Contemporary Mathematics
Volume 199, 1996

Stable K-theory and topological Hochschild homology of A_∞ rings

Roland Schwänzl, Ross Staffeldt, and Friedhelm Waldhausen

ABSTRACT. We present a program to prove the equivalence of stable K-theory and topological Hochschild homology for A_∞-rings. We explain the reductions of the problem to one crucial calculation and outline the approach to the calculation, which uses a generalization of the notion of generalized free product of rings and accompanying decomposition theorems in K-theory.

1. Stable K-theory and topological Hochschild homology

The purpose of this note is to expand on material first presented by the third-named author in a lecture at Northwestern University in 1988. The material that concerns us describes a program to prove that the trace map induces a natural homotopy equivalence of infinite loop spaces between stable K-theory and topological Hochschild homology of A_∞ ring spectra. B. Dundas and R. McCarthy [**3**] have given a proof of the result for simplicial rings, which are included in our category as the generalized Eilenberg-MacLane spectra, using a strategy totally different from what we will describe here. Also, B. Dundas has a recent preprint [**2**] in which he proves agreement of relative algebraic K-theory completed at a prime p with relative topological cyclic homology also completed at p, which is a generalization of the main result of Goodwillie's paper [**6**]. Dundas remarks that the ideas of his preprint can be used to provide a completely different proof of the result we are discussing here.

In this section we recall the definitions of these objects and prove a few of the basic properties. For us a spectrum will always be a connective strict Ω-spectrum [**8**, page 11]. The definition of an A_∞-ring spectrum is found in [**9**, page 248]. Expanding this definition in terms of diagrams satisfying certain properties suggests the definitions of A_∞-module spectra and of A_∞-bimodule spectra over a given A_∞-ring. The algebraic K-theory of an A_∞ ring was first defined in [**9**, page 272]; other definitions of the algebraic K-theory of A_∞-rings are found in [**14**]. In the rest of

1991 *Mathematics Subject Classification.* 19D10, 19D55, 55P42.

Key words and phrases. Stable K-theory, A_∞-ring spectra, topological Hochschild homology.

The second author gratefully acknowledges the support of the Sonderforschungsbereich 343 in Bielefeld for part of this work.

this paper we use the up-to-date terminology of [**4**] for ring spectra and direct the reader there for details. Thus, in this paper a ring R is a q-cofibrant S-algebra and a module over R is understood as a cell R-module in the sense of [**4**, Chapter III, section 2]. The algebraic K-theory of S-algebras is also discussed in [**4**, Chapter VI].

Our definition of stable K-theory is suggested by remarks concerning the stable K-theory of simplicial rings found in [**18**, page 388]. First come two pieces of notation. First, for any R-bimodule M we write $R\oplus M$ for the ring which has $R\vee M$ as underlying R-bimodule and which carries the unique multiplication compatible with the bimodule structure making M a square zero ideal. Writing $\lambda : R\wedge M \longrightarrow M$ and $\rho : M\wedge R \longrightarrow M$ for the bimodule structure maps and $\mu : R\wedge R \longrightarrow R$ for the multiplication on R, the multiplication $\mu' : (R\oplus M)\wedge(R\oplus M) \longrightarrow R\oplus M$ has the matrix representation

$$\mu' = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \lambda & \rho & 0 \end{pmatrix} : (R\wedge R)\vee(R\wedge M)\vee(M\wedge R)\vee(M\wedge M) \longrightarrow R\vee M.$$

Second, if M is a bimodule over R, then $M\langle m\rangle$ is the bimodule M shifted up by m dimensions. More precisely,

$$M\langle m\rangle = M\wedge S^m,$$

the smash product of the spectrum M with the pointed space S^m [**8**, page 16].

DEFINITION 1.1. We let R be a ring and let M be an R-bimodule. Define

$$K^s(R;M) = \lim_m \Omega^{m+1}\text{fiber}(K(R\oplus M\langle m\rangle) \longrightarrow K(R)).$$

In section 3 we will need to know how well $\Omega^{m+1}\text{fiber}(K(R\oplus M\langle m\rangle) \longrightarrow K(R))$ approximates $K^s(R;M)$. Proposition 1.3 of [**20**, page 38] generalizes to the A_∞ context and leads to the following result.

PROPOSITION 1.2. *Let*

$$F(R;M)(m) = \text{fiber}(K(R\oplus M\langle m\rangle) \longrightarrow K(R)).$$

Then the canonical inclusion

$$\Omega^{m+1}F(R;M)(m) \longrightarrow K^s(R;M)$$

is m-connected. □

We take the following definition of topological Hochschild homology of a ring R with coefficients in the bimodule M from [**4**, Section 2, Chapter IX].

DEFINITION 1.3. Let R be an S-algebra and M an R-bimodule. Then $THH(R;M)$ is the realization of the simplicial spectrum $THH_\bullet(R;M)$ whose k-simplices are

$$THH_k(R;M) = R^{\wedge k}\wedge M,$$

where the $\wedge$ denotes smash product of S-modules.

As is by now well-known, the technical point is to organize the smash products $R^{\wedge k}\wedge M$, $k\geq 0$, into a simplicial object. For instance, the first problem with a naive notion of smash product is that they are all *a priori* spectra defined over different universes. However, these problems are solved in [**4**], and the construction behaves as one would expect. For instance, we have the following result.

PROPOSITION 1.4. *Take $M = R \wedge R^{op}$, the free bimodule of rank one. We have a natural equivalence of spectra*

$$THH(R; R \wedge R^{op}) \xrightarrow{\simeq} R$$

for any ring R.

PROOF. The proof mimics in the category of spectra the usual manipulations in the category of rings. First we have for each $k \geq 0$ an isomorphism

$$R^{\wedge k} \wedge (R \wedge R^{op}) \cong R \wedge R^{\wedge k} \wedge R$$

which brings the R^{op} factor around in front. Filtering these isomorphisms through the construction which yields topological Hochschild homology, we obtain a natural homotopy equivalence

$$THH_\bullet(R; R \wedge R^{op}) \longrightarrow B_\bullet(R, R, R),$$

where the latter object is a two-sided bar construction in which R is acting on itself first by right multiplication and second by left multiplication. Then iterated multiplications induce a map of the bar construction to the constant simplicial object which is R in every dimension, and this is a simplicial homotopy equivalence. Therefore, passing to realizations and composing, we obtain the desired equivalence

$$THH(R; R \wedge R^{op}) \xrightarrow{\simeq} R.$$

□

The trace map

$$K^s(R; M) \longrightarrow HH(R; M)$$

from stable K-theory to Hochschild homology, where R and M are a simplicial ring and bimodule, respectively, and where $HH(R; M)$ is the realization of the usual Hochschild homology, is defined in [**18**, page 392]. In this paper we will not discuss details of the trace map in the world of A_∞-objects; we limit our present interest to a crucial calculation of stable K-theory, to which the proof of the following result is reduced.

THEOREM 1.5. *Let R be an S-algebra and let M be a bimodule over R. Then the trace map*

$$K^s(R; M) \longrightarrow THH(R; M)$$

is a homotopy equivalence.

The program to prove this result involves a number of reductions so that one is reduced to verifying the result in a special case. These reductions are described in the following propositions. Recall that for a discrete ring R flat over the integers one may define bi-free bimodules to be those bimodules isomorphic to $R \otimes L \otimes R$, where L is a free abelian group. Transferring the notion to A_∞ rings and bimodules in the most natural way, it becomes the following definition.

DEFINITION 1.6. For a ring R the bi-free bimodule of rank 1 is the bimodule $R \wedge R^{op}$. A bimodule M is bi-free if it is weakly homotopy equivalent to a bimodule of the form $R \wedge L \wedge R^{op}$ where L is a cell S-module. That is, L is an S-module with the property that L has a filtration by S-submodules in which the quotients of successive layers are wedges of sphere S-modules S_S^m for $m \geq 0$ [**4**, page 37].

We have the following result.

PROPOSITION 1.7. *If the theorem is true for M any bi-free bimodule over R, then it is true for every bimodule over R.*

PROOF. Every bimodule M has a functorial resolution by bi-free modules, given by combining the bi-free module construction with the bar construction. That $THH(R;M)$ for an arbitrary R-bimodule M may be computed through resolution of M by bi-free R-bimodules is a relatively simple argument based on the fact that the two ways of geometrically realizing a bisimplicial set yield the same space.

That $K^s(R;M)$ may also be retrieved from the stable K-theory of R with coefficients in bi-free R-modules is also true, but the argument is more difficult.

First let $U_\bullet \longrightarrow M$ be a resolution of M by bi-free R-modules. The realization of this map is a homotopy equivalence, and our definition of K-theory is homotopy invariant [14], so

$$K^s(R;|U_\bullet|) \longrightarrow K^s(R;M)$$

is also a homotopy equivalence.

What we have to do now is to commute the formation of the geometric realization and the formation of stable K-theory. Given the Volodin definition of K-theory for A_∞-rings [5] we may generalize Lemma 1.2.2 of [6] to prove that the relative K-theory of a surjection of simplicial A_∞-rings whose kernel has the property that its square is zero may be calculated degreewise. In our case, if we define simplicial spaces $F_\bullet(m)$ by

$$F_p(m) = \text{fibre}(K(R \oplus U_p\langle m\rangle) \longrightarrow K(R)),$$

then the result implies that there is a natural homotopy equivalence

$$|F_\bullet(m)| \simeq \text{fibre}(K(R \oplus |U_\bullet|\langle m\rangle) \longrightarrow K(R)).$$

Taking limits,

$$\lim_m \Omega^{m+1} F_p(m) = K^s(R;U_p).$$

Now we may assemble all these natural equivalences into one commuting diagram in which the vertical arrows are trace maps.

$$\begin{array}{ccc} |p \longmapsto K^s(R;U_p)| & \simeq & K^s(R;M) \\ \downarrow & & \downarrow \\ |p \longmapsto THH(R;U_p)| & \simeq & THH(R;M) \end{array}$$

So, by a wellknown principle, to deduce that the righthand vertical arrow is a homotopy equivalence, it suffices to show that for each p the lefthand vertical arrow is a homotopy equivalence. □

Using the following proposition, we reduce to the case of the bi-free bimodule of rank one.

PROPOSITION 1.8. *Let R be an S-algebra, and M be an R-bimodule. If L is a cell S-module, then there are canonical equivalences*

$$K^s(R;M) \wedge L \longrightarrow K^s(R;M \wedge L)$$

and

$$THH(R;M) \wedge L \longrightarrow THH(R;M \wedge L).$$

PROOF. For topological Hochschild homology this result is an immediate consequence of the definition. For stable K-theory, note that both sides are homology theories in the variable L after passage to homotopy groups. For $K^s(R; M \wedge L)$ this is exactly the universal property enjoyed by the stablization, and it is true by definition for $K^s(R; M) \wedge L$. Now both sides agree for S^0_S, and so an easy induction using the filtration of L ends the proof. □

Combining these reduction steps, one sees that it is enough to prove that the trace map induces an equivalence

$$K^s(R; R \wedge R^{op}) \longrightarrow THH(R; R \wedge R^{op}) \simeq R.$$

However, even to manage this, we have to change our ring R somewhat. In the following proposition, the notation $k \times R$ stands for the k-algebra with underlying spectrum $k \times R$, with the multiplication $\mu = \mu_k \times \mu_R$.

PROPOSITION 1.9. *The projection*

$$k \times R \longrightarrow R$$

induces isomorphisms

$$K^s(k \times R; 0 \times M) \longrightarrow K^s(R; M)$$

and

$$THH(k \times R; 0 \times M) \longrightarrow THH(R; M).$$

Thus, we may assume that the S-algebra R over a given commutative S-algebra k has k as a retract.

PROOF. For the first part of the argument we use a plus construction definition of the algebraic K-theory of an A_∞-ring as described in [**9**, page 272] and in [**14**, section 8]. However, our notation follows that of the definition given for the K-theory of simplicial rings found in [**20**].

We will always have rings with units

$$\eta : k \longrightarrow R$$

so that the natural way to view $k \times R$ is as a k-algebra with unit given by the diagonal. Then each of the projections from $k \times R$ to the factors is a k-algebra map, so that k is a retract of $k \times R$ as a k-algebra. Note also that using the projection to R to make M into a $k \times R$-bimodule delivers the advertised structure. Using the projections, then, the diagram

$$\begin{array}{ccc} B\widehat{GL}(k \times R \oplus M\langle n\rangle)^+ & \longrightarrow & B\widehat{GL}(R \oplus M\langle n\rangle)^+ \\ \downarrow & & \downarrow \\ B\widehat{GL}(k \times R)^+ & \longrightarrow & B\widehat{GL}(R)^+ \end{array}$$

is seen to be equivalent to the diagram

$$\begin{array}{ccc} B\widehat{GL}(k)^+ \times B\widehat{GL}(R \oplus M\langle n\rangle)^+ & \longrightarrow & B\widehat{GL}(R \oplus M\langle n\rangle)^+ \\ \downarrow & & \downarrow \\ B\widehat{GL}(k)^+ \times B\widehat{GL}(R)^+ & \longrightarrow & B\widehat{GL}(R)^+ \end{array}$$

which is obviously homotopy cartesian. This proves the part of the proposition dealing with stable K-theory.

The result we need follows from Proposition 4.20 of [**1**]. Alternatively, one may argue directly that the projections to the factors of a product ring $R_1 \times R_2$ induce a homotopy equivalence

$$THH(R_1 \times R_2; M_1 \times M_2) \longrightarrow THH(R_1; M_1) \times THH(R_2; M_2).$$

Or, if one assumes that R_1, M_1, R_2, and M_2 have integral homology which is finitely generated in each dimension (which covers many interesting cases), then it suffices to show that this arrow induces an isomorphism after passage to (spectral) homology with any field coefficients.

To see this one needs to use the spectral sequence arising from the skeletal filtration of the realization of a simplicial spectrum. The spectral sequence starts with

$$E^1_{p,q} = h_p(X_q) \Rightarrow h_*|X|,$$

where h is any homology theory. Since singular homology theory HF with coefficients in the field F satisfies the strict Künneth theorem, one has, in fact, that for $THH_\bullet(R; M)$ the E^1-term of the spectral sequence is the Hochschild complex for the graded ring HF_*R with coefficients in the bimodule HF_*M. In the case at hand, then, the spectral sequence may be pushed easily to the next term, giving us

$$E^2_{*,*} = HH_*(HF_*R; HF_*M) \Rightarrow HF_*THH(R; M).$$

Now we have

$$HF_*(R_1 \times R_2) = HF_*R_1 \times HF_*R_2$$

and

$$HF_*(M_1 \times M_2) = HF_*M_1 \times HF_*M_2,$$

so that the E^2-term of the spectral sequence associated with the domain is the Hochschild homology of a product ring with coefficients in a product module. Now the spectral sequence associated to the diagonal of the target has

$$E^2_{*,*} = HH_*(HF_*R_1; HF_*M_1) \times HH_*(HF_*R_2; HF_*M_2).$$

But the graded version of Theorem 6.2 of [**7**, page 295] says that projection to the factors of any F-algebra $A_1 \times A_2$ induces an isomorphism

$$HH_*(A_1 \times A_2; N_1 \times N_2) \cong HH_*(A_1; N_1) \times HH_*(A_2; N_2)$$

for any A_1-bimodule N_1 and any A_2-bimodule N_2, so that the induced map of spectral sequences is an isomorphism at E^2. The desired homology equivalence follows. □

We will make one more technical maneuver before we prove the theorem. Let $k \vee R$ denote the k-algebra obtained by forgetting that R has a unit $\eta : k \longrightarrow R$ and then adding a new unit. The underlying k-module is $k \vee R$ and the multiplication μ'' has the representation

$$\mu'' = \begin{pmatrix} \mu_k & 0 & 0 & 0 \\ 0 & \mu_R \circ (\eta \wedge 1) & \mu_R \circ (1 \wedge \eta) & \mu_R \end{pmatrix} :$$
$$(k \wedge k) \vee (k \wedge R) \vee (R \wedge k) \vee (R \wedge R) \longrightarrow k \vee R.$$

When R and k are real rings with elements, there is an isomorphism

$$c : k \vee R \longrightarrow k \times R$$

given by

$$(x, r) \longmapsto (x, x + r).$$

This map makes sense even for rings up to homotopy, and it is a map of k-algebras which is a homotopy equivalence. Since our definitions are homotopy invariant,

$$K^s(k \vee R; c^*(M)) \xrightarrow{\simeq} K^s(k \times R; M)$$

and similarly for topological Hochschild homology.

Now we can give a quick overview of the rest of the proof of the theorem, breaking it into three major steps. We intend to show that for a ring B of the form $B = S \vee R$ there is a natural equivalence

$$K^s(B; B \wedge B^{op}) \simeq B.$$

To obtain this stable K-theory calculation, we first use methods of abstract K-theory to analyze the K-theory of certain generalized free products which approximate the rings appearing in the definition of stable K-theory. The approximations and a partial analysis are explained in the next section. In the remaining part of the calculation, which we describe briefly in section 3, we go to work on the results of the general analysis, using the fact that the result is already true for $B = S$. Finally, we must show that the trace delivers compatibility with the equivalence

$$THH(B; B \wedge B^{op}) \simeq B.$$

of proposition 1.4. We then have the result for all B-bimodules by propositions 1.7 and 1.8. In particular, we have the result for the B-bimodule $c^*(M)$ pulled back from the R-bimodule M, which, in view of proposition 1.9, is just the result for R and M.

2. Generalized free products

In this section we give a definition of a generalized free product of rings-up-to-homotopy and present the example which is important for stable K-theory. We also state a decomposition theorem for the algebraic K-theory of generalized free products.

Definition 2.1. Let $\beta : A \longrightarrow B$ and $\gamma : A \longrightarrow C$ be a pair of maps of rings which are also cofibrations of S-modules. Let D be a ring and let $B \longrightarrow D$ and $C \longrightarrow D$ be homomorphisms and cofibrations such that

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

commutes. The ring D is a generalized free product of B and C over A if the diagram

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D, \end{array}$$

is homotopy cartesian in the category of spectra, where the objects in the corners are the smash products defined in [**4**, Chapter III].

Shorthand notation for such a ring D will be $D = C *_A B$.

The form of the definition is motivated by the problem of analysing the K-theory of such a ring $D = C *_A B$ in terms of its constituent rings. Later in the section we sketch the program for solving this problem. The presence of the cofibration conditions eases the homotopical analysis of the diagram of smash products and is easily verified in the examples we have in mind. Now we show that the free product can in some cases be written down following the recipe given for certain discrete rings in [**16**].

Let A denote a commutative ring. That is, let A be an E_∞-ring spectrum, or commutative S-algebra, and let B' be an A-algebra. Put $B = A \vee B'$. Generalizing the construction of the preceding section, B is then the A-algebra obtained by forgetting that B' has a unit and then adjoining a unit. Let $G(m)$ denote the realization of the Moore loop group of a simplicial $(m+1)$-sphere. $G(m)$ is a topological group homotopy equivalent to the loop space of an ordinary $(m+1)$-sphere. Let $C(m) = A \wedge G(m)_+$ be the A-algebra obtained by smashing together the spectrum A and the space $G(m)_+$. We think of $C(m)$ as the group algebra of $G(m)$ over A. Notice that if we put $C'(m) = A \wedge G(m)$, then there is an A-bimodule splitting

$$C(m) \cong A \vee C'(m).$$

Now we try out the formula

$$D(m) = A \vee B' \vee C'(m) \vee B' \wedge_A C'(m) \vee C'(m) \wedge_A B' \vee B' \wedge_A C'(m) \wedge_A B' \vee \dots$$

for $D(m) = C(m) *_A B$. All we are doing is altering the definition of free product of rings given in [**16**] by the substitution of smash products of A-bimodules for normal tensor products.

PROPOSITION 2.2. *The ring $D(m)$ defined above is a generalized free product of B and $C(m)$ over A.*

PROOF. For the purposes of verifying that $D(m)$ has the correct homotopy type, we need to rewrite $D(m)$ somewhat. To simplify things, write C for $C(m)$, D for $D(m)$, and so on. Collecting together all terms which have B' on the left we obtain an A-submodule B'', and doing likewise for C', obtaining C'', we may write $D = A \vee B'' \vee C''$. Collecting differently and using certain algebraic properties of smash products, we also obtain homotopy equivalences

$$D \simeq B \vee B \wedge_A C'' \text{ and } D \simeq C \vee C \wedge_A B''$$

of B- and C-modules, respectively. Now we examine the square

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D, \end{array}$$

and verify that it is homotopy cartesian. Plugging into this square the alternate expressions for D, we obtain

$$\begin{array}{ccc} D \wedge_A (A \vee B'' \vee C'') & \longrightarrow & D \wedge_B (B \vee B \wedge_A C'') \\ \downarrow & & \downarrow \\ D \wedge_C (C \vee C \wedge_A B'') & \longrightarrow & D \wedge_D D. \end{array}$$

Appealing again to natural algebraic properties of the smash products, one may verify that the canonical map between the vertical fibres is a homotopy equivalence. It follows that the original diagram is homotopy cartesian, as needed. □

For the next part of the discussion we adopt the framework of [**17**] for algebraic K-theory of categories with cofibrations and weak equivalences. For an S-algebra R we take $\mathcal{M}_f(R)$ to be the full subcategory of the category of R-modules whose objects are the finite cell R-modules. In $\mathcal{M}_f(R)$ a cofibration

$$M \rightarrowtail M',$$

denoted by a feathered arrow, is a map which is isomorphic to the inclusion of a subcomplex. The weak equivalences in $\mathcal{M}_f(R)$ are the homotopy equivalences. We refer the reader to [4, pages 125–128] for more discussion of these conventions. Applying to this module category the $S_\bullet$ construction of [**17**], one obtains a simplicial category $hS_\bullet\mathcal{M}_f(R)$, and we define the K-theory of R to be

$$K(R) = \Omega|hS_\bullet\mathcal{M}_f(R)|,$$

the loop space of the realization of the bisimplicial set obtained by taking the nerve of the simplicial category $hS_\bullet\mathcal{M}_f(R)$.

To analyze the K-theory of a free product D as defined above in terms of its constituent rings, we introduce the category $\mathcal{M}V$ of Mayer-Vietoris presentations. A Mayer-Vietoris presentation is a quaduple of modules, one from each of the categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(D)$, satisfying certain conditions, as follows.

DEFINITION 2.3. A Mayer-Vietoris presentation consists of four modules M_A, M_B, M_C, M_D, A-module maps $M_A \to M_B$ and $M_A \to M_C$, a B module map $M_B \to M_D$, and a C-module map $M_C \to M_D$ such that the diagram

$$\begin{array}{ccc} M_A & \longrightarrow & M_B \\ \downarrow & & \downarrow \\ M_C & \longrightarrow & M_D \end{array}$$

commutes. Moreover, we require that the cofibration condition in the appropriate category be satisfied for the extended structure arrows

$$M_A \wedge_A B \rightarrowtail M_B$$
$$M_A \wedge_A C \rightarrowtail M_C$$
$$M_B \wedge_B D \rightarrowtail M_D$$
$$M_C \wedge_C D \rightarrowtail M_D.$$

Finally we require that the diagram of D-modules

$$\begin{array}{ccc} M_A \wedge_A D & \longrightarrow & M_B \wedge_B D \\ \downarrow & & \downarrow \\ M_C \wedge_C D & \longrightarrow & M_D \end{array}$$

be homotopy cartesian.

A map of Mayer-Vietoris presentations is a quadruple $f_- = (f_A, f_B, f_C, f_D)$ of module maps such that the resulting cubical diagram commutes.

The first example which comes to mind is of course

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

One verifies that the category $\mathcal{M}V$ is also a category with cofibrations in the sense of [**17**], which involves a certain amount of routine work. To continue, we need to know that $\mathcal{M}V$ supports two notions of weak equivalence, called v-equivalences and w-equivalences, and supports a cylinder functor compatible with the equivalences. We say a map of Mayer-Vietoris presentations

$$f_{-} = (f_A, f_B, f_C, f_D)$$

is a w-equivalence, or a coarse equivalence, if f_D is a weak homotopy equivalence of D-modules. We say that f_{-} is a v-equivalence, or a fine equivalence, if the components f_A, f_B, and f_C are all weak homotopy equivalences in the respective module categories. By the homotopy cartesian property of a Mayer-Vietoris presentation, it follows that any v-equivalence is also a w-equivalence. Thus, the Fibration Theorem (Theorem 1.6.4) of [**17**, page 350] applies, yielding the following homotopy cartesian square

$$\begin{array}{ccc} vS_{\bullet}\mathcal{M}V^w & \longrightarrow & wS_{\bullet}\mathcal{M}V^w \\ \downarrow & & \downarrow \\ vS_{\bullet}\mathcal{M}V & \longrightarrow & wS_{\bullet}\mathcal{M}V \end{array}$$

in which the upper righthand term is contractible. In [**12**] we prove two identification theorems. The first one identifies the lower lefthand term of the diagram with the product of the K-theories of A, B, and C, essentially.

THEOREM 2.4. *The forgetful functors from $\mathcal{M}V$ to the module categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, and $\mathcal{M}_f(C)$ induce a homotopy equivalence*

$$u_* : vS_{\bullet}\mathcal{M}V \longrightarrow hS_{\bullet}\mathcal{M}_f(A) \times hS_{\bullet}\mathcal{M}_f(B) \times hS_{\bullet}\mathcal{M}_f(C).$$

□

The second one identifies the lower righthand term with the K-theory of D. At present the proof requires an extra hypothesis on the maps $\beta : A \longrightarrow B$ and $\gamma : A \longrightarrow C$. This hypothesis is clearly satisfied in our applications.

THEOREM 2.5. *Suppose that β and γ admit left inverses which are also ring maps. Then the forgetful functor*

$$u_D : \mathcal{M}V \longrightarrow \mathcal{M}_f(D)$$

induces a homotopy equivalence

$$wS_{\bullet}\mathcal{M}V \longrightarrow hS_{\bullet}\mathcal{M}_f(D).$$

□

The paper [**12**] develops the preceding results for A_∞ rings extending ideas of Pierre Vogel, [**15**]. The paper by Schwänzl and Staffeldt [**11**] may be viewed as a preview of the methods of abstract K-theory used in [**12**]. However, the preparations for the use of the methods of abstract K-theory in the case of A_∞-rings are very strenuous. Both of these results will be used in the next section.

3. Approximations to stable K-theory

In this section we outline the steps in the key calculation that for a ring $B = S \vee B'$ we have

$$K^s(B; B \wedge B^{op}) \simeq B.$$

To identify $K^s(B; B \wedge B^{op})$ in a stable range, it suffices to calculate

$$F(B; B \wedge B^{op})(m) = \text{fiber}(K(B \oplus B \wedge B^{op}\langle m \rangle) \longrightarrow K(B)$$

since the canonical map

$$\Omega^{m+1} F(B; B \wedge B^{op})(m) \longrightarrow K^s(B; B \wedge B^{op})$$

is m-connected by proposition 1.2.

Now, working with $B = S \vee B'$ enables us to reformulate the calculation of the relative K-theory in terms of the K-theory of a generalized free product. To do this, we specialize the example of the free product

$$D(m) = C(m) *_A B = (A \wedge G(m)_+) *_A B$$

given in the preceding section by taking $A = S$, the sphere spectrum.

LEMMA 3.1. *In a stable range (up through $\approx 2m$) we have a natural chain of equivalences*

$$F(B; B \wedge B^{op})(m) \simeq \text{fiber}(K((S \wedge G(m)_+) *_S B) \longrightarrow K(B)).$$

PROOF. There is a canonical $(2m+1)$-connected map

$$S^m \longrightarrow G(m)$$

analogous to the familiar map $S^m \longrightarrow \Omega S^{m+1}$. In the definition

$$B \oplus B \wedge B^{op}\langle m \rangle \cong B \oplus B \wedge S^m \wedge B^{op},$$

where multiplication is zero on $B \wedge S^m \wedge B$, one may substitute $G(m)$ for S^m obtaining a similar ring

$$D'(m) = B \oplus B \wedge G(m) \wedge B^{op}$$

and a $(2m+1)$-connected ring map

$$B \oplus B \wedge B^{op}\langle m \rangle \longrightarrow D'(m).$$

Writing $D(m) = (S \wedge G(m)_+) *_S B$ and using the recipe given in proposition 2.2, we see there is an obvious $(2m+1)$-connected retraction

$$D(m) \longrightarrow D'(m),$$

since $D'(m)$ is exactly the part of $D(m)$ displayed in the formula preceding proposition 2.2, and the parts of $D(m)$ that are not displayed all involve at least two smash factors $C'(m) = S \wedge G(m)$. Furthermore, using the plus-construction definition of K-theory, as in the proof of proposition 1.9, one sees that passage to K-theory preserves connectivity of ring maps. (Cf. proposition 1.1, [**20**, page 36].) But we are interested in K-theory of these rings relative to the fixed ground ring B, so we observe that passage to homotopy fibers preserves connectivity on the level of K-theory, and this finishes the proof of the lemma. □

Continue with the specializations $A = S$, $C(m) = S \wedge G(m)_+$, and $D(m) = C(m) *_S B$ introduced above, and write

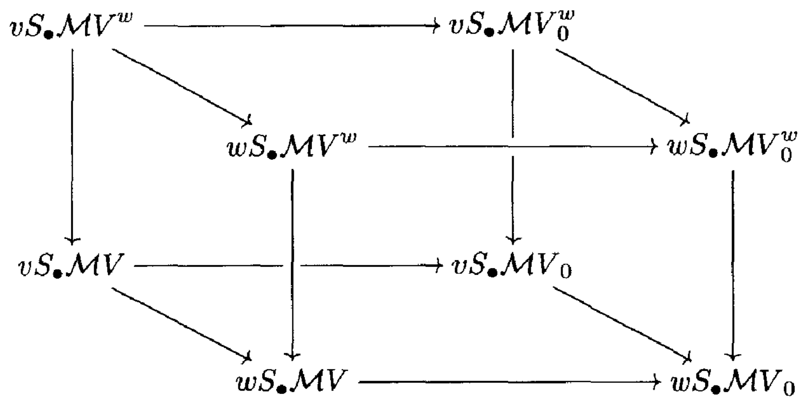
$$K(D(m), B) = \text{fiber}(K(D(m)) \longrightarrow K(B)),$$

just renaming our approximation to $F(B; B \wedge B^{op})$.

Next observe that the retraction $C(m) \longrightarrow S$ induces a map of generalized free product diagrams

$$\begin{array}{ccc} S & \longrightarrow & B \\ \downarrow & & \downarrow \\ C(m) & \longrightarrow & D(m) \end{array} \quad \longrightarrow \quad \begin{array}{ccc} S & \longrightarrow & B \\ \downarrow & & \downarrow \\ S & \longrightarrow & B. \end{array}$$

Write $\mathcal{M}V$ for the category of Mayer-Vietoris presentations associated to the left-hand diagram and $\mathcal{M}V_0$ for the Mayer-Vietoris presentations associated to the righthand diagram. We obtain a map of homotopy cartesian squares, displayed as follows.



Having defined

$$K(D(m), B) = \text{fiber}(K(D(m)) \longrightarrow K(B)),$$

we also define

$$K(C(m), S) = \text{fiber}(K(C(m)) \longrightarrow K(S)).$$

Now we take homotopy fibers of the map of cartesian squares above and apply theorem 2.4 and theorem 2.5 to interpret the spaces at the bottom of the squares. We obtain the following result.

PROPOSITION 3.2. *There is a fibration-up-to-homotopy*

$$\widetilde{K}(\text{Nil}(S; B', C'(m))) \longrightarrow K(C(m), S) \longrightarrow K(D(m), B),$$

where $\widetilde{K}(\text{Nil}(S; B', C'(m))) = \text{fiber}(\Omega|vS_\bullet \mathcal{M}V^w| \longrightarrow \Omega|vS_\bullet \mathcal{M}V_0^w)|)$.

□

From this fibration sequence one derives the sequence

$$\Omega^{m+1} K(C(m), S) \longrightarrow \Omega^{m+1} K(D(m), B) \longrightarrow \Omega^m \widetilde{K}(\text{Nil}(S; B', C'(m)))$$

in which the middle term is our approximation to $K^s(B; B \wedge B^{op})$. The following result identifies the other two terms in the fibration sequence.

THEOREM 3.3. *In a stable range of dimensions $\phi(m)$ tending to infinity with m we have two $\phi(m)$-equivalences*

$$\Omega^{m+1} K(C(m), S) \simeq S \text{ and } \Omega^m \widetilde{K}(\text{Nil}(S; B', C'(m))) \simeq B'.$$

REMARKS ON THE PROOF. The stable range equivalence $\Omega^{m+1}K(C(m), S) \simeq S$ is derived by combining the fact that stable homotopy theory splits off the stable algebraic K-theory of the one point space, the main result of [**18**], with the fact that the other term in the splitting is in fact trivial, the main result of [**19**]. The other identification follows after careful analysis of the reduced Nil-term $\widetilde{K}(\mathrm{Nil}(S; B', C'(m)))$ [**13**]. This part of the argument uses alternative descriptions of the Nil-term and manipulations patterned on those in [**18**], as well as two more applications of [**19**]. □

References

[1] M. Bökstedt, W.C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces,* Invent. Math, **111**(1993), 465–540.

[2] B. Dundas, *Relative K-theory and topological cyclic homology,* Preprint, 1995.

[3] B. Dundas and R. McCarthy, *Stable K-theory and topological Hochschild homology,* Ann. Math., **140**(1994), 685–701.

[4] A.D. Elmendorf, I. Kriz, J.P. May, and M. Mandell, *Rings, modules, and algebras in stable homotopy theory,* Preprint, 1995.

[5] Z. Fiedorowicz, C. Ogle, and R.M. Vogt, *Volodin K-theory of A_∞-ring spaces,* Topology, **32**(1993), 329–352.

[6] T. G. Goodwillie, *Relative algebraic K-theory and cyclic homology,* Ann. Math., **124** (1986), 347–402.

[7] S. MacLane, *Homology,* Third Corrected Printing. Springer Verlag, New York, 1975.

[8] L.G. Lewis, J.P. May, and M. Steinberger, *Equivariant Stable Homotopy Theory,* Springer Lecture Notes in Mathematics 1213, 1986.

[9] J.P. May, *A_∞ ring spaces and algebraic K-theory,* Springer Lecture Notes in Mathematics 658, (1978), 240–315.

[10] A. Robinson, *Derived tensor products in stable homotopy theory,* Topology **22**(1983), 1–18.

[11] R. Schwänzl and R.E. Staffeldt, *The approximation theorem and the K-theory of generalized free products,* Trans. Amer. Math. Soc., **347**(1995), 3319–3345.

[12] R. Schwänzl, R.E. Staffeldt, and F. Waldhausen, *Stable K-theory and generalized free products,* In preparation.

[13] R. Schwänzl, R.E. Staffeldt, and F. Waldhausen, *Stable K-theory and topological Hochschild homology,* In preparation.

[14] R. Schwänzl and R.M. Vogt, *Basic constructions in the K-theory of homotopy ring spaces,* Trans. Amer. Math. Soc. **341**(1994), 549–583.

[15] P. Vogel, *On the vanishing of* Nil-*groups,* Preprint, 1990.

[16] F. Waldhausen, *Algebraic K-theory of generalized free products,* Ann. Math., **108** (1978), 135–256.

[17] F. Waldhausen, *Algebraic K-theory of spaces,* Springer Lecture Notes in Mathematics 1126 (1985), 318–419.

[18] F. Waldhausen, *Algebraic K-theory of topological spaces, II,* Springer Lecture Notes in Mathematics 763 (1979), 356–394.

[19] F. Waldhausen, *Algebraic K-theory of spaces, concordance and stable homotopy theory,* Ann. Math. Study 113 (1987), 392–417.

[20] F. Waldhausen, *Algebraic K-theory of topological spaces, I,* Proc. Symp. Pure Math. **32**(1978), 35–60.

FACHBEREICH MATHEMATIK/INFORMATIK, UNIVERSITÄT OSNABRÜCK, 49069 OSNABRÜCK, FEDERAL REPUBLIC OF GERMANY

E-mail address: `roland@mathematik.uni-osnabrueck.de`

DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NM 88003, USA

E-mail address: `ross@nmsu.edu`

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, 33615 BIELEFELD, FEDERAL REPUBLIC OF GERMANY

E-mail address: `fw@mathematik.uni-bielefeld.de`