IMPROVED MONTE CARLO RENORMALIZATION GROUP METHODS

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Received 20 February 1984

The practical application of MCRG requires the flow lines to reach the renormalized trajectory after a small number of blocking steps. It is suggested to use optimized block transformations in order to shift the fixed point and the renormalized trajectory closer to a given action. In asymptotically free theories, perturbation theory can be used to find the improved block transformations. Another MCRG method, the improved ratio method, is discussed also. The methods are tested on $d = 2$, asymptotically free spin models.

Monte Carlo renormalization group (MCRG) is a powerful technique for the study of the critical properties of spin and gauge systems [1–5] [‡1]. In this paper, improved MCRG methods will be discussed which, although they are more general, will be formulated in the specific context of asymptotically free theories. The methods are tested here on $d = 2$ spin models, but all the steps are immediately generalizable to $d = 4$ gauge theories [‡2].

Consider an O(N) spin model on a periodic, square lattice. The partition function is given by the standard action

$$A(s) = -\beta \sum_n \sum_{\mu=1}^{2} s_n s_{n+\hat{\mu}} , \qquad (1)$$

as

$$Z = \int \mathrm{D}s \prod_n \delta(1 - s_n^2) \exp[-A(s)] . \qquad (2)$$

The basic quantity we are interested in is the β-function of the theory. This function describes the relation between the bare coupling constant β and the value of the cut-off, and it has a well-defined meaning in the vicinity of the ultraviolet fixed point, $\beta = \infty$. The β-function gives the way asymptotic scaling is approached, it connects numerical studies with perturbation theory, reveals the existence of possible phase transitions, and so on. In the MCRG approach, not the β-function itself, but a related quantity $\Delta\beta = \Delta\beta(\beta)$, is determined, which gives the change of the coupling $\beta \to \beta - \Delta\beta(\beta)$, when the (dimensionless) correlation length (or the cut-off) is decreased by a factor of b. Here b is the basic change of scale in a single renormalization group (RG) step ($b = 2$ in the following). At the couplings β and $\beta' = \beta - \Delta\beta(\beta)$ the model has identical long-distance properties, only the (dimensionless) correlation length ξ differs by a factor of 2.

Consider a specific block transformation, where the block spin μ_ℓ ($\mu_\ell^2 = 1$) is constructed as some kind of average from the 4 spins of the 2×2 block ℓ:

$$\mu_\ell = \mu_\ell(s_i, i \in \text{block } \ell) . \qquad (3)$$

The interaction between the block spins is described by a new action $A'(\mu)$, which, in general, will contain all kinds of interactions. $A'(\mu)$ can be represented as a point in a multidimensional space of different coupling constants. It is expected that the RG transformation defined by eq. (3) has a fixed point somewhere in the $\beta = \infty$ hyperplane of this multidimensional space and a single renormalized trajectory (RT) starts from this

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[‡1] A pedagogical discussion is also given in ref. [6].

[‡2] The results of this paper – together with some preliminary results for $d = 4$ SU(3) gauge theory – have been presented at the Lattice Coordinating Meeting (CERN, December 1983).

Table 1
Three-level perturbative coefficients of the block correlations (see eq. (9)) using the $P = \infty$ block transformations given in eq. (4).

Lattice size	Blocking step	α_{NN}	α_{DI}	α_{NNN}
64 × 64	0	0.249939	0.318188	0.363136
	1	0.162549	0.218690	0.271767
	2	0.137073	0.190322	0.243512
	3	0.127501	0.177596	0.224678
	4	0.113629	0.154199	0.168479
	5	0.059385	0.083494	–
32 × 32	0	0.249756	0.317822	0.362402
	1	0.161815	0.217234	0.268811
	2	0.134117	0.184583	0.231339
	3	0.115331	0.155989	0.170175
	4	0.059827	0.083984	–
16 × 16	0	0.249023	0.316369	0.359450
	1	0.158864	0.211507	0.256649
	2	0.121959	0.163045	0.176818
	3	0.061574	0.085938	–
8 × 8	0	0.246094	0.310706	0.347339
	1	0.146763	0.190290	0.202009
	2	0.068359	0.093750	–

point [3,4,6] (fig. 1). An important point to mention is that the position of the fixed point and of the RT is not universal; it depends on the details of the block transformation in eq. (3).

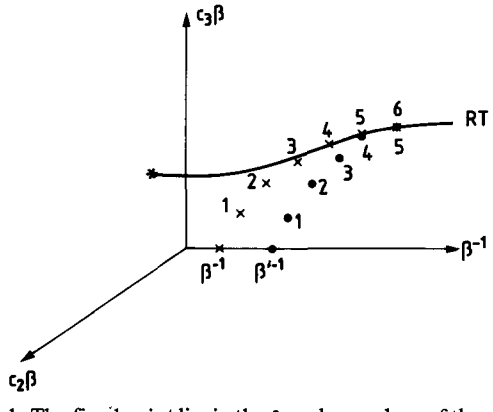

Fig. 1. The fixed point lies in the $\beta = \infty$ hyperplane of the multi-dimensional coupling constant space. $c_2, c_3, \ldots$ are the next-to-nearest neighbour, diagonal, ... couplings of the block spin actions. The standard action is represented by the points of the $c_2 = c_3 = \ldots = 0$ axes. The RT attracts the flow lines starting from the neighbourhood of the fixed point.

Starting with the standard action at a given β value (β is large), the effective actions obtained after a few RG steps will move along the RT. The same will happen if we start at some other coupling β'. By tuning β' it can be arranged that those points of this second sequence which lie on the RT coincide with the corresponding points of the first sequence, but one step behind (fig. 1). Then the models defined by the standard action at β and β' are identical concerning their long-distance properties, while their correlation lengths differ by a factor of 2. Therefore, $\Delta\beta(\beta) = \beta - \beta'(\beta)$ is the relation we are looking for.

At every blocking step the linear size of the lattice is reduced by a factor of 2. If the RT is far from the standard action and many blocking steps are required in order to match the two sequences of points in the multiparameter space of fig. 1, the procedure would require a prohibitively large starting lattice. Or, saying differently, if we can perform only an insufficient number of RG steps then no consistent matching will be possible (the matching of the different block spin expectation values would give different $\Delta\beta$ values). This is illustrated in fig. 2 for the case of the exactly solvable $O(N)_{N\to\infty}$ model [‡3] using the simple block transformation [‡4]

$$\mu_\ell = \frac{s_{i_1} + s_{i_2} + s_{i_3} + s_{i_4}}{\| s_{i_1} + s_{i_2} + s_{i_3} + s_{i_4} \|}, \quad i_1, i_2, i_3, i_4 \in \text{block } \ell\,. \quad (4)$$

As can be seen in fig. 2, the procedure breaks down completely at large correlation lengths. There is a significant deviation between the predicted and the exact value of $\Delta\beta$, and the deviation increases linearly with β.

A possible solution is to use an improved action which lies closer to the given RT [3,4,7]. There is another possibility, however: search for an *improved block transformation* whose fixed point and RT lie close to the *standard action* [‡5]. This procedure offers several advantages, especially in the case of $d = 4$ gauge theories. There is no need to simulate a com-

[‡3] In this figure, and everywhere in the following, $\beta \to \beta/N$ for the $O(N)_{N\to\infty}$ model.

[‡4] A RG study of the $O(N)$, $N \to \infty$, model using the block transformation in eq. (4) is given in ref. [7]. In this paper the effect of improving the action is also discussed.

[‡5] A similar idea has been put forward by Swendsen [8], in ref. [9] a systematic study of the Ising model is given along these lines.

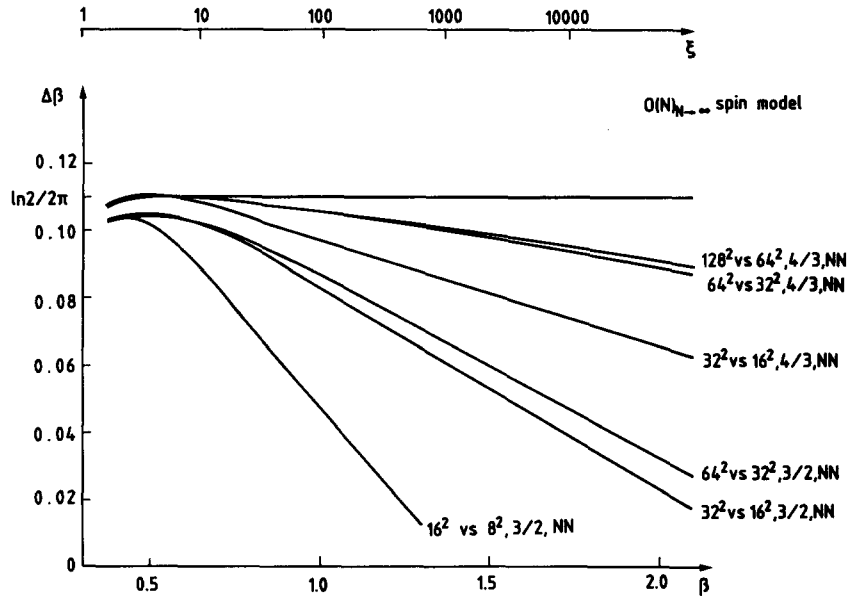

Fig. 2. The matching predictions of this figure were obtained for the standard O(n), $N \to \infty$, action using the simple block transformation in eq. (4). $\Delta\beta(\beta)$ is the change of β resulting in reducing the correlation length by a factor of 2. The exact result in the continuum limit is $\Delta\beta = (2\pi)^{-1} \ln 2$ (without corrections), $\xi(\beta)$ is the correlation length. The curve "128^2 versus 64^2, 4/3 blocking, NN" refers to a matching, where the nearest-neighbour block correlation after 4 RG steps on a 128×128 lattice was compared to that after 3 RG steps on a 64×64 lattice, and so on.

plicated action, given the configurations one can experiment easily with different block transformations, a lot of effort is invested already into studying the standard action, and so on. Additionally, the method seems to work well. In asymptotically-free theories one might use perturbation theory to find improved block transformations – just as was done previously in searching for improved actions [3,4,7,10].

Without some improvement, the error in $\Delta\beta$ increases linearly with β. This feature is easy to understand. In the matching procedure, block spin correlation functions (obtained after k and $(k-1)$ RG steps, starting from a lattice of size $L \times L$ and $L/2 \times L/2$ respectively) are compared. For large β these block correlation functions can be evaluated in perturbation theory. On the tree level this leads to the matching equation

$$1 - c/\beta + O(1/\beta^2) = 1 - c'/\beta' + O(1/\beta^2), \qquad (5)$$

giving

$$\Delta\beta \equiv \beta' - \beta = [(c - c')/c]\beta + O(1) . \qquad (6)$$

The contribution $(c - c')\beta/c$ in $\Delta\beta$ is an error: tree-level perturbation theory should give $\Delta\beta = 0$ ($c = c'$), since a non-trivial scale is generated only at the one-loop level. In this context "tree-level improvement" is a procedure to minimize $(c - c')/c$ in the matching conditions. This is true also in other methods, like the ratio method we discuss later. For block transformations this requirement is the same as that of starting close to the fixed point (since $c = c'$ at the fixed point).

As an example, consider the following one-parameter family of block transformations [‡6]: the probability that the ℓth block spin takes the value μ_ℓ is

$$\sim \exp\left[P\mu_\ell(s_{i_1} + s_{i_2} + s_{i_3} + s_{i_4})\right] ,$$
$$i_1, i_2, i_3, i_4 \in \text{block } \ell , \qquad (7)$$

where $\mu_\ell^2 = 1$, and P is a free parameter. For $P \to \infty$ we get back eq. (4). We shall use perturbation theory to find the value of P which gives a fixed point and RT lying closest to the standard action for large β.

Actually, P is not completely free. When $\beta \to \infty$

[‡6] This block transformation is easy to generalize to gauge theories as is discussed by Swendsen [6].

then $P \to \infty$ also is required, otherwise the block spin configuration completely "forgets" the content of the original configuration, and no fixed point is expected to occur. $P = c\beta$ is a consistent choice, where c is a constant, which should be determined.

Tree-level perturbation theory gives, for the block-spin–block-spin correlation function after the kth RG step on an $L \times L$ starting lattice:

$$\Gamma(n;\beta,P,L)_{(k)} = \Gamma(n;\beta,P=\infty,L)_{(k)} - (2/P)(1/4 + 1/4^2 + \ldots + 1/4^k)\,, \tag{8}$$

where $\Gamma(n;\beta,P=\infty,L)_{(k)}$ is obtained by using the block transformation eq. (4) and it has the general form

$$\Gamma(n;\beta,P=\infty,L)_{(k)} = 1-(N-1)\alpha(n;L)_{(k)}/\beta. \tag{9}$$

The numbers $\alpha(n;L)_{(k)}$ are given for $n = (0,1), (1,1), (0,2)$ for different lattice sizes L in table 1 [‡7]. The matching condition reads

$$\Gamma(n;\beta,P,L)_{(k)} = \Gamma(n;\beta',P,L/2)_{(k-1)}\,, \tag{10}$$

[‡7] A part of this table is given in ref. [4].

and the tree-level requirement $\Delta\beta = \beta - \beta' = 0$ determines the constant c in the relation $P = c\beta$:

$$P = c\beta = \frac{2}{4^k\,[\alpha(n;L/2)_{(k-1)} - \alpha(n;L)_{(k)}]} \cdot \left(\frac{\beta}{N-1}\right). \tag{11}$$

It is reassuring that for $k \geqslant 3$, c is only weakly dependent on k and n (the type of correlation considered). In our analysis we choose the value

$$P = 4\cdot 6\left(\frac{\beta}{N-1}\right). \tag{12}$$

Using this value of P, $\alpha(n;L/2)_{(k-1)} - \alpha(n;L)_{(k)}$ is small even for $k - 1 = 2$, showing that with this improved block transformation we get close to the fixed point after two RG transformation steps.

The matching results obtained by using this tree-level improved block transformation are given in figs. 3 and 4 for the $O(N)_{N\to\infty}$ and O(3) models, respectively. In fig. 3 $\Delta\beta$ was obtained by matching the nearest-neighbour (NN) correlations ($n = (0,1)$) starting on a 32^2 versus 16^2 lattice. In the continuum limit of the $O(N)_{N\to\infty}$ standard model the β-function con-

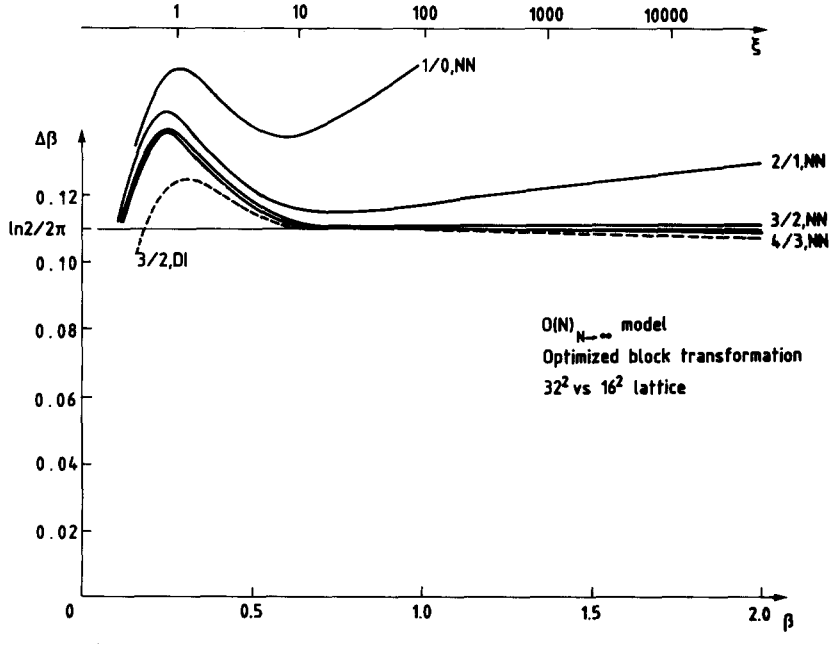

Fig. 3. The matching predictions for the standard $O(N), N \to \infty$, model using optimized block transformations. The disagreement between the NN and DI matchings below $\beta \approx 0.5$ indicates that this region is outside the scaling region and no unique β-function can be defined.

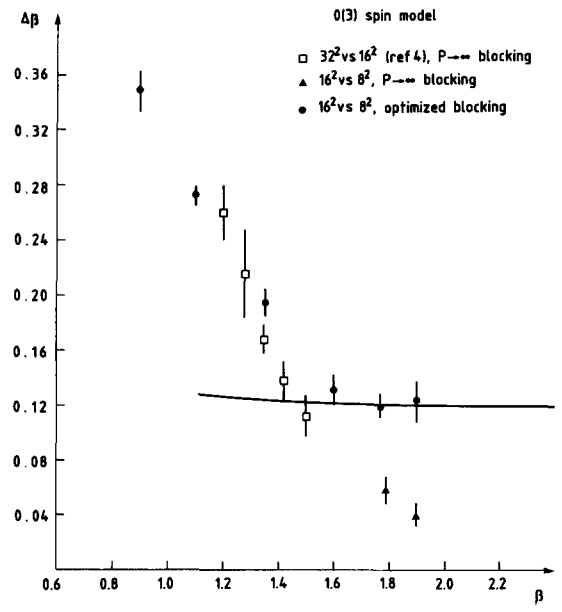

Fig. 4. Matching predictions for the O(3) standard model. At large correlation lengths the procedure breaks down if the simple block transformation of eq. (4) is used. The improved block transformation gives consistent results with the two-loop asymptotic β-function (solid line) in this region. In the case of optimized blocking the NN and DI matchings are consistent well within the statistical errors, except at $\beta = 0.9$, where we could not measure the DI matching with our statistics.

tains one term only, and $\Delta\beta = \ln 2/2\pi$ without higher order corrections. The 3/2 (= three steps on 32^2 versus two steps on 16^2) and 4/3 results are almost identical and give the correct result. Below $\Delta\beta = 0.5$ ($\xi \leqslant$ few lattice units), $\Delta\beta$ is different for the NN and DI ($n = (1, 1)$) matchings indicating that this region is outside the continuum limit already.

As is shown in fig. 4 the unimproved matching procedure breaks down for large β in the O(3) model also, as we expected. The improved block transformation gives results which are consistent with the asymptotic value $\Delta\beta = \ln 2/2\pi$ at large β and predicts a non-trivial β-function at intermediate couplings. For example at $\beta = 1.35$ ($\xi \approx 1$ few lattice units), $\Delta\beta = 0.19 \pm 0.01$, consistently from the NN, DI and NNN correlation functions.

Let us remark that the deviation of the β-function from its asymptotic value at intermediate couplings seems to explain the apparent "scaling violation" observed for the mass gap of the standard action [11]. Presumably, the mass gap scales correctly in the region $\xi \geqslant$ few lattice units, but the β-function cannot be replaced by the leading perturbative terms there. A more quantitative statement would require a systematic measurement of the β-function with good statistics, which is beyond the scope of this paper.

At the end let us discuss another matching method which we call the ratio method. Like the previous method it also has the property that the necessary tree-level improvement can be done easily, without changing the action.

Consider the ratio $f(n_1, n_2; \beta, L) = \langle S_0 S_{n_1}\rangle_\beta / \langle S_0 S_{n_2}\rangle_\beta$. The wave function renormalization factors cancel and f satisfies the homogeneous RG equation in the continuum limit, if n_1 and n_2 are large enough to avoid lattice artifacts. We get the matching condition:

$$f(2n_1, 2n_2; \beta, L) = f(n_1, n_2; \beta', L/2)\,, \qquad (13)$$

where – in order to minimize the finite size effects – the left- and right-hand sides of this equation are evaluated on a lattice of size $L \times L$ and $L/2 \times L/2$ respectively [‡8]. This method breaks down for larger correlation lengths for the same reason as the block procedure: three-level perturbation theory for eq. (13) gives a non-zero (and, in general, not small) $(c - c')/c$, in the notation of eq. (6). A simple way to avoid this problem is to take an appropriate linear combination of two ratios in such a way that $\Delta\beta = 0$ is obtained in tree-level perturbation theory. A large number of tree-level improved (or "mixed") ratios can be obtained in this way.

In fig. 5 the matching predictions are given which were obtained from different, tree-level improved, ratios at selected β values in the $O(N)_{N\to\infty}$ model. We investigated 66 different mixed ratios of which the first 10 and the last 6 are shown in fig. 5. The numbering 1–66 is arbitrary. The ratios 61–66 contain correlations at longer distances, therefore the effect of lattice artifacts is expected to be smaller for them than for the ratios 1–10. To give two examples the no. 1 result was obtained by matching

$$f((2,2),(2,0);\beta,L) + 2.32284 f((4,2),(2,2);\beta,L)$$

[‡8] An analogous ratio test was considered for Wilson loops in ref. [12] without the volume adjustment of eq. (13) and without the tree-level improvement discussed in the following.

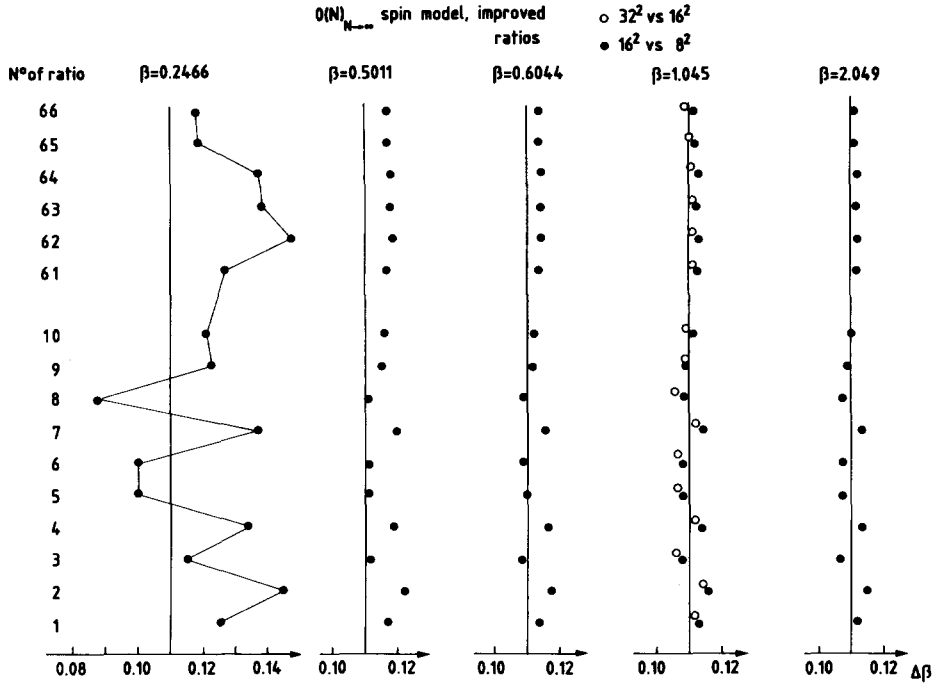

Fig. 5. Matching predictions for the O(N), $N \to \infty$, model obtained by using the improved ratio method.

with

$$f((1,1),(1,0);\beta',L/2) + 2.32284 f(2,1),(1,1);\beta',L/2),$$

while the no. 66 result was obtained by matching

$$f((8,4),(8,2);\beta,L) + 0.653535 f((8,2),(4,4);\beta,L)$$

with

$$f((4,2),(4,1);\beta',L/2) + 0.653535 f((4,1),(2,2);\beta',L/2) .$$

The mixing coefficients 2.32284, ..., 0.653535 were determined from the requirement of tree-level improvement, as we discussed above. At $\beta = 2.049$ ($\xi \approx 10^5$) and at $\beta = 1.045$ ($\xi \approx 10^2$) the ratio test reproduces the exact result within ≈2% error. At $\beta = 0.5011$ ($\xi \approx 4$) the matching is less consistent already, while at $\beta = 0.2466$ (strong coupling) no matching is found anymore. The comparison of the $L = 16$ and $L = 32$ results show an observable, but small finite size effect.

In fig. 6 the results of a single matching from $\beta = 1.90$ ($\xi \approx 10^2$) to β' is shown for the O(3) model. Those ratios are plotted whose statistical errors are

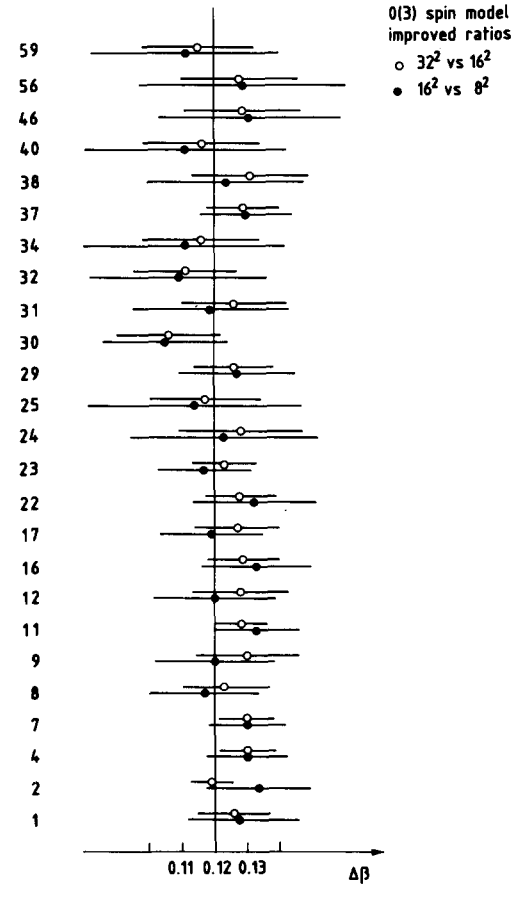

Fig. 6. A single matching for the O(3) model from $\beta = 1.90$ ($\xi \approx 10^2$) to $\beta' = \beta - \Delta\beta$ using the improved ratio method. The vertical line is the two-loop perturbative prediction.

acceptably small. Although it is clear that for a serious study a statistically improved measurement is necessary, the method seems to work well even on small lattices.

For both methods, one-loop perturbation theory would help significantly to understand the remaining systematic errors. For the ratio method this calculation is certainly feasible both for $d = 2$ spin and $d = 4$ gauge theories.

The authors are indebted to R.H. Swendsen for very useful discussions.

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