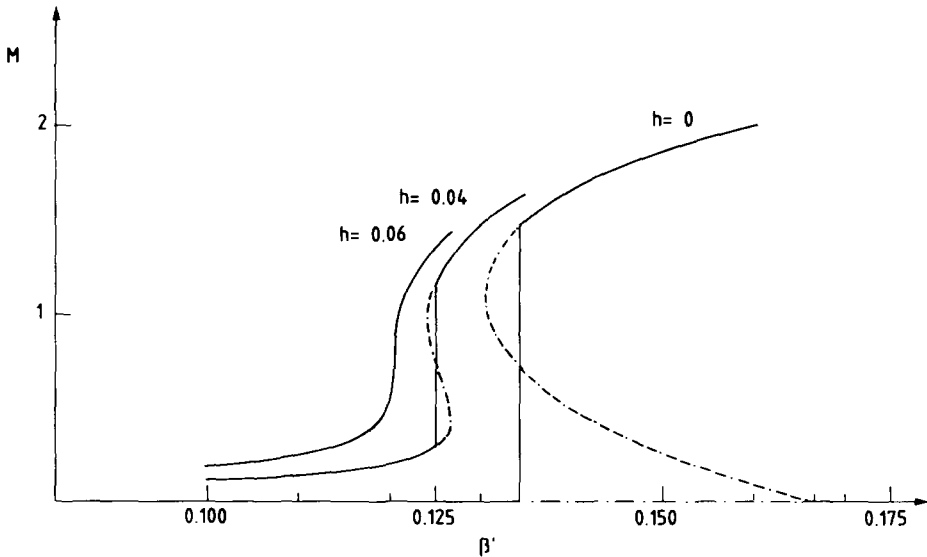

Fig. 3. The order parameter M as a function of β' for various values of h using the function f_1 [SU(3)] as an approximation to ln det Q. The solid lines give the order parameter, and the dotted lines indicate the unstable and metastable solutions to the MF equation. Numerical integration was used.

directly or via an analogue of the Clausius-Clapeyron equation, valid for small h [6],

$$\beta_c'(h) = \beta_c'(0) - \frac{(\Delta(\partial\mathscr{F}_{MF}/\partial h)|_{h=0})}{(\Delta(\partial\mathscr{F}_{MF}/\partial\beta)|_{h=0})} h \tag{3.6}$$

$$= \beta_c'(0) - h/(3\Delta\bar{M}|_{h=0}) , \tag{3.7}$$

where the Δ's denote the differences in the subsequent quantities between the two coexisting phases. As h_c turned out to be small, eq. (3.7) is valid up to h_c. At $h = 0$, $\Delta\bar{M} \cong 1.5$, so

$$\beta_c'(h) \simeq \beta_c'(0) - \tfrac{1}{4}h . \tag{3.8}$$

The use of f_2 does not appreciably change the results. We expect the same to be true for $N \geq 4$ for the analogue of f_2.

3.3. SU(N), $N \geq 4$

The series expansions of ref. [17] were used to obtain the values for $\beta_c(0)$, $\beta_c(h_c)$ and h_c listed in table 1, so they may be off by $\sim 10\%$. In all cases, we found *first-order* transitions for $N \geq 4$ with $h_c \neq 0$ as long as N was finite.

Interestingly, the large N limit (which, like $N = 2$ can be solved exactly [17]) predicts $h_c/N = 0$, even though the transition is first order. Since corrections to the large N group integrations are exponentially small [19], h_c itself should approach 0 faster than any power of N. Indeed, the finite N results indicate that h_c itself $\to 0$ as $N \to \infty$, not just h_c/N. We are thus faced with a typically pathological situation for $N \to \infty$: one could set det $Q = 1$ since internal quark loops give $1/N$ corrections. If this is done the phase transition *persists* for any finite quark mass ($h \neq 0$). But if the $1/N$ corrections are included, the transition *disappears* for any finite quark mass! The reason for this is that at β_c, the 'potential barrier' in $\mathscr{F}_{MF}$ is flat to all orders in N.

4. Comparison with Monte Carlo, and conclusions

From table 1 we can obtain numbers that can be compared with those obtained in Monte Carlo simulations. For SU(2) the character coefficient $z_{1;0}$ can be written in closed form as [$\beta \equiv 1/g^2$ for SU(2)],

$$z_{1;0} = \frac{I_2(4\beta)}{I_1(4\beta)}. \tag{4.1}$$

Using the critical value for $\beta_c' = z_{1;0}^{N_\tau}$ from table 1, we obtain for β_c the numbers 0.17, 0.46 and 0.71 for $N_\tau = 1$, 2 and 3, respectively. These are to be compared with the Monte Carlo results 0.19 [3], 0.47 and 0.55 [5], respectively, using our β normalizations. Since $N_\tau = 3$ brings us well into the intermediate coupling region,

it should not be surprising that the agreement is bad there. Our results can be trusted, apparently, only for $N_\tau \leqslant 2$. For these values of N_τ they are also roughly consistent with more precise strong coupling estimates ($\beta_c = 0.24$ for $N_\tau = 1$ and 0.49 for $N_\tau = 2$ [15]).

For SU(3), $z_{1;0}$ cannot be written in closed form, so we resort to an expansion. Including the first few corrections in $z_{1;0}$,

$$z_{1;0} = \beta + \tfrac{3}{2}\beta^2 - \tfrac{135}{24}\beta^4 + O(\beta^5) , \tag{4.2}$$

then

$$\beta' = \beta^2 + 3\beta^3 + \tfrac{9}{4}\beta^4 - \tfrac{135}{12}\beta^5 + O(\beta^6) , \tag{4.3}$$

for $N_\tau = 2$. Then for $N_\tau = 2$, table 1 yields $\beta_c(0) = 0.27$ and $\beta_c(h_c) = 0.26$. Again this is quite consistent with Monte Carlo data which give $\beta_c(0) = 0.28$ [20] and $\beta_c(h_c) = 0.27$ [6]. The agreement is also reasonable for h_c, which was found to be 0.055 in ref. [6] on a $8^3 \times 2$ lattice*.

Using eq. (2.10), we can try to relate h_c to a critical quark mass. We emphasize, once again, that this relation is reliable only when $K_c \ll K_{m_\pi = 0}$, where $K_{m_\pi = 0} = 0.25$ in the strong coupling limit [22]. One is of course most interested in $n_f = 3$. In this case we find $K_c = 0.05$ for $N_\tau = 2$, which is certainly small enough such that we can trust the K expansion. Thus, using eq. (2.14), we arrive at

$$m_c / T_c = \ln (h_c / 2n_f) = 4.6 , \tag{4.4}$$

for three flavours.

In conclusion, we have analyzed the effective theory for SU(N) lattice gauge theory in the presence of dynamical quarks at strong coupling. Mean field theory predicts a second-order phase transition for SU(2), and *first-order transitions for all other SU(N)'s* in the infinite quark mass limit. This contradicts previous claims for $N \geqslant 4$ [11]. Also via MF theory critical masses and temperatures were obtained which indicate that the deconfining transition already disappears at a very high quark mass. They are in excellent agreement with the existing SU(3) measurement at $N_\tau = 2$ [6], while our $N_\tau \geqslant 3$ results are not trustworthy. One may worry that the agreement between strong coupling and Monte Carlo implies that the Monte Carlo data are still in the strong coupling region. However, strong coupling results have previously been found to be consistent with continuum behaviour in the intermediate coupling regime [15, 23]. In any case this indicates that Monte Carlo simulations should be performed on larger lattices in order to probe the continuum more deeply.

The critical masses remain large as N increases above 3. It would be interesting to see if these $N \geqslant 4$ results are supported by Monte Carlo simulations.

---

* This is consistent with recent Monte Carlo results on a $8^3 \times 3$ lattice, where no first-order phase transition has been observed above $h_c > 0.054$ [21].

References

[1] A.M. Polyakov, Phys. Lett. 72B (1978) 477
[2] L. Susskind, Phys. Rev. D20 (1979) 2610
[3] L. McLerran and B. Svetitsky, Phys. Lett. 98B (1981) 195
[4] J. Kuti, J. Polónyi and K. Szlachányi, Phys. Lett. 98B (1981) 199
[5] J. Engels, F. Karsch, H. Satz and I. Montvay, Phys. Lett. 101B (1981) 89; Nucl. Phys. B205 [FS5] (1982) 545
[6] P. Hasenfratz, F. Karsch and I. O. Stamatescu, The SU(3) deconfinement phase transition in the presence of quarks, CERN preprint TH. 3636 (June 1983)
[7] T.A. DeGrand and C.E. DeTar, Phase structure of QCD at high temperature with massive quarks and finite quark density: a Z(3) paradigm, University of Colorado preprint COLO HEP66 (1983)
[8] T. Banks and A. Ukawa, Nucl. Phys. B225 [FS9] (1983) 145
[9] J. Polónyi and K. Szlachányi, Phys. Lett. 110B (1982) 395
[10] B. Svetitsky and L.G. Yaffe, Phys. Rev. D26 (1982) 963; Nucl. Phys. B210 [FS6] (1982) 423
[11] M. Gross, J. Bartholomew and D. Hochberg, SU(N) deconfinement transition and the N state clock model, University of Chicago preprint EFI 83–35 (July 1983)
[12] J. Bartholomew, D. Hochberg, P. H. Damgaard and M. Gross, Effect of quarks on SU(N) deconfinement phase transitions, University of Chicago preprint EFI 83–42 (August 1983)
[13] F.Y. Wu, Rev. Mod. Phys. 54 (1982) 235
[14] A. Patel, A flux tube model of the finite temperature deconfining transition in QCD, Caltech preprint (August 1983)
[15] F. Green, Nucl. Phys. B215 [FS7] (1983) 83
[16] F. Green and S. Samuel, Nucl. Phys. B190 [FS3] (1981) 113
[17] J.B. Kogut, M. Snow and M. Stone, Nucl. Phys. B200 [FS4] (1982) 211
[18] K. E. Eriksson, N. Svartholm and B.S. Skagerstam, J. Math. Phys. 22 (1981) 2276
[19] Y.Y. Goldschmidt, J. Math. Phys. 21 (1980) 1842
[20] J. Kogut, M. Stone, H.W. Wyld, W.R. Gibbs, J. Shigemitsu, S.H. Shenker and D.K. Sinclair, Phys. Rev. Lett. 50 (1983) 393;
    T. Celik, J. Engels and H. Satz, Phys. Lett. 125B (1983) 411
[21] T. Celik, J. Engels and H. Satz, Deconfinement and virtual quark loops, Bielefeld University preprint BI-TP 83/15 (August 1983)
[22] N. Kawamoto, Nucl. Phys. B190 [FS3] (1981) 617
[23] G. Münster, Nucl. Phys. B180 [FS2] (1981) 23