TOPOLOGICAL HOCHSCHILD HOMOLOGY

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1. *Introduction*

In the appendix to [20] Waldhausen discussed a trace map $\mathrm{tr}: K(R) \longrightarrow \mathrm{HH}(R)$, from the algebraic K-theory of a ring to its Hochschild homology, which can be used to obtain information about $K(R)$ from $\mathrm{HH}(R)$. In [1] Bökstedt described a factorization of this trace map. The intermediate functor $\mathrm{THH}(HR)$ is called the topological Hochschild homology of the Eilenberg–MacLane spectrum HR associated with R, because it is constructed similarly to Hochschild homology with the tensor product replaced by the smash product of spectra.

There has been considerable interest in computations of $\mathrm{THH}(HR)$, primarily because it coincides with the stable K-theory of R [5]. Bökstedt himself determined $\mathrm{THH}(H\mathbb{Z})$ and $\mathrm{THH}(H\mathbb{F}_p)$ [2] (see also Remark 4.7). Using a spectral sequence from [2], Lindenstrauss [9] and Pirashvili [13] showed that $\mathrm{THH}_*(HR)$ is the derived tensor product of ordinary Hochschild homology $\mathrm{HH}_*(R)$ and topological Hochschild homology of the respective ground ring if R is a certain ring extension of $\mathbb{Z}$ or $\mathbb{F}_p$. Their results can be summarized as follows.

If R is $\mathbb{Z}[X]/(X^n)$, $\mathbb{Z}[X]/(X^n - 1)$, a group ring $\mathbb{Z}[G]$, or an extension $\mathbb{F}_p[X]/(f)$ of $\mathbb{F}_p$, where f is a monic polynomial, then

$$\mathrm{THH}_n(HR) \cong \bigoplus_{p+q=n} \mathrm{HH}_p(R) \otimes_{\mathbb{Z}} \mathrm{THH}_q(H\mathbb{Z}) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{HH}_p(R), \mathrm{THH}_q(H\mathbb{Z})). \tag{1.1}$$

The objective of this paper is to provide a conceptual frame for such reductions: the existence of a lift of the extension to an extension of the sphere spectrum $\mathbb{S}$. We prove the following.

THEOREM 2.1. *Let K be a commutative ring and R be a K-algebra. Suppose there is a good $\mathbb{S}$-algebra spectrum E (for example, a cell algebra spectrum) and a weak equivalence of HK-algebra spectra $E \wedge_{\mathbb{S}} HK \longrightarrow HR$; then there is a homotopy equivalence of HK-module spectra*

$$\mathrm{THH}(HR) \simeq \mathrm{THH}^{HK}(HR) \wedge_{HK} \mathrm{THH}(HK).$$

The results (1.1) are fairly immediate consequences of this theorem (see Corollary 2.2 and the applications). If R is a commutative ring, $\mathrm{THH}(HR)$ is a ring spectrum and the above homotopy equivalence is multiplicative, providing information about the ring structure of $\mathrm{THH}_*(HR)$ (Corollary 2.3).

2. *Definitions and main results*

We recall the following definition of Hochschild homology, which easily generalizes to suitable spectra. Let K be a commutative ring, R be a K-algebra, and

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M be an R-bimodule. The *Hochschild homology space* $\mathrm{HH}^K(R;M)$ of R with coefficients in M over the ground ring K is the topological realization of the simplicial K-module

$$[n] \longmapsto R^{\otimes n} \otimes_K M$$

with the face and degeneracy maps

$$d_i(r_n \otimes_K \dots \otimes_K r_1 \otimes_K m) = \begin{cases} r_n \otimes_K \dots \otimes_K r_2 \otimes_K (r_1 \cdot m) & i = 0 \\ r_n \otimes_K \dots \otimes_K (r_{i+1} \cdot r_i) \otimes_K \dots \otimes_K m & 0 \leqslant i \leqslant n. \\ r_{n-1} \otimes_K \dots \otimes_K (m \cdot r_n) & i = n \end{cases}$$
$$s_i(r_n \otimes_K \dots \otimes_K r_1 \otimes_K m) = r_n \otimes_K \dots \otimes_K 1 \otimes_K r_i \otimes_K \dots \otimes_K m \qquad 0 \leqslant i \leqslant n. \tag{2.1}$$

The *Hochschild homology groups* are defined to be the homotopy groups of the topological K-module $\mathrm{HH}^K(R;M)$. We write $\mathrm{HH}^K(R)$ for $\mathrm{HH}^K(R;R)$ and $\mathrm{HH}(R)$ for $\mathrm{HH}^{\mathbb{Z}}(R)$.

Topological Hochschild homology is defined analogously. One has to replace rings, algebras, and bimodules by suitable ring spectra, algebra spectra, and bimodule spectra, and the tensor product by the smash product of spectra.

When Bökstedt constructed topological Hochschild homology in 1985, there was no known category of spectra with an associative, commutative and unital smash product functor. This situation has changed in recent years; there are competing constructions of symmetric monoidal smash product functors on suitable categories of spectra having the same formal properties as the classical tensor product [**6**, **8**, **10**]. In this paper we work in the setting and, unless stated otherwise, use the terminology of [**6**].

Let S denote the sphere spectrum, and let K be a commutative S-algebra, R be a K-algebra, and M be an R-bimodule (note that we consider spectra!). We write $R^{\wedge n}$ for the n-fold smash product $R \wedge_K \dots \wedge_K R$, if the ground S-algebra K is clear from the context. We define the *topological Hochschild homology spectrum* $\mathrm{THH}^K(R;M)$ of R with coefficients in M over K to be the realization in the category of spectra of the simplicial spectrum

$$[n] \longmapsto \mathrm{thh}_n^K(R;M) = R^{\wedge n} \wedge_K M$$

with face and degeneracy maps analogous to those of (2.1). Our notation differs from that of [**6**] where $\mathrm{THH}^K(R;M)$ and $\mathrm{thh}_n^K(R;M)$ are denoted by $\mathrm{thh}^K(R;M)$ and $\mathrm{thh}^K(R;M)_n$ respectively. The *topological Hochschild homology groups* are the homotopy groups of $\mathrm{THH}^K(R;M)$. We write $\mathrm{THH}^K(R)$ for $\mathrm{THH}^K(R;R)$ and $\mathrm{THH}(R)$ for $\mathrm{THH}^{\mathrm{S}}(R)$. The equivalence of our definition with the one of Bökstedt in the q-cofibrant case (see Section 3) can be deduced from the combined results of [**18**] and [**19**].

The 'inclusion of the 0-skeleton' defines a map of K-modules

$$\xi : M \longrightarrow \mathrm{THH}^K(A;M).$$

If R is a commutative K-algebra, $\mathrm{thh}_\bullet^K(R)$ is a simplicial commutative K-algebra; hence $\mathrm{THH}^K(R)$ is a commutative K-algebra. The map

$$\xi : R \longrightarrow \mathrm{THH}^K(R)$$

makes it a commutative R-algebra with unit ξ.

If K is a commutative ring, R is a K-algebra, and M is an R-bimodule in the classical sense, there are associated Eilenberg–MacLane spectra HK, HR and HM so that $\mathrm{THH}^{HK}(HR;HM)$ can be defined. Here some care is required; to make sure that the homotopy type of $\mathrm{THH}^{HK}(HR;HM)$ does not depend on the choice of the models for HK, HR and HM we have to take q-cofibrant models (see Remark 3.7). We will always assume this. For details on the notion of *q-cofibrancy* see Section 3. The Eilenberg–MacLane spectra constructed in [6] have this property.

Our main observation is the following result.

THEOREM 2.1. *Let K be a commutative ring and R be a K-algebra in the classical sense. Suppose there is a q-cofibrant* $\mathbf{S}$*-algebra E and a weak equivalence of HK-algebras* $E \wedge_{\mathbf{S}} HK \longrightarrow HR$*, where HR is represented as q-cofibrant HK-algebra; then there is a homotopy equivalence of HK-module spectra*

$$\mathrm{THH}(HR) \simeq \mathrm{THH}^{HK}(HR) \wedge_{HK} \mathrm{THH}(HK).$$

If R is commutative, E is a q-cofibrant commutative $\mathbf{S}$*-algebra, and HR is represented by a q-cofibrant commutative HK-algebra, this homotopy equivalence is a homotopy equivalence of commutative HK-algebra spectra.*

Given the assumptions of the theorem we in principle can calculate the homotopy of $\mathrm{THH}^{HK}(HR) \wedge_{HK} \mathrm{THH}(HK)$, and we obtain the following.

COROLLARY 2.2. *Suppose that K is a semi-hereditary commutative ring (see [3]), that R is K-flat, and that the assumptions of Theorem 2.1 hold. Then there is a non-natural isomorphism of graded K-modules*

$$\mathrm{THH}_n(HR) \cong \bigoplus_{p+q=n} \mathrm{HH}_p^K(R) \otimes_K \mathrm{THH}_q(HK) \otimes \bigoplus_{p+q=n-1} \mathrm{Tor}_1^K(\mathrm{HH}_p^K(R), \mathrm{THH}_q(HK)).$$

In the commutative case we obtain information about the multiplicative structure.

COROLLARY 2.3. *If R is a commutative K-algebra, the assumptions of Theorem* 2.1 *for the commutative case and the assumptions of Corollary* 2.2 *hold, and if* $\mathrm{HH}_p^K(R)$ *is K-flat for all p, then*

$$\mathrm{THH}_*(HR) \cong \mathrm{HH}_*^K(R) \otimes_K \mathrm{THH}_*(HK)$$

as graded K-algebras.

REMARK 2.4. Theorem 2.1 and its corollaries can be slightly generalized: we can replace the 'ground ring spectrum' $\mathbf{S}$ by any q-cofibrant commutative $\mathbf{S}$-algebra A throughout and obtain the analogous results by exactly the same proofs.

3. *Background on spectra and proofs*

Proof of Theorem 2.1. (1) We have a sequence of isomorphisms

$$\begin{aligned}\mathrm{THH}(E \wedge_{\mathbf{S}} HK) &\cong \mathrm{THH}(E) \wedge_{\mathbf{S}} \mathrm{THH}(HK) && \text{(by Lemma 3.1)}\\ &\cong \mathrm{THH}(E) \wedge_{\mathbf{S}} HK \wedge_{HK} \mathrm{THH}(HK)\\ &\cong \mathrm{THH}^{HK}(E \wedge_{\mathbf{S}} HK) \wedge_{HK} \mathrm{THH}(HK) && \text{(by Lemma 3.2).}\end{aligned}$$

(2) Given the assumptions of the theorem there are genuine homotopy equivalences of HK-modules (HK-algebras in the commutative case)

$$\mathrm{THH}(E \wedge_{\mathbf{S}} HK) \simeq \mathrm{THH}(HR) \quad \text{and} \quad \mathrm{THH}^{HK}(E \wedge_{\mathbf{S}} HK) \simeq \mathrm{THH}^{HK}(HR). \qquad \square$$

We first provide the steps of part (1) of the proof.

LEMMA 3.1. *Let K be a commutative* S*-algebra, and let Q and R be K-algebras. Then*

$$\mathrm{THH}^K(Q \wedge_K R) \cong \mathrm{THH}^K(Q) \wedge_K \mathrm{THH}^K(R)$$

as K-modules. If R is commutative, this is an isomorphism of R-modules, and if both Q and R are commutative K-algebras, this is an isomorphism of commutative $(Q \wedge_K R)$-algebras.

Proof. By [6, X.1.4] $\mathrm{THH}^K(Q) \wedge_K \mathrm{THH}^K(R)$ can be viewed as the realization of the simplicial K-module

$$[n] \longmapsto Q^{\wedge(n+1)} \wedge_K R^{\wedge(n+1)} \cong (Q \wedge_K R)^{\wedge(n+1)}.$$

If R is commutative, this is an isomorphism of R-modules, and if both Q and R are commutative, this is an isomorphism of $(Q \wedge_K R)$-algebras. □

LEMMA 3.2. *Let K be a commutative* S*-algebra, and let R be a K-algebra, and B be a commutative K-algebra. Then*

$$\mathrm{THH}^K(R) \wedge_K B \cong \mathrm{THH}^B(R \wedge_K B)$$

as B-modules. If also R is commutative, this is an isomorphism of commutative $(R \wedge_K B)$-algebras.

Proof. $\mathrm{THH}^B(R \wedge_K B)$ is the realization of the simplicial B-module

$$[n] \longmapsto (R \wedge_K B) \wedge_B \dots \wedge_B (R \wedge_K B).$$

Note that we have a B-module structure only if B is commutative. This simplicial B-module is isomorphic as simplicial B-module to

$$[n] \longmapsto R^{\wedge(n+1)} \wedge_K B.$$

Since $- \wedge_K B$ commutes with realizations, the result follows. If R is commutative, we have isomorphisms of $(R \wedge_K B)$-algebras. □

Part (2) of the proof of Theorem 2.1 is not a trivial consequence of the assumptions. If one tries to show directly that the weak equivalence $E \wedge_S HK \longrightarrow HR$ induces a weak equivalence $\mathrm{THH}^{HK}(E \wedge_S HK) \longrightarrow \mathrm{THH}^{HK}(HR)$, or if one tries to deduce Corollary 2.2 from the theorem by calculating the homotopy groups of $\mathrm{THH}^{HK}(HR) \wedge_{HK} \mathrm{THH}(HK)$, one runs into problems which can best be explained by an analogy: the situation for chain complexes and tensor products is similar. Taking homotopy groups corresponds to taking homology groups, and weak equivalences correspond to quasi-isomorphisms. If C is a chain complex of R-modules, the tensor product $C \otimes_R -$ preserves homotopy, but not quasi-isomorphisms unless C is flat in some sense. Similarly, the homology of the tensor product of two chain complexes can be computed by the Künneth spectral sequence provided one of the complexes satisfies a flatness condition [11, XII.12.3]. We are in an analogous situation:

(1) Let R be an S-algebra and let M be a q-cofibrant right R-module. Then $M \wedge_R - : {}_R\mathrm{Mod} \longrightarrow {}_S\mathrm{Mod}$ preserves weak equivalences [6, 3.8].

(2) Let R be an S-algebra, M be a right R-module and N be a left R-module. There is a strongly convergent natural right half-plane spectral sequence of differential $\pi_*(R)$-modules

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*(R)}_{p,q}(\pi_* M, \pi_* N) \Rightarrow \pi_{p+q}(M \wedge_R N)$$

provided that M or N is q-cofibrant as a right, respectively left, R-module [6, IV.4.1].

A projective resolution of a chain complex is certainly flat. A q-cofibrant object corresponds to a projectively resolved one in the context of closed model category structures in the sense of Quillen [**14**].

Let R be an S-algebra. The categories ${}_R\mathrm{Mod}$, Mod_R, Alg_R, and $\mathscr{C}\mathrm{Alg}_R$ of left and right R-modules, R-algebras, and commutative R-algebras admit topological closed model category structures. Their weak equivalences are the weak equivalences of underlying spectra, their cofibrations, called *q-cofibrations*, are retracts of relative cell objects in their categories, and all objects are fibrant. A relative cell object is a morphism $A \longrightarrow B$ obtained by successively attaching cells to A. The following results are a consequence of this structure:

(i) Each object X has a q-cofibrant resolution, that is, there is a weak equivalence $\Gamma X \longrightarrow X$ with ΓX q-cofibrant.

(ii) Since all objects are fibrant, any two weakly equivalent q-cofibrant objects are genuinely homotopy equivalent in their category.

(iii) Given a sequence of q-cofibrations $X_0 \longrightarrow X_1 \longrightarrow \dots$ then the induced map $X_0 \longrightarrow \operatorname{colim} X_n$ is a q-cofibration.

(iv) Our categories $\mathscr{C}$ are tensored and cotensored over Top, that is, we have continuous functors

$$\mathscr{C} \times \mathrm{Top} \longrightarrow \mathscr{C}, \quad (X, K) \longmapsto X \otimes K$$
$$\mathscr{X} \times \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathscr{C}, \quad (X, K) \longmapsto X^K$$

and natural homeomorphisms

$$\mathscr{C}(X \otimes K, Y) \cong \mathrm{Top}(K, \mathscr{C}(X, Y)) \cong \mathscr{C}(X, Y^K).$$

Moreover, if $A \longrightarrow B$ is a q-cofibration and $X \longrightarrow Y$ is a q-fibration and a weak equivalence in $\mathscr{C}$, then the induced map

$$\mathscr{C}(B, X) \longrightarrow \mathscr{C}(A, X) \times_{\mathscr{C}(A, Y)} \mathscr{C}(B, Y)$$

is a Serre fibration and a weak equivalence. Passing to adjoints, this implies that if $i: A \longrightarrow B$ is a q-cofibration and $K \subset L$ is a CW-pair, then $A \otimes L \bigcup_{A \otimes K} B \otimes K \longrightarrow B \otimes L$ is a q-cofibration.

(v) Let $Q \longrightarrow R$ be a map of (commutative) S-algebras. Since $R \wedge_Q -$ preserves cell objects it preserves q-cofibrant objects.

We are now in a position to prove part (2) of the proof of Theorem 2.1. Since E is a q-cofibrant (commutative) S-algebra, $E \wedge_{\mathrm{S}} HK$ is a q-cofibrant (commutative) HK-algebra by (v). Hence the weak equivalence $E \wedge_{\mathrm{S}} HK \longrightarrow HR$ is a genuine homotopy equivalence in Alg_{HK} (respectively $\mathscr{C}\mathrm{Alg}_{HK}$) by (ii). The induced maps of simplicial HK-modules (respectively commutative HK-algebras)

$$\mathrm{thh}^{\mathrm{S}}_{\bullet}(E \wedge_{\mathrm{S}} HK) \longrightarrow \mathrm{thh}^{\mathrm{S}}_{\bullet}(HR) \quad \text{and} \quad \mathrm{thh}^{HK}_{\bullet}(E \wedge_{\mathrm{S}} HK) \longrightarrow \mathrm{thh}^{HK}_{\bullet}(HR)$$

are dimensionwise homotopy equivalences. By [**6**, X.2.4] we obtain homotopy equivalences of HK-modules after realization, provided the simplicial objects involved are proper as simplicial HK-modules. This is proved in Lemma 3.3 below.

In the commutative case we want a homotopy equivalence in $\mathscr{C}\mathrm{Alg}_{HK}$. This follows from (ii) and Lemma 3.6 below.

LEMMA 3.3. *Let R be a q-cofibrant object in* Alg_K *or* $\mathscr{C}\mathrm{Alg}_K$. *Then* $\mathrm{thh}^K_{\bullet}(R)$ *is proper (the term will be explained in the proof) as simplicial K-module, and it is proper as simplicial R-module in the commutative case.*

We have to distinguish between the associative and the commutative case, because the forgetful functor $\mathscr{C}\mathrm{Alg}_K \longrightarrow \mathrm{Alg}_K$ does not preserve q-cofibrant objects. This is a well-known phenomenon: in ordinary algebra free associative resolutions use tensor algebras, while free associative and commutative resolutions use symmetric algebras.

Proof of Lemma 3.3. We use results from [**15**]. A map $f: X \longrightarrow Y$ of module or algebra spectra is called a strong cofibration if each Hurewicz fibration in the category has the relative homotopy lifting property with respect to f. By assumption, the unit map $\eta: K \longrightarrow R$ is a q-cofibration. We show in [**15**] that the forgetful functors from Alg_K or $\mathscr{C}\mathrm{Alg}_K$ to Mod_K map q-cofibrations to strong cofibrations. Hence $\eta: K \longrightarrow R$ is a strong cofibration of K-modules. Let $sR^{\wedge n}$ denote the 'union of the images' of the maps

$$\mathrm{id}^{n-i-1} \wedge \eta \wedge \mathrm{id}^i : R^{\wedge(n-1)} \longrightarrow R^{\wedge n}, \quad 0 \leqslant i \leqslant n-1,$$

that is, the appropriate iterated pushout in Mod_K. By the pairing theorem for strong cofibrations the induced map $sR^{\wedge n} \longrightarrow R^{\wedge n}$ is a strong cofibration [**15**]. The map $sR^{\wedge n} \wedge_K R \longrightarrow R^{\wedge(n+1)}$ is the 'inclusion' of the degenerate part $\deg(\mathrm{thh}_n^K(R))$ into $\mathrm{thh}_n^K(R)$. Recall that $\mathrm{thh}_n^K(R)$ is called *proper* if this map has the homotopy extension property for each n. Since $- \wedge_K R$ preserves strong cofibrations, the result follows. □

To compute the homotopy of $\mathrm{THH}^{HK}(HR) \wedge_{HK} \mathrm{THH}(HK)$ using (1) and (2), it suffices to know that a q-cofibrant resolution $\Gamma(\mathrm{THH}(HK)) \longrightarrow \mathrm{THH}(HK)$ of $\mathrm{THH}(HK)$ as HK-module induces a weak equivalence

$$\mathrm{THH}^{HK}(HR) \wedge_{HK} \Gamma(\mathrm{THH}(HK)) \longrightarrow \mathrm{THH}^{HK}(HR) \wedge_{HK} \mathrm{THH}(HK).$$

For the commutative case, where $\mathrm{THH}^{HK}(HR)$ and $\mathrm{THH}(HK)$ are commutative HK-algebras, this follows from [**6**, VII.6.5, VII.6.7] and Lemma 3.6. For the associative case this follows from Lemma 3.4 and a relative variant of (1), Lemma 3.5.

LEMMA 3.4. *Let R be q-cofibrant in* Alg_K. *Then* $\xi: R \longrightarrow \mathrm{THH}^K(R)$ *and its composite* $K \longrightarrow R \longrightarrow \mathrm{THH}^K(R)$ *with the unit of R are q-cofibrations in* Mod_K.

Proof. By [**6**, VII.6.2] the unit $\eta: K \longrightarrow R$ is a q-cofibration of K-modules. Hence (R, K) is a retract of a relative K-cell module $(\bar{R}, K)$. Let $\bar{\eta}: K \longrightarrow \bar{R}$ be the inclusion. Form the degenerate parts $\deg(\mathrm{thh}_n^K(R))$ and $\deg(\mathrm{thh}_n^K(\bar{R}))$ as in Lemma 3.3. Then $\deg(\mathrm{thh}_n^K(\bar{R}))$ is a subcomplex of $\mathrm{thh}_n^K(\bar{R})$ so that $\deg(\mathrm{thh}_n^K(\bar{R})) \longrightarrow \mathrm{thh}_n^K(\bar{R})$ is a q-cofibration of K-modules. Hence so is $\deg(\mathrm{thh}_n^K(R)) \longrightarrow \mathrm{thh}_n^K(R)$. Let $\mathrm{THH}^K(R)^{(n)}$ denote the n-skeleton of the realization. Then by (iv) the map $\mathrm{THH}^K(R)^{(n)} \longrightarrow \mathrm{THH}^K(R)^{(n+1)}$ is a q-cofibration of K-modules, and the statement follows from (iii). □

LEMMA 3.5. *If* $R \longrightarrow M$ *is a q-cofibration of right R-modules, then the functor* $M \wedge_R - : {}_R\mathrm{Mod} \longrightarrow {}_S\mathrm{Mod}$ *preserves weak equivalences.*

Proof. Since retractions preserve weak equivalences, we may assume that $R \longrightarrow M$ is a relative cell R-module. If $\phi: N_1 \longrightarrow N_2$ is a weak equivalence of R-modules, so is $R \wedge \phi: R \wedge_R N_1 \longrightarrow R \wedge_R N_2$. Now proceed as in the proof of [**6**, III.3.8]. □

LEMMA 3.6. *Let R be q-cofibrant as commutative K-algebra. Then* $\mathrm{THH}^K(R)$ *is q-cofibrant in* $\mathscr{C}\mathrm{Alg}_R$ *and hence also in* $\mathscr{C}\mathrm{Alg}_K$.

Proof. By [**12**, 4.5] $\mathrm{THH}^K(R)$ is isomorphic as R-algebra to the internal realization of the simplicial R-algebra $\mathrm{thh}^K_\bullet(R)$ in $\mathscr{C}\mathrm{Alg}_R$, constructed by using its tensor structure over Top (iv). We investigate this internal realization. The functor $-\wedge_K R:\mathscr{C}\mathrm{Alg}_K \longrightarrow \mathscr{C}\mathrm{Alg}_R$ preserves colimits and q-cofibrations. Applying it to the pushout

$$\begin{array}{ccc} K & \xrightarrow{\ \eta\ } & R \\ \Big\downarrow{\scriptstyle\eta} & & \Big\downarrow{\scriptstyle\eta\wedge\mathrm{id}} \\ R & \xrightarrow{\ \mathrm{id}\wedge\eta\ } & R\wedge_K R \end{array}$$

in $\mathscr{C}\mathrm{Alg}_K$ we obtain the pushout

$$\begin{array}{ccc} R & \longrightarrow & R\wedge_K R \\ \Big\downarrow & & \Big\downarrow \\ R\wedge_K R & \longrightarrow & R\wedge_K R\wedge_K R \end{array}$$

in $\mathscr{C}\mathrm{Alg}_R$ showing that $\mathrm{thh}^K_2(R)$ is degenerate. By the same argument $\mathrm{thh}^K_n(R)$ is degenerate for all $n \geqslant 2$. Hence we have a pushout

$$\begin{array}{ccc} R\otimes I\bigcup_{R\otimes\partial I}(R\wedge_K R)\otimes\partial I & \longrightarrow & (R\wedge_K R)\otimes I \\ \Big\downarrow & & \Big\downarrow \\ R & \xrightarrow{\ \xi\ } & \mathrm{THH}^K(R) \end{array}$$

in $\mathscr{C}\mathrm{Alg}_R$. Since R is q-cofibrant in $\mathscr{C}\mathrm{Alg}_K$, the map $\eta\wedge\mathrm{id}:R\longrightarrow R\wedge_K R$ is a q-cofibration in $\mathscr{C}\mathrm{Alg}_R$; hence so is

$$R\otimes I\bigcup_{R\otimes\partial I}(R\wedge_K R)\otimes\partial I\longrightarrow(R\wedge_K R)\otimes I$$

by (iv) and $\xi:R\longrightarrow\mathrm{THH}^K(R)$. □

Proof of Corollaries 2.2 *and* 2.3. If K is a semi-hereditary ring, $\mathrm{Tor}^K_{p,q}$ vanishes for $p > 1$ [**3**, VI.2.9]. Hence the spectral sequence (2) collapses, giving rise to a short split exact sequence [**3**, VI.3.2]. Since R is K-flat, we have a natural isomorphism $\mathrm{HH}^K_*(R) \cong \mathrm{THH}^{HK}_*(HR)$ by [**6**, IX.1.7].

In the commutative case the spectral sequence is multiplicative. Hence Corollary 2.3 follows from Corollary 2.2, because $\mathrm{Tor}^K_1(\mathrm{HH}^K_p(R), \mathrm{THH}_q(K))$ vanishes. □
□

REMARK 3.7. There are many possible models for Eilenberg–MacLane spectra HR of classical rings R, which are all weakly equivalent. We have assumed throughout that our models are q-cofibrant. In this case $\mathrm{THH}^{HK}(HR)$ is well defined up to weak equivalence. If R is a commutative K-algebra, we can represent HR as a q-cofibrant object HR_a in Alg_K or as a q-cofibrant object HR_c in $\mathscr{C}\mathrm{Alg}_K$ together with a weak equivalence $HR_a \longrightarrow HR_c$ in Alg_K. This map induces a weak equivalence of the associated topological Hochschild spectra, which is a homotopy equivalence of underlying spectra [6, VII §6].

4. *Applications*

Let R be a classical commutative ring and Π be a based monoid, that is, a discrete monoid with a unit and a multiplicative 0 which serves as base point. Let $R[\Pi]$ be the associated monoid ring, that is, the usual monoid ring with $0 \in R$ identified $0 \in \Pi$. In abuse of notation, but in accordance with common practice, we write $\mathrm{THH}(R)$ for $\mathrm{THH}(HR)$. We deduce the following theorem from Corollary 2.2.

THEOREM 4.1. *If R is a semi-hereditary ring, then, as R-modules,*

$$\mathrm{THH}_n(R[\Pi]) \cong \bigoplus_{p+q=n} \mathrm{HH}_p^R(R[\Pi]) \otimes_R \mathrm{THH}_q(R) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{HH}_p^R(R[\Pi]), \mathrm{THH}_q(R)).$$

Proof. Let $E \longrightarrow \Sigma^\infty(\Pi)$ be a q-cofibrant resolution of $\Sigma^\infty(\Pi)$ in the category of non-commutative S-algebras. Since HR is q-cofibrant, the unit $\mathbf{S} \longrightarrow HR$ is a q-cofibration in ${}_{\mathbf{S}}\mathrm{Mod}$, and we have a weak equivalence

$$E \wedge_{\mathbf{S}} HR \longrightarrow \Sigma^\infty(\Pi) \wedge_{\mathbf{S}} HR \cong HR \wedge \Pi \cong H(R[\Pi])$$

by Lemma 3.5. □

If Π is the based monoid $\Pi = \{0, 1, x, x^2, \ldots, x^{n-1}\}$ with the multiplication given by the rule $x^n = 0$, then $\mathbb{Z}[\Pi] \cong \mathbb{Z}[X]/(X^n)$.

COROLLARY 4.2.

$$\mathrm{THH}_k(\mathbb{Z}[X]/(X^n)) \cong \bigoplus_{p+q=k} \mathrm{HH}_p^{\mathbb{Z}}(\mathbb{Z}[X]/(X^n)) \otimes_{\mathbb{Z}} \mathrm{THH}_q(\mathbb{Z}) \oplus \bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{HH}_p^{\mathbb{Z}}(\mathbb{Z}[X]/(X^n)), \mathrm{THH}_q(\mathbb{Z}))$$

as abelian groups.

If G is a discrete group and $\Pi = G_+$, that is, G with an additional multiplicative 0, then $\mathbb{Z}[\Pi]$ is the usual group ring of G over $\mathbb{Z}$, whose classical Hochschild homology is known in terms of group homology; for example see [21, 9.7.5]. We obtain the following.

COROLLARY 4.3. *Let Λ be a set of representatives of conjugacy classes of elements of G and $C_G(x)$ be the centralizer subgroup of $x \in G$. Then, as abelian groups,*

$$\mathrm{THH}_k(\mathbb{Z}[G]) \cong \left(\bigoplus_{x\in\Lambda} \bigoplus_{p+q=k} H_p(C_G(x);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{THH}_q(\mathbb{Z})\right) \oplus \left(\bigoplus_{x\in\Lambda} \bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C_G(x);\mathbb{Z}), \mathrm{THH}_q(\mathbb{Z}))\right)$$

where $H_p(C_G(x);\mathbb{Z})$ is the usual group homology with trivial $\mathbb{Z}$-coefficients.

In particular, if G is abelian of order n then

$$\mathrm{THH}_k(\mathbb{Z}[G]) \cong \bigoplus_{p+q=k} H_p(G;\mathbb{Z})^n \otimes_{\mathbb{Z}} \mathrm{THH}_q(\mathbb{Z}) \oplus \bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(G;\mathbb{Z})^n, \mathrm{THH}_q(\mathbb{Z})).$$

This recovers the results of (1.1), except for the case of $\mathbb{F}_p[X]/(f)$.

REMARK 4.4. Theorem 4.1 and Corollary 4.2 were also observed by Hesselholt and Madsen as a corollary to [7, Theorem 6.1]. Recall that Bökstedt's original construction of topological Hochschild homology took functors with smash products as inputs. Let F be a functor with smash products. It has an associated S-algebra spectrum which we also denote by F and [7, Theorem 6.1] states that

$$\mathrm{THH}(F[\Pi]) \cong \mathrm{THH}(F) \wedge |N^{\mathrm{cy}}_{*\wedge}(\Pi)|$$

as spectra, where $N^{\mathrm{cy}}_{*\wedge}(\Pi)$ is the cyclic nerve of Π formed using the smash product. This result is a special case of Lemma 3.1:

$$\mathrm{THH}(F[\Pi]) \cong \mathrm{THH}(F \wedge_{\mathrm{S}} \mathrm{S}[\Pi]) \cong \mathrm{THH}(F) \wedge_{\mathrm{S}} \mathrm{THH}(\mathrm{S}[\Pi]).$$

Since $(\mathrm{S}[\Pi])^{\wedge n} \cong (\mathrm{S} \wedge \Pi)^{\wedge n} \cong \mathrm{S} \wedge \Pi^{\wedge n}$, we obtain $\mathrm{THH}(\mathrm{S}[\Pi]) \cong \Sigma^\infty |N^{\mathrm{cy}}_{*\wedge}(\Pi)|$. Hence

$$\mathrm{THH}(F[\Pi]) \cong \mathrm{THH}(F) \wedge_{\mathrm{S}} \Sigma^\infty |N^{\mathrm{cy}}_{*\wedge}(\Pi)| \cong \mathrm{THH}(F) \wedge |N^{\mathrm{cy}}_{*\wedge}(\Pi)|.$$

THEOREM 4.5. *Let R be a subring of the rationals $\mathbb{Q}$. Then, as graded rings,*

$$\mathrm{THH}_*(R) \cong R \otimes_{\mathbb{Z}} \mathrm{THH}_*(\mathbb{Z}).$$

Proof. Any subring $R \subset \mathbb{Q}$ is a localization $\mathbb{Z}[X^{-1}]$ where $X \subset \mathbb{Z}$ is a set of primes. Consider X as subset of $\pi_0(\mathrm{S}) \cong \mathbb{Z}$. By [6, VIII.4.2] the algebraic localization $\pi_*(\mathrm{S}) \longrightarrow \pi_*(\mathrm{S})[X^{-1}]$ can be realized by the unit $\mathrm{S} \longrightarrow \mathrm{S}[X^{-1}]$ of a commutative cell S-algebra. Then $\mathrm{S}[X^{-1}] \wedge_{\mathrm{S}} H(\mathbb{Z})$ is a q-cofibrant commutative $H\mathbb{Z}$-algebra with the same homotopy groups as $H(\mathbb{Z}[X^{-1}]) = HR$. Hence $\mathrm{S}[X^{-1}] \wedge_{\mathrm{S}} H(\mathbb{Z})$ is a model for HR and Corollary 2.3 applies. □

Theorem 4.5 is a special case of a more general result: one can prove that THH commutes with localizations [16].

Away from the characteristic it is possible to adjoin roots of unity to a ring spectrum [17]. We have the following.

THEOREM 4.6. *Let $R \subset \mathbb{Q}$ be a subring and ζ be a primitive nth root of unity such that the prime factors of n are invertible in R. Then*

$$\mathrm{THH}_*(R(\zeta)) \cong \mathrm{HH}^{\mathbb{Z}}_*(R(\zeta)) \otimes_{\mathbb{Z}} \mathrm{THH}_*(\mathbb{Z})$$

as graded rings.

REMARK 4.7. The following results allow the explicit computation of topological Hochschild homology groups [2, 4]:

$$\mathrm{HH}_k(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/i & k=2i-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x] \text{ with degree } (x)=2.$$

Let $f \in R[X]$ be a monic polynomial. Then

$$\mathrm{HH}^R_k(R[X]/(f)) = \begin{cases} R[X]/(f) & \text{if } k=0 \\ R[X]/(f,f') & \text{if } k \text{ odd} \\ \mathrm{Ann}(f') \text{ in } R[X]/(f) & \text{if } k \text{ even.} \end{cases}$$

In particular, $HH^{\mathbb{Z}}_*(R(\zeta))$ in Theorem 4.6 is additively free.

Proof of Theorem 4.6. We construct the spectrum E in steps. Let $n = p_1^{r_1} \cdot \ldots \cdot p_k^{r_k}$ be the factorization of n into distinct primes. Let X be the set of these primes, and $S[X^{-1}]$ be the S-algebra of the proof of Theorem 4.5. Let ζ_i be a $p_i^{r_i}$th primitive root of unity. By [17] there is a commutative q-cofibrant $S[X^{-1}]$-algebra $S[X^{-1}](\zeta_1)$ such that

$$S[X^{-1}](\zeta_1) \wedge_{S[X^{-1}]} H(\mathbb{Z}[X^{-1}])$$

is a model for $H(\mathbb{Z}[X^{-1}](\zeta_1))$ as q-cofibrant commutative $H(\mathbb{Z}[X^{-1}])$-algebra. As we have seen in the proof of Theorem 4.5, we can use $S[X^{-1}] \wedge_S H(\mathbb{Z})$ as a model for $H(\mathbb{Z}[X^{-1}])$. Hence

$$S[X^{-1}](\zeta_1) \wedge_{S[X^{-1}]} S[X^{-1}] \wedge_S H(\mathbb{Z}) \cong S[X^{-1}](\zeta_1) \wedge_S H(\mathbb{Z})$$

is a model for $H(\mathbb{Z}[X^{-1}](\zeta_1))$. We now iterate the process adjoining $\zeta_2, \ldots, \zeta_r$. □

Using this result we can give a simple proof of the last case of (1.1) including its multiplicative structure.

THEOREM 4.8. *Let $f \in \mathbb{F}_p[X]$ be an arbitrary monic polynomial. Then*

$$THH_*(\mathbb{F}_p[X]/(f)) \cong HH^{\mathbb{F}_p}_*(\mathbb{F}_p[X]/(f)) \otimes THH_*(\mathbb{F}_p)$$

as graded $\mathbb{F}_p$-algebras.

We need an algebraic lemma.

LEMMA 4.9. *If $f \in \mathbb{F}_p[X]$ is irreducible, then $\mathbb{F}_p[X]/(f^n) \cong (\mathbb{F}_p[X]/(f))[Y]/(Y^n)$.*

Proof. Now $A = \mathbb{F}_p[X]/(f^n)$ is a local ring with maximal ideal $\mathbf{m} = (\bar{f})$ and residue field $R = \mathbb{F}_p[X]/(f)$, where $\bar{g}$ denotes the image of $g \in \mathbb{F}_p[X]$ in A. Let $\pi: A \longrightarrow R$ be the projection. Also $\pi(\bar{X})$ is a simple root of f in R. Since $\mathbf{m}$ is nilpotent, Hensel's lemma applies. Hence there is a $z \in A$ such that $\pi(z) = \pi(\bar{X})$ and $f(z) = 0$, so the map

$$\mathbb{F}_p[X] \longrightarrow A, \quad X \longmapsto z$$

induces a ring homomorphism $\phi: R \longrightarrow A$. Since R is a field, ϕ is injective. We extend ϕ to a ring homomorphism

$$\theta: R[Y]/(Y^n) \longrightarrow A, \quad Y \longmapsto \bar{f}.$$

θ is injective:

$$0 = \theta\left(\sum_{i=k}^{n-1} r_i Y^i\right) = \sum_{i=k}^{n-1} \phi(r_i)\bar{f}^i \quad \text{implies that} \quad 0 = \left(1 + \sum_{i=1}^{n-k-1} \phi(r_k^{-1}) r_{k+i} \bar{f}^i\right) \cdot \bar{f}^k$$

if $r_k \neq 0$. Since the first factor is not in $\mathbf{m}$, it is a unit in A. Hence $\bar{f}^k = 0$ in A, a contradiction. Hence all $r_k = 0$. Since the source and target of θ have the same dimension as $\mathbb{F}_p$-vector spaces, θ is bijective. □

Proof of Theorem 4.8. By the Chinese remainder theorem $\mathbb{F}_p[X]/(f)$ is of the form

$$\mathbb{F}_p[X]/(f) \cong \mathbb{F}_p[X]/(f_1^{r_1}) \times \ldots \times \mathbb{F}_p[X]/(f_k^{r_k})$$

with $f_i \in \mathbb{F}_p[X]$ monic and irreducible.

Let E_1, E_2 be two commutative S-algebras. We have a canonical commutative diagram of $H\mathbb{F}_p$-modules

$$\begin{array}{ccc} (E_1 \vee E_2) \wedge_S H\mathbb{F}_p & \cong & (E_1 \wedge_S H\mathbb{F}_p) \vee (E_2 \wedge_S H\mathbb{F}_p) \\ \downarrow & & \downarrow \\ (E_1 \times E_2) \wedge_S H\mathbb{F}_p & \longrightarrow & (E_1 \wedge_S H\mathbb{F}_p) \times (E_2 \wedge_S H\mathbb{F}_p) \end{array}$$

whose vertical maps are weak equivalences, so that the canonical map

$$(E_1 \times E_2) \wedge_S H\mathbb{F}_p \longrightarrow (E_1 \wedge_S H\mathbb{F}_p) \times (E_2 \wedge_S H\mathbb{F}_p)$$

of commutative $H\mathbb{F}_p$-algebras is a weak equivalence. Hence it suffices to consider the case $\mathbb{F}_p[X]/(f^n)$ with f monic and irreducible.

Suppose we can construct a q-cofibrant S-algebra $E(f)$ for an irreducible polynomial f in $\mathbb{F}_p[X]$ such that $E(f) \wedge_S H\mathbb{F}_p \simeq H(\mathbb{F}_p[X]/(f))$ and Π is the based monoid $\{0, 1, x, \ldots, x^{n-1}\}$ with multiplication determined by $x^n = 0$; then $E(f)[\Pi]$, made q-cofibrant, is the S-algebra required for f^n and we are done.

If f is irreducible of degree k, then $\mathbb{F}_p[X]/(f)$ is a cyclotomic extension $\mathbb{F}_p(\zeta)$ of $\mathbb{F}_p$ by a $(p^k - 1)$st primitive root of unity ζ. The S-algebra $E(f)$ is the spectrum constructed in the proof of Theorem 4.6, for which $E(f) \wedge_S H\mathbb{Z} \simeq H(\mathbb{Z}[X^{-1}](\zeta))$, where X is the set of primes dividing $p^k - 1$. Note that $\mathbb{Z}[X^{-1}] \otimes \mathbb{F}_p \cong \mathbb{F}_p$. □

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