Thermodynamics of extended hadrons

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Hadronic systems are characterized by unrestricted conversion of energy into particles and by the spatial extension of the particles. Investigating a system of hard-core particles with free creation and annihilation, we show that with increasing energy density this leads to a phase transition from a state of abundant production and high mobility ("hadron gas") to a state in which both creation and mobility are strongly restricted ("hadronic solid"). Such hadronic solids could thus be considered in terms of weakly coupled quark matter.

I. INTRODUCTION

The essential feature of strongly interacting multiparticle systems is the presence of dominant particle production: it is the copious conversion of energy into additional particle degrees of freedom that distinguishes hadron physics from weak or electromagnetic interactions (at least at present energies). On the other hand, hadronic interactions also exhibit a basic dimensional scale, interpretable in a variety of ways: as universal Regge slope, through universal transverse-momentum spectra for secondaries, as the finite range of strong interactions, or perhaps simplest as the finite extent or "size" of hadrons in coordinate space. This aspect led Pomeranchuk,[1] already quite some time ago, to demand a minimum size for multihadron systems. More recently, and triggered by the advent of the quark infrastructure of hadrons, it led to the bag picture[2] of hadrons, suggesting the existence of hadronic matter up to a certain density, and quark matter beyond.[3]

In the present paper, we want to study in very simple terms the thermodynamic competition of the two basic hadronic features, abundant particle production and finite particle size.

The simplest model for hadronic matter is an ideal gas with free creation and annihilation,[4] similar to a photon gas. The nonvanishing rest mass of the lightest hadron assures, however, that the grand microcanonical density of states, $\sigma(P^2)$, contains at fixed overall four-momentum P only a finite number of terms. For a given system of pions in a box V we thus have

$$\sigma(P^2) \simeq \sum_{N=1}^{C(P^2)} \frac{V^N}{(2\pi)^{3N}N!} \int \prod_{i=1}^{N} d^3 p_i \delta^{(4)}\left(\sum_i p_i - P\right), \quad (1)$$

with $C(P^2) = [\sqrt{P^2}/m]$, where m is the rest mass of the pion: $p^2 = m^2$. For finite-sized hadrons, Eq. (1) remains a approximation valid at low density, since for any given hadron the reduction

of available coordinate space by the presence of all other hadrons is neglected. If we hold V fixed for large P^2, we have as high-density limit of $\sigma(P^2)$,

$$\sigma(P^2) \simeq \sum_{N=1}^{C(V)} \frac{V(N)}{(2\pi)^{3N}N!} \int \prod_{i=1}^{N} d^3 p_i \delta^{(4)}\left(\sum_i p_i - P\right), \quad (2)$$

where $V(N)$ is the coordinate-space integral for N hadrons of extent V_0 each. Now $C(V) \equiv [V/V_0]$ terminates the summation when "the box is full." In other words, hadronic matter exhibits two limiting forms: at low-energy density it is the rest mass of the pion that provides a bound on the number of particles possible and at high energy it is the size of the pion that does so.

Our aim will be to obtain the thermodynamic description of these two regimes (equations of state, temperature dependence of energy and particle density, etc.) as well as of their connection; in particular we shall show that the increase of energy density or temperature in general leads to a phase transition from a state of high mobility and copious production to one of low mobility and strongly suppressed production.

II. THE RELATIVISTIC HARD-CORE GAS

A. The general formalism

We consider here the density of states for a system of hard-sphere pions, each of intrinsic volume V_0 contained in a box of volume V,

$$\sigma(P^2, V) = \delta^{(4)}(P)$$

$$+ \sum_{N=1}^{\infty} \frac{V(N)}{(2\pi)^{3N}N!}$$

$$\times \int \prod_{i=1}^{N} d^3 p_i \delta^{(4)}\left(\sum p_i - P\right). \quad (3)$$

The first term corresponds to the vacuum, while $V(N)$ denotes the coordinate-space volume avail-

able to N pions. The highest possible value of
the number of pions is determined either by the
finite overall energy of the system or by the
finite overall volume, as discussed above.

The corresponding partition function $Z(\beta, V)$ is
given by

$$Z(\beta, V) \equiv \int d^4P \, e^{-\beta_\mu P^\mu} \sigma(P^2, V) , \qquad (4)$$

where $\beta = (\beta_\mu \beta^\mu)^{1/2}$, with $\beta^2 > 0$, $\beta_0 > 0$, denotes
the inverse temperature $\beta = 1/kT$. From Eq. (3)
this yields

$$Z(\beta, V) = \sum_{N=0}^{C} \frac{V(N)}{(2\pi)^{3N} N!} [\varphi_0(\beta)]^N , \qquad (5)$$

$$\varphi_0(\beta) \equiv \int d^3p \, e^{-\beta_\mu p^\mu} = \frac{4\pi m^2}{\beta} K_2(m\beta) , \qquad (6)$$

with $V(0) \equiv 1$, and again $C \equiv [V/V_0]$. The coordi-
nate-space volume[5] for N hard spheres is given by

$$V(N) = \int \prod_{i=1}^{N} d^3q_i \exp\left[-\beta \sum_{i<j=2}^{N} U(|\vec{q}_i - \vec{q}_j|)\right], \qquad (7)$$

$$U(r) = \begin{cases} 0, & r > R_0 \\ \infty, & r \leq R_0 \end{cases} \qquad (8)$$

where $V_0 = 4\pi R_0^3/3$.

We note that in contrast to nonrelativistic sys-
tems, where one has the particle number or
chemical potential as an additional external param-
eter, both $\sigma(P^2, V)$ and $Z(\beta, V)$ are functions of two
variables only. This feature is characteristic
for closed relativistic systems without conserved
quantum numbers, since here the interaction dy-
namics fixes the particle number density for a
given internal energy. For such systems—the
photon gas is a well-known example—all thermo-
dynamic quantities become, with $V \to \infty$, functions
of energy density or temperature only. In par-
ticular, from the Helmholtz free energy

$$\beta A(\beta, V) = -\ln Z(\beta, V), \qquad (9)$$

the pressure is obtained as

$$P\beta = \lim_{V \to \infty} [-\beta(\partial A/\partial V)_\beta]$$

$$= \lim_{V \to \infty} [\partial \ln Z(\beta, V)/\partial V]_\beta , \qquad (10)$$

while the particle number density is given by

$$n = \lim_{V \to \infty} \left[\frac{1}{V} \varphi_0 \left(\frac{\partial \ln Z(\beta, V)}{\partial \varphi_0}\right)_V\right]. \qquad (11)$$

Equations (10) and (11) take the place of the non-
relativistic virial equations. Although there is
no room now for a chemical potential as an inde-
pendent variable, we may nevertheless introduce an

analogous creation-annihilation potential μ to
describe the resistance or affinity of the system
towards particle production. The general parti-
tion function

$$Z(\beta, V) = \sum_{N=0}^{\infty} R_N(V, \beta) \frac{\varphi_0^N(\beta)}{(2\pi)^{3N} N!} , \qquad (12)$$

with $R_N(V, \beta)$ determined by the interaction, is
written as

$$Z(\beta, V) = \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{V\varphi_0(\beta) e^{\mu\beta}}{(2\pi)^3}\right)^N , \qquad (13)$$

in order to define μ in

$$e^{\mu\beta} = \lim_{V \to \infty} \frac{(2\pi)^3 \ln Z(\beta, V)}{V\varphi_0(\beta)} \qquad (14)$$

as a creation potential. From Eq. (14) one im-
mediately obtains $\mu = 0$ for a photon gas ($R_N = V^N$),
as expected for a system with "free" creation and
annihilation. Finally we observe that the require-
ment of equal free energy densities as an equilib-
rium condition implies automatically, by Eqs. (9)
and (14), also the equality of creation potentials.

Let us now return to the hard-sphere partition
function (5). The evaluation of the configuration-
space integral (7) is in general a highly nontrivial
mathematical problem.[6] A closed-form solution
exists in fact only for systems in one space di-
mension[7]; however, such systems cannot undergo
phase transitions[8] and therefore do not provide
even a qualitative approximation in the vicinity of
the critical density. Alternative approaches are
computer simulations,[9] mean-field models,[6] and
considerations of high- and low-density limits
together with equilibrium conditions.[10] We want
to concentrate mainly on the latter; but to illus-
trate the concepts and methods involved, we shall
start with some over-simplified soluble models,
including also the one-dimensional case.

B. The truncated ideal gas

We shall first look at the simplified form of the
partition function (5) resulting from the ansatz

$$N(V) = V^N , \qquad (15)$$

maintaining, however, $C = [V/V_0]$ as the upper
limit of the summation in Eq. (5). This amounts
to neglecting the effect of hadronic size on the
coordinate-space volume everywhere except in the
cutoff when the box is full. Using Eq. (15), we
can rewrite the partition function as

$$Z(\beta, V) = \sum_{N=0}^{C} \frac{(Cx)^N}{N!} \equiv Z(x, C) , \qquad (16)$$

$$x \equiv \frac{V_0 \varphi_0(\beta)}{(2\pi)^3} . \qquad (17)$$

The thermodynamic description of the system is then obtained through Eqs. (10) and (11). To evaluate $Z(x, C)$, we note that it is always bounded by $\exp\{Cx\}$. In the summation (16), two different cases are now possible: $\{Cx\}$ can be such that the largest term in the expansion of $\exp\{Cx\}$ occurs for some $\bar{N} \leq C$, or $\{Cx\}$ is such that $\bar{N} > C$. In the first case, we expect $\exp\{Cx\}$ to approximate $Z(x, C)$; in the second we do not. To determine the largest term in the expansion of $\exp\{Cx\}$, we write

$$(Cx)^N/N! \simeq (eCX/N)^N/\sqrt{2\pi N} = \frac{1}{\sqrt{2\pi N}} e^{N(\ln\{eCx\}-\ln N)} . \tag{18}$$

Differentiating with respect to N, we obtain

$$\ln(eCx/\bar{N}) = 1 + 1/2\bar{N} \tag{19}$$

as a condition for the maximum, which thus occurs when

$$Cx = \bar{N}e^{1/2\bar{N}}, \tag{20}$$

or, with $N \to \infty$, approximately for

$$Cx = \bar{N}. \tag{21}$$

Hence, if $x \leq 1$, the maximum term in the expansion $\exp\{Cx\}$ is included in the summation (16); if $x > 1$ it is not.

For $x \leq 1$, we have, therefore,

$$e^{Cx} \geq Z(x, C) \geq \frac{(Cx)^{Cx}}{C!} \simeq \frac{e^{Cx}}{(2\pi Cx)^{1/2}}, \tag{22}$$

using the largest term in the sum as lower bound. It follows that

$$x \geq \frac{1}{C}\ln Z(x, C) \geq x - \frac{1}{C}\ln(2\pi Cx)^{1/2}, \tag{23}$$

so that in the thermodynamical limit we have

$$\lim_{C \to \infty}\frac{1}{C}\ln Z(x, C) = x. \tag{24}$$

For $x > 1$, the largest term in Eq. (16) occurs at $N = C$, and hence

$$(C + 1)\frac{(Cx)^C}{C!} \geq Z(x, C) \geq \frac{(Cx)^C}{C!}. \tag{25}$$

Using again the Stirling approximation, this gives

$$\lim_{C \to \infty}\frac{1}{C}\ln Z(x, C) = 1 + \ln x \tag{26}$$

as the thermodynamic limit.

For the pressure, we then have from Eqs. (10), (24), and (26)

$$p\beta V_0 = \begin{cases} x, & x \leq 1 \\ 1 + \ln x, & x \geq 1 \end{cases} \tag{27}$$
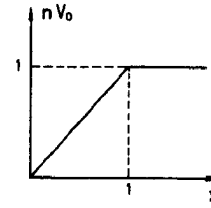
while Eqs. (11), (24), and (26) give

FIG. 1. The particle density for the truncated ideal gas.

$$nV_0 = \begin{cases} x, & x \leq 1 \\ 1, & x \geq 1 \end{cases} \tag{28}$$

for the particle density. The density of the system thus increases up to the critical value $n_c = 1/V_0$ of one hadron per cell of hardonic size ("full box") and then remains constant, as shown in Fig. 1. From Eqs. (27) and (28) it becomes clear that

$$x_c = \frac{V_0\varphi_0(\beta_c)}{(2\pi)^3} = 1 \tag{29}$$

defines a critical temperature associated with a phase transition; since $(\partial^2 A/\partial T^2)_V \sim \partial n/\partial x$ is discontinuous at β_c, while $(\partial A/\partial T)_V \sim n$ is continuous there, the transition is of second order.

The energy density of the system,

$$\epsilon = \frac{1}{V_0}\frac{\partial x}{\partial \beta}\lim_{C \to \infty} -\frac{1}{C}\frac{\partial \ln Z(x, C)}{\partial x}, \tag{30}$$

becomes similarly

$$\epsilon = -\frac{1}{V_0}\frac{\partial x}{\partial \beta}\times\begin{cases} 1, & x \leq 1 \\ 1/x, & x \geq 1 \end{cases} \tag{31}$$

and thus leads to a specific heat

$$V_0 C_V = \beta^2 \times \begin{cases} \dfrac{\partial^2 x}{\partial \beta^2}, & x \leq 1 \\ \dfrac{1}{x}\dfrac{\partial^2 x}{\partial \beta^2} - \dfrac{1}{x^2}\left(\dfrac{\partial x}{\partial \beta}\right)^2, & x \geq 1 \end{cases} \tag{32}$$

with a finite discontinuity at $x = 1$, as expected of a second-order phase transition. In the ultra-relativistic limit $(m \to 0)$,

$$\varphi_0(\beta) = 8\pi/\beta^3 \tag{33}$$

and hence at the critical temperature

$$kT_c = \pi^{2/3}V_0^{-1/3} \tag{34}$$

the jump in the specific heat becomes

$$\Delta C_V = C_V(T_c^+) - C_V(T_c^-) = -9, \tag{35}$$

with $T_c^+(T_c^-)$ denoting the approach to T_c from above (below), respectively.

Finally, we note that the creation potential (14) is given by

$$\mu\beta = \begin{cases} 0, & x \leq 1 \\ \ln\left(\dfrac{1+\ln x}{x}\right), & x \geq 1 \end{cases} \tag{36}$$

as shown in Fig. 2.

The nature of the two phases involved is by now quite clear: below T_c, we have essentially black-body radiation, with $\mu = 0$ and [from Eqs. (31) and (33)] the Stefan-Boltzmann relation

$$\epsilon = (3/\pi^2)(kT)^4 \tag{37}$$

between energy density and temperature. At $T = T_c$, the creation of further hadrons is stopped because of the finite size of the hadrons [hence T_c is, in Eq. (29) or (34), determined by V_0]. Beyond T_c we have a relativistic gas with a fixed number of constituents and hence an energy density

$$\epsilon = 3n_c kT, \tag{38}$$

as obtained from Eqs. (28) and (31). It is thus the size of the hadrons which assures that the limit of high energy density does not lead to a photon gas. Turning to the specific heat, we note in the low-density phase, an energy input is converted both into additional particles and internal motion, while in the high-density phase, it is all put into internal motion. Hence more energy is needed below T_c than above to increase the temperature, and so $C_V^- > C_V^+$.

In closing this section, we observe that the equation of state

$$p\beta/n = \begin{cases} 1, & x < 1 \\ 1 + \ln x, & x \geq 1 \end{cases} \tag{39}$$

has the form shown in Fig. 3.

C. The one-dimensional case

In Sec. II B, we considered the hard-sphere repulsion between hadrons only through the upper bound $N \leq C$ of the partition function. We shall now include to some degree the reduction of the coordinate-space volume $V(N)$ as well, replacing Eq. (15) by

$$V(N) = (V - NV_0)^N; \tag{40}$$

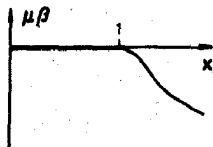this approximation becomes exact for hard-core

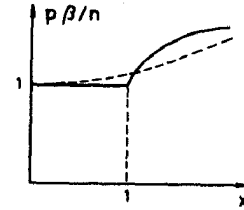FIG. 2. The creation potential for the truncated ideal gas.

FIG. 3. The equation of state for the truncated ideal gas (solid line) and for the one-dimensional hard-core gas (dashed line).

systems in one space dimension.[7]

Using the same notation as before, we find for the partition function (5)

$$Z(x, C) = \sum_{N=0}^{c} \frac{(Cx)^N}{N!} \left(1 - \frac{N}{C}\right)^N \equiv \sum_{N=0}^{c} g_N(x, C). \tag{41}$$

In contrast to Eq. (16) we no longer have two cases: the largest term of the sum always occurs for $N < C$, which—as we shall see—prevents the appearance of a phase transition. Denoting the largest term in the sum (41) by $g_{\bar{N}}(x, C)$, we have

$$g_{\bar{N}}(x, C) \leq Z(x, C) \leq C g_{\bar{N}}(x, C), \tag{42}$$

and hence

$$\frac{1}{C}\ln g_{\bar{N}} \leq \frac{1}{C}\ln Z \leq \frac{1}{C}\ln g_{\bar{N}} + \frac{1}{C}\ln C. \tag{43}$$

In the limit of $C \to \infty$, this implies

$$\lim_{C\to\infty}\frac{1}{C}\ln Z(x, C) = \lim_{C\to\infty}\frac{1}{C}\ln g_{\bar{N}}(x, C), \tag{44}$$

and similarly for the derivatives with respect to V or x; we thus need to determine only $g_{\bar{N}}(x, C)$. Using the Stirling approximation

$$g_N(x, C) \simeq \frac{1}{\sqrt{2\pi N}} e^{N[1+\ln x - \ln(N/C)+\ln(1-N/C)]}, \tag{45}$$

we find that $g_N(x, C)$ attains its maximum as a function of N at fixed x for $N = \bar{N}$, with $\bar{N} \sim V$ and

$$\ln x = \ln\frac{\bar{N}}{C - \bar{N}} + \frac{\bar{N}}{C - \bar{N}} \equiv \ln b(x) + b(x). \tag{46}$$

From Eq. (45) follows

$$\frac{1}{C}\ln g_{\bar{N}}(x, C) = b(x), \tag{47}$$

so that Eqs. (44) and (47) give

$$\lim_{C\to\infty}\frac{1}{C}\ln Z(x, C) = b(x), \tag{48}$$

with $b(x)$ defined by Eq. (46) as satisfying

$$x = be^b. \tag{49}$$

Consequently we obtain

$$p\beta V_0 = b(x) \tag{50}$$

for the pressure,

$$nV_0 = b(x)/[1 + b(x)] \qquad (51)$$

for the particle density, and

$$\mu\beta = \ln[b(x)/x] \qquad (52)$$

as the creation potential. Combining Eqs. (50) and (51), we recover with

$$p\beta/n = 1/(1 - nV_0) = 1 + b(x) \qquad (53)$$

the equation of state of a one-dimensional hard-core gas.[7]

Both Eq. (53) and the arbitrary differentiability of $b(x)$ show that no phase transition can occur. The low-temperature limit ($x \ll 1$),

$$p\beta V_0 \simeq x,$$
$$nV_0 \simeq x, \qquad (54)$$
$$\mu\beta \simeq 0,$$

as well as the high-temperature limit ($x \gg 1$),

$$p\beta V_0 \simeq \ln x,$$
$$nV_0 \simeq 1, \qquad (55)$$
$$\mu\beta \simeq \ln[b(x)/x],$$

agree with the results of the truncated ideal gas, but here the transition between the two limits is without any discontinuity. The functional form of the equation of state (53) is shown in Fig. 3.

We note that the absence of a phase transition is in accord with Van Hove's theorem,[8] which excludes phase transitions for one-dimensional systems with finite-range potentials. Although our Eq. (5) describes three-dimensional systems, the approximation (40) becomes exact only in the one-dimensional case and hence must lead to the corresponding result. We shall return to the physics of this situation a little later.

D. Gas-solid phase transitions

Let us now come back to the full three-dimensional coordinate-space integral [Eqs. (7) and (8)]. In spite of many attempts, it has so far not been possible to obtain a general analytic evaluation for it, nor for the corresponding two-dimensional case; the available results are either high- or low-density limiting forms, or numerical (Monte Carlo) simulations. Neither allows to show, in a mathematical sense, the existence of a phase transition.[11] Nevertheless, the computer experiments are quite generally taken as convincing evidence that such a transition does indeed occur. A further discussion of this problem is clearly beyond the scope of the present paper; see, e.g., Ref. 9 for an assessment of the case. Here we

shall instead concentrate on the limiting forms and on the physical nature of the two phases, as well as to the relation of the full problem to the simplified examples considered above.

For a sufficiently dilute hard-sphere gas, a given constituent can, in principle, reach almost any region of the available coordinate space—provided enough time and collisions. With increasing density, this mobility is more and more restrained by the extension of the other constituents. At some critical density n_c, clearly less than that of the full box ("close packing"), individual particle mobility over arbitrary distances ceases: the particle becomes essentially imprisoned by its neighbors (Fig. 4). Although motion as such is still possible, it resembles more an oscillation about a fixed lattice site, rather than Brownian motion throughout the system. Thus, the two phases on a qualitative level emerge quite naturally: a gas phase, in which an individual constituent can get around its fellow constituents by collisions and hence move over arbitrary distances in the system, and a solid phase, in which a given constituent can no longer get past his partners and hence becomes localized.

It is already clear at this point why ideal one-dimensional hard-core systems cannot exhibit phase transitions: the constituents can never get past each other and hence the system is always, even at low density, in a solidlike phase. We shall soon see this more quantitatively.

On the other hand, by neglecting the constituents' size, except in the upper limit of the summation (5), we always allow complete mobility, restricting by $N \leq C$ only the number of possible constituents. Hence the system of Sec. IIB is always a gas; the transition observed is from unlimited to limited particle number.

Thus neither of our simplified models provides an adequate description of the phase transition exhibited by a system of three-dimensional extended hadrons. Such a system passes, with increasing density, from a state of free mobility and creation/annihilation to one of bounded mobility and strongly damped production. In other words: particle creation, when combined with the spatial extension of the created objects, leads
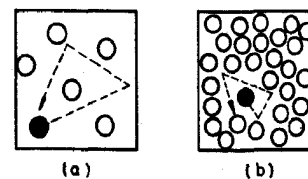

(a)   (b)

FIG. 4. Particle mobility in a dilute (a) and in a dense (b) system.

with increasing energy density to the occurrence of solidification (bounded mobility) as well as to the eventual suppression of further particle production.

We should emphasize here that the phase transition gas/solid arises without any attractive potential[6]; it is solely the difference in mobility, due to the constituents' extension, which defines the phases and provides the transition. The introduction of an additional potential can modify the critical temperature or lead to further transitions (such as gas/liquid), but it cannot remove the basic transition from mobility to imprisonment. As this transition corresponds to a fundamental change in the order of the system, we expect the transition to be of higher order.

Let us now consider the low- and high-density limits of the hard-sphere gas. If the free energy densities (or equivalently, the pressures) of these two limiting forms cross at some temperature T_c, then the system, which at equilibrium must always be in the state of lowest free energy, will experience its phase transition approximately at $T = T_c$. Clearly, the more precise a limiting form we obtain for each phase, the more precise will such a determination of T_c be. We expect, of course, the free energy to be lower for the gas phase when $T < T_c$ and lower for the solid when $T > T_c$.

At low density, we obtain the leading behavior by considering, as in the one-dimensional case, the reduction of the coordinate-space volume due to the presence of the constituents. For a repulsive hard-core potential, the presence of one particle removes from accessibility to the center of a second particle a region of the size

$$V_E(d) = \tfrac{1}{2}(2^d V_0),$$ (56)

where d denotes the spatial dimension of the system; the factor $\tfrac{1}{2}$ distributes the excluded volume equally among the two particles. If we neglect the effect of higher-particle-number interferences, we get

$$V(N) \simeq (V - NV_E)^N = V^N\left(1 - \frac{N}{V}V_E\right)^N$$ (57)

for the low-density coordinate-space volume. Proceeding as in Sec. II C, this results in expressions analogous to Eqs.(50), (51), (52), and (53), with V_0 replaced by V_E.

The high-density limit can be obtained by use of the "free volume" approach.[6] At sufficiently high density, a given constituent is essentially constrained to a cell of volume V/N; inside this cell, the motion of the constituent's center of mass is further restrained by the extension of the constituent. Adopting for simplicity cubic cells,

we see that the center of mass can move freely within a volume

$$[(V/N)^{1/d} - \tilde{V}_E^{1/d}]^d,$$ (58)

where $\tilde{V}_E$ denotes the volume occupied by the constituent in the limit of dense packing, and d again the space dimension. This gives

$$V(N) \simeq N![(V/N)^{1/d} - \tilde{V}_E^{1/d}]^{dN}$$ (59)

as the coordinate-space volume at high density; the $N!$ accounts for the possible permutations [or, equivalently, removes the $1/N!$ in Eq. (3), introduced there to avoid double counting]. Equations (5) and (59) lead to the partition function

$$Z(\beta, V) \simeq \sum_{N=0}^{V/V_E} \frac{[V\varphi_0(\beta)/e]^N}{(2\pi)^{3N}N!}\left[1 - \left(\frac{N}{V}\tilde{V}_E\right)^{1/d}\right]^{dN},$$ (60)

which can again be treated by the methods used in Sec. II C. The result is

$$p\beta\tilde{V}_E = b_s^d/(1 + b_s)^{d-1},$$ (61)

$$n\tilde{V}_E = [b_s/(1 + b_s)]^d,$$ (62)

$$p\beta/n = 1/\{1 - [n\tilde{V}_E(d)]^{1/d}\} = 1 + b_s,$$ (63)

for pressure, density, and equation of state. Here

$$b_s^d(x_s)e^b s^{(x_s)} = x_s \equiv \tilde{V}_E\varphi_0(\beta)/e$$ (64)

determines b_s as a function of the temperature.

In the three-dimensional case,[6] $\tilde{V}_E = 6V_0/\sqrt{2\pi^2} \simeq 1.35\ V_0$, yielding with Eq. (57),

$$p\beta/n = \begin{cases} (1 - 4nV_0)^{-1}, & \text{gas} & (65) \\ [1 - (1.35nV_0)^{1/3}]^{-1}, & \text{solid} & (66) \end{cases}$$

for the equations of state (see Fig. 5). When considered as a function of the temperature, the two limits of the pressure exhibit the behavior shown in Fig. 6: the pressure of the gas is largest at low temperature, that of the solid at high temperature; for the corresponding free energies, the inverse is true. Consequently, we can approximate the behavior of the system by the gas curve for $T < T_c$, the solid curve for $T > T_c$, with the phase transition at $T = T_c$. The corresponding
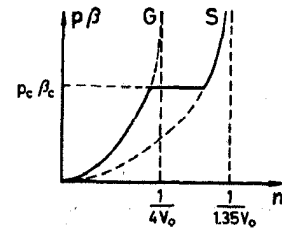

FIG. 5. The equation of state of the hard-sphere system (solid line) and its low- (G) and high- (S) density limits.
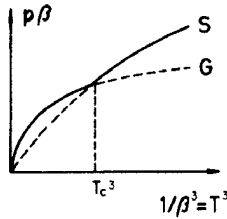
FIG. 6. Pressure vs temperature for the hard-sphere system (solid line), with low-(G) and high- (S) density limits.

equation of state is shown in Fig. 5: at the critical pressure p_c, as determined by the temperature T_c, the system undergoes the phase transition from gas to solid. The density exhibits the behavior shown in Fig. 7: approaching from below T_c, the free energy becomes at T_c lower for the solid than for the gas, and so the density jumps from the lower gas to the higher solid value. We note that the limiting values of the pressure become, from Eqs. (61), (64), and from the results of Eq. (57),

$$
p\beta V_0 = \begin{cases} V_0\varphi_0 > V_0\varphi_0/e, & x \ll 1 \qquad (67) \\ \dfrac{\ln V_0\varphi_0}{V_E/V_0} < \dfrac{\ln V_0\varphi_0}{\tilde{V}_E/V_0}, & x \gg 1. \qquad (68) \end{cases}
$$

The last inequality, i.e., the higher pressure for the solid as compared to that of the gas, follows whenever $V_E > \tilde{V}_E$, which is true except for the one-dimensional case, where $V_E = \tilde{V}_E = V_0$, excluding the crossover in the pressure and hence, as already mentioned, the phase transition.

The phase transition we have obtained by using the crossover in free energies (or pressures) is, as seen from Figs. 5 and 7, of first order: $(\partial a/\partial n)_T$ is discontinuous at T_c, with $a = \lim_{V \to \infty} A(T, V)/V$ denoting the free energy density. It should be emphasized, however, that this result is a consequence of our approach to finding the phase transition through limiting forms and crossover; it says nothing about the actual order of the phase transition, and hence cannot be used, e.g., to determine meaningful critical exponents.
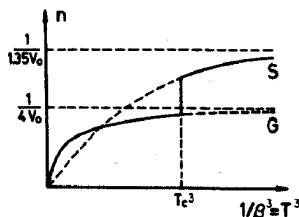

FIG. 7. The particle density for the hard-sphere system (solid line), with low- (G) and high- (S) density limits.

Finally, some remarks on the numerical value of the critical temperature. As already indicated, the precision in the determination of T depends sensitively on the quality of the approximation provided by the two limiting forms. The forms we have used here, Eqs. (57) and (59), correspond to one order beyond the asymptotic form [Eqs. (53) and (57) give the correct second-order virial coefficients]. Comparing our Fig. 5 with the result of computer simulations,[9] we find rather good agreement for the solid phase, but a considerable overestimate for the gas. As evident from Fig. 6, this result is an overestimate for T_c. To illustrate, using the simplest limiting form, Eq. (57), we obtain

$$
T_c \sim 7 \times 10^6 m, \quad V_0 = \frac{4\pi}{3} m^{-3}, \qquad (69)
$$

while the use of a more precise gas approximation[10] gives

$$
T_c \sim 350 m, \quad V_0 = \frac{4\pi}{3} m^{-3} \qquad (70)
$$

instead, showing the strong dependence of the value of the critical temperature on geometrical details in the vicinity of dense packing. The value (70) is still considerably higher than expected from either the results of Sec. IIB or from previous considerations of critical hadron temperatures.[12-14] It is not clear whether this is due to the method of determining T_c used here, or whether it calls for the introduction of an additional attractive potential, which would at the same time lower T_c and include the effect of hadronic resonances.

III. CONCLUSIONS

We have shown that hadron physics, because of the presence of a dimensional scale, the size of the hadron, contains an intrinsic "self-bounding" mechanism: the size of the constituents prevents, with increasing energy density, an ever increasing number of produced hadrons. In Fig. 7 we saw that the particle density approaches, as a function of the temperature, a limiting value of dense packing. If we want to interpret this in terms of interaction strength, then we must conclude that with increasing energy density the coupling between hadrons becomes weaker—in accord with the idea of asymptotic freedom.

In closing, we note some further points of interest.

(1) The behavior of the temperature as a function of the energy density is shown in Fig. 8. At low-energy density, the system is in the gas phase; it continues to remain there even beyond
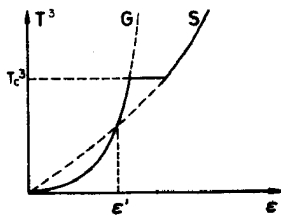
FIG. 8. Temperature vs energy density for the hard-sphere system (solid line), with low- (G) and high-(S) density limits.

FIG. 9. The creation potential for the hard-sphere system (solid line), with low- (G) and high- (S) density limits.

ϵ', although this leads to a more rapid temperature rise than the solid phase would cause. The reason is evident from Fig. 9, showing the creation potential; the system remains always in the phase of lowest resistance towards production, since more particles provide, roughly speaking, more states and hence a higher entropy. Up to the critical temperature T_c, this is best achieved in the gas phase; only beyond T_c does the ordering of the solid become more efficient.

(2) The results obtained here remain valid also for sufficiently singular soft-core potentials of the type $U(r) \sim r^{-n}$, $n \geq 4$, instead of Eq. (8).[15] Such a change could, however, greatly affect the numerical value of T_c.

(3) The critical behavior found here is due to the finite size of the hadrons. On the other hand, critical behavior is also obtained for pointlike hadrons with suitable interaction.[12-14,16] This suggests that these different ways to a similar conclusion are but two ways of stating the same thing,
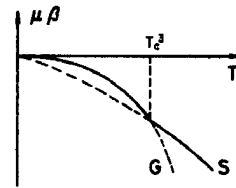
the presence of a dimensional hadronic scale. The equivalence[17] of the Pomeranchuk model[1] and the statistical bootstrap model,[12] as far as the density of states is concerned, supports this supposition. However, a picture of pointlike constituents in interaction does not, without additional (and apparently so far unaccountable) assumptions, provide the limited density, which arises naturally for extended hadrons.

We have, in the present paper, considered hard spheres in a vacuum—looking at hadrons from the outside, so to speak. As a complementary study, one may start from a world of confined quark-antiquark pairs of potential zero within a volume V_0, infinity outside of it. This look at hadrons from the inside will be taken up in a subsequent paper.

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