

Covariant structure of relativistic gases in equilibrium

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(Received 4 June 1981)

The dynamical formulation of many-particle relativistic systems using the constraint formalism is applied to the description of equilibrium statistical mechanics in such a way as to include the known properties of covariant thermodynamics. In order to investigate explicitly the thermodynamical functions from this formulation, we explain the development of a unique general prescription for the Lorentz-invariant phase-space measure of the relativistic gases, for which we look into some special cases of classical and quantum systems. In particular, by using this formulation we are able to calculate in detail a relativistic thermodynamical model for a dilute gas of two-particle systems bound together by Lorentz-invariant pair interactions.

I. INTRODUCTION

In this work we put together two recent important developments for the analysis of relativistic statistical systems. First we remark that the progress on relativistic thermodynamics can be related to the new work on the covariant structure of relativistic statistical systems (both for the special and general theories) which has been carried out in the past few years. Especially noteworthy in this context is the fact that the laws of relativistic thermodynamics¹ have been recently formulated in terms of statistical quantities arising from relativistic kinetic theory. Furthermore, the covariance of the thermodynamical quantities has been explicitly used for the investigation of equilibrium and non-equilibrium relativistic fluids.² The covariant description of the dynamics of interacting many-particle systems has lately brought some new prospects to relativistic statistical mechanics. The formulation by Todorov³ of the dynamics of relativistic point particles as a problem with constraints put together a number of important concepts for a Lorentz-invariant theory. Further independent work⁴ on interacting classical and quantum relativistic systems has demonstrated the power of this description. This approach has been united with generalized relativistic Hamiltonian dynamics.⁵

Although the classical thermodynamics for an ideal gas^{6,7} using the special theory of relativity

has been known for a long time, the structure of the phase space^{8,9} as well as the invariance of the thermodynamics¹⁰ have been often discussed since then. There are a number of rather difficult problems associated with the formulation of the theory for consistent thermodynamics and statistical mechanics using the special theory of relativity,¹⁰ which we shall not review extensively here. However, we will briefly present in the next section the basic approach to covariant thermodynamics.^{1,2} In the following section of this paper we shall discuss the covariant structure of relativistic phase space¹¹ as a basis for the statistical ensembles. In the first part of Sec. III we briefly elaborate on the description of the dynamics¹² of a relativistic many-particle system with constraints as a means of getting around the well-known fundamental problems¹³ associated with the dynamical variables for interacting relativistic particles. From here on we look into a covariant statistical formulation using a definite prescription for the phase-space measure.¹¹ In the following sections we demonstrate this theoretical framework by first working out¹⁴ some special cases. Section IV contains a new analysis of the ideal relativistic quantum gases arising out of this framework. Then in Sec. V we develop a thermodynamical model for a dilute gas with Lorentz-invariant scalar pair interactions. Finally we discuss our results in relation to some possible applications.

II. COVARIANT THERMODYNAMICS

Before we go into the details of the relativistic dynamics, which will provide for us the basis of the equilibrium statistical ensemble theory, we want to discuss briefly the fundamental thermodynamical quantities necessary for such a description. The transformation properties of these thermodynamical quantities have already been discussed in a general way which now may be brought into a satisfactory covariant formulation for the equilibrium state of a relativistic system.² Although this thermodynamical formulation is valid for more general metrics, we shall here confine ourselves to the special theory of relativity with the positive-time metric tensor $g^{\mu\nu}$, which includes only the diagonal terms $g^{\mu\mu}$ so that we have only the elements $+1, -1, -1, -1$ for $\mu=0, 1, 2, 3$. For the sake of simplicity in the equations we shall also take $c=1=\hbar$ in the following work.

The basic quantities for the description of thermodynamical systems (both for equilibrium and nonequilibrium processes) are the energy-momentum tensor $T^{\mu\nu}$, the entropy flux density s^μ , and the particle flux density n_a^μ for each constituent a of the system. The corresponding energy-momentum four-vector can be found in the usual manner by an integration over the spatial coordinates at a constant x_0 in the form

$$p^\mu = \int_{x_0} d^3x T^{\mu 0}. \quad (2.1)$$

In the general (nonequilibrium) situation the relative orientation of the four-vectors p^μ , n_a^μ , and s^μ is not strictly predetermined. However, in order to arrive at the true equilibrium situation, they must all be parallel to the single four-vector velocity u^μ of the entire system.² Therefore, we define the following scalar quantities:

$$p \equiv (p_\mu p^\mu)^{1/2} \quad (2.2)$$

is the total energy in the rest frame of the system, n_a is the number density of the particles of type a , s is the entropy density, and V is the usual three-dimensional volume. In equilibrium we have the relationships

$$p^\mu = pu^\mu, \quad (2.3a)$$

$$n_a^\mu = n_a u^\mu, \quad (2.3b)$$

$$s^\mu = su^\mu, \quad (2.3c)$$

$$V^\mu = Vu^\mu. \quad (2.3d)$$

The structure of the energy-momentum tensor $T^{\lambda\mu}$

relates to the total energy density ϵ and the scalar pressure P through the total four-velocities as

$$T^{\lambda\mu} = \epsilon u^\lambda u^\mu + P \Delta^{\lambda\mu}, \quad (2.4)$$

where the projection tensor $\Delta^{\lambda\mu}$ is defined so that

$$\Delta^{\lambda\mu} \equiv g^{\lambda\mu} - u^\lambda u^\mu. \quad (2.5)$$

For closed systems we must always have the energy-momentum conservation law

$$\partial_\mu T^{\lambda\mu} = 0. \quad (2.6)$$

This expression tells us that the macroscopic observables p^μ are conserved quantities.

It is important for us to note that in a general macroscopic state of the system the M reactions among the different types of particles may take place in such a way that the particle number densities n_a may not be conserved. Thus we may in such cases² have

$$\partial_\mu n_a^\mu = \sum_{i=1}^M c_{ai} r_i, \quad (2.7)$$

where c_{ai} is the stoichiometric coefficient for the i th reaction and r_i is the respective rate of reaction. However, in the state of equilibrium the rates of reaction will all vanish in such a way that we have a particle conservation law for each component

$$\partial_\mu n_a^\mu = 0. \quad (2.8)$$

Furthermore, the total particle flux density becomes

$$n^\mu = \sum_a n_a u^\mu = n u^\mu, \quad (2.9)$$

which also obeys this particle conservation law. These conditions mean that the corresponding macroscopic observables N_a and N (the particle numbers) defined by the spatial integrals on the time components are necessarily conserved quantities in equilibrium. However, because u^μ for the whole system is independent of each particle coordinate, the same conservation laws hold as for the densities n_a and n , so that we can write these macroscopic particle four-vectors in equilibrium N_a^μ and N^μ as

$$N_a^\mu = N_a u^\mu, \quad (2.10a)$$

$$N^\mu = N u^\mu. \quad (2.10b)$$

The general validity of the second law of thermodynamics for relativistic systems has been shown to be written in the form^{1,2}

$$\partial_\mu s^\mu \geq 0, \quad (2.11)$$

which becomes a strict equality in equilibrium. Furthermore, it is also possible to state some other properties of these systems in equilibrium such as the lack of viscous forces as well as of diffusion and of heat flow.²

Since in the equilibrium state the scalar entropy density s is a function of the total energy-momentum p and the number density n_a only, the conditions (2.6), (2.8), and (2.11) lead to the introduction of their respective equilibrium parameters.^{1,2,7} The four-vector analog of the usual inverse temperature β for the nonrelativistic systems takes the form

$$\beta^\mu = \frac{1}{kT} u^\mu, \quad (2.12)$$

This four-vector is defined so that

$$\frac{1}{kT} \equiv (\beta_\mu \beta^\mu)^{1/2} \quad (2.13)$$

simply relates to the temperature in the rest frame. In order to arrive at a manifestly covariant formulation of our ensembles, we may define i_a^μ as the four-vector equilibrium parameter relating to the relativistic chemical potential μ_a , which together with (2.12) yields

$$i_a^\nu = \mu_a \beta^\nu. \quad (2.14)$$

From this consideration we should also notice that, in contrast with Touschek's definition,⁸ the chemical potential alone is not a four-vector. μ_a is related to the usual nonrelativistic chemical potential μ_a' through

$$\mu_a = m_a + \mu_a', \quad (2.15)$$

where m_a is the rest mass of the particles of type a . Furthermore, in the case of chemical reactions the equilibrium conditions

$$\sum_a \mu_a c_{ai} = 0 \quad (2.16)$$

must be fulfilled for each reaction i . In particular for pairs of particles and antiparticles reacting with each other this equation has the consequence in equilibrium that when there is no total conserved charge then $\mu_a^- = -\mu_a^+$, where μ_a^+ and μ_a^- are the particle and antiparticle chemical potentials, respectively. Another motivation for our choice of equilibrium parameters in the form of four-vectors is that β^ν and i_a^ν relate closely to the Killing vector fields for thermal equilibrium in a gravitational field, which have a meaning when the metric is properly extended to general relativity.^{2,15}

III. RELATIVISTIC PHASE SPACE

Now that we have considered the basics of relativistic thermodynamics we want to bring these ideas together with the central concepts of the relativistic dynamics. There exist at present several new approaches to the relativistic N -particle problem³⁻⁵ using a formulation involving constraints all of which are based upon the structure of the particles' trajectories in the 8^N -dimensional phase space. The covariant structure of the equations of motion, however, introduces some additional problems^{5,12,13} which arise together with the presence of the time components of each particle. In this section we want to discuss the formulation of the relativistic dynamics in order to provide for the construction of the phase-space measure¹¹ as well as to enable us to carry out certain calculations for some special applications discussed in the following sections.

At this point we want to bring in the concept of the constraints as it first was used by Dirac¹² for his generalized Hamiltonian dynamics. He divided these constraints into two major types: the set of primary constraints $\{\varphi_i(p, q)\}$ and that of secondary constraints $\{\chi_j(p, q)\}$, both of which are in general dependent on the N -tuples of momenta and coordinates denoted by p and q , respectively. The primary constraints relates generally to the energy-momentum conservation, which appears as a mass-shell condition for the individual particles. We write these conditions in the general form

$$\varphi_i(p, q) = 0, \quad i = 1, \dots, M \quad (3.1)$$

where M is the number of primary constraints. The secondary constraints are devised so that each particle world line is counted only once for the dynamical system. Thus we write for the M secondary constraints

$$\chi_j(p, q) = 0, \quad j = 1, \dots, M \quad (3.2)$$

We may immediately write down the covariant generalization of the Poisson brackets⁴ in the form $[A(p, q), B(p, q)]$ for the functions A and B of the phase-space variables. Furthermore, one can formulate the equations of motion^{4,5,12} in a "weak form," where this expression refers to the weak equality " \approx " which has the meaning that all the algebraic operations with the Poisson brackets are carried out before equating the constraints (3.1) and (3.2) to zero. However, there is an important extra condition which assures the independence of the primary and the secondary constraints, which

may be expressed by the determinant

$$\Delta(p, q) = \det | [\chi_j, \varphi_i] | \neq 0. \quad (3.3)$$

Now we want to briefly illustrate the use of constraints through the simple case of classical free particles. The primary constraints are then with $M=N$

$$2\varphi_i \equiv m_i^2 - p_i^2 = 0 \quad (i = 1, \dots, N), \quad (3.4)$$

whereby the positive-energy solution $p_i^0 \geq 0$ is always taken. This condition now defines the positive mass shell from the energy-momentum conservation as a seven-dimension hypersurface for each particle. The six-dimensional reduced phase space may be gotten through the identification of all the points on the one-dimensional free-particle world line of each particle with a specified free parameter τ_i , which may be related to the proper time (actually the proper time per mass). Formally we may carry out this second reduction by introducing the secondary constraints of the form³

$$\chi_i \equiv \tau_i - \frac{q_{i\mu} p_i^\mu}{m_i^2} = 0 \quad (i = 1, \dots, N), \quad (3.5)$$

which themselves do not "commute" with the primary constraints since the usual relativistic brackets take the form

$$[q_i^\mu, q_j^\nu] = 0, \quad (3.6a)$$

$$[p_{i\mu}, p_{j\nu}] = 0, \quad (3.6b)$$

$$[q_i^\mu, p_{j\nu}] = -[p_{j\nu}, q_i^\mu] = \delta_{ij}^\mu \delta_{\nu\mu}. \quad (3.6c)$$

Thus the relativistic Poisson brackets for (3.4) and (3.5) are

$$[\chi_i, \varphi_j] \approx \delta_{ij}, \quad (3.7)$$

where this equality, as written, is taken in the weak sense of Dirac.¹² We would like to remark here that this case for free particles may be easily written in quantized form by simply rewriting these classical variables as their corresponding quantum operators, which we shall use for the quantum ideal gases in a later section.

As a generalization of this special case we discuss briefly the N -particle system with only two-particle interactions³⁻⁵ present. We know that in general we may write the expressions for each pair of interacting particles i and j in terms of the center of mass and relative coordinates for the momenta P_{ij} , p_{ij} and coordinates X_{ij} , x_{ij} , respectively, given as four-vectors by

$$P_{ij}^\mu = p_i^\mu + p_j^\mu, \quad p_{ij}^\mu = p_i^\mu - p_j^\mu, \quad (3.8)$$

$$X_{ij}^\mu = \frac{1}{2}(q_i^\mu + q_j^\mu), \quad x_{ij}^\mu = \frac{1}{2}(q_i^\mu - q_j^\mu).$$

Using these variables we may write a Lorentz-invariant scalar in the form

$$r_{ij} = 2 \left[\frac{(P_{ij} \cdot x_{ij})^2}{P_{ij}^2} - x_{ij}^2 \right]^{1/2} \quad (3.9)$$

and a reduced mass

$$\bar{m}_{ij} = m_i m_j / (m_i + m_j). \quad (3.10)$$

From the structure of these parameters we write our constraints for pair interactions between particles i and j with $i \neq j$ and $1 \leq i, j \leq N$ in the form

$$2\varphi_i \equiv p_i^2 - m_i^2 - 2 \sum_{j=1}^N \bar{m}_{ij} V(r_{ij}) = 0. \quad (3.11)$$

We see that the interaction must be symmetrical in the particles so that

$$\begin{aligned} \sum_{i=1}^N \Phi_i(X_{ij}, x_{ij}, P_{ij}, p_{ij}) &= \sum_{j=1}^N \Phi_j(X_{ij}, x_{ij}, P_{ij}, p_{ij}) \\ &= -2 \sum_{i,j=1}^N \bar{m}_{ij} V(r_{ij}), \end{aligned} \quad (3.12)$$

where $\Phi_i(p, q)$ represents a general interaction³ between particles, which in (3.12) is restricted to pairs. Furthermore, if we were to drop the demand of manifest covariance and take up the form of the generalized Hamiltonian H_i used in a many-time formulation,⁴ we would find from (3.11) that

$$H_i = \left[m_i^2 + \bar{p}_i^2 + 2 \sum_{j=1}^N \bar{m}_{ij} V(r_{ij}) \right]^{1/2}. \quad (3.13)$$

Here we have simply substituted H_i for p_{i0} , which gives the form of a spinless positive-energy classical Dirac equation. A general manifestly covariant form for the primary constraints is expressed by

$$2\varphi_i \equiv m_i^2 - p_i^2 - \Phi_i(p, q) = 0 \quad (i = 1, \dots, M). \quad (3.14)$$

The corresponding secondary constraints must be chosen to fulfill simultaneously the conditions (3.2) and (3.3).

Now we are in a position to discuss the construction of the phase-space measure for the interacting N -particle relativistic system. This construction has been carried out by Faddeev¹¹ in

terms of all the components p_i and q_i of the momenta and coordinates of each particle i , which may be written as

$$d\nu_N(q,p) = \prod_j \delta(\mathcal{X}_j) \delta(\varphi_j) \Delta(p,q) \prod_{i=1}^{4N} \frac{dp_i dq_i}{(2\pi)^{4N-M}}. \quad (3.15)$$

This particular form of $d\nu_N(q,p)$ was shown¹¹ to arise directly from the expression for the Lagrangian with constraints $\sum_i p_i \dot{q}_i - H(p,q) - \sum \lambda_i \varphi$, where the \dot{q}_i are the velocities and λ_i are the Lagrange undetermined multipliers. After an integration on the λ_i as variables it was seen that $d\nu_N$ in (3.15) comes as a restriction to the \mathcal{X}_j hypersurfaces written as the four-dimensional δ functions.

With the form (3.15) of the phase-space measure $d\nu_N(p,q)$ we are in a very good position for the construction of the ensembles in a covariant manner. Although we may most easily start from the microcanonical point of view, the technical aspects of the large hyperspace of 8^N dimensions together with the constraints lead to a very formidable problem which is not easily solved in covariant thermodynamical variables. However, there is a quite different situation for the canonical and grand canonical ensembles. For the canonical ensemble the number of particles N_a of each sort a is fixed. Thus the states are described by the states of fixed particle number and energy-momentum described by the four-vectors N_a^μ and p^μ discussed in Sec. II. This situation is well suited for our phase-space measure (3.15) for $d\nu_{N_a}(p,q)$. Thus for a fixed volume V_μ we may write the canonical partition function¹⁴

$$Q_{N_a}(V_\mu, \beta_\mu) = \int_{V_\mu} d\nu_{N_a}(p,q) e^{-\beta_\mu p^\mu}. \quad (3.16)$$

The grand canonical partition function can then be written for all types of particles in equilibrium as

$$\Xi(V_\mu, \beta_\mu, i_{a\mu}) = \sum_{\{N_a\}} e^{\sum_a i_{a\mu} N_a^\mu} Q_{N_a}(V_\mu, \beta_\mu). \quad (3.17)$$

The thermodynamics may now be arrived at through the normal procedure by evaluating the thermodynamical potential

$$\Omega(V_\mu, \beta_\mu, i_{a\mu}) = -kT \ln \Xi(V_\mu, \beta_\mu, i_{a\mu}) \quad (3.18)$$

in the large-volume limit.

The problem of quantization of the relativistic many-particle systems adds considerable complication to the evaluation of (3.16). Here we must re-

place the integral with discrete states $|\hat{p}^\mu, N_a^\mu\rangle$ of the operators \hat{p}^μ and \hat{N}_a^μ , respectively. Then (3.16) becomes a trace of the form

$$Q_{N_a}(V_\mu, \beta_\mu) = \sum_{\{p^\mu\}} \langle \hat{p}^\mu, N_a^\mu | e^{-\beta_\mu \hat{p}^\mu} | p^\mu, N_a^\mu \rangle. \quad (3.19)$$

However, this form generally loses many of the advantages gained from the phase-space measure. The one possible exception is the ideal quantum gases which we discuss in the next section.

IV. IDEAL RELATIVISTIC QUANTUM GASES

In this part of our work we look into the ideal relativistic quantum gases as a first application of the formalism developed in the preceding sections. Although the basic results for the ideal relativistic quantum gases are already very well known,^{6,8} it is, nevertheless, informative for us to show explicitly how these properties may be derived out of the above-developed formal structure. The detailed evaluation of the canonical partition function for the classical relativistic ideal gas using the constraint formalism may be found in Appendix A. Here we develop the general structure for many species of particles.

Since for now we are only dealing with free spinless particles, we are able to compute directly the trace in (3.19) by properly choosing a basis for our Fock space out of the simultaneous eigenstates of the operators \hat{p}^μ and \hat{N}^P built up out of the symmetrized or antisymmetrized products of the single-particle states for, respectively, the bosons (b) and the fermions (f) within the finite volume V . Furthermore the basic property of the ideal gas is expressed through the fact that the total grand partition function then splits up into products of the single species so that

$$\Xi(\beta_\mu, V_\mu, i_{a\mu}) = \prod_b \Xi(\beta_\mu, V_\mu, i_{b\mu}) \times \prod_f \Xi(\beta_\mu, V_\mu, i_{f\mu}), \quad (4.1)$$

where $\Xi(\beta_\mu, V_\mu, i_{b\mu})$ or $\Xi(\beta_\mu, V_\mu, i_{f\mu})$ represents a trace summed over the symmetrized (b) or the antisymmetrized (f) states. This trace can be explicitly evaluated for each type of particle in the form⁹

$$\Xi(\beta_\mu, V_\mu, i_{\mu\eta}) = \prod_\alpha (1 + \eta e^{k_{\mu\eta} N_a^\mu - k \beta_\mu p_{\alpha\eta}}) \quad (4.2)$$

with $\eta = \pm 1$ for the fermions and bosons, respectively. The eigenvalues N^μ and p_α^μ of the operators \hat{N}^μ and \hat{p}^μ , where α labels the eigenstates with these discrete eigenvalues, are given in the form

$$p_\alpha^\mu = ([(\vec{p}_\alpha)^2 + m^2]^{1/2}, \vec{p}_\alpha). \quad (4.3)$$

Furthermore, we define the relativistic activity (fugacity) A by

$$A^N = e^{i_\mu N^\mu}, \quad (4.4)$$

where from (2.10b) we have

$$N \equiv (N_\mu N^\mu)^{1/2}. \quad (4.5)$$

For the thermodynamics we take the logarithm of $\Xi(\beta_\mu, V_\mu, i_\eta^\mu)$ for the various species in Eq. (4.2) in order to find the thermodynamical potential $\Omega(\beta_\mu, V_\mu, i_\eta^\mu)$ in (3.18). Thus we get a sum over the eigenstates p_α^μ which may be replaced in most cases by an integral over the reduced single-particle phase space. The well-known exception to this replacement is the Bose gas at low temperature where a term is added^{16,17} for the ground state.

The use of the constraint formalism as discussed in Sec. III enables us readily to find the correct integration measure (3.15). We see that the chosen constraints for free particles written in Eqs. (3.3) to (3.5) together with the condition $p_0 > 0$ for the classical gas may be used analogously in Appendix A to yield the phase-space measure for the particle with the coordinates and momenta q_i, p_i of the form

$$d\nu(q_i, p_i) = \frac{2}{(2\pi)^3} \theta(p_{0i}) \delta(\chi_i) \delta(\varphi_i) d^4 p_i d^4 q_i. \quad (4.6)$$

If we perform the evaluation of the space-time integral in the volume V^μ with the given constraints, we find the invariant phase-space measure of Touschek,⁸

$$d\sigma(p_i) = 2V_\mu p_i^\mu \theta(p_{0i}) \delta(p_i^2 - m^2) \frac{d^4 p_i}{(2\pi)^3}. \quad (4.7)$$

The evaluation of $\Omega_\eta(\beta_\mu, V_\mu, A)$ demands at this point an approach different from its classical counterpart due to the presence of η in (4.2) arising from symmetry (antisymmetry) of the total wave function. This fact requires that we carry out this sum over the particles for $\Xi(\beta_\mu, V_\mu, A)$ in the same way as in the nonrelativistic quantum gases. When we apply (3.18) for this calculation for the Bose and Fermi gases, we find

$$\Omega(\beta_\mu, V_\mu, A) = -\beta^{-1} \eta \int d\sigma(p) \ln(1 + \eta A e^{-\beta_\mu p^\mu}), \quad (4.8)$$

where a term of the form $\beta^{-1} \ln(1 - A e^{-\beta m})$ must be added to (4.8) for the ground state of the Bose gas.¹⁶ In order to evaluate this integral analytically, we proceed to the rest frame of the system as in Appendix A for the relativistic classical ideal gas by using the Lorentz invariance of the free system. This evaluation leads to an integral very similar to the one in (A.7). We obtain the thermodynamical pressure P from $-\Omega/V$ in the large-volume limit, which is given by

$$P(\beta, A) = \frac{m^2}{2\pi^2} \sum_{k=1}^{\infty} \frac{(-\eta)^{k-1} A^k}{k^2} K_2(km\beta), \quad (4.9)$$

where $K_\nu(x)$ is the modified Bessel function of the second kind.¹⁸ Thus we see that the effect of the quantum statistics on the thermodynamics of the ideal relativistic quantum gases is represented by changing β to $k\beta$ for the single-particle system and then summing over all k with a factor $(-\eta)^{k-1}$ for the symmetry of the wave function.

The general case with many types of particles as expressed in (4.1) can be easily worked out under the equilibrium conditions in Sec. II. Therefore, the inclusion of the antiparticles for energies above the particle creation threshold is automatically built into this structure from (2.16).

V. PAIR INTERACTIONS IN RELATIVISTIC GASES

The treatment in the last section of the relativistic ideal gases is now basically well understood and requires little further elaboration. This statement is certainly not true for gases with interacting particles. In order to calculate anything at all for interacting systems, we must place very strong restrictions on the nature and type of the interactions as well as on the structure of the total system. Let us, therefore, consider the simplest extension of the ideal-gas structure, that is a classical relativistic gas of $2N$ particles forming a new system of N noninteracting pairs, for which the two particles inside each pair are bound together by the relativistic interaction $\Phi_2(r_{12})$.

We begin our investigation (preliminary results of this work have been reported by one of us in a conference¹⁹) of this "free gas of composite particles" by introducing the two-particle coordinates for each pair as given in (3.8). In terms of these coordinates we get the constraint equations for each interacting two-particle subsystem similar to the free-particle case (Appendix B) as follows:

$$\varphi_{12}^{\dagger} \equiv \frac{1}{4}(P_{12}^2 + p_{12}^2 - 4m^2) - \frac{a^2}{4}\Phi_2(r_{12}) = 0, \quad (5.1a)$$

$$\varphi_{12}^{-} \equiv \frac{1}{2}P_{12\mu}p_{12}^{\mu} = 0, \quad (5.1b)$$

$$\chi_{12}^{\dagger} \equiv X_{12\mu}u^{\mu} = 0, \quad (5.1c)$$

$$\chi_{12}^{-} \equiv x_{12\mu}u_{12}^{\mu} = 0, \quad (5.1d)$$

where u^{μ} is the four-velocity of the $2N$ -particle system, whereas u_{12}^{μ} is the four-velocity of the two-particle subsystem

$$u_{12}^{\mu} \equiv p_{12}^{\mu} / |P_{12}^{\mu}|. \quad (5.2)$$

In order to evaluate the partition function (3.16) for this system given by (5.1a) to (5.1d), we need the invariant phase-space measure (3.15), which may be written for our special case as

$$d\nu_{12}(P_{12}, p_{12}, X_{12}, x_{12}) = \theta(P_{12}^0)\theta(P_{12}^0 - p_{12}^0)\theta(R^2 - X_{12}^2)\theta(r_{12}^2 - x_{12}^2)\delta(\varphi_{12}^{\dagger})\delta(\varphi_{12}^{-})\delta(\chi_{12}^{\dagger})\delta(\chi_{12}^{-})\Delta_{12} \\ \times d^4P_{12}d^4p_{12}d^4X_{12}d^4x_{12}. \quad (5.3)$$

Here the first two θ functions assure the positivity of the energy while the third one fixes the thermodynamic volume $V = \frac{4}{3}\pi R^3$ and the last one defines the range of the relativistic two-particle interaction. The explicit evaluation of the covariant constraint determinant (3.3) yields

$$\Delta_{12}(P_{12}, p_{12}, r_{12}) = \frac{1}{4}P_{12\mu}u^{\mu} |P_{12}| - \frac{1}{4}(p_{12\mu}u^{\mu})(p_{12\mu}u_{12}^{\mu}) - \Psi(r_{12})(x_{12\mu}u_{12}^{\mu}), \quad (5.4)$$

where the last term does not contribute to the phase-space measure because of the constraint (5.1d) [$\Psi(r_{12})$ is given in Appendix C]. Now we have collected everything we need for the evaluation of the canonical partition function in the form

$$Q_N(V, \beta, \mu) = \frac{1}{(2\pi)^{6N}N!} \left[\int_{V_{\mu}} d\nu_{12} e^{-\beta p_{12}^{\mu}} \right]^N. \quad (5.5)$$

We first carry out the integrations over the zeroth components of the relative coordinates p_{12}^0 and x_{12}^0 in the rest frame of the two-particle subsystem. Then we rewrite the integrals over \vec{p}_{12} and \vec{x}_{12} as integrals (dropping the particle subscripts) over the Lorentz scalars r defined in (3.9) and k which is defined by

$$k \equiv \left[\frac{(P_{12\mu}p_{12}^{\mu})^2}{P_{12}^2} - p_{12}^2 \right]^{1/2}. \quad (5.6)$$

Next we perform the integrations over P_{12}^0 and X_{12} in the overall rest frame of the $2N$ -particle system using the constraints φ_{12}^{\dagger} and χ_{12}^{\dagger} in (5.1a) and (5.1c). The remaining integral over \vec{P}_{12} we then rewrite as an integral over the Lorentz scalar defined by

$$P \equiv [(P_{12\mu}u^{\mu}) - P_{12}^2]^{1/2}. \quad (5.7)$$

After all these simplifications have been carried out we finally may write the partition function in the overall rest frame as

$$Q_N(V, \beta) = \frac{V^N}{N!} \left[\int \int \int P^2 dP k^2 dk r^2 dr \exp\{-\beta[P^2 + k^2 + \Phi_2(r) + 4m^2]^{1/2}\} \right]^N. \quad (5.8)$$

Now we want to discuss the specific evaluation of the thermodynamical functions from our result. If we assume an interaction of the form

$$\Phi_2(r) = (ar^2)^s, \quad (5.9)$$

we can evaluate explicitly all the remaining integrals in (5.8) to find the partition function in the form

$$Q_N(V, \beta) = \frac{V^N}{\pi^{3N}N!} \left[\frac{(\pi/2)^{1/2} 2^{3/2} \Gamma(3/2s)(2m\beta)^{(7s+3)/2s}}{2sa^{3/2}\beta^{(6s+3)/s}} K_{(7s+3)/2s}(2m\beta) \right]^N. \quad (5.10)$$

From this analytical result for the partition function we can easily get all the thermodynamical functions. For the sake of our better understanding of the above result let us look into a few special cases of physical

relevance for the free-energy density

$$f(\beta, \rho) = - \lim_{N, v \rightarrow \infty} \frac{1}{V\beta} \ln Q_N(V, \beta), \tag{5.11}$$

where the thermodynamical limit is taken at a given particle density $\rho = N/V$. (a) The relativistic harmonic oscillator has the type of interaction (5.9) with $s=1$, so that we get from (5.11) using (5.10)

$$f(\beta, \rho) = - \frac{\rho}{\beta} \ln \left[\frac{2e(2m\beta)^5 K_5(2m\beta)}{(2\pi)^2 a^3 \rho \beta^9} \right]. \tag{5.12}$$

(b) The linear (confining) potential has $s = \frac{1}{2}$. Thus we find

$$f(\beta, \rho) = - \frac{\rho}{\beta} \ln \left[\frac{64e}{(2\pi)^{5/2} a^{3/2} \rho} \frac{(2m\beta)^{13/2} K_{13/2}(2m\beta)}{\beta^{12}} \right]. \tag{5.13}$$

(c) For very large s we can find a limiting form for $s \rightarrow \infty$, so that

$$f(\beta, \rho) = - \frac{\rho}{\beta} \left[\frac{4e(2m\beta)^{7/2}}{3(2\pi)^{5/2} a^{3/2} \beta^6} K_{7/2}(2m\beta) \right]. \tag{5.14}$$

This result may be interpreted as the extreme case of tightly bound pairs of particles, which is analogous to the nonrelativistic problem for two particles in an infinitely high wall potential (square well).

Finally we carry out a numerical evaluation of these formulas in order to gain a perspective on the effect of these interactions on the thermodynamics. For these evaluations it is convenient to introduce new variables in units of the rest mass of the particles; the mass inverse temperature $x = m\beta$ and the mass density $y = m\rho$. In terms of these variables we have drawn the free energy for the following three special cases: (a) the free relativistic ideal gas of $2N$ particles with the free-energy density for $y=1$,

$$f(x) = \frac{1}{x} \ln \left[\frac{e}{2\pi^2} \frac{K_2(x)}{x} \right], \tag{5.15}$$

and the energy per particle

$$\bar{\epsilon}(x) = x \frac{K_3(x)}{K_2(x)} - 1 - x; \tag{5.16}$$

(b) the tightly bound relativistic gas of N pairs with the free-energy density for $y=0.5$,

$$f(x) = - \frac{1}{2x} \ln \left[\frac{256e}{3(2\pi)^{5/2}} \frac{K_{7/2}(2x)}{(2x)^{5/2}} \right], \tag{5.17}$$

and the energy density per particle.

$$\bar{\epsilon}(x) = 2x \frac{K_{9/2}(2x)}{K_{7/2}(2x)} - 1 - 2x; \tag{5.18}$$

(c) the gas of N pairs with relativistic harmonic interactions for $y=0.5$,

$$f(x) = - \frac{1}{2x} \ln \left[\frac{32e}{\pi^2} \frac{K_5(2x)}{x^4} \right], \tag{5.19}$$

and the energy density per particle,

$$\bar{\epsilon}(x) = 2x \frac{K_6(2x)}{K_5(2x)} - 1 - 2x. \tag{5.20}$$

In Figs. 1 and 2 we show these cases (a), (b), and (c) for $f(x)$ and $\bar{\epsilon}(x)$, respectively.

As a conclusion for this section we may remark that the results of these evaluations for the thermodynamical functions show certain pronounced ten-

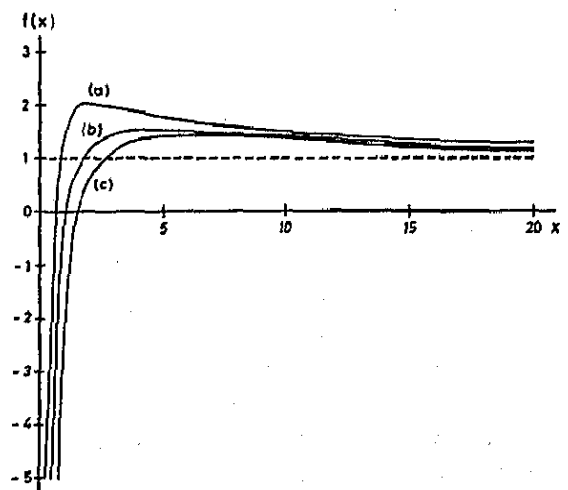


FIG. 1. The free-energy density in the three cases (a) free relativistic ideal gas, (b) tightly bound pairs, and (c) relativistic gas of pairs with harmonic interactions.

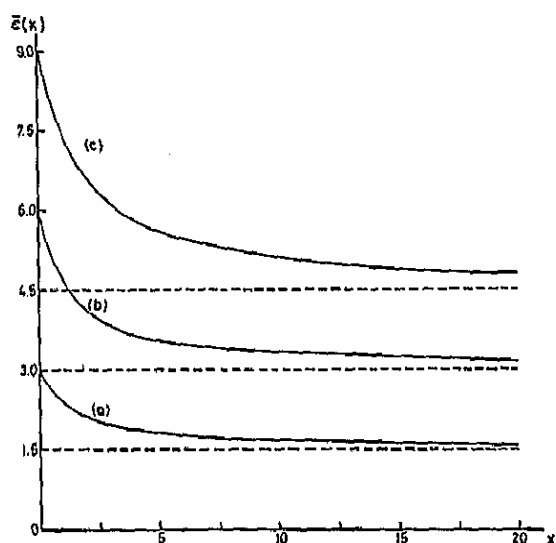


FIG. 2. The energy density per particle in the three cases (a), (b), and (c).

dencies in relation to the value of s in the pair interactions. In the first figure we see that the free-energy density $f(x)$ is generally lowered for smaller values of s . However, for the energy density per particle $\bar{\epsilon}(x)$ in the second figure the trend is the opposite. In fact, the values of $\bar{\epsilon}(x)$ at $x=0$ give the degrees of freedom of the corresponding ultrarelativistic gases in the sense of the equipartition law.²⁰ Likewise, the asymptotic values $\bar{\epsilon}(x)$ for large x give the corresponding expressions for the nonrelativistic gases. Thus we may summarize these results by stating that our given (model) form of the two-particle interaction (5.9) leads to a simple interpretation for gases with different numbers of degrees of freedom.

VI. SUMMARY AND DISCUSSION

In this work we have developed a method for calculating the thermodynamical functions for relativistic many-particle systems. First we saw how the Lorentz invariance of the thermodynamical variables in equilibrium may be constructed, especially for the temperature and chemical potential in terms of the four-vectors β^μ and i^μ , respectively. The relativistic dynamics has been expressed through the two types of manifestly covariant constraints for which we introduced the interactions in a Lorentz-invariant form $\Phi(r)$ where the Lorentz scalar r^2 replaced the distance (squared) in the moving systems. These ideas were then used in the formulation of a relativistic statistical mechanics with a Lorentz-invariant phase-space measure

(Faddeev measure). This formulation we have directly applied to the relativistic ideal quantum gases as well as to some special pairwise interacting relativistic gases.

Now we want briefly to analyze our approach to this problem. Throughout this work we have not brought into direct consideration the contributions from the angular momentum in the many-particle interactions even though they could have been included in the dynamics as mentioned in Sec. III. At the very minimum this oversight would give an incomplete dynamical description even for the pair interactions which may rotate around their center of mass analogous to the nonrelativistic diatomic molecule with its rotational degree of freedom. However, a possible justification in the special case of the rotationally symmetrical interactions $\Phi(r)$ as discussed in Sec. V may be that the dependence for large distances, as seen in the nonrelativistic limit, would be much weaker in the powers of r for the constraints than that of the central part of $\Phi(r)$. This statement would not be true in the presence of angular-momentum (spin) dependent forces, which we do not take into consideration here. Nevertheless, our special type of model with its obvious restrictions may well have a certain importance in relation to hadronic matter.

The dynamical form of our interactions has been previously proposed in a more general form for hadronic structure calculation. The model of hadronic structure developed by Feynman, Kislinger, and Ravndal²¹ (FKR) used a relativistic harmonic-oscillator model for the interactions of quarks for an explanation for the mass spectrum of the mesons and baryons. In a similar manner we could also discuss the thermodynamics of the meson as a relativistic composite system in the model of Sec. V with $s=2$, where we only considered relativistic pair interactions. However, we want to state clearly that this type of model is much too primitive to realistically hope to be able to say anything about the thermodynamics arising directly from the quark interactions since we have no way of including any of the effects coming from the gluons. As before¹⁹ we shall refrain from going into the details of the thermodynamics of systems for quantum chromodynamics. Thus in our calculations we only mention the relativistic thermodynamics arising from a classical pairwise interacting gas without further consideration of the internal structure.

Recent considerations of many-body forces^{22,23} investigate the direct-interaction dynamics of rela-

tivistic N -particle systems ($N > 2$) for the solution of the Todorov-Komar equations.^{3,4} These investigations go beyond the simplified dynamical structure from pair interactions used by us in our thermodynamical model or by FKR in their model for hadronic structure. Although it may eventually be possible to similarly generalize our model for the thermodynamics of larger clusters, the forms of the constraints become for many particles increasingly more difficult.

ACKNOWLEDGMENTS

We would like to thank H. Satz for his encouragement of this work. To our colleagues N. Bilić and V. V. Dixit we are grateful for many discussions. We would also like to thank W.-H. Steeb for critically reading and discussing the manuscript. This work has been supported in part by the Deutsche Forschungsgemeinschaft under Contract No. Mi 211/1.

APPENDIX A: CLASSICAL RELATIVISTIC IDEAL GAS

We want to evaluate the canonical partition function $Q_N(\beta, V)$ for the classical relativistic ideal

$$d\nu_N(q, p) = 2^N \prod_{i=1}^N \theta(p_{0i}) \delta(p_i^2 - m^2) \theta(R^2 - q_{i\mu}^2) \delta(q_{i\mu} u^\mu) \frac{d^4 p_i d^4 q_i}{(2\pi)^{3i}} \tag{A5}$$

We perform the integration over the coordinates using the secondary constraint condition and the similarity of the particles to give the result

$$Q_N(V, \beta, \mu) = \frac{2^N}{(2\pi)^{3N} N!} \left[\int d^4 p \delta(p^2 - m^2) p_\mu V^\mu \theta(p_0) e^{-\beta p^\mu} \right]^N, \tag{A6}$$

which is simply the Touschek measure.⁸ Since we have a Lorentz-invariant measure, we transform for convenience of our evaluation to the rest frame of the volume V . The integration over p_0 using the primary constraint with (p_0) yields

$$Q_N(\beta, V) = \frac{V^N}{(2\pi)^{3N} N!} \left[\int d^3 p e^{-\beta(p^2 + m^2)^{1/2}} \right]^N. \tag{A7}$$

After evaluating this integral we find the canonical partition function of the form⁶⁻¹⁰

$$Q_N(\beta, V) = \frac{V^N m^{2N}}{(2\pi^2)^N N! \beta^N} [K_2(m\beta)]^N. \tag{A8}$$

For the thermodynamics of the classical relativistic ideal gas we may quite simply replace this result

gas⁶ of similar particles as a simple example of how to use different secondary constraints. The primary constraint remains as in (3.9)

$$2\varphi_i \equiv p_i^2 - m^2 = 0, \tag{A1}$$

while we choose another gauge for χ_i , which is convenient for the gas in a box of volume V^μ moving with velocity u^μ as described in Sec. II, Eq. (2.3d). We write the secondary constraint

$$\chi_i \equiv q_\mu u^\mu = 0. \tag{A2}$$

We have the relativistic brackets of the form

$$[\chi_i, \varphi_j] = p_\mu u^\mu \neq 0. \tag{A3}$$

The classical canonical partition function is a special case of (4.7) with the correct Boltzmann counting factors which takes the form

$$Q_n(V_\mu, \beta_\mu) = \frac{1}{N!} \int d\nu_N(q, p) e^{-\beta_\mu p^\mu} \tag{A4}$$

with the Faddeev phase-space measure of the form (4.10), which reduces to a product of the form with the positive-energy condition

into (4.6) together with (4.2).

However, we may explicitly calculate the free-energy density $f(\beta)$ in the thermodynamic limit by keeping the density $N/V = \rho$ fixed. Thus we write the logarithm of (A8) as

$$f(\beta) = -\beta^{-1} \lim_{\substack{V \rightarrow \infty \\ N \rightarrow \infty}} \frac{1}{V} \ln Q_N(\beta, V). \tag{A9}$$

After we take the Stirling approximation, we find

$$f(\beta) = -\frac{\rho}{\beta} \ln \left[\frac{m^2 e K_2(m\beta)}{2\pi^2 \beta \rho} \right].$$

From $f(\beta)$ we can derive the other thermodynamical quantities such as the pressure, entropy, and so forth.

APPENDIX B: PHASE SPACE FOR TWO-PARTICLE SYSTEMS

In this appendix we want to carry out the technical details for the two-particle systems of Sec. III for the Faddeev phase-space measure. After using the transformations to the center-of-mass system (CMS) in (3.8) we may propose new constraints so as to replace the ones of the form (3.4). We define the two-particle constraints for the free system in the form

$$\varphi_{ij}^+ \equiv \varphi_i + \varphi_j = \frac{1}{4}(P_{ij}^2 + p_{ij}^2 - 4m^2), \tag{B1a}$$

$$\varphi_{ij}^- \equiv \varphi_i - \varphi_j = \frac{1}{2}P_{ij\mu}p_{ij}^\mu, \tag{B1b}$$

$$\chi_{ij}^+ \equiv X_{ij\mu}u^\mu, \tag{B1c}$$

$$\chi_{ij}^- \equiv x_{ij\mu}u^\mu, \tag{B1d}$$

where the unit vector (the relative velocity) gives the relative direction

$$u_{ij}^\mu = P_{ij}^\mu / |P_{ij}|. \tag{B2}$$

We evaluate the determinant in (4.9) as

$$\det \begin{vmatrix} [\chi_{ij}^+, \varphi_{ij}^+] [\chi_{ij}^+, \varphi_{ij}^-] \\ [\chi_{ij}^-, \varphi_{ij}^+] [\chi_{ij}^-, \varphi_{ij}^-] \end{vmatrix} = \frac{1}{4} |P_{ij}| P_{ij\mu}u^\mu - \frac{1}{4} (p_{ij\mu}u^\mu)(P_{ij\mu}u_{ij}^\mu). \tag{B3}$$

Now we may construct the Faddeev phase-space measure in (4.10) of the form for any two particles,

$$\begin{aligned} d\nu_{ij}(P_{ij}, p_{ij}, X_{ij}, x_{ij}) &= \theta(P_{ij0} - p_{ij0}) \delta(\varphi_{ij}^+) \delta(\varphi_{ij}^-) \delta(\chi_{ij}^+) \delta(\chi_{ij}^-) \\ &\times \frac{1}{4} [|P_{ij}| P_{ij\mu}u^\mu - (p_{ij\mu}u^\mu)(P_{ij\mu}u_{ij}^\mu)] \theta(R^2 - \chi_{ij}^2) \theta(R^2 - r_{ij}^2) \\ &\times d^4P_{ij} d^4p_{ij} d^4X_{ij} d^4x_{ij}, \end{aligned} \tag{B4}$$

where the two θ functions assure the positivity of the energy.

Now we want to evaluate the canonical partition function for N pairs containing two types of particles with this measure given by (B4), so that

$$Q_N(V, \beta, \mu) = \frac{1}{(2\pi)^{6N} (N!)^2} \left[\int d\nu_{ij} e^{-\beta_\mu P_{ij}^\mu} \right]^N, \tag{B5}$$

where we have reduced the products over the individual particles to the N pairs. Thus we may evaluate the integral by beginning as we did in Appendix A in the CMS for the P_{ij} and X_{ij} integrations in invariant form while transforming into the relative coordinates for the p_{ij} and x_{ij} (using the

Lorentz invariants k_{ij} and r_{ij}). Therefore, after performing the integrations over the coordinates X_{ij} and x_{ij} using the secondary constraints (B1c) and (B1d) with the corresponding θ functions in (B4) as well as integrating over p_{ij}^0 and k_{ij}^0 in the above states manner, we find that the second term in the determinant is removed by (B1d), so that

$$Q_N(V, \beta) = \frac{V^{2N}}{(2\pi)^{6N} (N!)^2} \left[\int d^3P_{ij} d^3k_{ij} e^{-\beta P_{ij}^0} \right]^N, \tag{B6}$$

where from (B1a) we get

$$P_{ij}^0 = (P_{ij}^2 + k_{ij}^2 - 4m^2)^{1/2}. \tag{B7}$$

The integral in (B6) can be evaluated in the form²⁴

$$(4\pi)^2 \int x_1^2 dx_1 x_2^2 dx_2 e^{-\beta(x_1^2 + x_2^2 + b^2)^{1/2}} = \frac{(4\pi)^2}{\beta} \int x_2^2 (x_2^2 + b^2) K_2(\beta(x_2^2 + b^2)^{1/2}) \tag{B8a}$$

$$= \frac{\sqrt{\pi} (4\pi)^2 b^{7/2}}{\beta^{5/2} \sqrt{2}} K_{7/2}(\beta b) \tag{B8b}$$

$$= \frac{2(2\pi)^{5/2} b^{7/2}}{\beta^{5/2}} K_{7/2}(\beta b). \tag{B8c}$$

Altogether we have

$$Q_N(V, \beta) = \frac{2^N V^{2N} (2m)^N}{(2\pi)^{5/2N} \beta^{5/2} (N!)^2} [K_{7/2}(2\beta m)]^N. \quad (\text{B9})$$

Then we use (3.17) and (3.18) again for the thermodynamics.

APPENDIX C: PHASE-SPACE MEASURE FOR PAIR INTERACTIONS

Here we want to indicate the calculation of the Faddeev phase-space measure. We begin with the manifestly covariant constraints (5.1a) to (5.1d), where only (5.1a) is changed from those in Appen-

dix B through the factor $\Phi_2(r_{ij})$. Thus we need only discuss $[\chi_{ij}^+, \varphi_{ij}^+]$ and $[\chi_{ij}^-, \varphi_{ij}^+]$, from which only the first term offers a new contribution. We find from (5.1a) and (5.1c) that

$$[\chi_{ij}^+, \varphi_{ij}^+] = \frac{1}{2} P_{ij\mu} u^\mu - \frac{a^2}{4} [\chi_{ij\mu}, \Phi(r_{ij})] u^\mu. \quad (\text{C1})$$

From the definition of r_{ij} in (3.9) the contribution comes from $(x_{ij\mu} P_{ij}^\mu)^2 / |P_{ij}|^2$, which always gives a term of the form $(x_{ij\mu} u_{ij}^\mu)$. Then the function $\Psi(r_{ij})$ represents the other terms. For example, for the harmonic oscillator we find

$$\Psi(r_{ij}) = \frac{1}{2} (x_{ij\mu} u^\mu) - \frac{1}{4} (r_{ij}^2 - x_{ij}^2). \quad (\text{C2})$$

However, because of the constraint (5.1d) this additional term always disappears in the evaluation of the integral in (5.5).

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