Contemporary Mathematics
Volume 239, 1999

Adjoining Roots of Unity to E_∞ Ring Spectra in Good Cases - A Remark.

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Dedicated to Michael Boardman on the occasion of his 60^{th} *birthday.*

Throughout this paper we work in categories of ring-, module-, and algebra spectra as constructed by Elmendorf, Kriz, Mandell, and May [2], who exploit a crucial observation by Hopkins [4].

We start by specifying what we mean by "adjoining roots of unity" to an E_∞ ring spectrum or, more precisely, to a commutative **S**-algebra E. For a ring R let $H(R)$ denote its associated Eilenberg-MacLane ring spectrum given as appropriate cell spectrum. If E is connective there is a map of **S**-algebras $E \to H(\pi_0 E)$ realizing the identity on π_0 [2, IV.3.1]. Hence $H(\pi_0 E)$ is an E-algebra.

Definition 1: Let E be a connective, commutative **S**-algebra, and $\pi_0(E) \subset R$ an extension of the commutative ring $\pi_0(E)$ in the usual algebraic sense. We say a map $E \to F$ of commutative **S**-algebras *lifts* the extension $\pi_0(E) \subset R$ if $\pi_0(F) \cong R$ and there is a weak equivalence of $H(\pi_0 E)$-algebras

$$F \wedge_E H(\pi_0 E) \to H(R)$$

Motivation: Let A be an S-algebra and B be an A-algebra. We can define topological Hochschild homology $THH^A(B)$ of B over A as the realization (in the category of A-module spectra) of the simplicial spectrum

$$[n] \mapsto B \wedge_A B \wedge_A \dots \wedge_A B \quad (n+1) \text{ factors}$$

1991 *Mathematics Subject Classification.* Primary 55P42.

with the well-known Hochschild structure maps. We have the following result from [6].

Theorem 1: Let K be a classical commutative ring and R a flat K-algebra. Assume there is a commutative $\mathbf{S}$-algebra A and an A-algebra E such that
(1) HK is a commutative algebra over A.
(2) there is a weak equivalence of HK-algebras $E \wedge_A HK \to HR$.
Then (modulo technical cofibrancy conditions)

$$THH_*^A(HR) \cong HH_*^K(R) \otimes_K^{\mathbb{L}} THH_*^A(HK),$$

as graded K-modules, where HH_*^K stands for the classical Hochschild homology over the ground ring K and $\otimes^{\mathbb{L}}$ for the total left derived of $\otimes$.

We want to investigate lifts of algebraic extensions by roots of unity. An investigation of more general "algebraic extensions" of algebra spectra is work in progress.

Proposition 2: In general there is no lift for extensions.

Proof: Take the sphere spectrum $\mathbf{S}$. Suppose we could adjoin a fourth root of unity to $\mathbf{S}$, then according to Theorem 1

$$THH_*^{\mathbf{S}}(\mathbb{Z}[i]) \cong HH_*^{\mathbb{Z}}(\mathbb{Z}[i]) \otimes_{\mathbb{Z}}^{\mathbb{L}} THH_*^{\mathbf{S}}(\mathbb{Z})$$

In particular, $THH_*^{\mathbf{S}}(\mathbb{Z})$ would be a direct summand of $THH_*^{\mathbf{S}}(\mathbb{Z}[i])$.

Calculations by Bökstedt and Lindenstrauss show that this is not the case: we use the following results from [1] and [5]

$$THH_k^{\mathbf{S}}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/i & \text{if } k = 2i-1 \\ 0 & \text{if } k \text{ is even} \end{cases}$$

$$THH_n^{\mathbf{S}}(\mathbb{Z}[i]) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0 \\ 0 & n > 0 \text{ even} \\ \mathbb{Z}/2j \oplus \mathbb{Z}/4j & n = 2j-1 \end{cases}$$

and the induced map $THH_n^{\mathbf{S}}(\mathbb{Z}) \to THH_n^{\mathbf{S}}(\mathbb{Z}[i])$ for $n = 2j-1$ comes from multiplication by 2 on $\mathbb{Z}/j \to \mathbb{Z}/2j$.

Our main result is

Theorem 3: Let E be a connective commutative $\mathbf{S}$-algebra spectrum and p a prime which is invertible in $\pi_0(E)$. Suppose the cyclotomic polynomial

$$X^{q(p-1)} + X^{q(p-2)} + \ldots + X^q + 1$$

with $q = p^{n-1}$ is irreducible in $\pi_0(E)[X]$ and ζ is a p^n-th primitive root of unity. Then there is a commutative E-algebra spectrum $E(\zeta)$ lifting the extension $\pi_0(E) \subset \pi_0(E)(\zeta)$.

The proof uses the simple observation, well known in algebra, that factoring idempotents $\varepsilon \in \pi_0 E$ can be done by localizing.

Lemma 4: Let E be a connective commutative $\mathbf{S}$-algebra and $\pi_0(E)[X^{-1}]$ the localization of the ring $\pi_0(E)$ where X is a subset of $\pi_0(E)$. Then there is a commutative cell E-algebra $E[X^{-1}]$ and a weak equivalence of $H(\pi_0 E)$-algebras

$$E[X^{-1}] \wedge_E H(\pi_0 E) \to H((\pi_0 E)[X^{-1}]) = H(\pi_0(E[X^{-1}])).$$

Proof: By [2, VIII.4.2] there is a cell E-algebra $E[X^{-1}]$ whose unit $\lambda : E \to E[X^{-1}]$ induces the localization

$$\lambda_* : \pi_*(E) \to \pi_*(E)[X^{-1}].$$

Then $\lambda \wedge_E id : E \wedge_E H(\pi_0 E) \to E[X^{-1}] \wedge_E H(\pi_0 E)$ is the localization of $H(\pi_0 E)$ by [2, VIII.4.1], and the claim follows.

Proposition 5: Let E be a connective commutative $\mathbf{S}$-algebra and $\varepsilon \in \pi_0 E$ an idempotent. Then there is a commutative E-algebra $E/\varepsilon E$ and a weak equivalence of $H(\pi_0 E)$-algebras

$$(E/\varepsilon E) \wedge_E H(\pi_0 E) \to H(\pi_0 E/\varepsilon \pi_0 E) = H(\pi_0(E/\varepsilon E))$$

Proof: Let $\eta = 1 - \varepsilon$ in $\pi_0 E$. Let $E \to E[\frac{1}{\eta}]$ be the localization. Since as rings

$$(\pi_0 E)\left[\frac{1}{\eta}\right] \cong \pi_0 E/\varepsilon \pi_0 E$$

we can take $E[\frac{1}{\eta}]$ for $E/\varepsilon E$.

Proof of Theorem 3: We have a commutative E-algebra group spectrum

$$E[\mathbb{Z}/p^n] = E \wedge (\mathbb{Z}/p^n)_+$$

by taking the small smash product with $(\mathbb{Z}/p^n)_+$. Let $t \in \mathbb{Z}/p^n$ be a generator and $x = t^q$, $q = p^{n-1}$. Then

$$\varepsilon = \frac{1}{p}(1 + x + \ldots + x^{p-1}) \in (\pi_0 E)[\mathbb{Z}/p^n] = \pi_0(E[\mathbb{Z}/p^n])$$

is an idempotent and $E(\zeta) = E[\mathbb{Z}/p^n]/\varepsilon(E[\mathbb{Z}/p^n])$ is the required spectrum.

Remark: The automorphism group G of $\mathbb{Z}/p^n$ acts on $E[\mathbb{Z}/p^n]$ leaving the idempotent ε fixed. Hence, if we work in the category of G-spectra in the naive sense, we have the Galois group of the extension operating on $E(\zeta)$, and the map $E \to E(\zeta)$ is equivariant with the trivial action on E.

Corollary 6: Let $R \subset \mathbb{Q}$ be a subring and ζ a primitive p^n-th root of unity, p a prime which is invertible in R. Then

$$THH_*^{\mathbf{S}}(R(\zeta)) \cong HH_*^{\mathbb{Z}}(R(\zeta)) \otimes_{\mathbb{Z}}^{\mathbb{L}} THH_*^{\mathbf{S}}(\mathbb{Z})$$

This determines $THH_*^{\mathbf{S}}(R(\zeta))$ by [1] and the following result from [3]:

$$HH_k^R(R[X]/(f)) = \begin{cases} R[X]/(f) & \text{if } k = 0 \\ R[X]/(f, f') & \text{if } k \text{ odd} \\ Ann(f') \text{ in } R[X]/(f) & \text{if } k > 0 \text{ even} \end{cases}$$

for a monic polynomial $f \in R[X]$ over a commutative ring R.

Acknowledgement: It is our pleasure to acknowledge the support of the Deutsche Forschungsgemeinschaft during the preparation of this paper.

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