

$G_2(2)$ as the automorphism group of the octonionic root system of E_7

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Abstract. A simple method is suggested for the construction of the seven-dimensional representation of the adjoint Chevalley group $G_2(2)$, the automorphism group of the octonionic root system of E_7 . The maximal subgroups of $G_2(2)$ preserving the octonionic root systems of the maximal subgroups of E_7 are identified. Possible implications in physics are discussed.

Dedicated to Feza Gürsey on the occasion of his 70th birthday.

1. Introduction

There is growing interest in the problem of deformations of conformal field theories [1]. Recently Zamolodchikov [2] has proved that the Ising model perturbed with a non-zero magnetic field remains integrable. He has also shown that the theory contains exactly eight massive particles and conjectured that the associated S -matrix describes the scaling limit of the Ising model at the critical point with non-zero magnetic field. It has since been shown [3] that this theory can be identified with the \hat{E}_8 Toda field theory [4]. The masses of the eight particles of the \hat{E}_8 Toda field theory turn out to be exactly those masses calculated by Zamolodchikov [2]. A similar calculation has been done for the \hat{E}_7 Toda field theory which describes the perturbed tricritical Ising model [5].

Recently one of us has shown that, within the context of the octonionic description of the E_8 root system, the simple roots (indeed all roots) of E_8 can be generated, by multiplication, from its three simple roots which can be associated with its $SU(4)$ subgroup whose Coxeter-Dynkin (CD) diagram is the incidence diagram of the Ising model [6, 7]. This suggests that there may arise a close connection between the octonionic presentation of the E_8 root system and the Ising model with non-zero magnetic field. If this conjecture can be justified, then octonions may play a prominent role in the formulation of the relevant field theories.

In this paper we study the automorphism group of the octonionic root system of E_7 represented by purely imaginary octonions. This group is known as the adjoint Chevalley group $G_2(2)$ [8] of order 12 096, which is the automorphic extension of the finite simple group $G_2'(2)$ of order 6048, also known as the derived Chevalley group [9]. $G_2'(2) \approx U_3(3)$ was first discovered by Dickson in 1901 [10]. Here we give a 7×7 matrix representation of $G_2(2)$ using a simple method. The paper is organized as follows.

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In section 2 we give a brief description of the octonionic presentation of the root systems of E_8 and E_7 and define the automorphism of the E_7 root system with the actions of the elements from the coset space $E_8/E_7 \times SU(2)$. This transformation is used to define the 7×7 matrices, acting on the imaginary unit octonions, which preserve the octonion algebra [7]. We show that the derived Chevalley group $G'_2(2)$ can be generated by three matrices associated with the simple roots describing the CD diagram of $SU(4)$. In section 3 we study the extension of $G'_2(2)$ to $G_2(2)$ by the outer automorphism of $G'_2(2)$. In section 4 we identify the maximal subgroups of $G_2(2)$ which leave the octonionic root systems of the maximal subgroups of E_7 invariant. We give explicit expressions of the matrices generating the maximal subgroups of G_2 . Finally, in section 5 we discuss our results and make remarks concerning the relations of $G_2(2)$ with the Hall-Janko group and the Weyl group of E_8 .

2. E_7 root system with pure imaginary integral octonions

The E_8 lattice can be described by integral octonions [11], the units of which form a closed non-associative algebra of order 240. There is a natural classification of the octonionic roots where ± 1 , pure imaginary units, and the units with non-zero scalar parts, respectively, describe the roots of $SU(2)$, E_7 and the coset space $E_8/E_7 \times SU(2)$ provided they are multiplied by $\sqrt{2}$:

$$\frac{SU(2)}{\pm 1} \quad \frac{E_7}{\pm e_i, \frac{1}{2}(\pm e_j \pm e_k \pm e_l \pm e_m)} \quad \frac{E_8/E_7 \times SU(2)}{\frac{1}{2}(\pm 1 \pm e_n \pm e_p \pm e_q)} \quad (2.1)$$

where the indices take the values

$$\begin{aligned} i &= 1, \dots, 7 \\ jklm &: 1246, 1257, 1345, 1367, 2356, 2347, 4567 \\ npq &: 123, 147, 165, 245, 267, 346, 357. \end{aligned} \quad (2.2)$$

The result (2.1) can be obtained from the CD diagram of E_8 (figure 1). The set of units of integral octonions of E_8 in (2.1) is closed under the octonionic multiplication. Hereafter, when we say octonionic roots, we mean the 240 units of integral octonions.

Let A , B and R be arbitrary octonionic roots of E_8 provided R belongs to the set of roots of the coset space $E_8/E_7 \times SU(2)$. It has been proved in [7] that the transformations

$$A' = RA\bar{R} \quad B' = RB\bar{R} \quad (\bar{R}: \text{octonionic conjugate of } R) \quad (2.3)$$

preserve the octonionic multiplication AB :

$$A'B' = (AB)' = R(AB)\bar{R}. \quad (2.4)$$

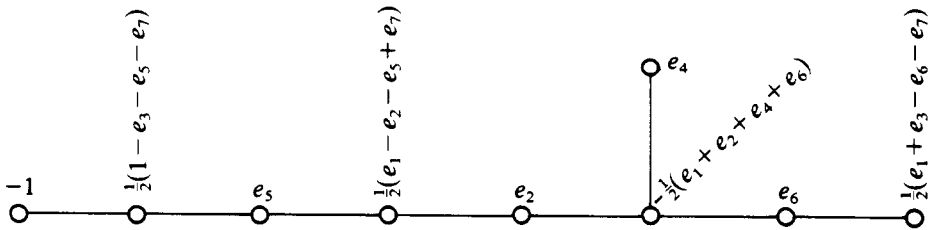


Figure 1. Extended Coxeter-Dynkin diagram of E_8 leading to the representation of the root system of E_7 with pure imaginary integral octonions.

Expressions (2.3) leave the scalar parts of the octonions unchanged so that the roots remain in their own sectors under this transformation. That is, if A and B are pure imaginary octonions they remain pure imaginary after the transformation. Therefore (2.3) not only preserves the root system of E_7 described by pure imaginary integral octonions, but also keep their algebraic relations. Let us define the matrix R_{ij} by the transformation

$$e'_i = \sum R_{ij} e_j = (\pm R) e_i (\pm \bar{R}) \quad i, j = 1, \dots, 7 \tag{2.5}$$

where e'_i satisfy the octonion algebra described by e_i . It is known that the automorphism group of the octonion algebra is the exceptional group G_2 [12]. Therefore the matrices defined by (2.5) should be the elements of some finite subgroup of G_2 . The matrices in (2.5) satisfy the relation $R^3 = 1$. As they are the elements of the coset space, there exist 28 pairs of the form $(\pm R, \pm \bar{R})$. We use the same notation for the root and the matrix which we associate with (2.5). We hope that the distinction between them is clear. By (2.5) we can define at most 28 matrices. Nevertheless, these matrices generate under multiplication new matrices which cannot be obtained by the method of (2.5). In fact, three matrices are sufficient to generate the whole group. For this purpose we have chosen three roots, from the coset space, which describe the simple roots of $SU(4)$ in E_8 . Let us choose them as follows:

$$\begin{aligned} P &= -\frac{1}{2}(1 - e_1 - e_5 + e_6) & Q &= \frac{1}{2}(1 - e_1 + e_2 + e_3) \\ R &= -\frac{1}{2}(1 + e_1 + e_2 + e_3). \end{aligned} \tag{2.6}$$

It is clear that they describe the simple roots of $SU(4)$ where Q is the root in the middle of its CD diagram. The matrices corresponding to the transformations of these roots with their conjugates can be calculated using (2.5):

$$P = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 \end{bmatrix} \quad Q = \frac{1}{2} \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}$$

$$R = \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} . \tag{2.7}$$

They satisfy the relations

$$P^3 = Q^3 = R^3 = (PQR)^7 = I \tag{2.8}$$

where I is the 7×7 unit matrix. It is straightforward to check that these matrices leave the root system of E_7 invariant. Note that the pair of roots in (2.6) (P, Q) , (Q, R) and (P, R) each generate a quaternionic root system of $SO(8)$ which form the binary

tetrahedral group. Analogously, the 7×7 matrices in (2.7) associated with these roots pairwise generate a group of order 24 isomorphic to the binary tetrahedral group [7]. It has been shown that, for a particular choice of a root from the coset space, the root system of E_8 can be arranged in such a way that there exist nine different constructions of the binary tetrahedral groups out of the roots of E_8 [7]. As we have 28 choices, the order of the group turns out to be $24 \times 9 \times 28 = 6048$. This is in accord with the order of the derived Chevalley group $G_2'(2)$. A computer calculation justifies this fact that the group generated by the matrices of (2.7) is of order 6048. We have displayed the period-trace correlations of the elements of $G_2'(2)$ in table 1 and compared with the result obtained by Dickson [13]. They are in agreement. We refer the reader to [14] for further discussions on the group $G_2'(2)$ and the other finite automorphism groups of octonions.

Table 1. Trace-period correlation of the elements of the derived Chevalley group $G_2'(2)$ (period of a matrix M is defined by $M^n = I$).

Trace	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 6$	$n = 7$	$n = 8$	$n = 12$
-2			56					
-1		63		378			1512	
0						1728		1008
1			672					
2					504			
3				126				
7	1							

3. $G_2(2)$ as the automorphic extension of $G_2'(2)$

The derived Chevalley group $G_2'(2)$ involves three diagonal 7×7 matrices other than the unit matrix. They are simply given by

$$\begin{aligned}
 M_1 &= (1, 1, 1, -1, -1, -1, -1) & M_2 &= (1, -1, -1, 1, -1, -1, 1) \\
 M_3 &= (1, -1, -1, -1, 1, 1, -1)
 \end{aligned}
 \tag{3.1}$$

where the elements corresponding to the respective sets of indices 123, 147 and 156 are positive. These matrices are in one-to-one correspondence with the occurrence of the Hurwitz integers [15] in (2.1) described by the three sets of quaternionic units 123, 147 and 156. The matrices in (3.1) satisfy the relations

$$M_1 M_2 = M_2 M_1 = M_3 \quad \text{cyclic permutations of } 1, 2, 3. \tag{3.2}$$

One can readily check that there are four more diagonal matrices whose positive elements correspond to the set of indices 246, 257, 345 and 367. They also leave the octonion algebra and the octonionic root system of E_7 invariant. But none of these latter matrices is an element of $G_2'(2)$.

Let $M_4 = (-1, 1, -1, 1, -1, 1, -1)$ denote the matrix with positive elements corresponding to the indices 246. The remaining diagonal matrices can be obtained as the products of M_4 with M_1 , M_2 , and $M_1 M_2 = M_3$ just in the manner the octonionic units are obtained from the quaternionic imaginary units by multiplying them with an

independent imaginary unit of octonion. Let us define the remaining diagonal matrices by

$$M_7 = M_4 M_1 \quad M_6 = M_4 M_2 \quad M_5 = M_4 M_3. \tag{3.3}$$

The seven diagonal matrices satisfy, under multiplication, the same algebraic structure of octonions except they are both associative and commutative. Their multiplicative structure can be written in a compact form

$$M_i M_j = M_j M_i = M_k$$

where i, j and k take the values of the sets of indices 123, 165, 147, 246, 257, 435 and 367. The outer automorphism of $G'_2(2)$ can be obtained by the action of M_4 , $M_4^2 = I$. Let $h \in G'_2(2)$ denote an arbitrary element of $G'_2(2)$. One can always find an element $g \in G'_2(2)$ such that

$$M_4 h M_4^{-1} = g. \tag{3.4}$$

It is sufficient to show that (3.4) holds when h is a generator of $G'_2(2)$. Thus the automorphic extension of $G'_2(2)$ can be made with M_4 . The products of the elements of $G'_2(2)$ with M_4 from left or right define the new sets of elements of the extended group $G_2(2)$, of order 12 096, which admits $G'_2(2)$ as the normal subgroup.

4. Maximal subgroups of $G_2(2)$

In this section we study the subgroups of $G_2(2)$ preserving the octonionic root systems of the regular maximal subgroups of E_7 . The regular maximal subgroups of E_7 are given by

$$\begin{aligned} E_7 &\supset E_6 \times U(1) & E_7 &\supset SU(8) \\ E_7 &\supset SU(2) \times SO(12) & E_7 &\supset SU(3) \times SU(6). \end{aligned} \tag{4.1}$$

For non-regular subgroups see, e.g., [16].

4.1. $E_6 \times U(1)$

$U(1)$ being in the Cartan subalgebra of E_7 , it is represented by a zero root. Therefore the roots of E_6 in E_7 are the root system of $E_6 \times U(1)$. Using figure 1, we obtain the roots of E_6 and the weights corresponding to its $27 + 27^*$ representations as follows:

E_6 roots:

$$\begin{aligned} \pm e_1, \pm e_2, \pm e_4, \pm e_6 & \quad \frac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6) \\ \frac{1}{2}(\pm e_1 \pm e_2 - e_5 + e_7) & \quad \frac{1}{2}(\pm e_1 \pm e_2 + e_5 - e_7) \\ \frac{1}{2}(\pm e_1 \pm e_4 - e_3 + e_5) & \quad \frac{1}{2}(\pm e_1 \pm e_4 + e_3 - e_5) \\ \frac{1}{2}(\pm e_1 \pm e_6 - e_7 + e_3) & \quad \frac{1}{2}(\pm e_1 \pm e_6 + e_7 - e_3) \\ \frac{1}{2}(\pm e_4 \pm e_6 - e_5 + e_7) & \quad \frac{1}{2}(\pm e_4 \pm e_6 + e_5 - e_7) \\ \frac{1}{2}(\pm e_2 \pm e_6 - e_3 + e_5) & \quad \frac{1}{2}(\pm e_2 \pm e_6 + e_3 - e_5) \\ \frac{1}{2}(\pm e_2 \pm e_4 - e_7 + e_3) & \quad \frac{1}{2}(\pm e_2 \pm e_4 + e_7 - e_3). \end{aligned} \tag{4.2}$$

The weights of $27 + 27^*$

$$\pm e_3, \pm e_5, \pm e_7$$

$$\begin{aligned} \frac{1}{2}(\pm e_1 \pm e_2 + e_5 + e_7) & \quad \frac{1}{2}(\pm e_1 \pm e_2 - e_5 - e_7) \\ \frac{1}{2}(\pm e_1 \pm e_4 + e_3 + e_5) & \quad \frac{1}{2}(\pm e_1 \pm e_4 - e_3 - e_5) \\ \frac{1}{2}(\pm e_1 \pm e_6 + e_7 + e_3) & \quad \frac{1}{2}(\pm e_1 \pm e_6 - e_7 - e_3) \\ \frac{1}{2}(\pm e_4 \pm e_6 + e_5 + e_7) & \quad \frac{1}{2}(\pm e_4 \pm e_6 - e_5 - e_7) \\ \frac{1}{2}(\pm e_2 \pm e_6 + e_3 + e_5) & \quad \frac{1}{2}(\pm e_2 \pm e_6 - e_3 - e_5) \\ \frac{1}{2}(\pm e_2 \pm e_4 + e_7 + e_3) & \quad \frac{1}{2}(\pm e_2 \pm e_4 - e_7 - e_3). \end{aligned} \tag{4.3}$$

To search for all the transformations leaving (4.2) and (4.3) invariant is rather lengthy. In principle, this could have been investigated with a computer calculation. We have, instead, used an intuitive method to find out a number of matrices which preserve the invariance of (4.2) and (4.3) separately. Using them we have generated all the rest. A simpler method can be described as follows.

When we decompose the root system of E_8 under $E_6 \times SU(3)$ (see figure 1) we observe that the $SU(3)$ roots, the orthogonal vectors to those in (4.2), are given by

$$C: \pm 1, \pm \frac{1}{2}(1 + e_3 + e_5 + e_7), \pm \frac{1}{2}(1 - e_3 - e_5 - e_7). \tag{4.4}$$

Any transformation on the imaginary octonionic units leaving the set of $SU(3)$ roots in (4.4) invariant also preserves the set of roots in (4.2). The imaginary units e_3, e_5 and e_7 constitute a non-associative triad. The transformation of these octonionic units determines the complete transformation matrix acting on seven imaginary units. We refer the reader to reference [7] for the details of the procedure. By selecting certain matrices from $G_2(2)$ preserving (4.4) we have generated the maximal number of elements of $G_2(2)$ leaving the root systems in (4.2), (4.3) and (4.4) unchanged. We have checked that they form the group $3_+^{1+2}:8$, of order 216, which is one of the maximal subgroups of $G_2(2)$ [9]. The transformation of the octonionic imaginary units by the matrix M_4 also leaves the root systems (4.2), (4.3) and (4.4) invariant. Therefore the automorphic extension of the group $3_+^{1+2}:8$ can be made with M_4 . The extended group $3_+^{1+2}:8:2$ of order 432, is one of the maximal subgroups of the adjoint Chevalley group $G_2(2)$. This is the maximal group which preserves the octonionic root system of E_6 . The group $3_+^{1+2}:8:2$ can be generated by the matrices

$$S = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \quad T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.5}$$

They satisfy the relation

$$T^6 = S^8 = (TS)^{12} = I. \tag{4.6}$$

We have calculated the periods and the traces of the 432 matrices which can be found in table 2.

Table 2. Trace-period correlation of the elements of the group of order 432 (automorphism group of the octonionic root system of E_6).

Trace	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 6$	$n = 7$	$n = 8$	$n = 12$
-2			2					
-1		45			72		108	
0								108
1			24					
2					18			
3				54				
7	1							

4.2. $SU(8)$

The root system of E_7 splits under its maximal subgroup $SU(8)$ as follows:

$$126 = 56 + 70 \tag{4.7}$$

where 56 represents the non-zero roots of $SU(8)$ and 70 stands for the weights of its 70-dimensional representation. The roots of $SU(8)$ are given by the set

$$\begin{aligned} &\pm e_2, \pm e_3, \pm e_5, \pm e_6, \pm \frac{1}{2}(e_1 \pm e_2 \pm e_5 + e_7), \pm \frac{1}{2}(e_1 \pm e_2 \pm e_4 \pm e_6) \\ &\pm \frac{1}{2}(e_1 \pm e_3 - e_4 \pm e_5), \pm \frac{1}{2}(e_1 \pm e_3 \pm e_6 - e_7), \pm \frac{1}{2}(\pm e_2 \pm e_3 + e_4 + e_7), \pm \frac{1}{2}(e_4 \pm e_5 \pm e_6 - e_7). \end{aligned} \tag{4.8}$$

One can guess several transformations which preserve (4.8) and, at the same time, leave the octonion algebra invariant. The matrices which we have selected from $G_2'(2)$ generated a group of order 168 isomorphic to $L_2(7)$ [9]. This group corresponds to one of the other maximal subgroups of $G_2'(2)$. There exist several other notations for $L_2(7)$ used in mathematical literature. One of the viable notations is $PSL(2, 7)$ [17]. The automorphic extension of $L_2(7)$ which is denoted by $SL(2, 7)$ can, in principle, be made with M_4 . However, as M_4 does not leave the $SU(8)$ root system in (4.8) invariant, the automorphic extension of $L_2(7)$ by M_4 is not the maximal subgroup of $G_2(2)$ preserving the set in (4.8). The following matrix, which neither belongs to $L_2(7)$ nor to $G_2'(2)$, leaves the root system of $SU(8)$ unchanged:

$$U = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \quad U^2 = I. \tag{4.9}$$

It can be checked that U constitutes the outer automorphism of $L_2(7)$. Therefore $L_2(7)$ can be automorphically extended by U where the extended group $SL(2, 7)$, of order 336, is the maximal subgroup of $G_2(2)$, which also preserves the octonionic root system of $SU(8)$. $SL(2, 7)$ is one of the modular groups. A general definition of this class of finite groups can be made as follows: $SL(2, p)$ ($p = \text{prime}$) is the special linear

homogeneous group of 2×2 matrices, of modulo p , of order $p(p^2 - 1)$. $SL(2, 7)$ has the subgroup of the binary octahedral group of index 7. The group preserving the octonionic root system of $SU(8)$ is isomorphic to $SL(2, 7)$ and can be generated by the matrices

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad X = \frac{1}{2} \begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix}. \quad (4.10)$$

They satisfy the relations

$$V^4 = X^6 = (VX)^8 = I. \quad (4.11)$$

The period-trace correlations of the matrices of the group $SL(2, 7)$ are given in table 3.

4.3. $SU(2) \times SO(12)$

From figure 1 we obtain the following roots:

$$SU(2): \pm e_5$$

$$SO(12): \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_6, \pm e_7, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6), \quad (4.12)$$

$$\frac{1}{2}(\pm e_1 \bullet e_3 \pm e_6 \pm e_7), \frac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_7).$$

It is clear from an inspection of (4.12) that the matrices satisfying $R_{55} = \pm 1$ may leave this system invariant. If we select these matrices from the elements of $G_2(2)$ we observe that they indeed preserve the root systems in (4.12) separately invariant. Moreover, they generate the group $4 \cdot S_4$, of order 96, which is also one of the maximal subgroups of $G_2(2)$. The matrix M_4 preserves the root system of $SU(2) \times SO(12)$. Therefore the automorphic extension of the group $4 \cdot S_4$ can be made by M_4 . Then the extended group $4 \cdot S_4:2$, of order 192, is one of the maximal subgroups of $G_2(2)$. It is obvious from these discussions that the set of matrices with $R_{55} = \pm 1$ in $G_2(2)$ form the

Table 3. Trace-period correlation of the elements of the group of order 336 (automorphism group of the octonionic root system of $SU(8)$).

Trace	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 6$	$n = 7$	$n = 8$	$n = 12$
-2								
-1		49		42	56			
0						48		
1			56				84	
2								
3								
7	1							

automorphism group of the root system of $SU(2) \times SO(12)$. One can check that the other sets of matrices with $R_{ii} = \pm 1$ ($i = 1, \dots, 7$) in $G_2(2)$ also form a group of order 192. However, there is one amusing distinction between the sets of matrices with $R_{11} = \pm 1$ and the others. The matrix elements of the set of matrices with $R_{11} = \pm 1$ are either 0 or ± 1 , in contrast to the other sets with $R_{ii} = \pm 1$ ($i \neq 1$) where the matrix elements of the matrices also include the numbers $\pm \frac{1}{2}$. The reason is clear. In the root system we have chosen in (2.1), e_1 plays a special role as it occurs repeatedly in the quaternionic units 123, 147 and 156. The group $4 \cdot S_4:2$ is generated by the matrices

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad Y = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \quad (4.13)$$

We have the relations

$$M_1^2 = Y^6 = (M_1 Y)^8 = I. \quad (4.14)$$

The trace-period correlation of this group is displayed in table 4.

4.4. $SU(3) \times SU(6)$

In this case figure 1 leads to the following root systems:

$$\begin{aligned} SU(3): & \pm e_5, \pm \frac{1}{2}(e_1 - e_2 - e_5 + e_7), \pm \frac{1}{2}(e_1 - e_2 + e_5 + e_7) \\ SU(6): & \pm e_3, \pm e_4, \pm e_6, \pm \frac{1}{2}(e_1 + e_2 \pm e_4 \pm e_6), \pm \frac{1}{2}(e_1 \pm e_3 \pm e_6 - e_7), \\ & \pm \frac{1}{2}(e_2 \pm e_3 \pm e_4 + e_7). \end{aligned} \quad (4.15)$$

The subgroup of $G_2(2)$ preserving this system is of order 18. The diagonal matrix M_4 does not preserve this root system. It can be checked that M_6 leaves them invariant. The outer automorphism of the group of order 18 can be made by M_6 . Then the maximal group preserving (4.15) is a subgroup of $G_2(2)$ of order 36. It can be generated

Table 4. Trace-period correlation of the elements of the group of order 192 (automorphism group of the octonionic root system of $SO(12) \times SU(2)$).

Trace	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 6$	$n = 7$	$n = 8$	$n = 12$
-2								
-1		43		18	32		24	
0								
1			32				24	
2								
3				18				
7	1							

Table 5. Trace-period correlation of the elements of the group of order 36 (automorphism group of the octonionic root system of $SU(6) \times SU(3)$).

Trace	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 6$	$n = 7$	$n = 8$	$n = 12$
-2			2					
-1		15			6			
0								
1			6					
2					6			
3								
7	1							

by the matrices

$$Z = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 & -1 \end{bmatrix} \quad W = \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \tag{4.16}$$

which satisfy the relations

$$Z^2 = W^6 = (ZW)^6 = I. \tag{4.17}$$

The trace-period correlation of this group is displayed in table 5. It seems that the groups we have generated in this case are not maximal in $G_2(2)$ and correspondingly in $G_2(2)$ [9], a surprising fact which we have not understood.

Before we conclude this section we would like to note the following interesting relations between the groups preserving the octonionic root systems and the Weyl groups of the relevant groups. The adjoint Chevalley group $G_2(2)$ is a subgroup of the Weyl group $W(E_7)$ with index 240, which is equal to the index of $W(E_7)$ in $W(E_8)$:

$$\frac{|W(E_8)|}{|W(E_7)|} = \frac{|W(E_7)|}{|G_2(2)|} = 240. \tag{4.18}$$

We have similar relations to (4.18) for the other groups

$$\begin{aligned} \frac{|W(E_6)|}{216} &= \frac{|W(SU(8))|}{168} = \frac{|W(SO(12))|}{96} \\ &= \frac{|W(SU(2) \times SO(12))|}{192} = \frac{|W(SU(3) \times SU(6))|}{18} = 240. \end{aligned} \tag{4.19}$$

5. Discussion and conclusion

Employing a simple method, we have given the explicit seven-dimensional representations of the groups $G_2(2)$, $G_2(2)$ and their maximal subgroups by identifying them as the automorphism groups of the octonionic root system of E_7 and its regular subgroups.

Our main aim was to generate the group $G_2'(2)$ using the roots (2.6), the simple roots of the $SU(4)$ subalgebra of E_8 , which lead to the matrices (2.7) satisfying the relation (2.8). One can check that (2.8) is not the proper generating relation for $G_2'(2)$ [9]. Similarly (4.6), (4.11), (4.14) and (4.17) do not represent the generating relations for the groups concerned. We should make it clear that the pure imaginary units of integral octonions (2.1), (4.2), (4.8), (4.12) and (4.15), representing respectively the roots of E_7 and the root systems of its regular maximal subalgebras, are not closed under octonionic multiplication as the closure occurs only for the 240 units of integral octonions. The $G(2)$ is the automorphism group of the 240 units of integral octonions leaving the scalar parts fixed, in other words it preserves the E_8 lattice decomposition in (2.1): $E_8 = SU(2) + E_7 + E_8 / (SU(2) + E_7)$. The full automorphism group of 240 unit octonions is $2^2 \cdot D_4(2)$, which is of the same order of the Weyl group $W(E_8)$ but not the same group [9].

As stated in the introduction, the importance of the adjoint Chevalley group $G_2(2)$ and its normal subgroup $G_2'(2)$ may turn out to be relevant if any sort of relations between the octonions and the statistical mechanical models associated with E_8 , E_7 , E_6 and their maximal subgroups are obtained. $G_2'(2)$ is one of the maximal subgroups of the sporadic Hall-Janko group J_2 of order 604 800 [18]. Analogously, $G_2(2)$ is the maximal subgroup of the covering group of J_2 . The Hall-Janko group J_2 is characterized by three sets of icosians, also establishing its connection with the Leech lattice [19]. The relations between the icosians and the octonions have been recently obtained [20]. It is obvious from these discussions that the extension of the automorphism group of the octonionic root system of E_7 in various directions is possible. Perhaps the generalizations of the statistical mechanical models associated with E_8 and E_7 to the models which can be associated with the Leech lattice can be made via this connection. The group $G_2(2)$ is nowhere discussed at length. We hope that the material discussed here may shed some light on the problems relevant to physics and mathematics.

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