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The "fundamental theorem" for the algebraic K-theory of spaces: II—the canonical involution

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Abstract

Let $X \mapsto A(X)$ denote the algebraic K-theory of spaces functor. In the first paper of this series, we showed $A(X \times S^1)$ decomposes into a product of a copy of $A(X)$, a delooped copy of $A(X)$ and two homeomorphic *nil terms*. The primary goal of this paper is to determine how the "canonical involution" acts on this splitting. A consequence of the main result is that the involution acts so as to transpose the nil terms. From a technical point of view, however, our purpose will be to give another description of the involution on $A(X)$ which arises as a (suitably modified) $\mathcal{S}_\cdot$-construction. The main result is proved using this alternative discription. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

The algebraic K-theory of spaces functor $X \mapsto A(X)$, defined by the fourth author, relates the stable concordance space $\mathscr{C}(X)$ to the higher algebraic K-theory of the integral group ring $\mathbb{Z}[\pi_1(X)]$ (see [5–7]).

In [1], we established a splitting of based spaces

$$A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times \mathscr{B}A^{\mathrm{fd}}(X) \times N_{-}A^{\mathrm{fd}}(X) \times N_{+}A^{\mathrm{fd}}(X),$$

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where $A^{\mathrm{fd}}(X)$ is a version of $A(X)$ which incorporates finitely dominated spaces. [1] In this splitting, $\mathscr{B}A^{\mathrm{fd}}(X)$ denotes certain non-connective delooping of $A^{\mathrm{fd}}(X)$, and the *nil-terms* $N_{-}A^{\mathrm{fd}}(X)$ and $N_{+}A^{\mathrm{fd}}(X)$ are canonically isomorphic. This decomposition should be viewed as the algebraic K-theory of spaces analogue of the "fundamental theorem" for the algebraic K-theory of rings.

In [4], it was shown how to modify $A(X)$ so as to equip it with an involution. This involution corresponds under linearization to the involution on the K-theory space of the group ring $\mathbb{Z}[\pi_1(X)]$ which is induced by mapping a matrix with $\mathbb{Z}[\pi_1(X)]$-coefficients to its conjugate transpose (the conjugate is defined using the anti-automorphism of $\pi_1(X)$ which maps an element to its inverse).

One interest in the involution on $A(X)$ is in its relation to the involution on the stable concordance space $\mathscr{C}(X)$ (cf. [4, Section 2]). The eigenspaces of the latter provide homotopy theoretic information about automorphism groups of manifolds (see e.g. [2,8]).

The main result of this paper is to identify how the involution on $A^{\mathrm{fd}}(X \times S^1)$ acts with respect to the decomposition provided by the "fundamental theorem":

Theorem (*Equivariant "Fundamental Theorem"*). *With respect to the above splitting of $A^{\mathrm{fd}}(X \times S^1)$, the involution acts as a product of the involution on $A^{\mathrm{fd}}(X)$, the delooped involution on $\mathscr{B}A^{\mathrm{fd}}(X)$ and an involution on $N_{-}A^{\mathrm{fd}}(X) \times N_{+}A^{\mathrm{fd}}(X)$ which transposes factors.*

(For the precise statement, see Theorem 10.3.2 below.)

From a technical point of view, however, our purpose will be to give another description of the involution on $A^{\mathrm{fd}}(X)$ which arises as a (suitably modified) $\mathscr{S}_{\bullet}$-construction. The main result is proved using this alternative description. It should also perhaps be mentioned here that Weiss and Williams [9] give yet another construction of a space with involution having the underlying unequivariant homotopy type of $A^{\mathrm{fd}}(X)$.

We now briefly outline the contents of this paper. Section 1 is preliminary; among other things it sets up equivariant duality. In Section 2 we extend the notion of duality to filtered objects. In Section 3 we define categories of filtered equivariant spaces which are equipped with duality data. These categories are equipped with a 'stabilization' functor, which is given by suspension, and a 'left forgetful' functor which forgets the duality data. Stabilizing our model to infinity, we show that the approximation defined by the left forgetful functor tends to a homotopy equivalence on realizations. In Section 4, by varying the lengths of our filtrations and including suitable quotient data, we assemble the categories defined in Section 3 into a simplicial category. We then use the realization theorem applied to the left forgetful functor to compare our simplicial category with the $\mathscr{S}_{\bullet}$-construction. In Section 5 we define 'dualization functors'. These are in turn used in Section 6 to define the canonical involution on the algebraic K-theory

[1] The higher homotopy groups of these spaces coincide. The essential difference between them is that the group of path components of the former is isomorphic to the projective class group $K_0(\mathbb{Z}[\pi_1(X)])$, whereas the group of path components of the latter is isomorphic to a cyclic group.

of spaces. Also in Section 6, compare our involution with the involution of [4]. In Section 7 we extend the theory to the projective line category of [1]. In Section 8 we construct an equivariant version of the 'canonical diagram' of [1, 4.13]. The material contained Section 9 is preparation for the proof of main result. The proof of the main result is the content of Section 10.

1. Preliminaries

1.1. Equivariant spaces. The term *space* in this paper refers to a topological space which has the compactly generated topology. Products are to be taken in the compactly generated sense, and function spaces are to be given the compact-open topology.

Let $M_\bullet$ be a simplicial monoid, and let $M = |M_\bullet|$ be the topological monoid which arises by taking the geometric realization of its underlying simplicial set. If X and Y are based, left M-spaces, we say that a based M-map $X \to Y$ is a *weak equivalence* if (and only if) it is a weak homotopy equivalence of the underlying topological spaces. A weak equivalence will often be indicated by the symbol $\xrightarrow{\sim}$.

Let $\mathbb{T}(M)$ denote the category whose objects are based M-spaces and whose morphisms are based M-maps. The *cell* of dimension n is the (unbased) M-space $D^n \times M$ with action defined by left translation. Similarly, one has the (unbased) equivariant sphere $S^{n-1} \times M$.

If Z is an object of $\mathbb{T}(M)$ and $\alpha : S^{n-1} \times M \to Z$ is an equivariant map, then attaching $D^n \times M$ to Z along α defines an object $Z \cup_\alpha (D^n \times M) \in \mathbb{T}(M)$. If an object Y is obtained from an object X up by a (possibly transfinite) sequence of cell attachments, then we say that the inclusion $X \to Y$ is a *cofibration.* More generally, we include in the class of cofibrations retracts of such inclusions. Observe that cofibrations have the equivariant homotopy extension property. A cofibration will often be specified by the symbol $\rightarrowtail$.

An object $Z \in \mathbb{T}(M)$ is said to be *cofibrant* if the map $* \to Z$ is a cofibration. We let $\mathbb{C}(M)$ denote the full category of $\mathbb{T}(M)$ consisting of the cofibrant objects.

An object $Y \in \mathbb{C}(M)$ is *finite* if it is isomorphic to a finite M–CW complex which is free away from the basepoint. It is *homotopy finite* if there exists a weak equivalence $Y \to Z$, where Z is finite. The full subcategory of $\mathbb{C}(M)$ whose objects are homotopy finite will be denoted $\mathbb{C}_{\mathrm{hf}}(M)$.

An object of $\mathbb{C}(M)$ is said to be *finitely dominated* if it is a retract of a homotopy finite object. Let $\mathbb{C}_{\mathrm{fd}}(M)$ denote the full subcategory of $\mathbb{C}(M)$ whose objects are finitely dominated.

Call a morphism in any of these subcategories a *cofibration* if it is one when considered in $\mathbb{T}(M)$. We shall let $h\mathbb{C}_?(M)$ denote the subcategory of $\mathbb{C}_?(M)$ consisting of the weak equivalences, where ? denotes one of the decorations hf, fd. With these conventions, $\mathbb{C}_?(M)$ is a category with cofibrations and weak equivalences.

1.2. The based equivariant sphere. Let $S^0_M \in \mathbb{C}(M)$ be M with the addition of a basepoint. Identify the n-sphere S^n with the smash product of n-copies of S^1. The based left M-space

$$S^n_M := S^n \wedge S^0_M$$

is also a left M^{op}-space, where M^{op} denotes the opposite monoid of M. The induced left $(M \times M^{\mathrm{op}})$-action is given in formulas as follows: if $(g, h^{\mathrm{op}}) \in M \times M^{\mathrm{op}}$ is an element, then the action of this element on S^n_M is given by

$$(v, x) \mapsto (v, gxh) \quad \text{for } v \in S^n,\ x \in S^0_M.$$

Another structure we shall require is the homeomorphism

$$S^n_M \xrightarrow{\iota} S^n_{M^{\mathrm{op}}}$$

defined as follows: with respect to the natural coordinates on the smash product, ι is defined by

$$(x_1, \dots, x_n, m) \mapsto (x_n, \dots, x_1, m^{\mathrm{op}}) \quad \text{for } (x_1, \dots, x_n) \in S^n,\ m \in S^0_M. \tag{1.2.1}$$

We remark that if M is the realization of a simplicial group, then there is a canonical isomorphism

$$M \cong M^{\mathrm{op}}$$

given by $m \mapsto (m^{-1})^{\mathrm{op}}$. With respect to the identification provided by this isomorphism, $\iota : S^n_M \to S^n_M$ is an involution ($\iota^2 = \mathrm{id}$).

1.3. Definition of equivariant duality. Let $Y \in \mathbb{C}(M)$ and $Z \in \mathbb{C}(M^{\mathrm{op}})$ be objects. Then in a natural way, the smash product $Y \wedge Z$ has the structure of an object of $\mathbb{C}(M \times M^{\mathrm{op}})$.

By an *m-pairing*, we mean an $(M \times M^{\mathrm{op}})$-map

$$Y \wedge Z \xrightarrow{u} S^m_M.$$

Suspending k-times, u defines an $(m + k)$-pairing

$$Y \wedge (\Sigma^k Z) = S^k \wedge (Y \wedge Z) \xrightarrow{\mathrm{id}_{S^k} \wedge u} S^k \wedge S^m_M = S^{m+k}_M.$$

Hence, for each $k \geqslant 0$, we obtain a formal adjoint

$$\Sigma^k Z \xrightarrow{\mathrm{adj}_k\, u} F_M(Y, S^{m+k}_M),$$

where $F_M(Y, S^{m+k}_M)$ is the function space of based M-maps from Y to S^{m+k}_M.

Observe that the action of M^{op} on S^{m+k}_M induces an action of M^{op} on the function space, and $\mathrm{adj}_k\, u$ is M^{op}-equivariant with respect to this choice.

1.3.1. Definition (*Cf. Vogell* [4, 1.10])**.** Suppose that $Y \in \mathbb{C}(M)$ and $Z \in \mathbb{C}(M^{\mathrm{op}})$ are finitely dominated. We say that an m-pairing

$$Y \wedge Z \xrightarrow{u} S^m_M$$

is a *duality* (more precisely, *m-duality*) provided that there exists a non-negative integer ε (possibly depending on m, Y and Z, but not on k), such that the map

$$\Sigma^k Z \xrightarrow{\mathrm{adj}_k\, u} F_M(Y, S^{m+k}_K)$$

defined above is $(2k - \varepsilon)$-connected, for all $k \geqslant 0$ sufficiently large.

The following is implicit in [4]—we omit the details.

1.3.2. Lemma. (1) *Duality pairings are compatible with suspension, i.e., if* $Y \wedge Z \to S^m_M$ *is a duality map, then so are the maps*

$$(\Sigma Y) \wedge Z \to S^{m+1}_M \quad \text{and} \quad Y \wedge (\Sigma Z) \to S^{m+1}_M$$

given by suspending once and shuffling the suspension coordinate.

(2) *An m-pairing* $u: Y \wedge Z \to S^m_M$ *of finitely dominated objects is a duality if and only if the induced map*

$$Q(Z) \to F_M(Y, Q(S^m_M))$$

is a weak homotopy equivalence, where $Q(-) = \Omega^\infty \Sigma^\infty$ *denotes ordinary stable homotopy.*

(3) *The duality condition is symmetric in the following sense: If* $u: Y \wedge Z \to S^m_M$ *is a duality, then so is*

$${}^t u : Z \wedge Y \xrightarrow{\tau} Y \wedge Z \xrightarrow{u} S^m_M \overset{\iota}{\cong} S^m_{M^{\mathrm{op}}},$$

where τ *is the map which permutes factors.*

It was proved in [4] that a homotopy finite object admits a homotopy finite dual (in the simplicial setting). The steps of the proof are as follows:

1. It is enough to show that a finite object admits a homotopy finite dual.
2. The map $S^0_M \wedge S^0_{M^{\mathrm{op}}} \to S^0_M$ given by $(m, n^{\mathrm{op}}) \mapsto mn$ is a duality map (note that $S^0_M \wedge S^0_{M^{\mathrm{op}}}$ is just $M \times M^{\mathrm{op}}$ with an additional basepoint).
3. Suspending, we have for each pair of non-negative integers k and ℓ, duality maps
$$S^k_M \wedge S^\ell_{M^{\mathrm{op}}} \to S^{k+\ell}_M.$$
4. Every finite object is given by attaching a finite number of cells, starting with a point. Induction plus the previous step enables one to construct duals inductively via cell attachments.

In summary, we have

1.3.3. Lemma (Cf. Vogell [4, 1.13]). *Let* $Y \in \mathbb{C}(M)$ *be a homotopy finite object. Then there exists a non-negative integer m, an object* $Z \in \mathbb{C}_{\mathrm{hf}}(M^{\mathrm{op}})$ *and a duality map*

$$Y \wedge Z \to S^m_M.$$

The next result is an extension of the previous one in that we show that a finitely dominated object of $\mathbb{C}(M)$ possesses a finitely dominated dual.

1.3.4. Lemma. *Let* $Y \in \mathbb{C}_{\mathrm{fd}}(M)$ *be an object. Then there exists an positive integer m, an object* $Z \in \mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})$ *and a duality map*

$$Y \wedge Z \xrightarrow{u} S^m_M.$$

Proof. If $f : A \to X$ is a morphism of $\mathbb{C}_{\mathrm{fd}}(M)$ and $u : A \wedge A^* \to S^m_M$ and $v : X \wedge X^* \to S^m_M$ are duality maps, then f induces an *umkehr map*

$$f^! : \Sigma^k X^* \to \Sigma^k A^*$$

(for k sufficiently large) which is unique up to homotopy. The morphism $f^!$ is obtained by applying elementary obstruction theory and the definition of duality (Definition 1.3.1) to solve the factorization problem

$$\begin{array}{ccc} \Sigma^k X^* & \xrightarrow{\mathrm{adj}_k v} & F_M(X, S^{m+k}_M) \\ {\scriptstyle f^!}\,\vdots & & \downarrow{\scriptstyle f^*} \\ \Sigma^k A^* & \xrightarrow[\mathrm{adj}_k u]{} & F_M(A, S^{m+k}_M)\,, \end{array}$$

where f^* is the map induced by f on function spaces.

If Y is finitely dominated, then there exists a homotopy finite object K, a cofibration $i : Y \rightarrowtail K$, and a morphism $r : K \to Y$ such that $r \circ i$ is the identity. Observe that the composite $i \circ r : K \to K$ is idempotent.

By Lemma 1.3.3 there exists a duality map $v : K \wedge L \to S^m_M$ if m is sufficiently large. Then the umkehr map

$$(i \circ r)^! : \Sigma^k L \to \Sigma^k L$$

is equivariantly homotopy idempotent (if k is large). Fix therefore a large integer k, and let Z denote the homotopy colimit of the sequence

$$\Sigma^k L \xrightarrow{(i \circ r)^!} \Sigma^k L \xrightarrow{(i \circ r)^!} \cdots .$$

Then Z is finitely dominated (since it is a homotopy retract of $\Sigma^k L$) and the duality map v determines a duality $u : Y \wedge Z \to S^{m+k}_M$, which is canonically defined up to equivariant homotopy. □

2. Filtered duality

2.1. Filtered objects. A *filtered object* of length n of $\mathbb{C}_{\mathrm{fd}}(M)$ is defined to be a sequence of cofibrations

$$A := A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$

of $\mathbb{C}_{\mathrm{fd}}(M)$. For $0 < i < j \leqslant n$, there are associated quotients

$$A_{i,j} := A_j / A_i,$$

which are well defined *up to isomorphism* (cf. [5, 1.1]; we will later include the choice of quotients when defining K-theory). A_0 will denote the zero object ($= *$). A *morphism* $A \to A'$ of filtered objects length n is a compatible collection of morphisms $A_i \to A'_i$ for $i \leqslant n$. A *weak equivalence* is a morphism $\to A'$ such that $A_i \to A'_i$ is a weak equivalence for all i.

2.2. Definition of filtered duality. We shall describe what it means to "dualize" a filtered object. Suppose that we have another filtered object

$$B{:=}B_1 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n,$$

this time of $\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})$.

2.2.1. Definition. A *filtered m-pairing* for A and B is an $(M \times M^{\mathrm{op}})$-equivariant map

$$u: A_n \wedge B_n \to S^m_M$$

such that its restriction to $A_i \wedge B_j$ is the trivial map to the basepoint whenever $i+j \leqslant n$ (for all $0 \leqslant i, j \leqslant n$).

(Since there is a cofibration $B_j \rightarrowtail B_{n-i}$, for $i+j \leqslant n$, a map $u: A_n \wedge B_n \to S^m_M$ is a filtered m-pairing if and only if its restriction $A_i \wedge B_{n-i}$ is trivial for all i.)

For a filtered m-pairing $u: A_n \wedge B_n \to S^m_M$, and integers i and j such that $0 \leqslant i < j \leqslant n$, there is an associated map

$$A_{i,j} \wedge B_{n-j,n-i} \xrightarrow{u_{i,j}} S^m_M$$

which is defined by the recipe

- restrict u to $A_j \wedge B_{n-i}$,
- observe that u restricted further to

  $$A_j \wedge B_{n-j} \bigcup_{A_i \wedge B_{n-j}} A_i \wedge B_{n-i}$$

  is trivial,
- take the induced map on the quotient

$$A_{i,j} \wedge B_{n-j,n-i} = (A_j \wedge B_{n-i}) \Bigg/ \left(A_j \wedge B_{n-j} \bigcup_{A_i \wedge B_{n-j}} A_i \wedge B_{n-i} \right).$$

2.2.2. Definition. A filtered m-pairing $u: A_n \wedge B_n \to S^m_M$ for A and B is said to be a *filtered duality* (more precisely *filtered m-duality*) provided that the induced maps

$$u_{i,j}: A_{i,j} \wedge B_{n-j,n-i} \to S^m_M$$

are duality maps, for all $0 \leqslant i < j \leqslant n$.

2.2.3. Remark. (i) To check that a filtered m-pairing $u: A_n \wedge B_n \to S^m_M$ is also a filtered duality, it is enough to check that duality holds on adjacent indices, i.e., the maps

$$u_{i,i+1}: A_{i,i+1} \wedge B_{n-i-1,n-i} \to S^m_M$$

are duality maps for $0 \leqslant i \leqslant n-1$. This assertion follows from an induction in j using the cofibration sequences $A_{i,j} \rightarrowtail A_{i,j+1} \to A_{j,j+1}$ and $B_{n-j-1,n-j} \rightarrowtail B_{n-j-1,n-i} \to B_{n-j,n-i}$.

(ii) The condition for u to be a filtered duality can be rephrased in yet another way. Set $B^{(i)}:=B_{n-i,n}$, and write

$$u_i : A_i \wedge B^{(i)} \to S^m_M$$

for the map $u_{0,i}$. Then u is a filtered duality if (and only if) u_i is a duality for $0 \leqslant i \leqslant n$. This follows by induction using the cofibration sequences $A_i \rightarrowtail A_j \to A_{i,j}$ and $B_{n-j,n-i} \rightarrowtail B^{(j)} \to B^{(i)}$.

We now have the filtered analogue of Lemma 1.3.4.

2.2.4. Proposition. *Let $A = (A_1 \rightarrowtail \cdots \rightarrowtail A_n)$ be a filtered object of $\mathbb{C}_{\mathrm{fd}}(M)$. If m is sufficiently large, then there exists a filtered object $B = (B_1 \rightarrowtail \cdots \rightarrowtail B_n)$ of $\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})$ and also a filtered duality map $u : A_n \wedge B_n \to S^m_M$.*

Proof. One proves this by induction on n. When $n = 1$ this is just Lemma 1.3.4. To avoid notational clutter, we will give the argument in the case $n = 2$ and omit the general case (which is similar).

Consider the filtered object $A_1 \rightarrowtail A_2$. Applying Lemma 1.3.4 to A_1, we obtain an equivariant duality map $A_1 \wedge C \to S^m_M$ for some choice of $C \in \mathbb{C}_{\mathrm{fd}}(M)$ and m sufficiently large. Applying Lemma 1.3.4 again, this time to A_2, we can find an equivariant duality map $A_2 \wedge Z \to S^m_M$.

Let $i : A_1 \rightarrowtail A_2$ denote the inclusion map. Then there is an umkehr map

$$\Sigma^j Z \xrightarrow{i^!} \Sigma^j C$$

provided that j is sufficiently large (cf. the proof of Lemma 1.3.4). Let U denote the mapping cylinder of $i^!$. Then there is a factorization of $i^!$ as

$$\Sigma^j Z \rightarrowtail U \xrightarrow{\sim} \Sigma^j C.$$

Let B_1 denote the quotient $U/\Sigma^j Z$, and let B_2 denote the map given by taking the mapping cone of the quotient map $U \to B_1$. Then we have a cofibration $B_1 \rightarrowtail B_2$. Observe that B_2 is weak equivalent to $\Sigma^{j+1} Z$ and that B_2/B_1 is weak equivalent to $\Sigma^{j+1} C$.

By construction of B_1 and B_2, the duality maps $A_1 \wedge C \to S^m_M$ and $A_2 \wedge Z \to S^m_M$ induce duality maps $A_1 \wedge (B_2/B_1) \to S^{m+j+1}_M$ and $A_2 \wedge B_2 \to S^{m+j+1}_M$, in such a way that the diagram of adjunctions

$$\begin{array}{ccc} A_1 & \longrightarrow & F_M(B_2/B_1, S^{m+j+1}_M) \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & F_M(B_2, S^{m+j+1}_M) \end{array}$$

is equivariantly homotopy commutative. Since $A_1 \rightarrowtail A_2$ is a cofibration, we can use the equivariant homotopy extension property to deform the map $A_2 \to F_M(B_2, S^{m+j+1}_M)$ through morphisms of $\mathbb{T}(M)$ to a new map so that the diagram becomes strictly commutative with respect to the new map. The adjoint of the new map is then a filtered duality $A_2 \wedge B_2 \to S_M^{m+j+1}$ (the triviality condition in Definition 2.2.1 satisfied since in this case it amounts to checking that the restriction of the map $A_2 \wedge B_2 \to S_M^{m+j+1}$ to $A_1 \wedge B_1$ is the constant map to the basepoint; this holds because the restriction to $A_1 \wedge B_2$ factors through $A_1 \wedge B_2/B_1$). □

3. The comparison theorem

3.1. Given integers $m, n \geqslant 0$, let

$$h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)$$

be the category in which an *object* is given by a triple

$$(A, B, u)$$

in which

- A is a filtered object of length n of $\mathbb{C}_{\mathrm{fd}}(M)$;
- B is a filtered object of length n of $\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})$;
- $u : A_n \wedge B_n \to S_M^m$ is a filtered m-duality.

A *morphism* $(A, B, u) \to (A', B', u')$ is specified by a weak equivalence (of filtered objects) $f : A \to A'$ and a weak equivalence $g : B' \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A_n \wedge B'_n & \xrightarrow{f\wedge \mathrm{id}} & A'_n \wedge B'_n \\ {\scriptstyle \mathrm{id}\wedge g}\downarrow & & \downarrow{\scriptstyle u'} \\ A_n \wedge B_n & \xrightarrow[u]{} & S_M^m. \end{array}$$

Notice that g is an umkehr map for f.

For non-negative integers k and ℓ, the *suspension* functor

$$h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M) \xrightarrow{\Sigma^{k,\ell}} h\mathscr{D}_{k+m+\ell}\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)$$

is defined by

$$(A, B, u) \mapsto (\Sigma^k A, \Sigma^\ell B, \Sigma^{k,\ell} u),$$

where

- $\Sigma^k A$ denotes the filtered object which is given by k-fold suspension of the terms of A and $\Sigma^\ell B$ is defined similarly;
- the filtered duality map $\Sigma^{k,\ell} u$ is the composite

$$(\Sigma^k A_n) \wedge (\Sigma^\ell B_n) \cong S^k \wedge (A_n \wedge B_n) \wedge S^\ell \xrightarrow{\mathrm{id}\wedge u \wedge \mathrm{id}} S^k \wedge S_M^m \wedge S^\ell \cong S_M^{k+m+\ell},$$

where the first homeomorphism is given by permuting S^ℓ with B_n in the smash product, and the last homeomorphism is given by concatenation.

3.2. Let $h\mathscr{D}\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)$ be the category which is defined by taking the colimit with respect to the suspension maps $\Sigma^{1,1}$, i.e.,

$$h\mathscr{D}\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M) := \lim_{\substack{m\to\infty \\ (\Sigma^{1,1})}} h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M).$$

Let $h\mathcal{S}_n\mathbb{C}_{\rm fd}(M)$ denote the category whose *objects* are filtered objects of $\mathbb{C}_{\rm fd}(M)$ of length n and whose *morphisms* are weak equivalences of filtered objects.

Define the *left forgetful functor*

$$h\mathcal{D}_m\mathcal{S}_n\mathbb{C}_{\rm fd}(M)\xrightarrow{\phi_L}h\mathcal{S}_n\mathbb{C}_{\rm fd}(M)$$

by $(A,B,u)\mapsto A$. Then we have

$$\phi_L\circ\Sigma^{1,1}=\Sigma\circ\phi_L,$$

where Σ on the right-hand side is induced by the suspension functor on the category $\mathbb{C}_{\rm fd}(M)$.

Taking the colimit with respect to the indexing sequence defined by suspension, we have an induced map

$$|h\mathcal{D}\mathcal{S}_n\mathbb{C}_{\rm fd}(M)|\xrightarrow{\phi_L}\left|\lim_{\to(\Sigma)}h\mathcal{S}_n\mathbb{C}_{\rm fd}(M)\right|.$$

The following is a variant of [4, 1.15].

3.2.1. Theorem (Comparison Theorem). *The map*

$$|h\mathcal{D}\mathcal{S}_n\mathbb{C}_{\rm fd}(M)|\xrightarrow{\phi_L}\left|\lim_{\to(\Sigma)}h\mathcal{S}_n\mathbb{C}_{\rm fd}(M)\right|$$

is a homotopy equivalence (*of unequivariant spaces*).

Proof. The goal is to use Theorem A of [3] to deduce the result. The strategy is to show that suitable categories arising as the right fiber are contractible. Our method of proof is similar to the proof of [5, 1.6.7].

Let A and B be categories, $f:A\to B$ a functor, and let $b\in B$ be an object. Recall that the *right fiber* over b is the category $b\setminus f$ whose objects are given by maps $x:b\to f(a)$. A morphism

$$(b\xrightarrow{x}f(a))\to(b\xrightarrow{y}f(a'))$$

is specified by a map $s:a\to a'$ such that x followed by $f(s):f(a)\to f(a')$ coincides with y. To show that $b\setminus f$ is contractible after realization, it is sufficient to show that every simplicial map $X\to N(b\setminus f)$ is null homotopic, where $Nb\setminus f$ denotes the nerve of the category $b\setminus f$ and where X ranges through *non-singular* simplicial sets (recall that a non-singular simplicial set is a simplicial set such that for each non-degenerate k-simplex the representing map $\Delta[k]\to X$ is an embedding).

To each simplicial set X', let $\operatorname{simp}X'$ denote its category of simplices. The 'last vertex' map $N\operatorname{simp}X'\to X'$ is always a weak homotopy equivalence, where $N\operatorname{simp}X'$ denotes the nerve of $\operatorname{simp}X'$. To each non-singular simplicial set X, let $\operatorname{simp}^{\rm nd}X$ denote the partially ordered set of non-degenerate simplices. The inclusion map $\operatorname{simp}^{\rm nd}X\to\operatorname{simp}X$ is an equivalence after realization (see e.g. [5, p. 359]).

Given a map $X\to N(b\setminus f)$, there is an associated sequence of functors

$$\operatorname{simp}^{\rm nd}X\xrightarrow{\simeq}\operatorname{simp}X\to\operatorname{simp}N(b\setminus f)\xrightarrow{\simeq}b\setminus f$$

and also a commutative diagram of simplicial sets

$$\begin{array}{ccc} N\,\mathrm{simp}\,X & \longrightarrow & N\,\mathrm{simp}(N(b\setminus f)) \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ X & \longrightarrow & N(b\setminus f) \end{array}$$

(cf. [5, p. 355]). Consequently, a map $X \to N(b \setminus f)$ is null homotopic if the induced map $\mathrm{simp}^{\mathrm{nd}} X \to b \setminus f$ is null homotopic. From this we see that to prove $b \setminus f$ is contractible, is sufficient to prove that a diagram D in $b \setminus f$ possesses a 'cone point' when D ranges through all finite partially ordered sets. By a *cone point*, we mean a chain of natural transformations to a constant diagram.

To prove the theorem, it is sufficient to show that the right fiber $Y \setminus \phi_L$ of the functor $\phi_L : h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M) \to h\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)$ becomes contractible when m tends to ∞. By Proposition 2.2.4, $Y \setminus \phi_L$ is non-empty if m is sufficiently large. For the moment, choose such an integer m.

Suppose that D is any finite, partially ordered diagram in the right fiber. Then D is represented by data of the kind $\{A^\alpha, B^\alpha, u^\alpha\}_{\alpha\in D}$ together with compatible maps $Y \to A^\alpha$. At the expense of varying D up to objectwise weak equivalence, and possibly increasing the value of m, we will show how to find a cone point for D in three steps. In order to avoid notational clutter, for the rest of the proof we will avoid specifying the duality maps when referring to vertices of the diagram. Thus, a vertex of D is to be denoted by (A^α, B^α).

Step 1: Replace D up to objectwise weak equivalence by the diagram whose vertices are

$$(Y, B^\alpha)$$

with associated filtered duality map given by the composite

$$Y_n \wedge B^\alpha_n \to A^\alpha_n \wedge B^\alpha_n \to S^m_M.$$

The edges of the new diagram are evident. By abuse of notation, we denote the new diagram by the same symbol D.

Step 2: For each index α, we set

$$V^\alpha := \operatorname*{hocolim}_{\alpha \leqslant \gamma} B^\gamma,$$

where the homotopy colimits are indexed by the sub-poset of objects $\geqslant \alpha$. Then we have a new diagram

$$D_0 := \{(Y, V^\alpha)\}_\alpha$$

such that D_0 maps to D by objectwise weak equivalence.

Step 3: Set

$$B = \operatorname*{colim}_{\alpha} V^\alpha.$$

By construction, the map from the *homotopy colimit* of the V^α to B is a weak equivalence. Moreover, B has the structure of a filtered object, and there is a map $B_n \to$

$F_M(Y_n, S_M^m)$ because there is a compatible family of maps $V_n^\alpha \to F_M(Y_n, S_M^m)$ (the adjoints to the given duality pairings). We *do not* assert that the adjoint $B_n \wedge Y_n \to, S_M^m$ is a filtered duality map. However, at the expense of increasing m, we can map B to a filtered dual for Y.

To see this, let Z denote any filtered dual for Y. Note that B_n is a finitely dominated object (since it is weak equivalent to the homotopy colimit of a finite diagram of finitely dominated objects). Consider the lifting problem

$$\begin{array}{ccc} & & B_n \\ & \swarrow^{\cdots} & \downarrow \\ Z_n & \longrightarrow & F(Y_n, S_M^m)\,. \end{array}$$

At the expense of suspending B_n and Z_n (and increasing m) a suitable number of times, the bottom map can be made highly connected in such a way that its connectivity exceeds the dimension of some finite domination of B_n (suspended that many times). Let us assume that this has been achieved.

By obstruction theory, we can fill in the dotted arrow up to homotopy. Moreover, the dotted arrow can be taken as a morphism $B \to Z$ (we omit the proof; it is a straightforward, albeit tedious induction akin to the one appearing in Proposition 2.2.4). Also, at the expense of replacing Z by a suitable mapping cylinder, we can assume that the above diagram is strictly commutative. Assume that all of this has been done.

Then the foregoing manipulations yield a compatible family of morphisms

$$(Y, B^\alpha) \to (Y, Z).$$

Consequently (Y, Z) is a cone point for D.

Thus as m tends to ∞ the right fiber becomes contractible, as claimed. Applying [3, Theorem A] then completes the proof. □

4. The $\mathscr{D}\mathscr{S}.$-construction

4.1. We now let the n vary and show how to modify $h\mathscr{D}\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)$ so that it becomes a category in degree n of a simplicial category. This is done by including choices of quotients for the filtered object data. Let us first recall the definition of the $\mathscr{S}.$-construction.

4.1.1. Definition. For a category with cofibrations and weak equivalences in the sense of [5], we let $h\mathscr{S}_n C$ be the category in which an *object* consists of

- A filtered object of C of length n

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$

  and

- for each $0 < i < j \leqslant n$, a specified choice of quotient object

$$A_{i,j} := A_j/A_i.$$

A *morphism* $A \to B$ of $h\mathcal{S}_n C$ is defined to be a compatible collection of weak equivalences $A_i \to B_i$ for $i \leqslant n$.

If A is a filtered object of length n, define $d_i(A) \in h\mathcal{S}_{n-1}C$ to be the object which for $i > 0$ is given by deleting A_i from the sequence which defines A. If $i = 0$, then $d_0(A)$ is defined to be

$$A_2/A_1 \rightarrowtail A_3/A_1 \rightarrowtail \cdots \rightarrowtail A_n/A_1.$$

Let $s_i(A) \in h\mathcal{S}_{n+1}C$ be the object given by inserting A_i into the sequence which defines A at stage i if $i > 0$; if $i = 0$, we let $s_0(A)$ be the object

$$* \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n.$$

The above equips the disjoint union of the $h\mathcal{S}_n C$ with the structure of a simplicial category (cf. [5, 1.3]). Its realization $|h\mathcal{S}_\bullet C|$ is the *$\mathcal{S}_\bullet$-construction* of C. The loop space $\Omega|h\mathcal{S}_\bullet C|$ is the *K-theory* of C ([5, p. 330]).

4.2. Let

$$h\mathcal{D}_m\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)$$

denote the simplicial category which in simplicial degree n is the category consisting of triples

$$(A, B, u),$$

where

- A is a filtered object of length n of $\mathbb{C}_{\mathrm{fd}}(M)$ together with a specified choice of quotient object $A_{i,j} = A_j/A_i$ for $0 < i < j \leqslant n$;
- B is a filtered object of length n of $\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})$ together with a specified choice of quotient object $B_{i,j} = B_j/B_i$ for $0 < i < j \leqslant n$;
- $u : A_n \wedge B_n \to S^m_M$ is a filtered m-duality.

Thus in simplicial degree n what is described above amounts to the category denoted by $h\mathcal{D}_m\mathcal{S}_n\mathbb{C}_{\mathrm{fd}}(M)$ in previous sections, except that now we are including the choices of quotient objects.

Henceforth, $h\mathcal{D}_m\mathcal{S}_n\mathbb{C}_{\mathrm{fd}}(M)$ stands for the category which includes the choice of quotient data.

The simplicial category structure is more-or-less determined by demanding that the forgetful map

$$h\mathcal{D}_m\mathcal{S}_n\mathbb{C}_{\mathrm{fd}}(M) \to h\mathcal{S}_n\mathbb{C}_{\mathrm{fd}}(M) \times h\mathcal{S}_n\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})^{\mathrm{op}}$$

$$(A, B, u) \mapsto (A, B)$$

be simplicial in the A-variable and *anti*-simplicial in the B-variable (for the definition of anti-simplicial maps, see 5.2 below).

Explicitly, define the jth *degeneracy* functor

$$s_j : h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M) \to h\mathscr{D}_m\mathscr{S}_{n+1}\mathbb{C}_{\mathrm{fd}}(M)$$

by $s_j(A,B,u) = (s_j(A), s_{n-j}(B), u)$ for all j (where $s_j(A)$ and $s_{n-j}(B)$ are defined as above, and $u : A_n \wedge B_n \to S^n_M$ is now considered to be a filtered duality map for filtered objects of length $n+1$).

Define the jth *face* functor

$$d_j : h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M) \to h\mathscr{D}_m\mathscr{S}_{n-1}\mathbb{C}_{\mathrm{fd}}(M)$$

by $d_j(A,B,u) := (d_j(A), d_{n-j}(B), d_j(u))$, where the description of $d_j(u)$ will require a case distinction: $\neq 0,\ n; 0$ or n.

- if $j \neq 0, n$, then $d_j(u) := u$;
- $d_0(u)$ is the filtered duality map $u_{1,n} : A_{1,n} \wedge B_{0,n-1} \to S^m_M$ induced by $u : A_n \wedge B_n \to S^m_M$ (where $B_{n-1} := B_{0,n-1}$);
- $d_n(u)$ is the filtered duality map $u_{0,n-1} : A_{0,n-1} \wedge B_{1,n} \to S^m_M$ induced by u.

We omit the verification of the simplicial identities. In summary, we have

4.2.1. Lemma. *With the face and degeneracy functors described above, $h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)$ is a simplicial category.*

4.3. The suspension functors $\Sigma^{k,\ell}$ extend in this context to simplicial functors

$$h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M) \to h\mathscr{D}_{k+m+\ell}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M).$$

Taking the colimit of the indexing sequence defined by $\Sigma^{1,1}$, we obtain a simplicial category

$$h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M).$$

4.3.1. Theorem. *The left forgetful functor ϕ_L defines a homotopy equivalence of unequivariant spaces*

$$|h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)| \xrightarrow{\simeq} \left|\lim_{\to(\Sigma)} h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)\right| \simeq |h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)|.$$

Proof. In each simplicial degree n, we have by Theorem 3.2.1 a homotopy equivalence of unequivariant spaces

$$|h\mathscr{D}\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)| \xrightarrow{\simeq} \left|\lim_{\to(\Sigma)} h\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)\right|$$

induced by ϕ_L (the extra choice of quotient data does not change the homotopy type of the realization of the categories in question). The theorem now follows by the realization lemma (the second homotopy equivalence results from the observation that suspension induces a homotopy equivalence on $\mathscr{S}_\bullet$-constructions by Waldhausen [5, 1.6.2]). □

5. Dualization

5.1. Define a contravariant functor

$$h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)\xrightarrow{T_n^M}h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})$$

by

$$(A,B,u)\mapsto(B,A,{}^tu)$$

where we recall that ${}^tu : B_n\wedge A_n\to S^m_{M^{\mathrm{op}}}$ means the filtered duality map given by the composite

$$B_n\wedge A_n\xrightarrow{\tau}A_n\wedge B_n\xrightarrow{u}S^m_M\xrightarrow{\iota}S^m_{M^{\mathrm{op}}},$$

in which τ is the map which switches factors and ι is the homeomorphism defined in (1.2.1). Call T_n^M the *dualization functor.*

The composite $T_n^{M^{\mathrm{op}}}\circ T_n^M$ is given by

$$(A,B,u)\mapsto(B,A,{}^tu)\mapsto(A,B,{}^t({}^tu))=(A,B,u)$$

and is therefore the identity. Hence,

5.1.1. Lemma. *The contravariant functor T_n^M is invertible with inverse $T_n^{M^{\mathrm{op}}}$. Furthermore, T_n^M is compatible with suspension in the sense that*

$$\Sigma^{k,\ell}\circ T_n^M=T_n^M\circ\Sigma^{\ell,k}.$$

(*In particular, T_n^M commutes with $\Sigma^{1,1}$.*)

As n is allowed to vary, the T_n^M assemble to define an *anti*-simplicial, contravariant functor

$$h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)\xrightarrow{T_\bullet^M}h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}}),$$

where the term *anti-simplicial* means that

$$T_{n-1}^M(d_i(c))=d_{n-i}(T_n^M(c))\quad\text{and}\quad T_{n+1}^M(s_i(c))=s_{n-i}(T_n^M(c))$$

for all objects $c\in h\mathscr{D}_m\mathscr{S}_n\mathbb{C}_{\mathrm{fd}}(M)$.

By Lemma 5.1.1, we have that $T_\bullet^M$ commutes with the suspension functor $\Sigma^{1,1}$, and therefore induces an anti-simplicial, contravariant functor on colimits

$$h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)\xrightarrow{T_\bullet^M}h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}}).$$

5.2. Before proceeding any further, we need to explain how anti-simplicial maps give rise to maps of spaces on realization.

The topological standard simplex Δ^n has vertices given by the ordered set $\{0<1<\cdots<n\}$, and therefore comes equipped with a homeomorphism $\phi_n:\Delta^n\xrightarrow{\cong}\Delta^n$ which is induced by linearly extending $i\mapsto n-i$.

If $f : X_\bullet \to Y_\bullet$ is an *anti*-simplicial map of simplicial sets (i.e., $f(s_i(x)) = s_{n-i}(f(x))$ and $f(d_i(x)) = d_{n-i}(f(x))$ for $x \in X_n$), then f induces a map of realizations $|X_\bullet| \to |Y_\bullet|$ which is obtained by gluing together the set maps $X_n \times \Delta^n \to Y_n \times \Delta^n$ defined by $(x, t) \mapsto (f(x), \phi_n(t))$.

In particular, if $C \to D$ is a contravariant functor, then we obtain an anti-simplicial map of nerves, and hence an induced map $|C| \to |D|$.

Let $f : C_\bullet \to D_\bullet$ be an anti-simplicial (covariant) functor of simplicial categories. The nerve $N_\bullet C_\bullet$ is a bi-simplicial set. The realization of $C_\bullet$ is constructed from the spaces

$$N_n C_k \times \Delta^n \times \Delta^k$$

modulo the gluing relations. Then f induces a map of spaces

$$|C_\bullet| \to |D_\bullet|$$

via the map $N_n C_k \times \Delta^n \times \Delta^k \to N_n D_k \times \Delta^n \times \Delta^k$ given by $(x, s, t) \mapsto (f(x), \phi_n(s), t)$.

We can also combine these constructions: If $f : C_\bullet \to D_\bullet$ is an *anti*-simplicial, *contravariant* functor, then f induces a map $|C_\bullet| \to |D_\bullet|$. In this case the map is induced by $(x, s, t) \mapsto (f(x), \phi_n(s), \phi_k(t))$.

5.3. Applying the proceeding paragraph to the (anti-simplicial, contravariant) dualization functor $T_\bullet^M$, we obtain a map of based spaces

$$|h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M)| \xrightarrow{T^M} |h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(M^{\mathrm{op}})|,$$

which we term the *dualization map*. We now list the properties of T^M, which are an immediate consequence of the definitions and Lemma 5.1.1.

5.3.1. Lemma. *The dualization map T^M is a homeomorphism whose inverse is $T^{M^{\mathrm{op}}}$. If M is the realization of a simplicial group, then $T^M = T^{M^{\mathrm{op}}}$ with respect to the identification $M \cong M^{\mathrm{op}}$. Hence, T^M is an involution in this case (i.e., $T^M \circ T^M = id$).*

6. The canonical involution

6.1. Actions on loop coordinates. If Y is a based $\mathbb{Z}/2$-space, then its loop space can be equipped with two different $\mathbb{Z}/2$-actions:

- Let $\underset{\sim}{S}^1$ denote the circle with based action given by reflection. Define

  $$\underset{\sim}{\Omega} Y := \mathrm{Map}_*(\underset{\sim}{S}^1, Y),$$

  where the mapping space is given the $\mathbb{Z}/2$ action defined by conjugation of functions: $(\tau * f)(x) = \tau f(\tau x)$.
- The action given by letting $\mathbb{Z}/2$ act trivially on the loop coordinate: $(\tau * f)(x) = \tau f(x)$. We denote this $\mathbb{Z}/2$-space by ΩY.

6.2. Definition of the canonical involution. We now define a $\mathbb{Z}/2$-equivariant model for the functor $X \mapsto A^{\mathrm{fd}}(X)$. Let X be a connected based space. Let $G_{\bullet}$ denote the Kan loop group of the simplicial total singular complex of X. Setting $G{:=}|G_{\bullet}|$, we have that the classifying space BG is weak homotopy equivalent to X.

Using the involution on $|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$ defined by the dualization map T^G, we now have the definition of the 'canonical involution':

6.2.1. Definition. Let $\mathscr{D}A^{\mathrm{fd}}(X)$ denote the based $\mathbb{Z}/2$-space

$$\underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|.$$

6.2.2. Corollary. *There is a homotopy equivalence of based* (*unequivariant*) *spaces*

$$\mathscr{D}A^{\mathrm{fd}}(X) \xrightarrow{\simeq} A^{\mathrm{fd}}(X).$$

Proof. A model for $A^{\mathrm{fd}}(X)$ is given by $\Omega|h\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$ (cf. [1, 1.8(3)]). The result is therefore a consequence of Theorem 4.3.1. □

6.3. Comparison with the involution of [4]. We now discuss without details how the $\mathbb{Z}/2$-space $\mathscr{D}A^{\mathrm{fd}}(X)$ defined above relates to one of the descriptions of the 'canonical involution' on $A(X)$ which was given in [4].

Consider the category $h\mathscr{D}\mathbb{C}^n(M)$ whose *objects* are triples (Y, Z, u) such that

- Y is weak equivalent to a k-fold wedge of S^n_M, for some (unspecified) non-negative integer k.
- Z is weak equivalent to a k-fold wedge of $S^n_{M^{\mathrm{op}}}$, and
- $u: Y \wedge Z \to S^{2n}_M$ is a duality map.

One defines a *morphism* so that $h\mathscr{D}\mathbb{C}^n(M) \subset h\mathscr{D}_{2n}\mathscr{S}_1\mathbb{C}_{\mathrm{fd}}(M)$ is a full subcategory.

When M is the realization of a simplicial group, the involution on $h\mathscr{D}\mathbb{C}^n(M)$ is defined by $(A, B, u) \mapsto (B, A, {}^t u)$. The suspension functor

$$h\mathscr{D}\mathbb{C}^n(M) \xrightarrow{\Sigma^{1,1}} h\mathscr{D}\mathbb{C}^{n+1}(M)$$

given by $(A, B, u) \mapsto (\Sigma A, \Sigma B, \Sigma^{1,1} u)$ is therefore equivariant.

There is also a based equivariant map

$$|h\mathscr{D}\mathbb{C}^n(M)| \to \underset{\sim}{\Omega}|h\mathscr{D}_{2n}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(M)|,$$

which is induced by inclusion of the '1-skeleton' (cf. [5, p. 329]). The latter is compatible with suspension $\Sigma^{1,1}$, and therefore defines upon passage to limits an equivariant map

$$\lim_n |h\mathscr{D}\mathbb{C}^n(M)| \to \underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(M)|.$$

The source of this last map has the structure of a topological monoid whose multiplication is induced by the categorical sum operation. The involution of [4] was defined by taking the induced involution on the group completion

$$\lim_n \Omega B|h\mathscr{D}\mathbb{C}^n(M)|.$$

It is known that $A(X)$ (with $M =$ realization of the Kan loop group of the total singular complex of X) is homotopy equivalent to the latter by means of the left forgetful functor (cf. [4, 1.16]). In particular, on homotopy groups in positive degrees, our involution agrees with the one of [4].

7. Duality on the projective line

The aim of this section is to do for the projective line category $\mathbb{P}_{\mathrm{fd}}(G)$ of [1] what was done above for the category $\mathbb{C}_{\mathrm{fd}}(M)$.

We begin by recalling the definition of the projective line. We next define duality for objects of the projective line, and following that, we consider the case of filtered objects. We conclude the section with the analogue of Theorem 4.3.1.

7.1. The projective line revisited. Let $\mathbb{N}_-$ denote the monoid of negative natural numbers (including 0) with generator t^{-1}, and let $\mathbb{N}_+$ denote the monoid of positive natural numbers with generator t. In the sequel, we will be using the identification

$$\mathbb{N}_+ \cong \mathbb{N}_-^{\mathrm{op}},$$

which is induced by $t \mapsto (t^{-1})^{\mathrm{op}}$.

Let G be the realization of a simplicial group $G_\bullet$. If $M_\bullet$ denotes the simplicial monoid $G_\bullet \times \mathbb{N}_-$ and $M = |M_\bullet|$, then $M = G \times \mathbb{N}_-$, and $M^{\mathrm{op}} = G \times \mathbb{N}_+$.

If $U \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+)$ is an object, we may associate to it its *telescope* $U(t^{-1}) \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})$, given by taking the categorical colimit of the $\mathbb{Z}$-indexed sequence

$$\cdots \xrightarrow{t} U \xrightarrow{t} U \xrightarrow{t} \cdots,$$

where $t : U \to U$ denotes the translation by t map. Observe that the inclusion $U \subset U(t^{-1})$ is $(G \times \mathbb{N}_+)$-equivariant. If t acts by homeomorphism, then the inclusion is also an isomorphism.

Similarly, an object $V \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ has a telescope $V(t) \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})$ given by taking the colimit of the sequence

$$\cdots \xrightarrow{t^{-1}} V \xrightarrow{t^{-1}} V \xrightarrow{t^{-1}} \cdots.$$

Let $\mathbb{P}_{\mathrm{fd}}(G)$ be the category whose *objects* are diagrams

$$Y_- \to Y \leftarrow Y_+$$

in which $Y_- \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)$, $\in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})$ and $Y_+ \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+)$, and where the maps $Y_- \to Y$ and $Y_+ \to Y$ are required to be based and equivariant (where we restrict the action of $G \times \mathbb{Z}$ to its submonoids $G \times \mathbb{N}_{\pm}$). Moreover, the induced maps of telescopes

$$Y_{-}(t) \to Y(t) \cong Y \quad \text{and} \quad Y_{+}(t^{-1}) \to Y(t^{-1}) \cong Y$$

are required to be both cofibrations and weak equivalences of $\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})$.

We allow ourselves the liberty of specifying the object as the diagram $Y_{-} \to Y \leftarrow Y_{+}$, as the triple (Y_{-}, Y, Y_{+}), or as the corresponding lower case letter y. The terms $Y_{\pm}$ and Y are called the *components* of the object.

A *morphism* $(Y_{-}, Y, Y_{+}) \to (Z_{+}, Z, Z_{+})$ of $\mathbb{P}_{\mathrm{fd}}(G)$ consists of morphisms $Y_{-} \to Z_{-}$, $Y \to Z$ and $Y_{+} \to Z_{+}$ in such a way that the evident diagram is commutative. A *cofibration* is defined to be a morphism consisting of a triple of cofibrations with the additional property that the induced maps

$$Y \cup_{Y_{-}(t)} Z_{-}(t) \to Z \quad \text{and} \quad Y \cup_{Y_{-}(t^{-1})} Z_{+}(t^{-1}) \to Z$$

are cofibrations. A *weak equivalence* is defined to be a morphism such that each of its components is a weak homotopy equivalence of underlying spaces. The above conventions equip $\mathbb{P}_{\mathrm{fd}}(G)$ with the structure of a category with cofibrations and weak equivalences.

7.2. Definition of duality in the projective line. Consider a pair of objects $y = (Y_{-}, Y, Y_{+})$ and $z = (Z_{-}, Z, Z_{+})$ of $\mathbb{P}_{\mathrm{fd}}(G)$ equipped with a triple $u = (u_{-}, u_{\mathbb{Z}}, u_{+})$, the latter consisting of

- an m-pairing $u_{-} : Y_{-} \wedge Z_{+} \to S^m_{G \times \mathbb{N}_{-}}$,
- an m-pairing $u_{\mathbb{Z}} : Y \wedge Z \to S^m_{G \times \mathbb{Z}}$, and
- an m-pairing $u_{+} : Y_{+} \wedge Z_{-} \to S^m_{G \times \mathbb{N}_{+}}$.

These data are required to be *compatible* in the sense that the diagrams

$$\begin{array}{ccc} Y_{-} \wedge Z_{+} & \xrightarrow{\subset} & Y \wedge Z \\ {\scriptstyle u_{-}}\downarrow & & \downarrow{\scriptstyle u} \\ S^m_{G \times \mathbb{N}_{-}} & \xrightarrow{\subset} & S^m_{G \times \mathbb{Z}} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_{+} \wedge Z_{-} & \xrightarrow{\subset} & Y \wedge Z \\ {\scriptstyle u_{+}}\downarrow & & \downarrow{\scriptstyle u} \\ S^m_{G \times \mathbb{N}_{+}} & \xrightarrow{\subset} & S^m_{G \times \mathbb{Z}} \end{array}$$

are required to commute. We specify these data as a triple (y, z, u).

7.2.1. Definition. Given such a triple (y, z, u), one says that u is a *duality map* (more precisely, *m-duality map*) for y and z if each of the pairings u_{-}, $u_{\mathbb{Z}}$ and u_{+} is a duality map.

The following is obtained by applying Lemma 1.3.4 to each of the components of y. We omit the details.

7.2.2. Lemma. *If y is an object of $\mathbb{P}_{\mathrm{fd}}(G)$, then there exists an object z and an m-duality map $u = (u_{-}, u_{\mathbb{Z}}, u_{+})$ for y and z, provided that m is sufficiently large.*

The next step is to consider filtered duality in the context of the projective line. Let y and z denote filtered objects of $\mathbb{P}_{\mathrm{fd}}(G)$ of length n. By a *filtered duality* (more precisely, *filtered m-duality*) for y and z, we mean a compatible pairing $u=(u_-,u_{\mathbb{Z}},u_+)$ in which

- $u_-:(Y_-)_n\wedge(Z_+)_n\to S^m_{G\times\mathbb{N}_-}$ is a filtered duality for Y_- and Z_+,
- $u_{\mathbb{Z}}:Y_n\wedge Z_n\to S^m_{G\times\mathbb{Z}}$ is a filtered duality for Y and Z, and
- $u_+:(Y_+)_n\wedge(Z_-)_n\to S^m_{G\times\mathbb{N}_+}$ is a filtered duality for Y_+ and Z_-.

The following is obtained by applying Proposition 2.2.4 to the components of a filtered object y of the projective line (we again omit the details).

7.2.3. Lemma. *Let m be large. Given a filtered object y of length n of* $\mathbb{P}_{\mathrm{fd}}(G)$, *there exists another filtered object z of length n together with a filtered m-duality u for y and z.*

7.2.4. Definition. For integers $m,n\geqslant 0$, let $h\mathscr{D}_m\mathscr{S}_n\mathbb{P}_{\mathrm{fd}}(G)$ denote the category whose *objects* are specified by triples

$$(y,z,u),$$

where y and z are objects of $h\mathscr{S}_n\mathbb{P}_{\mathrm{fd}}(G)$ and $u=(u_-,u_{\mathbb{Z}},u_+)$ denotes a filtered m-duality for y and z.

A *morphism*

$$(y,z,u)\to(y',z',u')$$

is given by morphisms $y\to y'$ and $z'\to z$ of $h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)$ which are compatible with the duality data in the sense that they give rise to a morphism $(Y_-,Z_+,u_-)\to(Y'_-,Z'_+,u'_-)$ (of $h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G\times\mathbb{N}_-)$), a morphism $(Y,Z,u_{\mathbb{Z}})\to(Y',Z',u'_{\mathbb{Z}})$ and a morphism $(Y_+,Z_-,u_+)\to(Y'_+,Z'_-,u'_+)$.

As n varies, we obtain a simplicial category

$$h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)$$

by defining the face and degeneracy functors as follows: if (y,z,u) denotes an object in $h\mathscr{D}_m\mathscr{S}_n\mathbb{P}_{\mathrm{fd}}(G)$, then $s_i(y,z,u)=(s_i(y),s_{n-i}(z),u)$. Similarly, $d_i(y,z,u)$ is given by $(d_i(y),d_{n-i}(z),d_j(u))$ where if $u=(u_-,u_{\mathbb{Z}},u_+)$, then $d_j(u)$ denotes $(d_j(u_-),d_j(u_{\mathbb{Z}}),d_j(u_+))$, where $d_j(u_-)$, etc., is defined above in 4.2.

7.3. Dualization. As in Section 5, there is a (anti-simplicial, contravariant) dualization functor

$$h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)\xrightarrow{T_\bullet^G}h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G),$$

which is given by the operation

$$(y,z,u)\mapsto(z,y,{}^tu),$$

where ${}^tu:=({}^tu_+,{}^tu_{\mathbb{Z}},{}^tu_-)$.

Taking realization (using the discussion of 5.2), we obtain an involution T^G on $|h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|$.

7.4. Stabilization. As in 3.1, we have a *suspension functor*

$$h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)\xrightarrow{\Sigma^{k,\ell}}h\mathscr{D}_{k+m+\ell}\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G),$$

which is defined by $(y,z,u)\mapsto(\Sigma^k y,\Sigma^\ell z,\Sigma^{k+\ell}u)$. We also have a *left forgetful* functor

$$h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)\xrightarrow{\phi_L}h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)$$

defined by $(y,z,u)\mapsto y$. The latter is compatible with suspension. We therefore obtain an induced functor on colimits

$$h\mathscr{D}\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)\xrightarrow{\phi_L}\lim_{\to(\Sigma)}h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G), \tag{7.4.1}$$

where the source $h\mathscr{D}\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)$ denotes the colimit of the $h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)$ taken with respect to the indexing sequence defined by $\Sigma^{1,1}$.

The following is the analogue of Theorem 4.3.1 for the projective line. Its proof follows from Theorem 3.2.1. We omit the details.

7.4.2. Theorem. *The functor* (7.4.1) *induces a homotopy equivalence* (*of unequivariant spaces*) *on realizations.*

Since the involution T^G on $|h\mathscr{D}_m\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|$ is compatible the suspension map $\Sigma^{1,1}$, it induces an involution on $|h\mathscr{D}\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|$. The involution on the latter will also be denoted by T^G.

8. The canonical diagram

8.1. Let L denote one of the monoids $\mathbb{N}_-$, $\mathbb{Z}$ or $\mathbb{N}_+$. In [1], we defined a category with cofibrations and weak equivalences $\mathbb{D}_{\mathrm{fd}}(G\times L)$ which contains $\mathbb{P}_{\mathrm{fd}}(G)$ as a full subcategory. It was subsequently shown that a suitably defined forgetful functor

$$\mathbb{D}_{\mathrm{fd}}(G\times L)\to\mathbb{C}_{\mathrm{fd}}(G\times L)$$

induces a homotopy equivalence $|h\mathscr{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times L)|\xrightarrow{\simeq}|h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G\times L)|$. The idea of introducing this category was to obtain a commutative diagram of based spaces

$$\begin{array}{ccc}|h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| & \longrightarrow & |h\mathscr{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)| \\ \downarrow & & \downarrow \\ |h\mathscr{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)| & \longrightarrow & |h\mathscr{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|,\end{array}$$

which after looping once becomes homotopy cartesian. The unlooped diagram fails to be homotopy cartesian by a discrete set in the following sense: the universal map

$$|h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|\to\mathscr{P}_G$$

is a homotopy equivalence onto the basepoint component of the group-like H-space $\mathscr{P}_G$ which is defined to be the homotopy pullback of the diagram given by deleting the initial vertex from the above square. The group of path components of $\mathscr{P}_G$ is isomorphic to the negative K-group $K_{-1}(\mathbb{Z}[\pi_0(G)])$, so translation by the H-space multiplication gives rise to a homotopy equivalence

$$|h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \times K_{-1}(\mathbb{Z}[\pi_0(G)]) \xrightarrow{\simeq} \mathscr{P}_G.$$

We now briefly recall the definition of $\mathbb{D}_{\mathrm{fd}}(G \times L)$. First, suppose that $L = \mathbb{N}_-$. An *object* of $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ is specified by a triple (Y_-, Y, Y_+), as in $\mathbb{P}_{\mathrm{fd}}(G)$, the only difference now being that we do *not* require that the induced cofibration $Y_+(t^{-1}) \rightarrowtail Y$ to be a weak equivalence (although we still require the cofibration $Y_-(t) \rightarrowtail Y$ to be a weak equivalence). *Morphisms* and *cofibrations* of $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ are defined in the same way that we defined them for $\mathbb{P}_{\mathrm{fd}}(G)$. A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ is a *weak equivalence* if (and only if) the map $Y_- \to Z_-$ is a weak homotopy equivalence. We have the forgetful functor

$$\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-) \to \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)$$

defined by $(Y_-, Y, Y_+) \mapsto Y_-$.

The category $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)$ is defined similarly, i.e., an object is specified by a triple (Y_-, Y, Y_+), where this time we only require the map $Y_+(t^{-1}) \to Y$ to be a weak equivalence. The forgetful functor in this case is defined by $(Y_-, Y, Y_+) \mapsto Y_+$.

Lastly, the category $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})$ is defined so that its objects are specified by diagrams $Y_- \to Y \leftarrow Y_+$ with no condition imposed on the induced maps $Y_-(t) \to Y$ and $Y_+(t^{-1}) \to Y$ (except that they be cofibrations). A map $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ is a weak equivalence if (and only if) $Y \to Z$ is a weak equivalence. The forgetful functor is defined by $(Y_-, Y, Y_+) \mapsto Y$.

8.1.1. Lemma (Huttemann et al. [1, 4.10]). *Let L be $\mathbb{N}_-$, $\mathbb{Z}$ or $\mathbb{N}_+$. Then the forgetful functor induces a homotopy equivalence of unequivariant spaces*

$$|h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)| \xrightarrow{\simeq} |h\mathscr{S}.\mathbb{C}_{\mathrm{fd}}(G \times L)|.$$

8.2. We now define a version of the simplicial category $h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)$ which incorporates duality data.

8.2.1. Definition. With $L = \mathbb{N}_-, \mathbb{Z}, \mathbb{N}_+$, we let

$$h\mathscr{D}_m\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)$$

denote the simplicial category which is constructed in the same way as $h\mathscr{D}_m\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)$, with the exception that objects (y, z, u) are now defined so that y is an object of $h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)$, and z is an object of $h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L^{\mathrm{op}})$.

There is again a suspension functor

$$h\mathscr{D}_m\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L) \xrightarrow{\Sigma^{k,\ell}} h\mathscr{D}_{k+m+\ell}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L),$$

so we may take the colimit with respect $\Sigma^{1,1}$. Denote the resulting simplicial category by $h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)$. It is equipped with a contravariant, anti-simplicial dualization functor

$$h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times L) \xrightarrow{T_{\bullet}^{G\times L}} h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times L^{\mathrm{op}}),$$

which is defined by the same formula was used to define the dualization functor $T_{\bullet}^{G}$ on $h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)$. It therefore induces a dualization map

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times L)| \xrightarrow{T^{G\times L}} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times L^{\mathrm{op}})|,$$

which is an involution if $L = \mathbb{Z}$.

From the construction, we have

8.2.2. Lemma. *The diagram of based spaces induced by the inclusions*

$$\begin{array}{ccc} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| & \longrightarrow & |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)| \\ \downarrow & & \downarrow \\ |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| & \longrightarrow & |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \end{array}$$

is commutative.

8.2.3. Remark. Let $\mathscr{P}_G$ be the homotopy pullback of the diagram

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \leftarrow |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|.$$

Then the lemma shows that there is a preferred map

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| \to \mathscr{P}_G.$$

We can equip $\mathscr{P}_G$ with an involution as follows: a point in $\mathscr{P}_G$ consists of a triple (a, b, λ) in which $a \in |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|$, $b \in |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|$ and $\lambda : [0, 1] \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|$ is a path from the image of a to the image of b. In terms of this description the involution on $\mathscr{P}_G$ is given by

$$(a, b, \lambda) \mapsto (T(b), T(a), \lambda^*),$$

where T in each case denotes the appropriate dualization map and λ^* is the path given by $\lambda^*(t) = T\lambda(1 - t)$.

With respect to this involution, the preferred map $|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| \to \mathscr{P}_G$ is equivariant.

We now compare $h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times L)$ with $h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G \times L)$.

8.2.4. Definition. A map $f : A \to B$ of based $\mathbb{Z}/2$-spaces is said to be an *equivariant weak equivalence* if its underlying map of unequivariant spaces is a weak homotopy equivalence. More generally, based $\mathbb{Z}/2$-spaces A and B are said to be *equivariantly weak equivalent* if there exists a finite chain of morphisms from A to B, such that each such morphism is an equivariant weak equivalence. In this instance we write

$$A \simeq_{\mathbb{Z}/2} B.$$

8.2.5. Lemma. *For $L = \mathbb{N}_-, \mathbb{N}_+, \mathbb{Z}$, the forgetful functor induces an equivariant weak equivalence of $\mathbb{Z}/2$-spaces*

$$|h\mathscr{D}\mathscr{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times L)| \xrightarrow{\sim} |h\mathscr{D}\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G \times L)|.$$

Moreover, the forgetful functor is compatible with the dualization map $T^{G\times L}$.

Proof. For the first part, we give the argument when $L = \mathbb{N}_-$ and leave the remaining cases to the reader. The forgetful functor in this case, call it ϕ_-, is induced by $(y,z,u) \mapsto (Y_-, Z_+, u_+)$.

Define a functor

$$g : h\mathscr{D}\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-) \to h\mathscr{D}\mathscr{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)$$

by $(Y_-, Z_+, u_+) \mapsto (y,z,u)$, where

- $y := (Y_-, Y_-(t), *)$, and $z = (*, Z_+(t^{-1}), Z_+)$, where $Y_- \subset Y_-(t)$, and $Z_+ \subset Z_+(t^{-1})$ are given by the inclusions.
- The filtered duality map $u = (u_-, u_{\mathbb{Z}}, u_+)$ is defined so that u_- is trivial and

  $$u_{\mathbb{Z}} : Y_-(t) \wedge Z_+(t^{-1}) \to S^m_{G\times\mathbb{Z}}$$

  is the map which u_+ induces on telescopes.

It follows that there is a chain of equivalences of exact functors from $\phi_- \circ g$ and $g \circ \phi_-$ to the identity. Consequently, ϕ_- induces an equivariant weak equivalence.

The forgetful functor $\phi_+ : h\mathscr{D}\mathscr{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+) \to h\mathscr{D}\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+)$ is given by $(y,z,u) \mapsto (Y_+, Z_-, u_+)$. By construction, we have

$$T^{G\times\mathbb{N}_-} \circ \phi_- = \phi_+ \circ T^{G\times\mathbb{N}_-}.$$

Similarly, the forgetful map $\phi_{\mathbb{Z}}$ is induced by $(y,z,u) \mapsto (Y, Z, u_{\mathbb{Z}})$ and is equivariant with respect to $T^{G\times\mathbb{Z}}$ (i.e., $\phi_{\mathbb{Z}} \circ T^{G\times\mathbb{Z}} = T^{G\times\mathbb{Z}} \circ \phi_{\mathbb{Z}}$). Hence, the forgetful functors are compatible with the stabilization maps. □

9. Augmentation

9.1. We shall continue to let L denote either $\mathbb{N}_-$, $\mathbb{Z}$ or $\mathbb{N}_+$. In [1, 7.1] we introduced the *augmentation* functor $\varepsilon : \mathbb{D}_{\mathrm{fd}}(G \times L) \to \mathbb{C}_{\mathrm{fd}}(G)$ given by

$$(Y_-, Y, Y_+) \mapsto Y/\mathbb{Z},$$

where $Y/\mathbb{Z}$ means the orbit space with respect to the action of $\mathbb{Z}$.

This construction extends to a functor $h\mathscr{D}\mathscr{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times L) \to h\mathscr{D}\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)$, defined by

$$(y,z,u) \mapsto (\varepsilon(Y), \varepsilon(Z), \varepsilon(u)),$$

where

$$\varepsilon(u) : \varepsilon(Y) \wedge \varepsilon(Z) \to S^m_G$$

is the filtered duality map induced by u by means of taking orbits under the $(\mathbb{Z} \times \mathbb{Z})$-action.

We will denote this functor by ε_L, for $L = \mathbb{N}_-, \mathbb{Z}, \mathbb{N}_+$. Then we have

$$\varepsilon_{L^{\mathrm{op}}} \circ T_{\bullet}^{G\times L} = T_{\bullet}^{G} \circ \varepsilon_L,$$

where $T_{\bullet}^{G}, T_{\bullet}^{G\times L}$ are the dualization functors. Similar remarks apply to define an augmentation functor $\varepsilon : h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G) \to h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)$ which is compatible with the dualization functors.

9.2. The splitting defined by augmentation. For $L = \mathbb{N}_-, \mathbb{Z}$ or $\mathbb{N}_+$, let

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times L)|^{\varepsilon}$$

denote the homotopy fiber of $\varepsilon_L : |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times L)| \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$. Similarly, we let

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon}$$

denote the homotopy fiber of the augmentation map $\varepsilon : |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$.

9.2.1. Lemma. *There is an $\mathbb{Z}/2$-equivariant weak equivalence*

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})| \simeq_{\mathbb{Z}/2} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\varepsilon} \times |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G)|$$

Proof. There is an equivariant map

$$\sigma : |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G)| \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|,$$

which is induced by the operation $U \mapsto (U\times\mathbb{Z})/(*\times\mathbb{Z})$ ("extension of scalars"). This map, followed by augmentation, is induced up to isomorphism by the forgetful functor $\mathbb{D}_{\mathrm{fd}}(G) \to \mathbb{C}_{\mathrm{fd}}(G)$. Consequently, $\varepsilon\circ\sigma$ is an (unequivariant) weak equivalence.

Let

$$i_{\varepsilon} : |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\varepsilon} \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|$$

denote the structure map of the homotopy fiber of the augmentation map. We would like to use the map

$$\oplus : |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\times 2} \to |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|$$

induced by the categorical sum operation to add i_{ε} and σ. The sum of these maps is automatically a weak homotopy equivalence. However, the categorical sum is only equivariant up to unique isomorphism.

To get around this difficulty, we introduce a new simplicial category with involution $h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})_2$ whose objects consist of triples (a,b,z) in which a and b are objects of $h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})$ having the same simplicial degree and z is a representative of the sum of a and b. Define an involution on this simplicial category by $T(a,b,z) = (T(a),T(b),T(z))$. The forgetful functor $h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})_2 \to h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\times 2}$ defined by $(a,b,z)\mapsto(a,b)$ is a degreewise equivalence of categories; it is also involution preserving if we give the cartesian product the involution defined by $T(a,b) = (T(a),T(b))$. We also have an involution preserving functor

$$\oplus_2 : h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})_2 \to h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})$$

given by $(a,b,z)\mapsto z$.

Let Z denote the homotopy pullback of the diagram of $\mathbb{Z}/2$-spaces

$$\begin{array}{ccc} & & |h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\varepsilon}\times|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G)| \\ & & \downarrow (i_\varepsilon,\sigma) \\ |h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})_2| & \xrightarrow{\sim} & |h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|\times|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|. \end{array}$$

Then the map $Z\to|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\varepsilon}\times|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G)|$ is an equivariant weak equivalence. We also have an equivariant weak equivalence defined by the composite

$$Z\to|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})_2|\xrightarrow{\oplus_2}|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|.$$

Assembling the last two equivariant weak equivalences completes the proof. □

10. Proof of the main theorem

10.1. The diagram of Lemma 8.2.2 is compatible with augmentation, and we therefore obtain a commutative diagram of homotopy fibers

$$\begin{array}{ccc} |h\mathscr{D}\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} & \longrightarrow & |h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)|^{\varepsilon} \\ \downarrow & & \downarrow \\ |h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)|^{\varepsilon} & \longrightarrow & |h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\varepsilon} \end{array} \tag{10.1.1}$$

such that the induced map

$$|h\mathscr{D}\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon}\to\mathscr{P}_G^{\varepsilon}$$

is a weak homotopy equivalence onto the connected component of the basepoint, where $\mathscr{P}_G^{\varepsilon}$ is given by taking the homotopy pullback of the diagram obtained by deleting the initial vertex from the above square.

Moreover, the maps

$$\mathscr{P}_G^{\varepsilon}\to|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)|^{\varepsilon}\quad\text{and}\quad\mathscr{P}_G^{\varepsilon}\to|h\mathscr{D}\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)|^{\varepsilon}$$

are known to be null homotopic: this is a direct consequence of [1, Lemma 7.6].

10.2. Digression. Suppose that A, B and Z are connected, based spaces. Assume that Z is equipped with a based involution T. Let $i:A\to Z$ and $\phi:A\to B$ be based maps such that ϕ is a homeomorphism. Define a map $j:B\to Z$ by

$$j(b):=T(i(\phi^{-1}(b))).$$

Define an involution on $A\times B$, also denoted T, by the rule $T(a,b):=(\phi^{-1}(b),\phi(a))$. Suppose that we are given a based $\mathbb{Z}/2$-space and an equivariant map $P\to A\times B$. Assume further that the diagram

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow j \\ A & \xrightarrow[i]{} & Z. \end{array}$$

is homotopy cartesian.

10.2.1. Lemma. *In addition to the assumptions above, suppose that the map $P \to A$ is null homotopic (unequivariantly). Then there is an equivariant weak equivalence*

$$\underset{\sim}{\Omega} Z \simeq_{\mathbb{Z}/2} P \times \underset{\sim}{\Omega}(A \times B).$$

Proof. There is a homotopy cartesian square of $\mathbb{Z}/2$-spaces

$$\begin{array}{ccc} P & \longrightarrow & A \times B \\ \downarrow & & \downarrow \\ Z & \underset{\Delta}{\longrightarrow} & Z \times Z \end{array}$$

where the bottom map is the diagonal and the upper map is the evident one. The homotopy fiber of Δ is identified with $\underset{\sim}{\Omega} Z$. Consequently we have a homotopy fiber sequence of $\mathbb{Z}/2$-spaces

$$\underset{\sim}{\Omega} Z \to P \to A \times B.$$

By hypothesis, the map $P \to A$ is null homotopic. This implies that the equivariant map $P \to A \times B$ is equivariantly null homotopic. A choice of equivariant null homotopy together with the above homotopy fiber sequence (shifted once to the left) then gives the desired splitting. □

We will be applying Lemma 10.2.1 to the homotopy cartesian square

$$\begin{array}{ccc} \mathscr{P}_G^{\varepsilon} & \longrightarrow & |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^{\varepsilon} \\ \downarrow & & \downarrow \\ |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^{\varepsilon} & \longrightarrow & |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|^{\varepsilon}. \end{array}$$

The homeomorphism which switches the lower left hand and upper right hand vertices is given by the dualization maps.

Consequently, if we apply Lemma 10.2.1, we obtain a splitting of $\mathbb{Z}/2$-spaces.

(10.2.1)

$$\underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|^{\varepsilon} \simeq_{\mathbb{Z}/2} \mathscr{P}_G^{\varepsilon} \times \underset{\sim}{\Omega}(|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^{\varepsilon} \times |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^{\varepsilon}).$$

Recall that the $\mathbb{Z}/2$-space $\mathscr{D}A^{\mathrm{fd}}(X \times S^1)$ is equivariantly weak equivalent to $\underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|$, where G denotes the realization of the Kan loop group of the total singular complex of X.

Notation. As in the proof of [1, 7.9], we define the *nil-terms*

$$\mathscr{D}N_{-}A^{\mathrm{fd}}(X) := \Omega|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^{\varepsilon}$$

and

$$\mathscr{D}N_{+}A^{\mathrm{fd}}(X) := \Omega|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^{\varepsilon}.$$

Give the product

$$|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)|^\varepsilon\times|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)|^\varepsilon,$$

the $\mathbb{Z}/2$-action defined by $(x,y)\mapsto(T(y),T(x))$, where T denotes the dualization map. Then

$$\mathcal{D}N_-A^{\mathrm{fd}}(X)\times\mathcal{D}N_+A^{\mathrm{fd}}(X)=\underset{\sim}{\Omega}(|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)|^\varepsilon\times|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)|^\varepsilon)$$

has the structure of a based $\mathbb{Z}/2$-space.

10.2.3. Corollary. *There is an equivariant weak equivalence of* $\mathbb{Z}/2$*-spaces*

$$\mathcal{D}A^{\mathrm{fd}}(X\times S^1)\simeq_{\mathbb{Z}/2}\mathcal{D}A^{\mathrm{fd}}(X)\times\mathcal{P}_G^\varepsilon\times\mathcal{D}N_-A^{\mathrm{fd}}(X)\times\mathcal{D}N_+A^{\mathrm{fd}}(X),$$

where, in particular, the involution on the right side acts so as to switch the nil-terms.

Proof. By Lemma 10.2.1, we have an equivariant weak equivalence

$$\underset{\sim}{\Omega}|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^\varepsilon\simeq_{\mathbb{Z}/2}\mathcal{P}_G^\varepsilon\times\mathcal{D}N_-A^{\mathrm{fd}}(X)\times\mathcal{D}N_+A^{\mathrm{fd}}(X).$$

Take the cartesian product of this with $\mathcal{D}A^{\mathrm{fd}}(X)$, and use the equivariant weak equivalence of Lemma 9.2.1

$$|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|\simeq_{\mathbb{Z}/2}|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^\varepsilon\times|h\mathcal{D}\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G)|. \qquad \square$$

10.3. Using Corollary 10.2.3, we see that to complete the proof of the equivariant fundamental theorem, we must determine the equivariant weak homotopy type of $\mathcal{P}_G^\varepsilon$. This is accomplished by following the statement:

10.3.1. Proposition. *There is an equivariant weak equivalence*

$$\mathcal{D}A^{\mathrm{fd}}(X)\simeq_{\mathbb{Z}/2}\underset{\sim}{\Omega}\mathcal{P}_G^\varepsilon.$$

In particular, $\mathcal{P}_G^\varepsilon$ *is an equivariant non-connective delooping of* $\mathcal{D}A^{\mathrm{fd}}(X)$.

Using Proposition 10.3.1 and Corollary 10.2.3, we immediately obtain the main theorem:

10.3.2. Theorem. *There is an equivariant weak equivalence*

$$\mathcal{D}A^{\mathrm{fd}}(X\times S^1)\simeq_{\mathbb{Z}/2}\mathcal{D}A^{\mathrm{fd}}(X)\times\mathcal{B}\mathcal{D}A^{\mathrm{fd}}(X)\times\mathcal{D}N_-A^{\mathrm{fd}}(X)\times\mathcal{D}N_+A^{\mathrm{fd}}(X),$$

such that $\underset{\sim}{\Omega}\mathcal{B}\mathcal{D}A^{\mathrm{fd}}(X)$ *is equivariantly weak equivalent to* $\mathcal{D}A^{\mathrm{fd}}(X)$. *The action of* $\mathbb{Z}/2$ *on the splitting permutes the nil-terms.*

10.4. Thus it remains to prove Proposition 10.3.1. Recall that the equivariant map

$$|h\mathcal{D}\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|^\varepsilon\to\mathcal{P}_G^\varepsilon$$

is a weak equivalence onto the connected component of the basepoint of $\mathcal{P}_G^\varepsilon$. Consequently, Proposition 10.3.1 is equivalent to the statement.

10.4.1. Proposition. *There is an equivariant weak equivalence*

$$\mathscr{D}A^{\mathrm{fd}}(X) \simeq_{\mathbb{Z}/2} \underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon}.$$

The proof of Proposition 10.4.1 will require some preparation.

10.5. Global sections (unequivariant). Recall from [1, 5.1] that the *global sections* functor

$$\Gamma : \mathbb{P}(G) \to \mathbb{C}(G)$$

is defined by

$$(Y_-, Y, Y_+) \mapsto CY_- \cup_{Y_-} Y \cup_{Y_+} CY_+,$$

where CY_- denotes the cone on Y_-. It was shown in [1, Lemma 5.2] that Γ maps finitely dominated objects to stably finitely dominated objects. Consequently, Γ yields a map

$$\Gamma : |h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{sfd}}(G)|.$$

Since the suspension functor can be iterated, the map induced by inclusion $|h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \to |h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{sfd}}(G)|$ is a homotopy equivalence (cf. [1, Lemma 1.8(2)].

10.5.1. Lemma. *The composite map*

$$|h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \xrightarrow{i_\varepsilon} |h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma} |h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{sfd}}(G)|$$

is a homotopy equivalence, where i_ε denotes the structure map for the homotopy fiber of the augmentation map.

Proof. By Hüttemann et al. [1, Corollary 7.6], there is a homotopy equivalence

$$\psi_{-1} \ominus \psi_0 : |h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\sim} |h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon}$$

such that the composite

$$|h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\psi_{-1}\ominus\psi_0} |h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \xrightarrow{i_\varepsilon} |h\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma} |h\mathscr{S}_\bullet\mathbb{C}_{\mathrm{sfd}}(G)|$$

is homotopic to the map induced by the suspension functor (cf. the discussion prior to [1, Corollary 6.8]). But the suspension functor induces a weak equivalence [5, 1.6.2]. The result follows. □

10.6. Global sections (equivariant). Define an involution preserving functor

$$\Gamma_{\mathscr{D}} : h\mathscr{D}\mathscr{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G) \to h\mathscr{D}\mathscr{S}_\bullet\mathbb{C}_{\mathrm{sfd}}(G)$$

by

$$(y, z, u) \mapsto (\Gamma(y), \Gamma(z), u').$$

Here, $y = (Y_-, Y, Y_+)$, $z = (Z_+, Z, Z_-)$ and u denotes filtered duality pairing data. The map u' is induced by u.

With respect to the left forgetful functor, $\Gamma_{\mathscr{D}}$ corresponds to Γ. Hence, using Theorem 7.4.2 we obtain

10.6.1. Corollary. *The composite*

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \xrightarrow{i_{\varepsilon}} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma_{\mathscr{D}}} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{sfd}}(G)|$$

is an equivariant weak equivalence.

Proof of Proposition 10.4.1. By Corollary 10.6.1 and the discussion prior to Lemma 10.5.1, we have equivariant weak equivalences

$$|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \underset{\sim}{\xrightarrow{\Gamma_{\mathscr{D}} \circ i_{\varepsilon}}} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{sfd}}(G)| \xleftarrow{\simeq} |h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|.$$

Taking loop spaces, we obtain an equivariant weak equivalence

$$\underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \simeq_{\mathbb{Z}/2} \underset{\sim}{\Omega}|h\mathscr{D}\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)| =: \mathscr{D}A^{\mathrm{fd}}(X). \qquad \square$$

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