QCD THERMODYNAMICS WITH LIGHT QUARKS. QUANTUM CORRECTIONS TO THE FERMIONIC ANISOTROPY PARAMETER

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We determine the $O(g^2)$ corrections to the anisotropy parameter in the fermionic action, which are required to guarantee a rotational invariant continuum limit on anisotropic lattices. We show the importance of these quantum corrections for QCD thermodynamics on isotropic lattices. Only after including these corrections do we find, on large lattices, agreement between lattice and continuum perturbation theory at finite temperature. The implied renormalization of the operators used in Monte Carlo simulations to measure the energy density leads to a 20% reduction of the fermionic part of the energy density which, to a large extent, compensates the previously found overshooting in the gluonic sector. We reanalyze existing Monte Carlo data for the thermodynamics of QCD with light quarks and extract the entropy density. We find that immediately above the chiral transition the entropy density is already close to the ideal gas value.

1. Introduction

Lattice gauge theories at finite temperature are conveniently formulated on anisotropic lattices, i.e. lattices with different lattice spacings a (a_τ) in spatial (temporal) directions. This allows us to keep track of the temperature (T) and volume (V) dependence of the partition function and makes an independent variation of T and V possible. In order to ensure a rotationally invariant continuum limit even on anisotropic lattices a renormalization of the bare couplings appearing in the lattice action is required [1]. The couplings then become functions of a and the *anisotropy parameter* $\xi = a/a_\tau$. For pure SU(N) gauge theories the renormalization of the bare couplings, necessary to guarantee a rotational invariant continuum limit on anisotropic lattices, has been calculated to $O(g^2)$ in weak coupling perturbation theory [1,2]. Modifications due to dynamical fermions have also been determined [3]. A similar analysis of the renormalization of the fermionic part of the lattice QCD action on anisotropic lattices, however, is missing at present. First results, relevant for QCD thermodynamics, will be presented here.

Thermodynamic observables like, for instance the energy or entropy density involve derivatives of the couplings with respect to ξ [2,4]. Thus, even if thermodynamic quantities are calculated on isotropic lattices, knowledge about the renormalization of the bare couplings on anisotropic lattices is required. So far only the leading classical anisotropy effects have been taken into account in the fermionic sector of the QCD partition function. Expressions for energy or entropy density are thus approximate and may not give the correct temperature dependence of these quantities. An understanding of the influence of quantum corrections of the derivatives of the bare couplings thus becomes important, in particular, in view of the recently observed differences in behaviour of the energy density in pure SU(N) gauge theory and QCD with light quarks [5]. In fact, we will show that these quantum corrections have to be taken into account properly in order to recover, on large lattices, the correct high temperature perturbative behaviour found in continuum perturbation theory. Without them one finds the wrong sign and magnitude for the $O(g^2)$ corrections to the high temperature ideal gas behaviour of the QCD energy density. This gives a qualitative understanding of the large overshooting of the ideal gas limit found in Monte

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Carlo calculations of the energy density for QCD with light quarks [5], which were based on an expression for the energy density, which did not fully take into account the effect of quantum corrections to the bare couplings.

We present here a calculation of the $O(g^2)$ quantum corrections to the anisotropy parameter appearing in the fermionic part of the QCD action. We have calculated these quantum corrections in weak coupling perturbation theory demanding rotational invariance of the fermion propagator in the continuum limit. A detailed presentation of our calculations for staggered and Wilson fermions will be given elsewhere [6]. In this letter we present results for the staggered fermion action which are of immediate importance for thermodynamic calculations with dynamical fermions.

This paper is organized as follows. We briefly review in the next section the finite temperature formalism on anisotropic lattices. In section 3 we discuss the fermion self-energy on such lattices and extract the $O(g^2)$ corrections to the anisotropy parameter. Section 4 gives a comparison between the complete $O(g^2)$ correction to the QCD energy density calculated on the lattice and the corresponding continuum result. In section 5 we discuss the relevance of the quantum corrections to the fermionic couplings for Monte Carlo simulations. Finally we give our conclusions in section 6.

2. Finite temperature formalism

For simplicity lattice simulations are generally performed on isotropic lattices, i.e. lattices with identical lattice spacings in temporal and spatial directions ($a_\tau = a$). However, in order to discuss the finite temperature formalism on the lattice it is convenient to work on an anisotropic lattice ($a_\tau \neq a$) and introduce the *anisotropy parameter* [4]

$$\xi = a/a_\tau . \tag{1}$$

This allows us to keep track of the dependence on temperature, T, which enters the lattice action in a complicated way. While T appears in the continuum formulation explicitly only as an integration limit for the integration over the time components of the fields, on the lattice it appears through the finite number of lattice sites in time direction (N_τ) as well as through the time-like lattice cut-off (a_τ). The continuum four-volume VT^{-1} is replaced by a lattice of size $N_\tau \times N_\sigma^3$ with lattice spacings a_τ (a) in the temporal (spatial) directions such that

$$1/T = N_\tau a_\tau , \quad V^{1/3} = N_\sigma a . \tag{2}$$

The lattice spacings a, a_τ enter the lattice action only indirectly through the renormalization of the bare dimensionless couplings in the gluonic and fermionic parts of the action. An anisotropy can be introduced by choosing different couplings for space–space and space–time plaquettes in the gluonic action and similarly by introducing a new coupling in the 0th component of the Dirac operator. The gluonic action then becomes

$$S_G = \beta_\sigma \left(\sum_{n;0<\mu<\nu\leq 3} P_{n,\mu\nu} + \gamma_G^2 \sum_{n;0<\mu\leq 3} P_{n,0\mu} \right),$$

$$P_{n,\mu\nu} = 1 - \frac{1}{N} \mathrm{Re\,Tr}\, U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu} , \tag{3}$$

and the fermion action for staggered fermions is given by

$$S_F = \bar{\chi}_n Q_{n,m} \chi_m , \tag{4}$$

with the fermion matrix Q defined as

$$Q_{n,m} = \sum_{j=1}^{3} D_{n,m;j} + \gamma_F D_{n,m;0} + m_f a \delta_{n,m} ,$$

$$D_{n,m;\mu} = \tfrac{1}{2} \eta_\mu(n) \left(U_{n,\mu} \delta_{n,m-\hat{\mu}} - U^\dagger_{m,\mu} \delta_{n,m+\hat{\mu}} \right). \tag{5}$$

Here $\eta_\mu(n)$ denotes the phase factors $\eta_\mu(n) = (-1)^{n_1+\dots+n_\mu}$ for $\mu>0$ and $\eta_0(n)=1$ and m_f is the bare quark mass in units of the lattice spacing a.

Let us look at these expressions in some more detail. The QCD action on anisotropic lattices depends on three couplings β_σ, γ_G and γ_F. In the following we will ignore any further dependence on the quark masses, i.e. we assume $m_f a = 0$. The couplings β_σ, γ_G and γ_F have to be tuned on anisotropic lattice in order to ensure rotational invariance in the continuum limit [1,2]. The renormalization of the coupling in front of the spatial plaquettes, β_σ, may be parametrized as

$$\beta_\sigma = \xi^{-1} [\beta + 2N c_\sigma(\xi) + O(g^2)] . \tag{6}$$

Here $\beta = 2N/g^2$, with g^2 denoting the bare coupling constant on an isotropic euclidean lattice. It is related to the lattice cut-off $1/a$ through the usual QCD renormalization group equation. Similarly, the coupling for the temporal plaquettes may be written as $\beta_\tau=\xi[\beta+2Nc_\tau(\xi)+O(g^2)]$. However, rather than using the couplings β_σ and β_τ in the definition of the gluonic action, eq. (3), we have introduced the ratio $\gamma_G=\sqrt{\beta_\tau/\beta_\sigma}$ in order to stress the similarity of the modifications in the gluonic and fermionic parts of the action required on anisotropies lattices. The couplings γ_G and γ_F determine the relative weights of space- and time-like components in the gluonic and fermionic action. These couplings are equal to unity on an isotropic lattice. However, if the continuum limit is to be taken for fixed anisotropy ξ, one has to adjust both couplings such that rotational invariance is guaranteed in that limit. The naive classical continuum limit demands $\gamma_G=\gamma_F=\xi$. Quantum corrections to this leading order relation have been calculated for γ_G in the pure gauge theory [1,2] as well as for QCD[3] by demanding rotational invariance of the effective action. Here we want to discuss the quantum corrections to the fermionic coupling γ_F, which have not been determined so far and also never have been taken into account in Monte Carlo simulations of QCD thermodynamics. We will show that its inclusion is important for recovering the correct continuum limit of thermodynamic quantities.

Consider for instance the calculation of the energy density ϵ on the lattice. It is obtained from the partition function,

$$Z=\int\prod_{n,\mu}\mathrm{d}U_{n,\mu}\prod_n \mathrm{d}\chi_n\,\mathrm{d}\bar\chi_n\exp(-S_G-S_F), \qquad (7)$$

by taking a derivative of ln Z with respect to $1/T$ at fixed V. On a lattice of fixed size $N_\tau\times N_\sigma^3$ the temperature derivative can be written as

$$\frac{\partial}{\partial(1/T)}=-\frac{\xi^2}{N_\tau a}\frac{\partial}{\partial\xi}. \qquad (8)$$

One obtains then for the energy density

$$\frac{\epsilon}{T^4}=\frac{\epsilon_G+\epsilon_F}{T^4}=\left(\frac{N_\tau}{N_\sigma}\right)^3\frac{\partial}{\partial\xi}\ln Z\bigg|_{\xi=1}. \qquad (9)$$

Thus even in the isotropic limit derivatives of the couplings β_σ, γ_G and γ_F with respect to ξ evaluated for $\xi=1$ are needed to obtain the correct expression for the energy density. We have for the gluonic contribution, ϵ_G,

$$\frac{\epsilon_G}{T^4}=-3N_\tau^4\frac{\mathrm{d}\beta_\sigma}{\mathrm{d}\xi}\bigg|_{\xi=1}(\bar P_\sigma+\bar P_\tau)-6\beta N_\tau^4\frac{\mathrm{d}\gamma_G}{\mathrm{d}\xi}\bigg|_{\xi=1}\bar P_\tau, \qquad (10)$$

and the corresponding fermionic part, ϵ_F, for n_f massless flavours is given by

$$\frac{\epsilon_F}{T^4}=\frac{\mathrm{d}\gamma_F}{\mathrm{d}\xi}\bigg|_{\xi=1}N_\tau^4\frac{n_f}{4}(\langle\mathrm{Tr}\,D_0Q\rangle-\tfrac{1}{4}N). \qquad (11)$$

In eq. (10) we have introduced the normalized plaquette variables $\bar P_{\sigma(\tau)}$, which are defined by

$$\bar P_{\sigma(\tau)}=\langle P_{\sigma(\tau)}\rangle_{N_\tau<N_\sigma}-\langle P_{\sigma(\tau)}\rangle_{N_\tau=N_\sigma}. \qquad (12)$$

This takes care of the subtraction of divergent vacuum contributions and normalizes the gluonic part of the energy density to zero at $T=0$. The subtraction of the zero temperature contribution in ϵ_F is given explicitly by the last term in eq. (11). We note that the distinction between ϵ_G and ϵ_F reflects only the fact that the expectation values of gluonic and fermionic parts of the action contribute to these quantities. Also ϵ_G depends on the presence of light quarks through the statistical ensemble used to evaluate the expectation values and the explicit dependence of β'_σ and γ'_F on n_f [4].

The derivatives of the couplings appearing in eqs. (10) and (11) can be calculated perturbatively when the corresponding renormalizations on anisotropic lattices are known. For β_σ these are given by eq. (6). Similarly, the leading order quantum correction to the $\gamma_{G(F)}$ can be parametrized as

$$\gamma_G=\xi[1+c_G(\xi)g^2+O(g^4)],$$
$$\gamma_F=\xi[1+c_F(\xi)g^2+O(g^4)]. \qquad (13)$$

In general, beyond the validity regime of perturbation theory, one can determine the ξ-dependence of these couplings by demanding rotational invariance of physical observables, like for instance the heavy quark potential, in the continuum limit [2,7].

In eq. (13) we have introduced the parameter c_G. In terms of the functions c_σ and c_τ [2] that renormalize the spatial and temporal couplings β_σ and β_τ this is given by

$$c_G(\xi)=\tfrac{1}{2}[c_\tau(\xi)-c_\sigma(\xi)]. \qquad (14)$$

The derivatives of c_σ and c_τ have been calculated previously in weak coupling perturbation theory [2,3] and a first attempt of a non-perturbative determination has been undertaken in ref. [7]. The coupling γ_F, however, has been approximated so far in all numerical and analytic calculations by its classical approximation, $\gamma_F=\xi$. In the next section we present an explicit calculation of the $O(g^2)$ corrections and determine the derivative c'_F needed in the evaluation of the fermionic part of the energy density, eq. (11).

3. Fermion self-energy on anisotropic lattices

Let us consider the free fermion propagator on an anisotropic lattice

$$\langle \bar\chi_n \chi_m \rangle = \int d^4p\, S_{F,0}^{-1}(p)\exp[-ip(n-m)] \qquad (15)$$

with $S_{F,0}(p)$ given by

$$S_{F,0}(p) = \sum_{i=1}^{3} \sin(p_i a) + \gamma_F \sin(p_0 a_\tau) + m_f a. \qquad (16)$$

Demanding rotational invariance in the continuum limit, a, $a_\tau \to 0$ at fixed $\xi = a/a_\tau$, determines the relation between γ_F and the isotropy parameter ξ. We find $\gamma_F=\xi$. Including interactions, the fermion propagator receives self-energy corrections,

$$S_F(p) = S_{F,0}(p) + \Sigma(p). \qquad (17)$$

These fermion self-energy corrections have been calculated previously to $O(g^2)$ on isotropic lattices for Wilson fermions [8,9] as well as staggered fermions [10]. We have extended these calculations to the case of anisotropic lattices [5]. Here we will give our results for staggered fermions as far as they are relevant for the discussion of QCD thermodynamics.

The fermion self-energy on anisotropic lattices has the form

$$\Sigma(p) = g^2 \frac{N^2-1}{2N}\left(m_f a \Sigma_2 + \sum_{i=1}^{3} p_i a \Sigma_{1,\sigma} + p_0 a \frac{\gamma_F}{\sigma}\Sigma_{1,\tau}\right) + O(g^4). \qquad (18)$$

Here Σ_2 as well as $\Sigma_{1,\sigma(\tau)}$ depend on ξ and γ_F. Using for γ_F the ansatz given in eq. (13) and demanding again a rotational invariant continuum limit for the full propagator, eq. (17), we find that the $O(g^2)$ correction to $\gamma_F(\xi)$ is given by eq. (13) with $c_F(\xi)$ determined as

$$c_F(\xi) = \frac{N^2-1}{2N}[\Sigma_{1,\sigma}(\xi) - \Sigma_{1,\tau}(\xi)]. \qquad (19)$$

The $O(g^2)$ contributions to $\Sigma(p)$ can be obtained from the two diagrams shown in fig. 1. Here we are interested in particular in the results for $\Sigma_{1,\sigma(\tau)}$ which contribute to the renormalization of γ_F. We find

$$\Sigma_{1,\sigma} = \int d^4q \left[-\frac{\sin^2 q_1}{2\Delta_1(\xi)\Delta_2(\xi)} \times \left(\frac{1}{2} + \frac{1}{\Delta_1(\xi)}[3 - \cos q_1 - \tfrac{1}{2}\Delta_1(1)]\right) + \frac{1}{4\Delta_1(\xi)}\right],$$

$$\Sigma_{1,\tau} = \int d^4q \left[-\frac{\sin^2 q_0}{2\Delta_1(\xi)\Delta_2(\xi)} \times \left(\frac{1}{2} + \frac{1}{\Delta_1(\xi)}\xi^2[3 - \cos q_0 - \tfrac{1}{2}\Delta_1(1)]\right) + \frac{1}{4\xi^2\Delta_1(\xi)}\right]. \qquad (20)$$

where we have used the abbreviations

$$\Delta_1(\xi) = \sum_{i=1}^{3}(1-\cos q_i) + \xi^2(1-\cos q_4),$$

$$\Delta_2(\xi) = \sum_{i=1}^{3}\sin^2 q_i + \xi^2 \sin^2 q_0, \qquad (21)$$

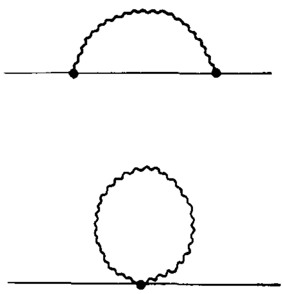

Fig. 1. Feynman diagrams contributing to the fermion self-energy at $O(g^2)$.

$$\int d^4q = \frac{1}{(2\pi)^4} \iiiint_{q_\mu \in [-\pi,\pi]} d^4q. \qquad (21\text{ cont'd})$$

The above formulae are given for $m_f = 0$. Corrections due to finite fermion masses are $O(m_f^2)$ for $\Sigma_{1,\sigma}$ and $\Sigma_{1,\tau}$. For non-zero m_f there is also an additional contribution to the energy density, eq. (11), which comes from the multiplicative, ξ-dependent renormalization of m_f for staggered fermions. However, we found this contribution to be numerically small [6] and we will thus neglect it here.

We note that both integrals in eq. (20) are logarithmically divergent. The divergent piece, however, is ξ-independent and identical in both integrals; the difference, contributing to eq. (19), is thus finite. Furthermore, we have, of course, $\Sigma_{1,\sigma}(1) = \Sigma_{1,\tau}(1)$. In the numerical evaluation of $\Sigma_{1,\sigma}$ and $\Sigma_{1,\tau}$, however, some care has to be taken in order to ensure the correct cancellation of the divergent part in the difference $\Sigma_{1,\sigma} - \Sigma_{1,\tau}$. We have transformed the integrals appearing in eq. (20) to the interval $[-\pi/2, \pi/2]$ for the three spatial momenta and to $[-\pi\xi/2, \pi\xi/2]$ for q_0. The divergent part is then cancelled by first calculating the finite differences $\Sigma_{1,\sigma(\tau)}(\xi) - \Sigma_{1,\sigma(\tau)}(1)$. From this we find the derivatives by taking the limit $\xi \to 1$ numerically. We then obtain for the derivative at $\xi = 1$

$$c'_F = -\frac{N^2-1}{2N} 0.1599. \qquad (22)$$

It is interesting to compare this with the corresponding result for the derivative of the gluonic anisotropy parameter γ_G. The derivative of the $O(g^2)$ corrections gives in this case

$$c'_G = -\frac{N^2-1}{2N} 0.1466 + 0.0096N - 0.0018n_f. \qquad (23)$$

The quantum corrections to the derivatives of γ_G and γ_F are thus very similar in magnitude for QCD.

4. Comparison with continuum perturbation theory

From perturbative calculations of the energy density for massless QCD we would expect that at high temperature corrections to the leading ideal gas behaviour are negative. To $O(g^2)$ one finds for the energy density, ϵ, in an SU(N) gauge theory with n_f massless quark flavours [11]

$$\begin{aligned}\epsilon/T^4 &= \epsilon_0^G(N) + n_f \epsilon_0^F(N) \\ &\quad + g^2(N^2-1)[\bar\epsilon_1^G(N) + n_f \bar\epsilon_1^F(N)] \\ &= \tfrac{1}{15}(N^2-1)\pi^2 + n_f \cdot \tfrac{7}{60}N \\ &\quad + g^2(N^2-1)(-\tfrac{1}{48}N - n_f \cdot \tfrac{5}{192}). \end{aligned} \qquad (24)$$

We note that the splitting into a fermionic and gluonic contribution to the energy density as indicated in eq. (24) is different from the conventions used to define ϵ_G and ϵ_F in eqs. (10) and (11). In fact, $\bar\epsilon_1^F$ receives contributions from the perturbative expansion of ϵ_G as well as ϵ_F.

The $O(g^2)$ corrections to the energy density have been calculated in weak coupling perturbation theory [12,13]. It has been found there that even on rather small lattices the purely gluonic corrections approximate well the continuum result $\bar\epsilon_1^G/T^4 = -\frac{1}{48}N$, whereas the corrections to the fermionic part showed much larger finite size effects and did not indicate any convergence to the continuum result. In fact, the $O(g^2)$ corrections turned out to be positive rather than reproducing the continuum result $\bar\epsilon_1^F/T^4 = -\frac{5}{192}$. These calculations, however, neglected the $O(g^2)$ contribution coming from the derivative of γ_F, and these terms have also not been included in the expressions used for the Monte Carlo simulations. In the previous section we found that the $O(g^2)$ correction to the derivative of γ_F is large and negative for $\xi = 1$. We thus expect that this leads to a compensation of the positive $O(g^2)$ correction to the energy density found in previous perturbative calculations [12,13]. In fact, we find that the continuum result is well reproduced on large lattices when the contribution coming from the $O(g^2)$ corrections to the coupling γ_F is taken into account correctly.

We will concentrate in the following on a discussion of the fermionic part, ϵ_1^F, of the $O(g^2)$ corrections [#1]

[#1] We use the same notations as in refs. [12,13]. Explicit formulas for all perturbative terms, except c'_F, which is given by eq. (22), can be found in these references.

$$\frac{\bar{\epsilon}_1^{\rm F}(N)}{T^4} = N_\tau^4 \bigg(6(P_\sigma^{(4f)} - P_\tau^{(4f)}) + P_f^{(2)} + 3[c'_{\sigma,\rm F}(P_{\rm sym}^{(2)} - P_\sigma^{(2)}) + c'_{\tau,\rm F}(P_{\rm sym}^{(2)} - P_\tau^{(2)})] + \frac{1}{N^2-1} c'_{\rm F}(P_f^{(1)} - \tfrac{1}{16}) \bigg). \qquad (25)$$

All but the last term appearing in eq. (25) have been calculated in ref. [12] and ref. [13]. We have denoted by $c'_{\sigma(\tau),\rm F}$ the n_f dependent part of $c'_{\sigma(\tau)}$ [3], i.e. $c'_{\sigma,\rm F} = -0.00031$ and $c'_{\tau,\rm F} = -0.00391$.

In fig. 2 we show results for $\bar{\epsilon}_1^{\rm F}/T^4$ as a function of N_τ. The perturbative contributions $P_\sigma^{(4f)}$, $P_\tau^{(4f)}$, $P_f^{(1)}$ and $P_f^{(2)}$ have been evaluated on infinite spatial lattices with varying temporal extent N_τ. With increasing N_τ larger numerical accuracy is required as the contributions to $\epsilon_1^{\rm F}$ drop like N_τ^{-4}. We therefore had to restrict our analysis to $N_\tau \leqslant 32$. As can be seen from fig. 2 the approach to the infinite N_τ limit is rather non-uniform for small N_τ. This has been noted previously also for the lowest order ideal gas contributions $\epsilon_0^{\rm G}$ and $\epsilon_0^{\rm F}$ on finite lattices [14]. The first term in eq. (25) gives for all N_τ a large positive contribution to $\bar{\epsilon}_1^{\rm F}$ whereas the other contributions resulting from the derivatives of the couplings give negative contributions, the largest coming from the term proportional to $c'_{\rm F}$. For $N_\tau = 32$, we find for instance $\bar{\epsilon}_1^{\rm F}/T^4 = 0.0665 - 0.0012 - 0.0939 = -0.0286$ for the contributions from the three different terms in eq. (25). For large N_τ the finite volume effects turn out to be proportional to N_τ^{-3}. An extrapolation to infinite N_τ gives $\bar{\epsilon}_1^{\rm F}/T^4 = -0.027$, which is in good agreement with the result obtained in continuum perturbation theory, $\bar{\epsilon}_1^{\rm F}/T^4 = -\frac{5}{192} = -0.0260$. Note that the contribution coming from $c'_{\rm F}$ is essential for getting agreement with continuum perturbation theory. In fact there is a strong cancellation between the negative contribution coming from $c'_{\rm F}$ and the positive contribution from the perturbative expansion of the lattice operators entering the expectation values in eqs. (10) and (11). In fig. 3 we show perturbative results on smaller lattices with finite spatial extent. As can be seen also in this case the $O(g^2)$ corrections turn out to be negative. The dependence on the spatial size turns out to be proportional to $1/N_\sigma$ once $N_\sigma > 2N_\tau$.

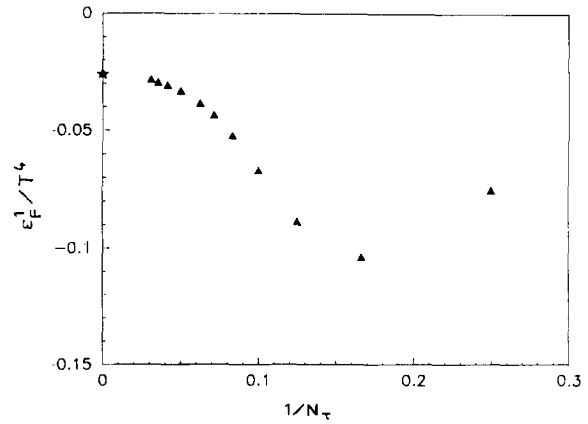

Fig. 2. $\bar{\epsilon}_1^{\rm F}/T^4$ on $N_\tau \times (\infty)^3$ lattices as function of $1/N_\tau$. The star on the ordinate gives the continuum result $\bar{\epsilon}_1^{\rm F}/T^4 = -5/192$.

5. Monte Carlo results for the energy density

The thermodynamics of QCD with light quarks has been studied in much detail in numerical simulations [5]. Already in the first SU(2) calculations [13] it has been observed that, unlike in the pure gauge sector, the energy density overshoots the ideal gas limit in the plasma phase. This effect increases with increasing number of flavours [15–17]. For instance in the case of QCD with $n_f = 4$ light flavours it has been found that for temperatures in the interval $(1–2)T_c$ the energy density is about a factor (1.5–2.0) higher than the ideal gas value for a massless quark–gluon gas. It has been noted that this over-

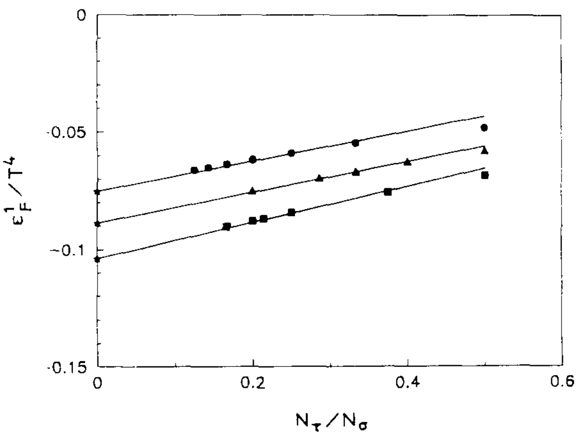

Fig. 3. $\bar{\epsilon}_1^{\rm F}/T^4$ on $N_\tau \times N_\sigma^3$ lattices as function of N_τ/N_σ for $N_\tau = 4$ (dots), 6 (squares) and 8 (triangles). The stars on the ordinate give the infinite spatial volume results shown in fig. 2.

shooting mainly results from a "too" large contribution of the gluonic sector.

In all the above-mentioned numerical work modifications of ϵ_F due to quantum corrections to the fermionic couplings have been neglected, i.e. $d\gamma_F/d\xi|_{\xi=1} \equiv 1$ has been used in eq. (11). We can use our perturbative results to get improved formulas for the energy density. This merely amounts to a multiplicative renormalization of ϵ_F, see eq. (11). We thus can easily take this into account and reanalyze the existing Monte Carlo data. As we have seen in the previous chapters the inclusion of these corrections will lead to a large reduction of the contribution of ϵ_F to the total energy density. One thus may expect that the previously observed overshooting will be reduced.

When reanalyzing the existing data we also should take into account that in most of the existing calculations the zero temperature contribution in the gluonic sector has not been subtracted and consequently also the derivatives of β_σ and γ_G have been approximated by the leading classical term [≠2]. It has been noted that this effectively means that one calculates the entropy density rather than the energy density [18]. We have used the data of refs. [16,17] to calculate the entropy density

$$\frac{S}{T^3} = \frac{8N}{g^2} N_\tau^4 \left(1 - \frac{c'_\sigma - c'_\tau}{2} g^2\right) (\bar{P}_\sigma - \bar{P}_\tau) + \frac{4}{3} \frac{\epsilon_F}{T^4}, \tag{26}$$

with ϵ_F/T^4 given by eq. (11). In fig. 4 we show results for a lattice with $N_\tau=4$, $n_f=2$ and 4. As can be seen the overshooting previously found in ϵ_G is now, to a large extent, compensated by the reduction of ϵ_F. The entropy density turns out to be in good agreement with the ideal gas result. For $n_f=4$ there still remains a peak close to T_c in S/T^3 which may be physically significant also in the continuum limit. In order to clarify this the scaling of results obtained on lattices with different N_τ should be checked in the future. It is interesting to note that a similar behaviour is also found in recent instanton liquid calculations [19].

[≠2] A proper subtraction of the vacuum contributions has been performed in ref. [15]. However, in this case only the total energy density has been published. We thus cannot use these data to correct the part coming from the fermionic sector.

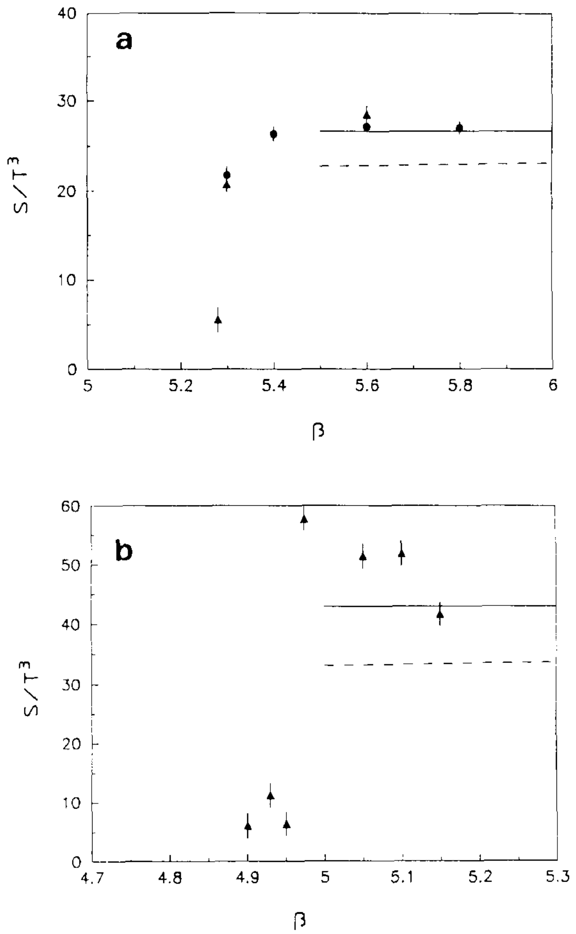

Fig. 4. Entropy density for QCD with quarks of mass $m/T=0.1$ on lattices of size $4\times N_\sigma^3$ versus β. Monte Carlo data from simulations for $n_f=2$ and $N_\sigma=8$ (triangles), 12 (dots) shown in (a) are taken from ref. [16]. Those for $n_f=4$ and $N_\sigma=8$ shown in (b) are taken from ref. [17]. The data have been reanalyzed using eq. (26) to extract the entropy density.

6. Conclusions

We have studied the $O(g^2)$ corrections to thermodynamic quantities calculated for QCD with light quarks on euclidean lattices. We find that quantum corrections to fermionic couplings are crucial in order to reproduce the correct perturbative results in the continuum limit. These perturbative corrections reduce the fermionic contribution to the total energy calculated in Monte Carlo simulation by about 20%. It thus leads to a reduction of the total energy density by 10% for $n_f=2$ and 15% for $n_f=4$ compared to earlier calculations [5]. Including all $O(g^2)$ corrections to the derivatives of the couplings appearing in the QCD action on anisotropic lattices we find good agreement of the entropy density calculated in Monte Carlo simulations with the high temperature ideal gas result on finite lattices. To what extent the remaining overshooting of the ideal gas limit observed for $n_f=4$ is a finite lattice artifact or can be attributed to truly non-perturbative effects requires the analysis of the entropy density on larger lattices.

Finally we would like to stress again that a truly non-perturbative calculation of thermodynamic quantities on the lattice also has to include a non-perturbative determination of the derivatives of the couplings β_σ, γ_G and γ_F. For the fermionic coupling γ_F this can be achieved by comparing fermionic correlation functions on anisotropic lattices measured in different directions and demanding rotational invariance. A similar procedure has been used for β_σ and $\gamma_G(\xi)$ by demanding rotational invariance of the heavy quark potential [7].

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