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The “fundamental theorem” for the algebraic K-theory of spaces: I

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Abstract

Let $X \mapsto A(X)$ denote the algebraic K-theory of spaces functor. The main objective of this paper is to show that $A(X \times S^1)$ admits a functorial splitting. The splitting has four factors: a copy of $A(X)$, a delooped copy of $A(X)$ and two homeomorphic *nil terms*. One should view the decomposition as the algebraic K-theory of spaces version of the Bass-Heller-Swan theorem. In deducing this splitting, we introduce a new tool: a “non-linear” analogue of the projective line.

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0. Introduction

The “fundamental theorem” of the algebraic K-theory of rings states that there is a decomposition

$$K_n(R[t,t^{-1}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus N_-K_n(R) \oplus N_+K_n(R),$$

where $R[t,t^{-1}]$ is the Laurent ring in one indeterminate over a ring R. The “nil-term” $N_+K_n(R)$ is defined to be the complementary summand of $K_n(R)$ in $K_n(R[t])$ (the former is a summand of the latter since $R[t]$ is an augmented R-algebra) and the nil-term $N_-K_n(R)$ is defined similarly by replacing t with t^{-1} (see [1,2,4]).

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In this paper we establish a corresponding result for the functor $X \mapsto A(X)$ of [8]. We prove that there is a functorial decomposition

$$A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times \mathscr{B}A^{\mathrm{fd}}(X) \times N_-A^{\mathrm{fd}}(X) \times N_+A^{\mathrm{fd}}(X).$$

Here, $A^{\mathrm{fd}}(X)$ is the version of $A(X)$ that is based on finitely dominated spaces and $\mathscr{B}A^{\mathrm{fd}}(X)$ is a certain canonical non-connective delooping of $A^{\mathrm{fd}}(X)$. The remaining functors $N_-A^{\mathrm{fd}}(X)$ and $N_+A^{\mathrm{fd}}(X)$ are *nil-terms*.

The above splitting of $A^{\mathrm{fd}}(X \times S^1)$ may be regarded as a special case of a "fundamental theorem" for the K-theory of "brave new rings" which has been sketched in [5]. From that point of view, the present study considers in effect "Laurent rings" over "group rings", i.e.,

$$Q_+(\Omega X)[t, t^{-1}] := Q_+(\Omega(X \times S^1)),$$

where Q_+ is unreduced stable homotopy.

There are two reasons for providing a different proof in the present context. The first (and perhaps minor) reason is to introduce the *projective line* category in connection with the algebraic K-theory of spaces, to identify its K-theory, and to deduce the fundamental theorem from this identification; this is very similar to Quillen's treatment in [4]. The second (and more weighty) reason is that the projective line approach as opposed to that of [5] provides a convenient framework for studying the action of the "canonical involution" of [7] on the splitting of $A^{\mathrm{fd}}(X \times S^1)$. This study shall be undertaken in part II of this series of papers.

In a recent preprint, Hughes and Prassidis [3] formulate and prove a geometric version of the "fundamental theorem" for the PL Whitehead space $\mathrm{Wh}^{\mathrm{PL}}(X)$. It would be of some interest to relate their result with ours.

Outline. Section 1 is foundational. In Section 2 we define the telescope in the context of the K-theory of spaces. In Section 3 we introduce the projective line category associated to G, the realization of a simplicial group. In Section 4 we express the K-theory of the projective line as a certain homotopy pullback. In Section 5 we define some exact functors to be used in the identification of the K-theory of the projective line. In Section 6 we establish the equivalence between the K-theory of the projective line of G and the cartesian product of two copies $A^{\mathrm{fd}}(BG)$. In Section 7 we assemble the material of the previous sections to prove the main result.

1. Preliminaries

1.1. Notational conventions. Let us recall that if C is a category with *cofibrations* co$C$ (usually not specified) and a category of *weak equivalences* wC, then there is a connected, based space $|w\mathscr{S}.C|$, the *$\mathscr{S}.$-construction* of C, whose loop space is taken as the definition of K-theory in this situation (see [8, 1.3]). We will sometimes employ the symbol $\rightarrowtail$ to indicate that a morphism of C is a cofibration. To indicate that a morphism is a weak equivalence, we use the symbol $\xrightarrow{\sim}$.

Given an *exact functor* $F:C \to D$ (cf. [8, p. 327]), a convention we shall often employ is to denote the induced map on $\mathcal{S}_\bullet$-constructions and K-theories by the same name. Thus, we write $F : |w\mathcal{S}_\bullet C| \to |w\mathcal{S}_\bullet D|$. Given a pair of exact functors $F_0, F_1 : C \to D$, an *equivalence* is a natural transformation $F_0 \to F_1$ whose value on any object $x \in C$ gives a weak equivalence $F_0(x) \xrightarrow{\sim} F_1(x)$. If this is the case, then F_0 and F_1 induce homotopic maps $|w\mathcal{S}_\bullet C| \to |w\mathcal{S}_\bullet D|$ (cf. [8, 1.3.1]).

1.2. Equivariant spaces. By *space* we mean a compactly generated topological space. Products are to be formed using the compactly generated topology. We now review the various categories with cofibrations and weak equivalences which arise in connection with the algebraic K-theory of spaces.

Let $M_\bullet$ be a simplicial monoid whose realization $|M_\bullet|$ we denote by M. With respect to our convention regarding products, M has the structure of a topological monoid.

Let $\mathbb{T}(M)$ denote the category whose *objects* are *based M-spaces*, i.e., based spaces Y equipped with a based left action $M \times Y \to Y$ (i.e., the action leaves the base point of Y invariant). *Morphisms* of $\mathbb{T}(M)$ are the based equivariant maps.

The *cell* of dimension n is defined to be

$$D^n \times M.$$

This is an (unbased) M-space, where the action of M is given by left translation (i.e., the effect of $m \in M$ acting on the point $(x,n) \in D^n \times M$ is the point $(x, m \cdot n)$). Similarly, we have the (unbased) equivariant sphere $S^{n-1} \times M$.

If Z is an object of $\mathbb{T}(M)$, and $\alpha : S^{n-1} \times M \to Z$ is an equivariant map, then we can form the object

$$Z \cup_\alpha (D^n \times M)$$

by attaching $D^n \times M$ to Z along α. We call this operation the effect of *attaching a cell*. A morphism $Y \to Z$ of $\mathbb{T}(M)$ is said to be a *cofibration* if either: (1) Z is obtained from Y up to isomorphism by a (possibly transfinite) sequence of cell attachments, or (2) it is a retract of the foregoing. Observe that cofibrations satisfy the equivariant homotopy extension property. An object Z is said to be *cofibrant* if the inclusion of the basepoint $* \to Z$ is a cofibration. We let $\mathbb{C}(M) \subset \mathbb{T}(M)$ denote the full subcategory of cofibrant objects.

1.3. Remark. Given $g \in M$, there is an associated morphism $g : M \amalg \mathrm{pt} \to M \amalg \mathrm{pt}$ defined by left translation (where $M \amalg \mathrm{pt}$ is just M with a disjoint basepoint added). This map is a cofibration of $\mathbb{C}(M)$ if and only if g is invertible in M. For the quotient space $(M \amalg \mathrm{pt})/g(M \amalg \mathrm{pt})$, if non-trivial, is never cofibrant (since multiplication by g acts trivially on it). This is even true when M has the left cancellation property, which shows that there might be equivariant inclusions of an elementary kind which fail to be cofibrations.

1.4. Finiteness. An object of $\mathbb{C}(M)$ is said to be *finite* if it is isomorphic to a finite free based M–CW complex, i.e., it is built up from a point by a finite number of cell attachments, where the order of attachment is compatible with the dimension of the cells. A *cellular* morphism of finite objects is a morphism of $\mathbb{C}(M)$ which preserves skeleta. The subcategory of $\mathbb{C}(M)$ consisting of the finite objects and their cellular morphisms will be denoted $\mathbb{C}_{\mathrm{f}}(M)$.

An object $Y \in \mathbb{C}(M)$ is *homotopy finite* if there is a finite object Z and a morphism $f : Y \to Z$ which, when considered as a map of ordinary spaces, is a weak homotopy equivalence (in the sense that the induced map of homotopy groups $\pi_n(Y, y) \to \pi_n(Z, f(y))$ is an isomorphism for all $n \geq 0$ for any choice of base point $y \in Y$).[1] The full subcategory of $\mathbb{C}(M)$ consisting of the homotopy finite objects is denoted $\mathbb{C}_{\mathrm{hf}}(M)$.

Call an object $Y \in \mathbb{C}(M)$ *finitely dominated* if it is a retract of a homotopy finite object, i.e., there is a factorization of the identity morphism of Y through a homotopy finite object (equivalently, the identity map of Y admits a factorization up to homotopy through a finite object). The full subcategory of $\mathbb{C}(M)$ consisting of finitely dominated objects is denoted $\mathbb{C}_{\mathrm{fd}}(M)$.

Lastly, an object Y of $\mathbb{C}(M)$ is said to be *s-finitely dominated* if $\Sigma^k Y$ is finitely dominated for some $k \in \mathbb{N}$, where the latter object denotes the k-fold reduced suspension of Y (given the structure of an M-space by letting M act trivially on the suspension coordinate). We denote the full subcategory consisting of s-finitely dominated objects of $\mathbb{C}(M)$ by $\mathbb{C}_{\mathrm{sfd}}(M)$.

We have thus defined a sequence of categories

$$\mathbb{C}_{\mathrm{f}}(M) \to \mathbb{C}_{\mathrm{hf}}(M) \xrightarrow{\subseteq} \mathbb{C}_{\mathrm{fd}}(M) \xrightarrow{\subseteq} \mathbb{C}_{\mathrm{sfd}}(M) \quad (\subset \mathbb{C}(M)), \tag{1.1}$$

where the functors are all given by inclusion, and all but the first of these is full. A morphism of $\mathbb{C}_{\mathrm{f}}(M)$ is a *cofibration* if it is isomorphic to a skeletal inclusion. Call a morphism of $\mathbb{C}_{\mathrm{hf}}(M), \mathbb{C}_{\mathrm{fd}}(M)$ or $\mathbb{C}_{\mathrm{sfd}}(M)$ a *cofibration* if it is so when considered in $\mathbb{C}(M)$.

Call a morphism in any of these categories a *weak equivalence* if it is a weak homotopy equivalence of underlying spaces. These notions equip $\mathbb{C}_?(M)$ with the structure of a category with cofibrations and weak equivalences, where ? denotes any of the decorations $\mathrm{f}, \mathrm{hf}, \mathrm{fd}, \mathrm{sfd}$.

1.5. Notation. With $M = |M_\bullet|$ as above, the K-theories of the categories appearing in (1.1) are correspondingly denoted by

$$A^{\mathrm{f}}(*, M) \to A^{\mathrm{hf}}(*, M) \to A^{\mathrm{fd}}(*, M) \to A^{\mathrm{sfd}}(*, M)$$

with the displayed maps induced by the inclusions.

1.6. Remark. When $G = |G_\bullet|$ is the realization of a simplicial group, then $A^{\mathrm{f}}(*, G)$ is one of the definitions of $A(BG)$, where BG denotes the classifying space of G (see [8, pp. 382–383]).

[1] In view of the cofibrancy condition, a weak homotopy equivalence is the same thing as an M-homotopy equivalence in the strong sense, by the equivariant Whitehead theorem.

The following relates the various notions of finiteness (see [8, 2.1]).

1.7. Lemma. (1) *The inclusion* $\mathbb{C}_{\mathrm{f}}(M) \subset \mathbb{C}_{\mathrm{hf}}(M)$ *induces a homotopy equivalence*

$$|h\mathcal{S}.\mathbb{C}_{\mathrm{f}}(M)| \xrightarrow{\simeq} |h\mathcal{S}.\mathbb{C}_{\mathrm{hf}}(M)|$$

and consequently, a homotopy equivalence $A^{\mathrm{f}}(*,M) \xrightarrow{\simeq} A^{\mathrm{hf}}(*,M)$.

(2) *The inclusion* $\mathbb{C}_{\mathrm{fd}}(M) \subset \mathbb{C}_{\mathrm{sfd}}(M)$ *induces a homotopy equivalence*

$$|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(M)| \xrightarrow{\simeq} |h\mathcal{S}.\mathbb{C}_{\mathrm{sfd}}(M)|$$

and consequently, a homotopy equivalence $A^{\mathrm{fd}}(*,M) \xrightarrow{\simeq} A^{\mathrm{sfd}}(*,M)$.

(3) *There is a homotopy equivalence*

$$A^{\mathrm{fd}}(*,M) \simeq \tilde{K}_0(\mathbb{Z}[\pi_0(M)]) \times A^{\mathrm{hf}}(*,M),$$

where $\tilde{K}_0(\mathbb{Z}[\pi_0(M)])$ *is the reduced class group of the integral monoid ring* $\mathbb{Z}[\pi_0(M)]$.

Proof. (1) This is just a part of [8, 2.1.5].

(2) Filter $\mathbb{C}_{\mathrm{sfd}}(M)$ by full subcategories $\mathbb{C}_{\mathrm{sfd}}(M,k)$ in which an object is in the latter if its k-fold suspension is finitely dominated. These subcategories inherit the structure of a category with cofibrations and weak equivalences. Notice that $\mathbb{C}_{\mathrm{fd}}(M) = \mathbb{C}_{\mathrm{sfd}}(M,0)$ and that $\mathbb{C}_{\mathrm{sfd}}(M)$ is the colimit of the sequence of inclusions

$$\cdots \xrightarrow{\subseteq} \mathbb{C}_{\mathrm{sfd}}(M,k) \xrightarrow{\subseteq} \mathbb{C}_{\mathrm{sfd}}(M,k+1) \xrightarrow{\subseteq} \cdots .$$

It will therefore be sufficient to show that the inclusion

$$\mathbb{C}_{\mathrm{sfd}}(M,k) \xrightarrow{i} \mathbb{C}_{\mathrm{sfd}}(M,k+1)$$

induces a homotopy equivalence on $\mathcal{S}.$-constructions for all $k \geq 0$.

Suspension defines a functor

$$\mathbb{C}_{\mathrm{sfd}}(M,k+1) \xrightarrow{\Sigma} \mathbb{C}_{\mathrm{sfd}}(M,k).$$

The composite $\Sigma \circ i$ is the suspension functor for $\mathbb{C}_{\mathrm{sfd}}(M,k)$ and the composite $i \circ \Sigma$ is the suspension functor for $\mathbb{C}_{\mathrm{sfd}}(M,k+1)$. We infer that $\Sigma \circ i$ and $i \circ \Sigma$ induce homotopy equivalences on $\mathcal{S}.$-constructions by Waldhausen [8, 1.6.2]. Consequently, i induces a homotopy equivalence on $\mathcal{S}.$-constructions, as was to be proved.

(3) See the remark on [8, p. 389]. One can also deduce this from the *cofinality theorem* of Thomason [6, 1.10.1] (see the argument used to prove Proposition 3.3(2) below). □

2. The telescope

2.1. Let $\mathbb{N}_+$ and $\mathbb{N}_-$, respectively, denote the monoids of nonnegative and nonpositive natural numbers (including 0) with generators t and t^{-1}. It will be typical in the sequel that M is the geometric realization of a simplicial monoid of the form

$$G., \quad G. \times \mathbb{N}_-, \quad G. \times \mathbb{N}_+ \quad \text{or} \quad G. \times \mathbb{Z}$$

for a fixed simplicial group $G_\bullet$. Consequently, if we write $G = |G_\bullet|$, then M has the form G, $G \times \mathbb{N}_-$, $G \times \mathbb{N}_+$ or $G \times \mathbb{Z}$.

We shall define functors

$$\begin{array}{ll} \mathbb{C}(G \times \mathbb{N}_-) \to \mathbb{C}(G \times \mathbb{Z}) \quad \text{and} & \mathbb{C}(G \times \mathbb{N}_+) \to \mathbb{C}(G \times \mathbb{Z}), \\ U \mapsto U(t), & V \mapsto V(t^{-1}), \end{array}$$

which assign to an object its *telescope*, and confirm that the telecope functors preserve cofibrations, weak equivalences and finiteness conditions. Although it is perhaps more precise to write $U((t^{-1})^{-1})$ for $U(t)$ to indicate in the first case that t^{-1} is to be inverted, we prefer the simpler notation.

Let $t: V \to V$ be defined by the action of t. We shall take $V(t^{-1})$ to be the categorical colimit of the sequence

$$\cdots \xrightarrow{t} V \xrightarrow{t} V \xrightarrow{t} \cdots$$

with the evident action of $G \times \mathbb{Z}$. Actually, this only defines $V(t^{-1})$ up to isomorphism. To get an explicit model for it, we shall define it as the quotient space of $V \times \mathbb{Z}$ in which a pair (v,n) is identified with the pair $(t(v), n+1)$. In the latter representation, the action of t^k on a pair (v,n) yields the pair $(v, n-k)$, for $k \in \mathbb{Z}$. Observe that the map

$$V \to V(t^{-1}), \qquad v \mapsto (v,0)$$

is a $(G \times \mathbb{N}_+)$-equivariant *inclusion*. If the action of t on V was invertible to begin with, then this inclusion is an isomorphism of $(G \times \mathbb{Z})$-spaces (the inverse map is defined by $(v,n) \mapsto (t^{-n}(v), 0)$).

Similarly, for $U \in \mathbb{C}(G \times \mathbb{N}_-)$, we define $U(t)$ by taking the categorical colimit of the sequence

$$\cdots \xrightarrow{t^{-1}} U \xrightarrow{t^{-1}} U \xrightarrow{t^{-1}} \cdots .$$

2.2. Lemma. (1) *The telescope functor preserves cofibrations and weak homotopy equivalences.*

(2) *If $V \in \mathbb{C}_?(G \times \mathbb{N}_+)$ is an object, where ? is one of the decorations f, hf, fd or sfd, then $V(t^{-1})$ is an object of $\mathbb{C}_?(G \times \mathbb{Z})$.*

Proof. (1) Observe that the telescope maps the pair $(D^n_{G\times\mathbb{N}_+}, S^{n-1}_{G\times\mathbb{N}_+})$ isomorphically to the pair $(D^n_{G\times\mathbb{Z}}, S^{n-1}_{G\times\mathbb{Z}})$. Also, since it is a kind of colimit, the telescope is stable under cobase change. The assertion follows, since an arbitrary cofibration is a sequence of cobase changes with respect to such pairs.

The map $t : V \to V$ is a cofibration of G-spaces (though *not* of $G \times \mathbb{N}_+$-spaces; note also that V is cofibrant when considered as a G-space). It follows that the categorical colimit is weak homotopy invariant (for it is weak homotopy equivalent to the homotopy colimit).

(2) This follows for finite objects by the argument used to establish the first part. The rest is evident. □

3. The projective line

3.1. The telescope constructions of Section 2 provide a pair of maps

$$A^{\mathrm{fd}}(*, G \times \mathbb{N}_-) \to A^{\mathrm{fd}}(*, G \times \mathbb{Z}) \leftarrow A^{\mathrm{fd}}(*, G \times \mathbb{N}_+).$$

It will be the purpose of the next sections to study the homotopy pullback of this diagram.

As the first step in our study, we define a *projective line* category $\mathbb{P}(G)$ which is equipped with forgetful functors to $\mathbb{C}(G \times \mathbb{N}_-)$, $\mathbb{C}(G \times \mathbb{N}_+)$ and $\mathbb{C}(G \times \mathbb{Z})$. We give $\mathbb{P}(G)$ the structure of a category with cofibrations and weak equivalences and define finiteness conditions parallel to those already defined for the categories $\mathbb{C}(M)$. Beyond giving the definitions the main purpose of the section is to prove results comparing the K-theories that result from the various finiteness conditions.

The category $\mathbb{P}(G)$ is defined as follows: an *object* is specified by a triple of objects Y_- of $\mathbb{C}(G \times \mathbb{N}_-)$, Y of $\mathbb{C}(G \times \mathbb{Z})$, and Y_+ of $\mathbb{C}(G \times \mathbb{N}_+)$, together with a pair of maps

$$Y_- \xrightarrow{a_-} Y \quad \text{and} \quad Y \xleftarrow{a_+} Y_+$$

such that a_- is $G \times \mathbb{N}_-$-equivariant and a_+ is $G \times \mathbb{N}_+$-equivariant (where we restrict the action of $G \times \mathbb{Z}$ on Y to $G \times \mathbb{N}_\pm$). Moreover, the data are required to satisfy the following auxiliary conditions:

- The induced maps of telescopes

$$Y_-(t) \xrightarrow{a_-(t)} Y(t) \cong Y \quad \text{and} \quad Y \cong Y(t^{-1}) \xleftarrow{a_+(t^{-1})} Y_+(t^{-1})$$

  are both *cofibrations* and *weak equivalences*.

We allow ourselves the liberty of using more than one notation to refer to objects of $\mathbb{P}(G)$: an object will be specified either as a triple (Y_-, Y, Y_+), or as a diagram $Y_- \to Y \leftarrow Y_+$. The terms Y_-, Y and Y_+ are called the *components* of the given object.

A *morphism* of $\mathbb{P}(G)$ is given by three morphisms $f_- \in \mathbb{C}(G \times \mathbb{N}_-)$, $f \in \mathbb{C}(G \times \mathbb{Z})$, and $f_+ \in \mathbb{C}(G \times \mathbb{N}_+)$ which satisfy a commutative diagram

$$\begin{array}{ccccc} Y_- & \longrightarrow & Y & \longleftarrow & Y_+ \\ {\scriptstyle f_-}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_+} \\ Z_- & \longrightarrow & Z & \longleftarrow & Z_+ \end{array}$$

3.2. Finiteness in the projective line. An object (Y_-, Y, Y_+) of $\mathbb{P}(G)$ is said to be (*locally*) *finite* if the objects Y_-, Y and Y_+ are finite in their respective categories. A *morphism of finite objects* of $\mathbb{P}(G)$ is defined so that the map on each component is cellular. We let $\mathbb{P}_{\mathrm{f}}(G)$ denote the (non-full) subcategory of $\mathbb{P}(\mathbb{G})$ given by finite objects and finite morphisms.

An object (Y_-, Y, Y_+) is said to be *homotopy finite* if each of its components is. It is said to be *finitely dominated* if it is a retract of a homotopy finite object. Similarly, an object is *stably finitely dominated* if some suspension of it is finitely dominated. We let $\mathbb{P}_{\mathrm{hf}}(G)$, $\mathbb{P}_{\mathrm{fd}}(G)$ and $\mathbb{P}_{\mathrm{sfd}}(G)$ denote the full subcategories of $\mathbb{P}(G)$ consisting of the homotopy finite, finitely dominated, and stably finitely dominated objects, respectively.

A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{P}(G)$ is a *cofibration* if

- each of the maps $Y_- \to Z_-$, $Y_+ \to Z_+$ and $Y \to Z$ is a cofibration, and
- the induced maps

$$Y \cup_{Y_-(t)} Z_-(t) \to Z \quad \text{and} \quad Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) \to Z \tag{$*$}$$

are cofibrations of $\mathbb{C}(G \times \mathbb{Z})$.

The cofibrations of the subcategory $\mathbb{P}_?(G)$ for each of the decorations ?= hf, fd, sfd are given by those morphisms which are cofibrations when considered in $\mathbb{P}(G)$. In the finite case, a morphism of $Y \to Z$ of $\mathbb{P}_{\mathrm{f}}(G)$ is a cofibration if the induced maps $(*)$ above are cofibrations of $\mathbb{C}(G \times \mathbb{Z})$.

A morphism in one of the above categories will be a *weak equivalence* if each of its component maps is a weak homotopy equivalence of underlying spaces.

We let $K(\mathbb{P}_?(G), h)$, with ? denoting one of the decorations f, hf, fd, sfd be the K-theory of the projective line with respect to the above notions of cofibration and weak equivalence.

3.3. Proposition. (1) *The canonical map* $|h\mathscr{S}.\mathbb{P}_{\mathrm{f}}(G)| \to |h\mathscr{S}.\mathbb{P}_{\mathrm{hf}}(G)|$ *is a homotopy equivalence.*

(2) *The canonical map* $|h\mathscr{S}.\mathbb{P}_{\mathrm{hf}}(G)| \to |h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)|$ *induces an isomorphism on homotopy groups in degrees* > 1.

The proof of the first part of the proposition will be a consequence of the *approximation theorem* [8, 1.6.7].

Before beginning the proof, we briefly recall the set-up for the approximation theorem. Suppose we are given an exact functor $F: C \to D$ of categories with cofibrations and weak equivalences such that C and D satisfy the saturation axiom [8, p. 327] and C admits a cylinder functor [8, p. 348] so that the weak equivalences of C satisfy the cylinder axiom. We say that F has the *approximation property* if

App 1. F *reflects* weak equivalences, i.e., a morphism of C is a weak equivalence if (and only if) its image in D is a weak equivalence.

App 2. Given any object c of C and any morphism $x: F(c) \to d$ in D, then there exists a morphism $y: c \to c'$ and a weak equivalence $z: F(c') \xrightarrow{\sim} d$ such that the composite $z \circ F(y): F(c) \to d$ equals x.

The approximation theorem says that if F has the approximation property, then the induced map $w\mathscr{S}.C \to w\mathscr{S}.D$ is an equivalence on realizations.

Proof of Proposition 3.3. (1) The proof will consist of several steps.

Step 1: Define a full subcategory $\mathbb{P}_f(G)' \subset \mathbb{P}_f(G)$ whose objects (Y_-, Y, Y_+) satisfy the additional property that the associated map

$$Y_-(t) \vee Y_+(t^{-1}) \to Y$$

is a cofibration of $\mathbb{C}_f(G \times \mathbb{Z})$. Call a morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{P}_f(G)'$ a *cofibration* if (and only if) its components are cofibrations, and moreover the induced map

$$Z_-(t) \cup_{Y_-(t)} Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) \to Z$$

is a cofibration of $\mathbb{C}_f(G \times \mathbb{Z})$. Call a morphism a *weak equivalence* if it is so when considered in $\mathbb{P}_f(G)$. With these definitions, $\mathbb{P}_f(G)'$ is a category with cofibrations and weak equivalences. Observe that the inclusion functor $i : \mathbb{P}_f(G)' \subset \mathbb{P}_f(G)$ is exact.

Define a functor $T : \mathbb{P}_f(G) \to \mathbb{P}_f(G)'$ by

$$(Y_-, Y, Y_+) \mapsto (Y_-, \bar{Y}, Y_+),$$

where $\bar{Y}$ is the mapping cylinder of the map $Y_-(t) \vee Y_+(t^{-1}) \to Y$ and the maps $Y_- \to \bar{Y}$ and $Y_+ \to \bar{Y}$ are the inclusions into the mapping cylinder. Then T is exact (the nontrivial thing to be verified is that T preserves cofibrations — we omit the details).

The canonical weak equivalence $\bar{Y} \xrightarrow{\sim} Y$ shows that the composites $T \circ i$ and $i \circ T$ are equivalent to the identity. Consequently, the map

$$|h\mathcal{S}.\mathbb{P}_f(G)'| \to |h\mathcal{S}.\mathbb{P}_f(G)|$$

is a homotopy equivalence.

In the homotopy finite case, we also have a subcategory $\mathbb{P}_{hf}(G)' \subset \mathbb{P}_{hf}(G)$ whose objects are defined analogously. The inclusion functor induces a homotopy equivalence $|h\mathcal{S}.\mathbb{P}_{hf}(G)'| \xrightarrow{\simeq} |h\mathcal{S}.\mathbb{P}_{hf}(G)|$.

Thus, we are reduced to the problem of showing that the inclusion functor

$$\mathbb{P}_f(G)' \to \mathbb{P}_{hf}(G)'$$

induces a homotopy equivalence on $\mathcal{S}.$-constructions.

Step 2 (*Assertion*): If $y = (Y_-, Y, Y_+) \in \mathbb{P}_{hf}(G)'$ is an object, then there exists an object $z = (Z_-, Z, Z_+) \in \mathbb{P}_f(G)'$ and mutually homotopy inverse weak equivalences

$$z \xrightarrow{\sim} y \quad \text{and} \quad y \xrightarrow{\sim} z.$$

To prove this, let $W_- \xrightarrow{\sim} Y_-$, $W \xrightarrow{\sim} Y$ and $W_+ \xrightarrow{\sim} Y_+$ be weak equivalences, where W_-, W and W_+ are, respectively, finite. Then, by the equivariant Whitehead theorem, there exist weak equivalences $W_-(t) \xrightarrow{\sim} W$ and $W_+(t^{-1}) \xrightarrow{\sim} W$ such that the induced diagram

$$\begin{array}{ccc} W_-(t) \vee W_+(t^{-1}) & \longrightarrow & W \\ \downarrow & & \downarrow \\ Y_-(t) \vee Y_+(t^{-1}) & \longrightarrow & Y \end{array}$$

is $(G\times\mathbb{Z})$-equivariantly *homotopy* commutative. Set $Z_-=W_-$, $Z_+=W_+$ and let Z be the mapping cylinder of the map $W_-(t)\vee W(t^{-1})\to W$. Then a choice of homotopy for the diagram defines a weak equivalence $(Z_-,Z,Z_+)\xrightarrow{\sim}(Y_-,Y,Y_+)$.

The inverse homotopy equivalence $(Y_-,Y,Y_+)\to(Z_-,Z,Z_+)$ is constructed as follows: choose inverse equivariant homotopy equivalences $Y_-\to Z_-$, $Y\to Z$ and $Y_+\to Z_+$. Then the resulting diagram

$$\begin{array}{ccc} Y_-(t)\vee Y_+(t^{-1}) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z_-(t)\vee Z_+(t^{-1}) & \longrightarrow & Z \end{array}$$

is equivariantly homotopy commutative. As the top horizontal map is a cofibration, it follows that we can deform (using the equivariant homotopy extension property) the map $Y\to Z$ to obtain a strictly commutative diagram. The latter defines the inverse equivalence.

A straightforward application of the equivariant homotopy extension property also shows that the composite $(Y_-,Y,Y_+)\to(Z_-,Z,Z_+)\to(Y_-,Y,Y_+)$ is homotopic to the identity. The same applies to the other composite.

This completes the proof of the assertion.

Step 3: The mapping cylinder construction applied component-wise equips $\mathbb{P}_{\mathrm{f}}(G)'$ with a cylinder functor. As condition App 1 holds for the inclusion functor $\mathbb{P}_{\mathrm{f}}(G)'\to\mathbb{P}_{\mathrm{hf}}(G)'$, we need only to verify condition App 2.

Let $y=(Y_+,Y,Y_+)\in\mathbb{P}_{\mathrm{f}}(G)'$ be an object and let $y\to z$ be a morphism of $\mathbb{P}_{\mathrm{hf}}(G)'$, where $z=(Z_-,Z,Z_+)$. We need to show that there exists a factorization $y\to u\xrightarrow{\sim}z$, with $u\in\mathbb{P}_{\mathrm{f}}(G)'$.

By the above assertion, we may choose an object $k=(K_-,K,K_+)\in\mathbb{P}_{\mathrm{f}}(G)'$ and weak equivalences $k\xrightarrow{\sim}z$ and $z\xrightarrow{\sim}k$ which are mutually inverse to each other.

Choose a morphism of finite objects $y\to k$ which is homotopic to the composite

$$y\to z\xrightarrow{\sim}k$$

(this is possible by the equivariant homotopy extension property since $Y_-(t)\vee Y_+(t)\to Y$ is a cofibration), and define $u=(U_-,U,U_+)$ to be its mapping cylinder. Then there exists a map $u\to z$ which extends the given maps on y and k. Consequently, we obtain a factorization $y\rightarrowtail u\xrightarrow{\sim}z$.

Thus the second approximation property holds, and we conclude that the map $|h\mathscr{S}.\mathbb{P}_{\mathrm{f}}(G)'|\to|h\mathscr{S}.\mathbb{P}_{\mathrm{hf}}(G)'|$ is a homotopy equivalence. This completes the proof of Proposition 3.3(1).

(2) This will be a special case of the *cofinality theorem* of Thomason [6, 1.10.1]. Let A_- be the abelian group given by taking the cokernel of the homomorphism

$$\pi_1(|h\mathscr{S}.\mathbb{C}_{\mathrm{hf}}(G\times\mathbb{N}_-)|)\to\pi_1(|h\mathscr{S}.\mathbb{C}_{\mathrm{fd}}(G\times\mathbb{N}_-)|)$$

and define A_+ by taking the cokernel of the homomorphism given by using $\mathbb{N}_+$ instead of $\mathbb{N}_-$ above.

Recall that an object $X \in \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ gives rise to an element of

$$\pi_1(|h\mathscr{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)|)$$

(cf. [8, p. 329]). The obstruction for X to be homotopy finite is given by the image of this class in A_- under the quotient homomorphism. Similar remarks apply to an object of $\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+)$.

The forgetful functor

$$\mathbb{P}_{\mathrm{fd}}(G) \to \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-) \times \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+),$$
$$(Y_-, Y, Y_+) \mapsto (Y_-, Y_+)$$

induces a map on $\mathscr{S}.$-constructions and consequently on their fundamental groups. Composing with the product of the quotient homomorphisms, we get a homomorphism

$$\pi_1(|h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)|) \to A_- \times A_+$$

such that an object $y = (Y_-, Y, Y_+) \in \mathbb{P}_{\mathrm{fd}}(G)$ is homotopy finite if and only if the associated class in the fundamental group gets mapped to zero in $A_- \times A_+$. The cofinality theorem then says that there is a homotopy fiber sequence

$$|h\mathscr{S}.\mathbb{P}_{\mathrm{hf}}(G)| \to |h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to A_- \times A_+$$

(where $A_- \times A_+$ is given the discrete topology). We infer that the map $|h\mathscr{S}.\mathbb{P}_{\mathrm{hf}}(G)| \to |h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)|$ induces an isomorphism on homotopy groups in degrees > 1. □

The following says in effect that any object of $\mathbb{C}_{\mathrm{f}}(G \times \mathbb{Z})$ admits a certain kind of 'Mayer–Vietoris decomposition'. Note that it is asserted only in the finite case.

3.4. Lemma. *Each finite object $Y \in \mathbb{C}_{\mathrm{f}}(G \times \mathbb{Z})$ may be taken as a constituent of an object $y := (Y_-, Y, Y_+) \in \mathbb{P}_{\mathrm{f}}(G)$. In fact, we may choose y so that $Y_-(t) \cong Y \cong Y_+(t^{-1})$.*

Proof. We proceed by induction. Assume that the result is true for some finite object Z where $Y = Z \cup_\alpha D^n_{G\times\mathbb{Z}}$ along an attaching map $\alpha : S^{n-1}_{G\times\mathbb{Z}} \to Z$. Hence there exists a finite object of the projective line of the form (Z_-, Z, Z_+), with $Z_-(t) \cong Z \cong Z_+(t^{-1})$.

Using the inclusion $S^{n-1}_{G\times\mathbb{N}_+} \subset S^{n-1}_{G\times\mathbb{Z}}$ and compactness of S^{n-1}, there exists a $k \geq 0$ such that

$$t^k \circ \alpha(S^{n-1}_{G\times\mathbb{N}_+}) \subset Z_+.$$

Let Y_+ be the effect of attaching $D^n_{G\times\mathbb{N}_+}$ to Z_+ along $t^k \circ \alpha$. This gives an inclusion $Y_+ \subset Y$ which induces an isomorphism $Y_+(t^{-1}) \cong Y$. A similar argument constructs Y_-. □

Given objects $Y_-, Z_- \in \mathbb{C}_{\mathrm{f}}(G \times \mathbb{N}_-)$, and a $(G \times \mathbb{Z})$-map $Y_-(t) \to Z_-(t)$, one may ask whether it comes from a map $Y_- \to Z_-$. The next result in effect says that this is indeed the case up to a translation. As its proof is similar to the proof of Lemma 3.4, we omit the details.

3.5. Translation Lemma. *Suppose that we are given objects* $Y_-, Z_- \in \mathbb{C}_\mathrm{f}(G \times \mathbb{N}_-)$, *an object* Y of $\mathbb{C}_\mathrm{f}(G \times \mathbb{Z})$ *and cellular morphisms* $\beta : Y_-(t) \to Y$ *and* $f : Y \to Z_-(t)$. *Then there exists an integer* $k \geq 0$ *and a cellular morphism* $g_- : Y_- \to Z_-$, *such that the following diagram of morphisms commutes*:

$$\begin{array}{ccc} Y_-(t) & \xrightarrow{g_-(t)} & Z_-(t) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle t^k} \\ Y & \xrightarrow[f]{} & Z_-(t) \end{array}$$

4. The K-theory of the projective line as a pullback

4.1. Let

$$\mathbb{P}_\mathrm{f}(G) \to \mathbb{C}_\mathrm{f}(G \times \mathbb{N}_-) \quad \text{and} \quad \mathbb{P}_\mathrm{f}(G) \to \mathbb{C}_\mathrm{f}(G \times \mathbb{N}_+)$$

be the forgetful functors which are defined on objects by

$$(Y_-, Y, Y_+) \mapsto Y_- \quad \text{and} \quad (Y_-, Y, Y_+) \mapsto Y_+,$$

respectively. Similarly, there is a forgetful functor

$$\mathbb{P}_\mathrm{f}(G) \to \mathbb{C}_\mathrm{f}(G \times \mathbb{Z}),$$

which is given by $(Y_-, Y, Y_+) \mapsto Y$. These functors are cofibration-preserving.

However, the diagram

$$\begin{array}{ccc} \mathbb{P}_\mathrm{f}(G) & \longrightarrow & \mathbb{C}_\mathrm{f}(G \times \mathbb{N}_+) \\ \downarrow & \searrow & \downarrow \\ \mathbb{C}_\mathrm{f}(G \times \mathbb{N}_-) & \longrightarrow & \mathbb{C}_\mathrm{f}(G \times \mathbb{Z}) \end{array}$$

in which the lower and right-hand arrows arise from the telescope construction, commutes only up to equivalence. Additional categories $\mathbb{D}(G \times \mathbb{N}_-), \mathbb{D}(G \times \mathbb{N}_+)$ and $\mathbb{D}(G \times \mathbb{Z})$ substituting for $\mathbb{C}_\mathrm{f}(G \times \mathbb{N}_-), \mathbb{C}_\mathrm{f}(G \times \mathbb{N}_+)$ and $\mathbb{C}_\mathrm{f}(G \times \mathbb{Z})$, respectively, and new notions of weak equivalence on the variants of $\mathbb{P}_?(G)$ are required to get around this difficulty. Once the technical difficulties are resolved, we exhibit in Corollary 4.14 a homotopy cartesian square

$$\begin{array}{ccc} K(\mathbb{P}_\mathrm{fd}(G), h) & \longrightarrow & A^\mathrm{fd}(*, G \times \mathbb{N}_+) \\ \downarrow & & \downarrow \\ A^\mathrm{fd}(*, G \times \mathbb{N}_-) & \longrightarrow & A^\mathrm{fd}(*, G \times \mathbb{Z}). \end{array}$$

More useful, however (as will be seen in Section 7), is a delooped version of this square, called the *canonical diagram* of the projective line. This is defined in 4.13

below as the homotopy cartesian square

$$\begin{array}{ccc} \mathscr{P}_G & \longrightarrow & |h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)| \\ \downarrow & & \downarrow \\ |h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| & \longrightarrow & |h\mathscr{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \end{array}$$

together with a specific identification

$$\mathscr{P}_G \simeq |h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \times K_{-1}(\mathbb{Z}[\pi_0(G)]).$$

To proceed, we now define three new notions of weak equivalence on $\mathbb{P}_{\mathrm{f}}(G)$, in addition to the h-notion. We will say that a morphism $(f_-, f, f_+) : (Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{P}_{\mathrm{f}}(G)$ is an $h_{\mathbb{N}_-}$*-equivalence* if (and only if) f_- is a weak equivalence (hence also f, but not necessarily f_+). Similarly, (f_-, f, f_+) will be called an $h_{\mathbb{Z}}$*-equivalence* (resp. $h_{\mathbb{N}_+}$*-equivalence*) if (and only if) f (resp., f_+) is a weak equivalence.

These forgetful functors induce maps

$$|h_L\mathscr{S}.\mathbb{P}_{\mathrm{f}}(G)| \to |h\mathscr{S}.\mathbb{C}_{\mathrm{f}}(G \times L)|, \tag{4.1}$$

where L denotes either $\mathbb{N}_-$, $\mathbb{Z}$, or $\mathbb{N}_+$.

4.2. Proposition. *The map* (4.1) *is a homotopy equivalence, for* $L = \mathbb{N}_-$, $\mathbb{Z}$ *or* $\mathbb{N}_+$.

It is perhaps worth noting here that the result is asserted only for the 'f' decoration.

Proof of Proposition 4.2. We give the argument for the map

$$|h_{\mathbb{N}_+}\mathscr{S}.\mathbb{P}_{\mathrm{f}}(G)| \to |h\mathscr{S}.\mathbb{C}_{\mathrm{f}}(G \times \mathbb{N}_+)|,$$

as the proofs in the other cases are analogous. To this end, we shall apply the approximation theorem [8, 1.6.7].

We want to check that the exact functor

$$\mathbb{P}_{\mathrm{f}}(G), h_{\mathbb{N}_+}\mathbb{P}_{\mathrm{f}}(G)) \to (\mathbb{C}_{\mathrm{f}}(G \times \mathbb{N}_+), h\mathbb{C}_{\mathrm{f}}(G \times \mathbb{N}_+)),$$
$$Y_- \to YY_+ \mapsto Y_+$$

satisfies the two approximation properties.

The functor evidently reflects weak equivalences (cf. App 1 before the proof of Proposition 3.3(1)). Thus we only need to check App 2, i.e.,

Assertion. Given an object (Y_-, Y, Y_+) of $\mathbb{P}_{\mathrm{f}}(G)$ and a morphism $f_+ : Y_+ \to Z_+$ of $\mathbb{C}_{\mathrm{f}}(G \times \mathbb{N}_+)$, there exists a cofibration

$$(Y_-, Y, Y_+) \rightarrowtail (W_-, W, W_+)$$

and a weak equivalence $W_+ \xrightarrow{\simeq} Z_+$ in $\mathbb{C}_{\mathrm{f}}(G \times \mathbb{N}_+)$ such that the composite

$$Y_+ \rightarrowtail W_+ \xrightarrow{\simeq} Z_+$$

is identical to f_+.

By replacing f_+ by its mapping cylinder if necessary, we can assume, without any loss in generality, that f_+ is a cofibration. Set Z_1 equal to the amalgamated union

$$Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}).$$

By Lemma 3.4, there exists an object Z_- of $\mathbb{C}_f(G \times \mathbb{N}_-)$ and an isomorphism $Z_1 \xrightarrow{\cong} Z_-(t)$. Consequently, the maps fit together to yield a commutative diagram

$$\begin{array}{ccccc} Y_- & \longrightarrow & Y & \longleftarrow & Y_+ \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_+} \\ & & Z_-(t) & \longleftarrow & Z_+ \end{array} \tag{4.2}$$

where f is the composite $Y \xrightarrow{\subset} Z_1 \xrightarrow{\cong} Z_-(t)$.

By Translation Lemma 3.5, there exists an integer k and a morphism $g_- : Y_- \to Z_-$ so that

$$\begin{array}{ccc} Y_- & \longrightarrow & Y \\ {\scriptstyle g_-}\downarrow & & \downarrow{\scriptstyle f} \\ Z_- & \underset{t^k}{\longleftarrow} & Z_-(t) \end{array} \tag{4.3}$$

commutes. We infer that there is a morphism

$$(Y_-, Y, Y_+) \to (Z_-, Z_-(t), Z_+),$$

which has the given map $Y_+ \to Z_+$ as a component. Set (W_-, W, W_+) equal to the mapping cylinder of (g_-, f, f_+). Then diagrams (4.2)–(4.3) show that the object (W_-, W, W_+) fulfills the assertion. □

The four notions of weak equivalence on the projective line induce a commutative square of based spaces

$$\begin{array}{ccc} |h\mathscr{S}.\mathbb{P}_?(G)| & \longrightarrow & |h_{\mathbb{N}_+}S.\mathbb{P}_?(G)| \\ \downarrow & & \downarrow \\ |h_{\mathbb{N}_-}S.\mathbb{P}_?(G)| & \longrightarrow & |h_{\mathbb{Z}}S.\mathbb{P}_?(G)| \end{array} \tag{4.4}$$

for each of the decorations $? = \mathrm{f}, \mathrm{hf}, \mathrm{fd}, \mathrm{sfd}$.

4.3. Proposition. *With respect to the decorations* hf, fd, sfd, *square* (4.4) *is homotopy cartesian.*

Proof. We will apply the *fibration theorem* [8, 1.6.4] to each of the horizontal arrows to show that (4.4) has contractible iterated homotopy fiber. This will be sufficient to conclude that the diagram is homotopy cartesian since each of its vertices is connected.

We give the proof in the finitely dominated case only (the proofs in the other cases are similar). Let

$$\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G) \subset \mathbb{P}_{\mathrm{fd}}(G)$$

denote the full subcategory whose objects (Y_-, Y, Y_+) satisfy the condition that Y_+ is *acyclic* (i.e., the morphism from the zero object to Y_+ is a weak equivalence).

By the fibration theorem, the commutative square

$$\begin{array}{ccc} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G)| & \longrightarrow & |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \\ \downarrow & & \downarrow \\ |h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G)| & \longrightarrow & |h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \end{array} \tag{4.5}$$

is homotopy cartesian. Moreover, the term in the lower left-hand corner is contractible.

Similarly, let $\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{Z}}}(G) \subset \mathbb{P}_{\mathrm{fd}}(G)$ be the full subcategory with objects (Y_-, Y, Y_+), where Y is an acyclic object of $\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})$. Again by the fibration theorem, one has a commutative homotopy cartesian square

$$\begin{array}{ccc} |h_{\mathbb{N}_-}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{Z}}}(G)| & \longrightarrow & |h_{\mathbb{N}_-}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \\ \downarrow & & \downarrow \\ |h_{\mathbb{Z}}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{Z}}}(G)| & \longrightarrow & |h_{\mathbb{Z}}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \end{array}$$

in which the lower left-hand term is contractible. Moreover, square (4.5) maps via inclusion into the latter square. Hence, to show that (4.4) has contractible iterated homotopy fibre, it will suffice to show that the map

$$|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G)| \to |h_{\mathbb{N}_-}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{Z}}}(G)|$$

is a homotopy equivalence.

Consider the inclusion functor $\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G) \subset \mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{Z}}}(G)$ which induces this map. We shall apply the approximation theorem [8, 1.6.7] to show that this functor induces a homotopy equivalence on $\mathcal{S}.$-constructions. By definition, the functor reflects weak equivalences (cf. App 1 before the proof of Proposition 3.3(1)). We therefore seek to establish condition App 2 of the approximation theorem.

Suppose that

$$y := (Y_-, Y, Y_+) \to (Z_-, Z, Z_+) =: z$$

is a morphism of $\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{Z}}}(G)$ such that the source is an object of $\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G)$.

Assertion. There exists an object $w \in \mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G)$ and a factorization

$$y \rightarrowtail w \xrightarrow{\sim} z.$$

By applying the cylinder functor if necessary, we may assume, without loss in generality that $y \to z$ is a cofibration. Define an object $w := (W_-, W, W_+)$ of $\mathbb{P}_{\mathrm{fd}}^{h_{\mathbb{N}_+}}(G)$

by setting $W_- := Z_-$, $W := Z$, and $W_+ := Y_+$, where the map $W_+ \to W$ is taken to be the composite $Y_+ \to Y \to Z$. Then there is an evident factorization $y \rightarrowtail w \xrightarrow{\sim} z$, so the assertion holds. □

4.4. The canonical diagram. We now restrict our discussion to the finitely dominated case. The diagram

$$\begin{array}{ccc} \mathbb{P}_{\mathrm{fd}}(G) & \longrightarrow & \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+) \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-) & \longrightarrow & \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z}) \end{array}$$

in which the upper horizontal map is given by $(Y_-, Y, Y_+) \mapsto Y_+$, the left vertical map is given by $(Y_-, Y, Y_+) \mapsto Y_-$, and the maps into the terminal vertex are given by the telescope construction is *not* commutative. However, it is commutative up to a canonical chain of natural transformations which is described by the chain of weak equivalences

$$Y_-(t) \xrightarrow{\sim} Y \xleftarrow{\sim} Y_+(t^{-1}).$$

The lack of commutativity will be rectified below by introducing another model for $\mathbb{C}_{\mathrm{fd}}(G \times L)$, where L denotes $\mathbb{N}_-$, $\mathbb{Z}$ or $\mathbb{N}_+$. We will define a category $\mathbb{D}_{\mathrm{fd}}(G \times L)$ with cofibrations and weak equivalences and a factorization by exact functors

$$\mathbb{P}_{\mathrm{fd}}(G) \to \mathbb{D}_{\mathrm{fd}}(G \times L) \to \mathbb{C}_{\mathrm{fd}}(G \times L).$$

The functor $\mathbb{D}_{\mathrm{fd}}(G \times L) \to \mathbb{C}_{\mathrm{fd}}(G \times L)$ will induce an equivalence on $\mathscr{S}_\bullet$-constructions.

Suppose first that $L = \mathbb{N}_-$. The category $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ is defined so that

- An *object* is specified by a diagram $Y_- \to Y \leftarrow Y_+$, as in $\mathbb{P}_{\mathrm{fd}}(G)$, except that *we do not require* the induced cofibration $Y_+(t^{-1}) \to Y$ to be a weak equivalence (although we do require the other cofibration $Y_-(t) \to Y$ to be a weak equivalence).
- *Morphisms* and *cofibrations* of $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ are defined in the same way that we defined them for $\mathbb{P}_{\mathrm{fd}}(G)$.
- A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ is a *weak equivalence* if (and only if) the map $Y_- \to Z_-$ is a weak homotopy equivalence.

The category $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)$ is defined similarly, i.e., an object is specified by a diagram $Y_- \to Y \leftarrow Y_+$, where this time we only require the map $Y_+(t^{-1}) \to Y$ to be a weak equivalence. A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ in this instance is a weak equivalence if (and only if) the map $Y_+ \to Z_+$ is a weak homotopy equivalence.

Lastly, $\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})$ is defined to be the category whose objects are $Y_- \to Y \leftarrow Y_+$ with no condition imposed on the induced maps $Y_-(t) \to Y$ and $Y_+(t^{-1}) \to Y$ except that they should be cofibrations. A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ is specified to be a weak equivalence if (and only if) $Y \to Z$ is a weak homotopy equivalence.

It follows that $\mathbb{P}_{\mathrm{fd}}(G)$ is a (full) subcategory of $\mathbb{D}_{\mathrm{fd}}(G \times L)$ for $L = \mathbb{N}_-, \mathbb{N}_+, \mathbb{Z}$.

4.5. Lemma. *Let L be $\mathbb{N}_-$ ($\mathbb{Z}$ or $\mathbb{N}_+$). Then the forgetful functor*

$$\mathbb{D}_{\mathrm{fd}}(G \times L) \to \mathbb{C}_{\mathrm{fd}}(G \times L),$$
$$(Y_-, Y, Y_+) \mapsto Y_- \quad (\text{resp. } Y, Y_+)$$

induces a homotopy equivalence $|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)| \xrightarrow{\simeq} |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times L)|$.

Proof. We prove the lemma only when $L = \mathbb{N}_-$ as the other cases are similar. Let f denote the forgetful functor. Define an exact functor $g : \mathbb{C}_{\mathrm{fd}}(G \times L) \to \mathbb{D}_{\mathrm{fd}}(G \times L)$ by $g(Y_-) = Y_- \to Y_-(t) \leftarrow *$. Then $f \circ g$ and $g \circ f$ are equivalent to the identity in an evident way. $\square$

There is an inclusion of categories $\mathbb{P}_{\mathrm{fd}}(G) \subset \mathbb{D}_{\mathrm{fd}}(G \times L)$ which gives rise to a commutative diagram

$$\begin{array}{ccc} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)| \\ \downarrow & & \downarrow \\ |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \end{array} . \tag{4.6}$$

Let $\mathcal{P}_G$ denote the homotopy pullback of the diagram

$$|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| \to |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \leftarrow |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|.$$

The commutativity of (4.11) shows that there is a preferred map $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to \mathcal{P}_G$.

4.6. Theorem. *The map $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to \mathcal{P}_G$ induces an isomorphism on homotopy groups in positive degrees. In particular, there is a homotopy equivalence*

$$\mathcal{P}_G \simeq |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \times K_{-1}(\mathbb{Z}[\pi_0(G)]),$$

where the second factor denotes the cokernel of the homomorphism

$$K_0(\mathbb{Z}[\pi_0(G)](t^{-1})) \times K_0(\mathbb{Z}[\pi_0(G)](t)) \to K_0(\mathbb{Z}[\pi_0(G)](t, t^{-1}))$$

given on each summand by the map induced by inclusion.

4.7. Terminology. The homotopy cartesian square

$$\begin{array}{ccc} \mathcal{P}_G & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)| \\ \downarrow & & \downarrow \\ |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \end{array}$$

together with the identification

$$\mathcal{P}_G \simeq |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \times K_{-1}(\mathbb{Z}[\pi_0(G)])$$

of Theorem 4.6 will be called the *canonical diagram* of the projective line.

Proof of Theorem 4.6. We first deduce the second assertion from the first one. The first assertion implies that $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|$ is homotopy equivalent to the component of the base point of $\mathcal{P}_G$. The space $\mathcal{P}_G$ is the homotopy pullback of maps of group-like H-spaces, by [8, 1.6.2]. Hence $\mathcal{P}_G$ is also a group-like H-space, and $\mathcal{P}_G$ is homotopy equivalent to the cartesian product of its base point component with $\pi_0(\mathcal{P}_G)$. Using Lemma 1.7(3) and the long exact sequence for the homotopy groups in a homotopy cartesian square, it follows that $\pi_0(\mathcal{P}_G)$ is isomorphic to $K_{-1}(\mathbb{Z}[\pi_0(G)])$. This gives the second assertion.

We now prove the first assertion. By Lemma 1.7(3), Proposition 4.2 and Lemma 4.5 we infer that the maps

$$|h_L\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)|$$

induce isomorphisms on homotopy groups in degrees > 1 for $L = \mathbb{N}_-, \mathbb{Z}$ or $\mathbb{N}_-$. Using Proposition 4.3, we infer that the map $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to \mathcal{P}_G$ also induces an isomorphism on homotopy groups in dimensions > 1. We are therefore reduced to showing that the map induces an isomorphism on fundamental groups. Surjectivity will be a consequence of the canonical presentation of K_0 of a category with cofibrations and weak equivalences. The injectivity part will be a consequence of the results of Sections 6 and 7 which are independent of this section.

Surjectivity. Recall that $\pi_1(|h\mathcal{S}.C|)$ for a category C with cofibrations and weak equivalences is an abelian group equipped with generators $[c]$ so that c is an object of C, with relations of two kinds:
(1) A cofibration sequence $c \rightarrowtail d \to d/c$ gives rise to the relation $[c] + [d/c] = [d]$.
(2) A weak equivalence $c \xrightarrow{\sim} d$ gives rise to the relation $[c] = [d]$.

Using the homotopy equivalences (Lemma 4.5), we see that $\mathcal{P}_G$ is homotopy equivalent to the homotopy pullback of the diagram

$$|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)| \to |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})| \leftarrow |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+)|.$$

Let $x \in \pi_1(\mathcal{P}_G)$ be an element. Then x is represented by objects $U_L \in \mathbb{C}_{\mathrm{fd}}(G \times L)$ for $L = \mathbb{N}_-, \mathbb{N}_+, \mathbb{Z}$ which are subject to the condition that $U_{\mathbb{N}_-}(t)$, $U_{\mathbb{N}_+}(t^{-1})$ and $U_{\mathbb{Z}}$ represent the same element of $\pi_1(|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})|)$. Using relations (1) and (2) above, we can assume that $U_{\mathbb{N}_-}$ is a retract up to homotopy of a j-*spherical object*, i.e., an object which is a finite coproduct of objects of the form $S^j_{\mathbb{N}_-}$ for some fixed positive integer j. Similar considerations apply to the objects $U_{\mathbb{N}_+}$ and $U_{\mathbb{Z}}$. Consequently, x is represented by V_L in which the latter is a j-spherical homotopy retract for $L = \mathbb{N}_-, \mathbb{N}_+, \mathbb{Z}$.

By wedging on a further j-spherical object (if necessary), we may conclude that x is represented by V_L in which there are weak homotopy equivalences

$$V_{\mathbb{N}_-}(t) \simeq V \simeq V_{\mathbb{N}_+}(t^{-1}).$$

Let V' be the result of converting the induced map $V_{\mathbb{N}_-}(t) \vee V_{\mathbb{N}_+}(t^{-1}) \to V$ into a cofibration. It follows that the triple $(V_{\mathbb{N}_-}, V', V_{\mathbb{N}_+})$ represents an element of $\pi_1(|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|)$ which maps to x. This establishes surjectivity.

Injectivity. In Section 6 we identify $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|$; we show in 6.8 that there is a certain homotopy equivalence

$$|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\simeq} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|,$$

such that the composite with the map

$$|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to \mathcal{P}_G \to |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)| \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_+)|$$

is a co-retract up to homotopy (the retraction property is a consequence of 7.1 and Lemma 7.2). This implies the injectivity of the map $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to \mathcal{P}_G$ on the level of fundamental groups. □

4.5. Corollary. *The commutative diagram* (4.6) *induces a homotopy cartesian square*

$$\begin{array}{ccc} K(\mathbb{P}_{\mathrm{fd}}(G), h) & \longrightarrow & A^{\mathrm{fd}}(*, G \times \mathbb{N}_+) \\ \downarrow & & \downarrow \\ A^{\mathrm{fd}}(*, G \times \mathbb{N}_-) & \longrightarrow & A^{\mathrm{fd}}(*, G \times \mathbb{Z}). \end{array}$$

Proof. The homotopy equivalence $\mathcal{P}_G \simeq |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \times K_{-1}(\mathbb{Z}[\pi_0(G)])$ looped once gives a homotopy equivalence

$$\Omega\mathcal{P}_G \simeq \Omega|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| =: K(\mathbb{P}_{\mathrm{fd}}(G), h).$$

The assertion now follows by applying the loop functor to the canonical diagram. □

5. Auxiliary functors

Our goal in the next section (Section 6) is to produce a homotopy equivalence

$$|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \simeq |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)|$$

which, of course, loops to a homotopy equivalence $K(\mathbb{P}_{\mathrm{fd}}(G), h) \simeq A^{\mathrm{fd}}(*, G) \times A^{\mathrm{fd}}(*, G)$. The proof of this result will be modeled on Quillen's proof of an analogous result for rings [4, Chapter 8, Theorem 3.1] and requires auxiliary functors $\Gamma : \mathbb{P}(G) \to \mathbb{C}(G)$ and, for $n \in \mathbb{Z}$, $\psi_n : \mathbb{C}_{\mathrm{fd}}(G) \to \mathbb{P}_{\mathrm{fd}}(G)$. Definitions of these functors and the basic identities they satisfy are given in this section.

5.1. Global sections. Define a functor Γ: $\mathbb{P}(G) \to \mathbb{C}(G)$ by the rule

$$Y_- \to Y \leftarrow Y_+ \mapsto CY_- \cup_{Y_-} Y \cup_{Y_+} CY_+,$$

where CY_- denotes the cone of Y_-, and CY_+ is the cone on Y_+.[2] This construction preserves cofibrations and weak equivalences.

[2] The use of the term 'global sections' here is actually a misnomer. However, it conveys a similar idea: $\Gamma(Y_-, Y, Y_+)$ is a model for the homotopy cofibre of the evident map $Y_- \vee Y_+ \to Y$, which we think of as *stably* representing the 'overlap' of Y_- with Y_+ inside Y up to a suspension.

5.2. Lemma. *The functor Γ maps finitely dominated objects to s-finitely dominated objects. Hence Γ induces a map*

$$|h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathscr{S}.\mathbb{C}_{\mathrm{sfd}}(G)| \simeq |h\mathscr{S}.\mathbb{C}_{\mathrm{fd}}(G)|.$$

Proof. As Γ preserves retractions, it is sufficient to show that Γ applied to a finite object is s-finitely dominated. We shall show that Γ applied to a finite object is finitely dominated after one suspension.

Call a finite object $z := (Z_-, Z, Z_+)$ *globally finite* if its skeletal filtration $z^j := (Z^j_-, Z^j, Z^j_+)$ is such that z^j is obtained from z^{j-1} by attaching a finite coproduct of objects of the kind

$$D^j_{G\times\mathbb{N}_-} \xrightarrow{t^r} D^j_{G\times\mathbb{Z}} \xleftarrow{t^s} D^j_{G\times\mathbb{N}_+} \quad \text{for some } r,s \in \mathbb{Z},$$

where the amalgamation is taken along morphisms

$$(S^{j-1}_{G\times\mathbb{N}_-} \xrightarrow{t^r} S^{j-1}_{G\times\mathbb{Z}} \xleftarrow{t^s} S^{j-1}_{G\times\mathbb{N}_+}) \to z^{j-1}.$$

By a straightforward induction which we omit, one sees that Γ applied to a globally finite object yields a homotopy finite object of $\mathbb{C}(G)$.

Let $y := (Y_-, Y, Y_+)$ be an arbitrary (locally) finite object of $\mathbb{P}(G)$. Choose a globally finite object $z := Z_- \xrightarrow{\alpha_-} Z \xleftarrow{\alpha_+} Z_+$ together with a weak equivalence $f : Y \xrightarrow{\sim} Z$ of $\mathbb{C}_f(G\times \mathbb{Z})$ (as is guaranteed, say, by Lemma 3.4). For integers $k, \ell \in \mathbb{Z}$, let $z_{k,\ell}$ be the object

$$Z_- \xrightarrow{t^k\circ\alpha_-} Z \xrightarrow{t^\ell\circ\alpha_+} Z_+.$$

Then, by induction, $z_{k,\ell}$ is also globally finite.

By translation Lemma 3.5 applied twice, there exist integers k, ℓ, and morphisms $g_- : Y_- \to Z_-$ and $g_+ : Y_+ \to Z_+$ which satisfy a commutative diagram

$$\begin{array}{ccccc} Y_- & \longrightarrow & Y & \longleftarrow & Y_+ \\ \Big\downarrow g_- & & f \Big\downarrow & & \Big\downarrow g_+ \\ Z_- & \xrightarrow[t^k\circ\alpha_-]{} & Z & \xleftarrow[t^\ell\circ\alpha_+]{} & Z_+ \end{array}$$

i.e., a morphism $(g_-, f, g_+) : y \to z_{k,\ell}$. Let $c := C_- \to C \leftarrow C_+$ denote the mapping cone of (g_-, f, g_+), given by the component-wise mapping cone of each of the maps g_-, f and g_+. Then C is acyclic, because $f : Y \to Z$ is a weak equivalence. This implies $C_+(t^{-1})$ and $C_-(t)$ are acyclic.

However, since $C_+(t^{-1})$ is acyclic, and C_+ is finite in $\mathbb{C}(G \times \mathbb{N}_+)$, there exists an integer $m \geq 0$ so that the map

$$C_+ \xrightarrow{t^m} C_+$$

is homotopic through morphisms of $\mathbb{C}(G)$ to the constant map to the base point.

This implies that the identity map id : $C_+ \to C_+$ factors up to homotopy in $\mathbb{C}(G)$ through the quotient map $C_+ \to C_+/t^m(C_+)$, where $C_+/t^m(C_+)$ makes sense because $t^m : C_+ \to C_+$ is a cofibration when considered as a morphism of $\mathbb{C}(G)$. Moreover, a straightforward induction on the number of cells of C_+ shows that $C_+/t^m(C_+)$ is a finite object of $\mathbb{C}(G)$. It follows that C_+ is a finitely dominated object of $\mathbb{C}(G)$. Similarly, C_- is a finitely dominated object of $\mathbb{C}(G)$.

As C is acyclic, it follows that $\Gamma(c)$ is finitely dominated. The evident cofibration sequence

$$\Gamma(y) \rightarrowtail \Gamma(T(y \to z_{k,\ell})) \to \Gamma(c)$$

(where $T(y \to z_{k,\ell})$ denotes the mapping cylinder of $y \to z_{k,\ell}$) then shows that $\Gamma(y)$ is stably finitely dominated. □

5.3. Extension by scalars. Let L be a monoid and M the realization of a simplicial monoid. The *extension by scalars* functor is given by

$$\begin{aligned} \mathbb{C}_{\mathrm{fd}}(M) &\to \mathbb{C}_{\mathrm{fd}}(M \times L) \\ K &\mapsto K \otimes L, \end{aligned}$$

where $K \otimes L$ denotes $(K \times L)/(* \times L)$.

5.4. The twists. For each integer n, define an exact functor

$$\theta_n : \mathbb{P}_{\mathrm{fd}}(G) \to \mathbb{P}_{\mathrm{fd}}(G),$$

where if $y = (Y_-, Y, Y_+)$ is an object of $\mathbb{P}_{\mathrm{fd}}(G)$, then $\theta_n(y)$ is the object

$$Y_- \xrightarrow{t^n} Y \leftarrow Y_+,$$

where the map $Y_+ \to Y$ is as before, and the map t^n is shorthand notation for the composite

$$Y_- \to Y \xrightarrow{t^n} Y$$

(inclusion followed by left translation).

5.5. The canonical sheaves. For $n \in \mathbb{Z}$ we define an exact functor

$$\psi_n : \mathbb{C}_{\mathrm{fd}}(G) \to \mathbb{P}_{\mathrm{fd}}(G)$$

called the *canonical sheaf twisted by n*.

To define it, let K be an object of $\mathbb{C}_{\mathrm{fd}}(G)$. Let $\psi_0(K)$ be the object of $\mathbb{P}_{\mathrm{fd}}(G)$ given by

$$K \otimes \mathbb{N}_- \xrightarrow{\subseteq} K \otimes \mathbb{Z} \xleftarrow{\supseteq} K \otimes \mathbb{N}_+,$$

where $K \otimes \mathbb{N}_-$, etc., are as in 5.3 (recall that $\mathbb{N}_-$ and $\mathbb{N}_+$ are, respectively, the negative and positive integers with 0 included).

Define $\psi_n(K)$ by

$$\psi_n(K) := \theta_n \circ \psi_0(K),$$

where θ_n is as in 5.4.

By a straightforward argument which we omit, it is readily verified that

5.6. Lemma. *If $n \geq 0$, then $\Gamma \circ \psi_n$ is equivalent as an exact functor to*

$$K \mapsto \underbrace{\Sigma K \oplus \cdots \oplus \Sigma K}_{(n+1)\ \text{times}} \quad ;$$

if $n < 0$, then $\Gamma \circ \psi_n$ is equivalent as an exact functor to

$$K \mapsto \underbrace{K \oplus \cdots \oplus K}_{-(n+1)\ \text{times}},$$

where in each case $\oplus$ means the coproduct operation (i.e., wedge sum).

6. Identification of the K-theory of the projective line

In this section we apply the constructions of the preceding section. First, we prove that the global sections functor Γ and the twist θ_1 combine to induce a homotopy equivalence

$$|h\mathscr{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)| \overset{(\Gamma\circ\theta_1,\Gamma)}{\longrightarrow} |h\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)|.$$

In particular, there will be a homotopy equivalence

$$K(\mathbb{P}_{\mathrm{fd}}(G), h) \simeq A^{\mathrm{fd}}(*, G) \times A^{\mathrm{fd}}(*, G).$$

We also prove that the canonical sheaf functors ψ_0 and ψ_{-1} induce a homotopy equivalence

$$|h\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \overset{\psi_0\oplus\psi_{-1}}{\longrightarrow} |h\mathscr{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)|.$$

6.1. Define a coarser category $h_\Gamma \mathbb{P}_{\mathrm{fd}}(G)$ of weak equivalences in $\mathbb{P}_{\mathrm{fd}}(G)$ by declaring a morphism $x \to y$ to be an *h_Γ-equivalence* if (and only if) $\Gamma(x) \to \Gamma(y)$ is a weak equivalence in $\mathbb{C}_{\mathrm{sfd}}(G)$. Correspondingly, we have $\mathbb{P}^\Gamma_{\mathrm{fd}}(G)$, the full subcategory of $\mathbb{P}_{\mathrm{fd}}(G)$ consisting of those objects x of the latter for which $\Gamma(x)$ is acyclic.

6.2. Proposition. *The square*

$$\begin{array}{ccc} |h\mathscr{S}_\bullet \mathbb{P}^\Gamma_{\mathrm{fd}}(G)| & \longrightarrow & |h\mathscr{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)| \\ \big\downarrow & & \big\downarrow \Gamma \\ |h_\Gamma\mathscr{S}_\bullet \mathbb{P}^\Gamma_{\mathrm{fd}}(G)| & \underset{\Gamma}{\longrightarrow} & |h\mathscr{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \end{array}$$

is homotopy cartesian. The lower left corner is contractible. Moreover, the right vertical map admits a section up to homotopy.

Proof. Application of the fibration theorem [8, 1.6.4] shows that the square (induced by the evident inclusions)

$$\begin{array}{ccc} |h\mathcal{S}.\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| & \longrightarrow & |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \\ \downarrow & & \downarrow \\ |h_{\Gamma}\mathcal{S}.\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| & \longrightarrow & |h_{\Gamma}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \end{array} \tag{6.1}$$

is homotopy cartesian. We will show that the map

$$|h_{\Gamma}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma} |h\mathcal{S}.\mathbb{C}_{\mathrm{sfd}}(G)|$$

is a homotopy equivalence. This will establish that the square in the statement of Proposition 6.2 is homotopy cartesian. The section to the right vertical map in this square is provided by ψ_0, since Lemma 5.6 shows that the composite

$$|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\psi_0} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to |h_{\Gamma}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma} |h\mathcal{S}.\mathbb{C}_{\mathrm{sfd}}(G)|$$

is a homotopy equivalence. Hence, Γ is a surjection on homotopy in all degrees. We are reduced to showing that Γ is an injection on homotopy in all degrees.

Let $\mathbb{R}^+$ denote the real numbers with the addition of a disjoint basepoint. Translation defines a $\mathbb{Z}$-action on $\mathbb{R}^+$ which is free in the based sense. Similarly, let $\mathbb{R}^+_{\leq 0}$ and $\mathbb{R}^+_{\geq 0}$ respectively denote the nonpositive and nonnegative real numbers equipped with their evident $\mathbb{N}_-$ and $\mathbb{N}_+$ actions.

Let $\Sigma' : \mathbb{P}_{\mathrm{fd}}(G) \to \mathbb{P}_{\mathrm{fd}}(G)$ be the functor which is given by mapping an object $y = (Y_-, Y, Y_+)$ to the object

$$\mathbb{R}^+_{\leq 0} \wedge (CY_- \cup_{Y_-} CY) \to \mathbb{R}^+ \wedge \Sigma Y \leftarrow \mathbb{R}^+_{\geq 0} \wedge (CY \cup_{Y_+} CY_+)$$

(with evident structure maps and the action on each smash product is the diagonal one). Each term in the above expression is naturally equivalent to a suspension, i.e., Σ' is equivalent, as an exact functor, to the suspension functor.

If we set $K{:=}\Gamma(y)$, it follows that there is a commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & \mathbb{R}^+_{\geq 0} \wedge (CY \cup_{Y_+} CY_+) \\ \downarrow & & \downarrow \\ \mathbb{R}^+_{\leq 0} \wedge (CY_- \cup_{Y_-} CY) & \longrightarrow & \mathbb{R}^+ \wedge \Sigma Y. \end{array}$$

The map $K \to \mathbb{R}^+_{\leq 0} \wedge (CY_- \cup_{Y_-} CY)$ may be extended to a map

$$K \otimes \mathbb{N}_- \to \mathbb{R}^+_{\leq 0} \wedge (CY_- \cup_{Y_-} CY)$$

by forcing equivariance. Similarly, we may define maps $K \otimes \mathbb{Z} \to \mathbb{R}^+ \wedge \Sigma Y$ and $K \otimes \mathbb{N}_+ \to \mathbb{R}^+_{\geq 0} \wedge (CY_+ \cup_{Y_+} CY)$, which, taken together, provide an h_{Γ}-equivalence

$$\psi_0(K) \xrightarrow{\sim} \Sigma' y.$$

This procedure describes an exact natural transformations (with respect to the h_{Γ}- equivalences) from $\psi_0 \circ \Gamma$ to Σ'. But Σ' is an equivalence on $\mathcal{S}.$-constructions, so Γ gives an injection on homotopy in all degrees. □

The next step is to identify the initial vertex in the square of Proposition 6.2 with $|h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|$.

6.6. Proposition. *The exact functor*

$$\Gamma \circ \theta_1 \,:\, \mathbb{P}^{\Gamma}_{\mathrm{fd}}(G) \to \mathbb{C}_{\mathrm{fd}}(G)$$

induces a homotopy equivalence

$$|h\mathcal{S}_\bullet\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \xrightarrow{\simeq} |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|.$$

This will be proven below. Let us first note that application of 6.6 yields the main result of this section, namely

6.7. Corollary. *The map*

$$|h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{(\Gamma\circ\theta_1,\Gamma)} |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence. In particular, there is a homotopy equivalence

$$K(\mathbb{P}_{\mathrm{fd}}(G), h) \simeq A^{\mathrm{fd}}(*, G) \times A^{\mathrm{fd}}(*, G).$$

Proof. From the cartesian square (Proposition 6.2), there is a homotopy fiber sequence

$$|h\mathcal{S}_\bullet\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma} |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|.$$

By Proposition 6.6, we find that the composite

$$|h\mathcal{S}_\bullet\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma\circ\theta_1} |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence. It follows that

$$(\Gamma \circ \theta_1, \Gamma) \,:\, |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence. □

For the proof of the main result of this paper, we will also need another corollary of Proposition 6.6 which gives a homotopy equivalence in the other direction. Consider the composite map

$$\begin{aligned}\psi_{-1} \oplus \psi_0 \,:\, |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| &\xrightarrow{\psi_{-1}\times\psi_0} |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \times |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)| \\ &\xrightarrow{\oplus} |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|,\end{aligned}$$

where $\oplus$ is induced by coproduct operation on $\mathbb{P}_{\mathrm{fd}}(G)$ (this gives $|h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|$ the structure of an H-space, by [8, p. 330]). Using Lemma 5.6, one sees that the composite

$$(\Gamma \circ \theta_1, \Gamma) \circ (\psi_{-1} \oplus \psi_0)$$

is the map which is given (up to homotopy) by the matrix

$$\begin{pmatrix} \Sigma^{\oplus 2} & \Sigma \\ \Sigma & 0 \end{pmatrix},$$

where $\Sigma^{\oplus 2}(K) = \Sigma K \oplus \Sigma K$ is the coproduct of two copies of ΣK. Since Σ induces a homotopy equivalence on $|h\mathcal{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$ (it is a homotopy inverse for the H-space structure), it follows that the map is homotopy invertible. Hence,

6.8. Corollary. [cf. Quillen [4, Chapter 8, Theorem 3.1]]*The map*

$$|h\mathcal{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\psi_{-1}\oplus\psi_0} |h\mathcal{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence.

With these corollaries out of the way, we now proceed with the proof of Proposition 6.6. Our first step is to define yet another collection of subcategories of weak equivalences on $\mathbb{P}_{\mathrm{fd}}(G)$.

For $j \in \mathbb{N} \cup \infty$, let $h_{\Gamma_{\leq j}}$ denote the (coarser) notion of weak equivalence on $\mathbb{P}_{\mathrm{fd}}(G)$ whereby a morphism $x \to y$ is a weak equivalence if and only if the induced map

$$\Gamma(\theta_i(x)) \to \Gamma(\theta_i(y))$$

is weak homotopy equivalence for $0 \leq i \leq j$, where $\theta_i(x)$ denotes the ith twist of x, defined in 5.4. This also restricts to a notion of weak equivalence on $\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)$.

Proof of Proposition 6.6. By Corollary 6.11 below, the canonical map

$$|h\mathcal{S}_{\bullet}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \to |h_{\Gamma_{\leq 1}}\mathcal{S}_{\bullet}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence. To prove that the map $\Gamma \circ \theta_1 : |h\mathcal{S}_{\bullet}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$ is a homotopy equivalence, it is therefore sufficient to prove that $\Gamma \circ \theta_1$ induces a homotopy equivalence

$$|h_{\Gamma_{\leq 1}}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|.$$

On the one hand, the functor $\psi_{-1} : \mathbb{C}_{\mathrm{fd}}(G) \to \mathbb{P}_{\mathrm{fd}}(G)$ factors through $\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)$ and the factorization $\psi_{-1} : \mathbb{C}_{\mathrm{fd}}(G) \to \mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)$ is exact with respect to the notion of weak equivalence given by $h_{\Gamma_{\leq 1}}$ on the codomain. Then we have

$$\begin{aligned}(\Gamma \circ \theta_1) \circ \psi_{-1} &= \Gamma \circ (\theta_1 \circ \psi_{-1}) \\ &= \Gamma \circ \psi_0 \quad \text{by (5.5)}, \\ &\simeq \Sigma \quad \text{by Lemma 5.6.}\end{aligned}$$

Consequently, $\Gamma \circ \theta_1$ induces a surjection on homotopy groups.

On the other hand, we assert that the composition

$$\psi_{-1} \circ (\Gamma \circ \theta_1) : (\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G), h_{\Gamma_{\leq 1}}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)) \to (\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G), h_{\Gamma_{\leq 1}}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G))$$

is equivalent by a chain of equivalences of exact functors to the suspension functor Σ. To see this, recall that there is a chain of equivalences from the exact functor $\psi_0 \circ \Gamma$ to the suspension functor Σ with respect to the h-notion of weak equivalence (cf. the discussion before Proposition 6.6). As the $h_{\Gamma_{\leq 1}}$-notion of weak equivalence is coarser, it follows that there is a chain of equivalences from $\psi_0 \circ \Gamma$ to Σ with respect to the $h_{\Gamma_{\leq 1}}$-notion of weak equivalence. Apply θ_1 to the right and θ_{-1} to the left of $\psi_0 \circ \Gamma$. It follows that there is a chain of equivalences of exact functors from $\theta_{-1} \circ \psi_0 \circ \Gamma \circ \theta_1 = \psi_{-1} \circ \Gamma \circ \theta_1$ to $\theta_{-1} \circ \Sigma \circ \theta_1 = \Sigma$ (with respect to the $h_{\Gamma_{\leq 1}}$-notion of weak equivalence).

In particular, $\Gamma \circ \theta_1$ induces an injection on homotopy groups. It follows that

$$|h_{\Gamma_{\leq 1}} \mathscr{S}.\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \xrightarrow{\Gamma \circ \theta_1} |h\mathscr{S}.\mathbb{C}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence. This completes the proof of Proposition 6.6. □

6.9. Lemma. *The evident map*

$$|h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \to |h_{\Gamma_{\leq\infty}} \mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence.

Proof. Let $\mathbb{P}^{\Gamma_{\leq\infty}}_{\mathrm{fd}}(G)$ be the full subcategory of $\mathbb{P}_{\mathrm{fd}}(G)$ consisting of objects x for which $\Gamma(\theta_n(x))$ are acyclic objects of $\mathbb{C}_{\mathrm{fd}}(G)$, for all $n \geq 0$.

By the fibration theorem [8, 1.6.4], there is a homotopy cartesian square

$$\begin{array}{ccc} |h\mathscr{S}.\mathbb{P}^{\Gamma_{\leq\infty}}_{\mathrm{fd}}(G)| & \longrightarrow & |h\mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \\ \downarrow & & \downarrow \\ |h_{\Gamma_{\leq\infty}} \mathscr{S}.\mathbb{P}^{\Gamma_{\leq\infty}}_{\mathrm{fd}}(G)| & \longrightarrow & |h_{\Gamma_{\leq\infty}} \mathscr{S}.\mathbb{P}_{\mathrm{fd}}(G)| \end{array}$$

in which the lower left corner is contractible. It is therefore sufficient to show that $|h\mathscr{S}.\mathbb{P}^{\Gamma_{\leq\infty}}_{\mathrm{fd}}(G)|$ is contractible. This will be true if every object of $\mathbb{P}^{\Gamma_{\leq\infty}}_{\mathrm{fd}}(G)$, after a suitable number of suspensions, is weak equivalent (with respect to the h-notion of weak equivalence) to the 0-object $(*,*,*)$.

Let $y{:=}(Y_-, Y, Y_+)$ be an object of $\mathbb{P}^{\Gamma_{\leq\infty}}_{\mathrm{fd}}(G)$. By suspending if necessary, we can assume that the components of y are simply connected. By the equivariant Whitehead theorem, it is then sufficient to prove that Y_- and Y_+ have trivial reduced singular homology. We will show that the groups $H_*(Y_+)$ are trivial (and hence also $H_*(Y)$); an analogous argument will show that the groups $H_*(Y_-)$ are trivial.

Denote the structure map $Y_- \to Y$ by α_- and the structure map $Y_+ \to Y$ by α_+. Let $[v_+] \in H_i(Y_+)$ be a homology class with representing cycle v_+. By the compactness of v_+, there exists an integer n so that $[\alpha_+(v_+)] \in H_i(Y)$ is the image under $t^n\alpha_-$ of a class $[v_-] \in H_i(Y_-)$.

Using the Mayer–Vietoris sequence,

$$\cdots \to H_{i+1}(\Gamma(\theta_n(y))) \to H_i(Y_-) \oplus H_i(Y_+) \xrightarrow{t^n\alpha_- \oplus \alpha_+} H_i(Y) \to H_i(\Gamma(\theta_n(y)) \to \cdots,$$

it follows that the element $-[v_-] \oplus [v_+] \in H_i(Y_-) \oplus H_i(Y_+)$ maps to zero. Using the assumption $H_i(\Gamma(\theta_n(y))){=}0$ for all $i \geq 0$, it follows that $-[v_-]{\oplus}[v_+]{=}0$. Consequently, $[v_+]$ is also zero. This shows that $H_i(Y_+)$ is trivial, since $[v_+]$ was arbitrarily chosen. □

6.10. Lemma. *The inclusions of subcategories of weak equivalences $h_{\Gamma_{\le i+1}}\mathbb{P}_{\mathrm{fd}}(G) \subset h_{\Gamma_{\le i}}\mathbb{P}_{\mathrm{fd}}(G)$ yield homotopy equivalences*

$$|h_{\Gamma_{\le 1}}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| \xleftarrow{\simeq} |h_{\Gamma_{\le 2}}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)| \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} |h_{\Gamma_{\le\infty}}\mathscr{S}_{\bullet}\mathbb{P}_{\mathrm{fd}}(G)|.$$

Proof. Suspension induces a homotopy equivalence on these $\mathscr{S}_{\bullet}$-constructions, so it will suffice to show that a morphism of $\mathbb{P}_{\mathrm{fd}}(G)$ which suspends to an $h_{\Gamma_{\le n-1}}$-equivalence (for $n \ge 2$) also suspends to an $h_{\Gamma_{\le n}}$-equivalence.

Let $y := (Y_-, Y, Y_+)$ be an object of $\mathbb{P}_{\mathrm{fd}}(G)$. For each nonnegative integer n, there is a homotopy pushout square in $\mathbb{C}_{\mathrm{sfd}}(G)$ of the form

$$\begin{array}{ccc} \Gamma(\theta_{n-2}(y)) & \xrightarrow{a_1} & \Gamma(\theta_{n-1}(y)) \\ {\scriptstyle b_1}\downarrow & & \downarrow{\scriptstyle b_2} \\ \Gamma(\theta_{n-1}(y)) & \xrightarrow[a_2]{} & \Gamma(\theta_n(y)) \end{array},$$

where $\theta_n(y)$ is the nth twist of y as in 5.4, and

- the map a_2 is induced by the applying Γ to top and bottom in the diagram

$$\begin{array}{ccccc} Y_- & \xrightarrow{t^{n-1}} & Y & \longleftarrow & Y_+ \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ Y_- & \xrightarrow{t^{n}} & Y & \longleftarrow & Y_+ \end{array}$$

  and a_1 is defined similarly.
- The map b_2 is defined by applying Γ to top and bottom in the diagram

$$\begin{array}{ccccc} Y_- & \xrightarrow{t^{n-1}} & Y & \longleftarrow & Y_+ \\ {\scriptstyle t^{-1}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ Y_- & \xrightarrow{t^{n}} & Y & \longleftarrow & Y_+ \end{array}$$

  and b_1 is defined similarly.

Using the square, a straightforward induction shows that if a suitable suspension of y is acyclic with respect to the $\Gamma_{\le n-1}$ notion of equivalence (for $n \ge 2$) then it is also acyclic with respect to the $\Gamma_{\le n}$ notion of equivalence. This implies the result, since a morphism suspends to a weak equivalence if and only if its mapping cone is stably acyclic. □

6.11. Corollary. *The map*

$$|h\mathscr{S}_{\bullet}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)| \to |h_{\Gamma_{\le 1}}\mathscr{S}_{\bullet}\mathbb{P}^{\Gamma}_{\mathrm{fd}}(G)|$$

is a homotopy equivalence.

Proof. We have a commutative square,

$$\begin{array}{ccc} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| & \longrightarrow & |h_\Gamma \mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \\ \simeq \big\downarrow & & \big\| \\ |h_{\Gamma_{\le 1}}\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| & \longrightarrow & |h_\Gamma \mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \end{array}$$

which by Lemmas 6.9 and 6.10 is homotopy cartesian. The result then follows by application of the fibration theorem [8, 1.6.4] to its horizontal arrows. □

7. The "fundamental theorem"

In this section we complete the proof of the "fundamental theorem": we will establish a splitting

$$A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times \mathcal{B}A^{\mathrm{fd}}(X) \times N_-A^{\mathrm{fd}}(X) \times N_+A^{\mathrm{fd}}(X),$$

where $N_-A^{\mathrm{fd}}(X)$ and $N_+A^{\mathrm{fd}}(X)$ are naturally isomorphic and $\mathcal{B}A^{\mathrm{fd}}(X)$ denotes a canonical non-connective delooping of $A^{\mathrm{fd}}(X)$.

7.1. Augmentation. Let L be one of the monoids $\mathbb{N}_-$, $\mathbb{Z}$ or $\mathbb{N}_+$. We define a functor $\varepsilon: \mathbb{D}_{\mathrm{fd}}(G \times L) \to \mathbb{C}_{\mathrm{fd}}(G)$ which will enable us to factor out a copy of $|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)|$ from $|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)|$.

For $y = (Y_-, Y, Y_+) \in \mathbb{D}_{\mathrm{fd}}(G \times L)$, we take $\varepsilon(y)$ to be

$$Y/\mathbb{Z},$$

i.e., the orbit space under the action of $\mathbb{Z}$. The same description defines an exact functor $\varepsilon: \mathbb{P}_{\mathrm{fd}}(G) \to \mathbb{C}_{\mathrm{fd}}(G)$.

7.2. Lemma. *If $K \in \mathbb{C}_{\mathrm{fd}}(G)$, then there is a natural isomorphism*

$$\varepsilon(\psi_n(K)) \xrightarrow{\cong} K$$

for every $n \in \mathbb{Z}$, where *ψ_n is the canonical sheaf functor twisted by n* (*see* 5.5).

Proof. $\psi_n(K)$ has the form $(K \otimes \mathbb{N}_-, K \otimes \mathbb{Z}, K \otimes \mathbb{N}_+)$, and $K \otimes \mathbb{Z}$ is a $\mathbb{Z}$-fold wedge of copies of K. Then $(K \otimes \mathbb{Z})/\mathbb{Z}$ is isomorphic to K. □

To state the next lemma, we introduce some notation: If $g: A \to B$ is a map of based spaces, we let A^g denote its homotopy fiber; it comes equipped with a map $A^g \to A$.

7.3. Lemma. *The canonical map $|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)|^\varepsilon \to |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)|$ admits a left homotopy inverse. Moreover, there is a homotopy equivalence*

$$|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)| \simeq |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G \times L)|^\varepsilon \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)|.$$

Similarly, the canonical map $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \to |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|$ *admits a left homotopy inverse, and there is a homotopy equivalence*

$$|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \simeq |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)|.$$

Proof. Using the previous lemma, it is readily verified that the composition

$$\begin{aligned}&|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times L)|^{\varepsilon} \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \\ &\quad\to |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times L)| \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\mathrm{id}\oplus\psi_0} |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times L)|\end{aligned}$$

gives the homotopy equivalence. Choosing a homotopy inverse and then following with the projection $|h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times L)|^{\varepsilon} \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \to |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times L)|^{\varepsilon}$ gives the left homotopy inverse. The argument for the other map is the same. □

7.4. The main result. Recall from 4.7 that there is a homotopy cartesian square

$$\begin{array}{ccc} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)| \times K_{-1}(\mathbb{Z}[\pi_0(G)]) & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)| \\ \downarrow & & \downarrow \\ |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)| & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|. \end{array}$$

By taking homotopy fibres of the maps ε, we obtain another homotopy cartesian square

$$\begin{array}{ccc} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \times K_{-1} & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_+)|^{\varepsilon} \\ \downarrow & & \downarrow \\ |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{N}_-)|^{\varepsilon} & \longrightarrow & |h\mathcal{S}.\mathbb{D}_{\mathrm{fd}}(G\times\mathbb{Z})|^{\varepsilon}, \end{array} \tag{7.1}$$

where K_{-1} is shorthand notation for $K_{-1}(\mathbb{Z}[\pi_0(G)])$.

From Corollary 6.8 we have a homotopy equivalence $\psi_{-1}\oplus\psi_0 : |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\simeq} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|$. As ε equalizes ψ_0 and ψ_{-1}, there exists a map

$$(\psi_{-1}\ominus\psi_0)^{\varepsilon} : |h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \to |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon}$$

whose composite with the map $|h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \to |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|$ is homotopic to $\psi_{-1}\ominus\psi_0$, where the latter denotes $\psi_{-1}\oplus\Sigma\psi_0$ (the notation is intended so as to recall the fact that suspension represents a choice of homotopy inverse to the H-multiplication).

By the elementary properties of Γ, we know that $\Gamma\circ\psi_{-1}$ is null homotopic and $\Gamma\circ\psi_0$ is equivalent to the identity. This implies that $\Gamma\circ(\psi_{-1}\ominus\psi_0)^{\varepsilon}$ is homotopic to the identity map. Consequently,

7.5. Lemma. *The map*

$$|h\mathcal{S}.\mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{(\psi_{-1}\ominus\psi_0)^{\varepsilon}} |h\mathcal{S}.\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon}$$

is a homotopy equivalence.

7.6. Lemma. *The map*

$$|h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)|^\varepsilon \times K_{-1} \to |h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^\varepsilon$$

is null homotopic. Similarly, the map $|h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)|^\varepsilon \times K_{-1} \to |h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^\varepsilon$ *is also null homotopic.*

Proof. It suffices by symmetry to prove the first part. Since the map is a morphism of group-like H-spaces, it is sufficient to show that the restriction to its identity component is null homotopic, i.e., it suffices to prove that the map

$$|h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)|^\varepsilon \to |h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^\varepsilon$$

is null homotopic.

Since $(\psi_{-1} \ominus \psi_0)^\varepsilon : |h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)|^\varepsilon$ is a homotopy equivalence (Lemma 7.5), it will be sufficient to show that the composite

$$|h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{(\psi_{-1}\ominus\psi_0)^\varepsilon} |h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)|^\varepsilon \to |h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^\varepsilon$$

is null homotopic.

By Lemma 7.2, the map $|h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^\varepsilon \to |h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|$ is a co-retraction up to homotopy. Hence, we are further reduced to showing that composition with this map is null homotopic. But this composite is homotopic to

$$|h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\psi_{-1}\ominus\psi_0} |h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|,$$

by the way $(\psi_{-1} \ominus \psi_0)^\varepsilon$ was defined. Since the map $|h\mathcal{S}_\bullet \mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)| \to |h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)|$ is a homotopy equivalence (by Lemma 4.5), we are even further reduced to showing that

$$|h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \xrightarrow{\psi_{-1}\ominus\psi_0} |h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)| \to |h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)|$$

is null homotopic, where the second of these maps is induced by the forgetful functor.

Let K be an object of $\mathbb{C}_{\mathrm{fd}}(G)$. Then $\psi_0(K)$ is given by

$$K \otimes \mathbb{N}_- \xrightarrow{\subseteq} K \otimes \mathbb{Z} \xleftarrow{\supseteq} K \otimes \mathbb{N}_+.$$

Similarly, $\psi_{-1}(K)$ is given by

$$K \otimes \mathbb{N}_- \xrightarrow{t^{-1}} K \otimes \mathbb{Z} \xleftarrow{\supseteq} K \otimes \mathbb{N}_+.$$

It follows from the definitions that the $\mathbb{N}_-$-components of $\psi_0(K)$ and $\psi_{-1}(K)$ are identical. This means that the forgetful functor $\mathbb{P}_{\mathrm{fd}}(G) \to \mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)$ equalizes $\psi_0(K)$ and $\psi_{-1}(K)$. We infer that the composite

$$\begin{aligned} |h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| &\xrightarrow{(\Sigma,\mathrm{id})} |h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \times |h\mathcal{S}_\bullet \mathbb{C}_{\mathrm{fd}}(G)| \\ &\xrightarrow{\psi_{-1}\ominus\psi_0} |h\mathcal{S}_\bullet \mathbb{P}_{\mathrm{fd}}(G)| \to |\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{N}_-)| \end{aligned}$$

is null homotopic, as was to be shown. □

For the proof of the main result, we shall need one last technical lemma.

7.7. Lemma. *Suppose that $f : X \to Z$ and $g : Y \to Z$ are maps of connected, based spaces which have the homotopy type of CW complexes. Let P be the homotopy pullback of f and g and suppose that the natural map $P \to X \times Y$ is null homotopic. Then there exists a homotopy equivalence of based spaces*

$$\Omega Z \simeq P \times \Omega X \times \Omega Y.$$

Proof. There is a homotopy fibre sequence $\Omega X \times \Omega Y \to \Omega Z \to P$ which is induced by the null homotopic map $P \to X \times Y$ (note that the second map in this sequence is surjective on π_0 because X and Y are connected).

Since the homotopy fibre sequence is induced by a null homotopic map, it is homotopically trivial. A choice of trivialization gives the conclusion in the statement of the lemma. □

We are now ready to prove the main result. Let X be a connected, based space. Let $G_\bullet$ denote the Kan loop group of the total singular complex of X, and set $G = |G_\bullet|$. Then $A^{\mathrm{fd}}(X)$ is given by $\Omega|h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)|$ (cf. Remark 1.6, Lemma 1.7(3)).

7.8. "Fundamental Theorem". *There is a splitting,*

$$A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times \mathscr{B}A^{\mathrm{fd}}(X) \times N_- A^{\mathrm{fd}}(X) \times N_+ A^{\mathrm{fd}}(X),$$

where $N_- A^{\mathrm{fd}}(X)$ and $N_+ A^{\mathrm{fd}}(X)$ are naturally isomorphic and $\mathscr{B}A^{\mathrm{fd}}(X)$ denotes a canonical non-connective delooping of $A^{\mathrm{fd}}(X)$.

Proof. Recall that by fibering the canonical diagram of the projective line (4.7) over the augmentation map (of 7.1), we obtained the homotopy cartesian square (7.1)

$$\begin{array}{ccc} |h\mathcal{S}_\bullet\mathbb{P}_{\mathrm{fd}}(G)|^{\varepsilon} \times K_{-1} & \longrightarrow & |h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^{\varepsilon} \\ \downarrow & & \downarrow \\ |h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^{\varepsilon} & \longrightarrow & |h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|^{\varepsilon} \end{array}$$

whose initial vertex is homotopy equivalent to $|h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \times K_{-1}$, by Lemma 7.5.

Using Lemma 7.6, we may apply Lemma 7.7 to conclude that there is a homotopy equivalence

$$\Omega|h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|^{\varepsilon} \simeq \mathscr{B}A^{\mathrm{fd}}(X) \times \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^{\varepsilon} \times \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^{\varepsilon},$$

where

$$\mathscr{B}A^{\mathrm{fd}}(X) := |h\mathcal{S}_\bullet\mathbb{C}_{\mathrm{fd}}(G)| \times K_{-1}$$

is a nonconnective one-fold delooping of $A^{\mathrm{fd}}(X)$.

We now define the *nil-terms.* Set $N_- A^{\mathrm{fd}}(X)$ equal to $\Omega|h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_-)|^{\varepsilon}$, and similarly, set $N_+ A^{\mathrm{fd}}(X)$ equal to $\Omega|h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{N}_+)|^{\varepsilon}$. With these notational changes, we have

$$\Omega|h\mathcal{S}_\bullet\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|^{\varepsilon} \simeq \mathscr{B}A^{\mathrm{fd}}(X) \times N_- A^{\mathrm{fd}}(X) \times N_+ A^{\mathrm{fd}}(X).$$

Take the cartesian product of both sides with $A^{\mathrm{fd}}(X) = \Omega|h\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)|$. Using the equivalence

$$\begin{aligned} A^{\mathrm{fd}}(X \times S^1) &= \Omega|h\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G \times \mathbb{Z})| \\ &\simeq \Omega|h\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})| \\ &\simeq \Omega|h\mathscr{S}_{\bullet}\mathbb{C}_{\mathrm{fd}}(G)| \times \Omega|h\mathscr{S}_{\bullet}\mathbb{D}_{\mathrm{fd}}(G \times \mathbb{Z})|^{\varepsilon} \end{aligned}$$

(by Lemmas 7.3 and 4.5), we obtain the splitting

$$A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times \mathscr{B}A^{\mathrm{fd}}(X) \times N_{-}A^{\mathrm{fd}}(X) \times N_{+}A^{\mathrm{fd}}(X). \qquad \square$$

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