Pricing Derivatives Under a Gains Tax Regime: New Impacts

Christoph Wöster
Bielefeld University

October 2006

BIELEFELD UNIVERSITY
Department of Business Administration and Economics
P.O. Box 10 01 31
D-33501 Bielefeld
Germany

Discussion Paper No. 559

Pricing Derivatives Under a Gains Tax Regime: New Impacts *

Christoph Wöster

Department of Business
Administration and Economics
Bielefeld University
P.O. Box 10 01 31
33 501 Bielefeld
Germany
cwoester@uni-bielefeld.de

October 2006

Abstract

Three years after the seminal work of Black and Scholes [3] on the pricing of European options, Scholes [18] presented a paper in which the impact of taxation on the value of an option is analyzed. We restart this discussion in a simple binomial setting emphasizing the economic principles of replicating strategies under taxation. Two perspectives will be introduced. The first one focuses on replicating payoff structures if the underlying assets are taxed. The second one discusses the influence of a tax system on a given contract specification. The limit results lead to a pricing formula in closed form suggesting a modification of the partial differential equation derived by Scholes [18]. We claim that the value of the option is influenced by taxation even if gains of all relevant assets are taxed with the same rate. Furthermore, we demonstrate that the difference between numerical and closed form solutions are negligible in acceptable computing time. Thus the algorithmic schemes can be used as a base for pricing of complex options under taxation.

Keywords: derivative, gains tax, option pricing, Black-Scholes model, binomial model, hedging

JEL Classification: G13, C60

* I like to thank Julia Frerking and Dennis Kirchhoff for valuable comments. All errors remain with the author.

1 Introduction

The theory of pricing derivatives is methodologically sound if the payoff of the claim can be replicated by assets on frictionless markets. Most publications in this area abstract from trading fees, taxes and regulatory restrictions, though one can expect a considerable influence of theses imperfections on the value of derivatives. There have been several approaches to relax the strict assumptions of standard derivative pricing. Some attempts have been made to integrate taxes, transaction costs and short-sale restrictions, nevertheless literature in this field remained rare.

The discussion of the influence of taxes on the value of derivatives began in 1976 with a paper of Scholes [18] presented on the 34^{th} annual meeting of the American Finance Association. Ross [17] analyzed taxes in the context of the arbitrage theory. His discussion was performed in a rather abstract framework, though he was able to point out parallels to the main ideas of Miller and Scholes [14], Constantinides [4, 5], Ross [16] and Constantinides and Ingersoll [6] ranging from the *debt/equity decision* via *tax options* to *dividends and taxes*.

In the 90s, there have been some new approaches to embed tax schemes into financial applications. However, most of these publications are directed to analyzing the impact of taxation in a portfolio optimization context. Dammon and Spatt [8] focus on asymmetries in a tax system, in which tax rates on capital gains differ with respect to the holding period of the investment. The approach extends similar models by Constantinides [4], Williams [21] and Schultz [19].

Dammon, Spatt, and Zhang [9] investigated the optimal consumption/investment decision of a private investor over his lifetime. The tax base was calculated as the weighted average purchase price, which kept the dimensionality of the problem low. Nevertheless, they were not able to obtain analytical solutions and had to perform numerical analysis to get results. However, they illustrate that the tax system can have a substantial impact on the investor's decision, particularly if asymmetric tax rules are considered. Similar models have been developed by Gallmeyer, Kaniel, and Tompaidis [10] and Garlappi, Naik, and Slive [11].

The overview shows that there is active research on the impact of taxation on decisions in a dynamic portfolio problem. Moreover, we have seen various analysis in all kinds of financial applications. In contrast, papers integrating tax issues into a derivative pricing framework are rare. This has motivated us to review the problem on a sound foundation.

This paper is organized as follows. In section 2 the tax system is modelled in a discrete-time framework. The idea is based on the famous binomial model by Cox, Ross, and Rubinstein [7] but performed in a modern version using numeraire processes and martingales. The tax terminology is introduced and the general concepts of replicating payoffs are adapted to taxation. Although taxation can induce some path-dependencies, the value of the derivative can always be computed in efficient recombining trees.

In section 3 limit results are obtained. The closed form solution can be considered as a Black-Scholes formula under taxation with different tax rates. The special case of equal tax rates — often discussed against the background of decision neutral tax schemes — is reviewed. Furthermore, sensitivity functions used for approximating hedging in discrete time are derived. They confirm the partial differential equation that follows from the no-arbitrage or martingale condition of the price process of the standardized tradeable asset.

Numerical results are presented in section 4. The converging results of the discrete models are compared with the value of the closed form solution. We show that acceptable results are obtained within a computation time of no more than 1 second.

2 Capital gains taxes in a discrete-time model

In this section a capital gains tax scheme is modelled in a discrete-time framework. There are several reasons for using this simplified approach. First, we can introduce the basic terms and the tax-specific modifications in a traceable environment. Secondly, we can analyze different aspects of taxation and compare the limit of each version with the continuous-time model. Finally, discrete-time models are very popular among practitioners — and taxes are a highly relevant aspect in real-life problems. Binomial models can easily be modified and adapted to a wide range of derivative contracts. Though most scientists would recommend taking partial differential equations as the starting point for modifications, the binomial model is still a widely-used approach.

2.1 The modelling framework

Assume there is a market that is open at certain discretely spaced trading dates contained in the naturally ordered set

$$\mathbb{T} := \{t_0, t_1, \ldots, t_N\} .$$

The uncertainty about the evolution of future prices on this market is captured by a probability space

$$(\Omega, \mathcal{F}, P) ,$$

where Ω is the (countable) sample space, $\mathcal{F}$ a σ-algebra, and P a probability measure. The probability space is equipped with a filtration $(\mathcal{F}_{t_i})_{i=0}^{N}$ representing the evolution of information over time. The filtration satisfies the usual conditions.

The basic principle of pricing derivatives is to determine a portfolio strategy formed of elementary assets whose price dynamics can be described with high precision. These assets are combined such that the resulting portfolio payoff replicates exactly the derivative payoff in any state at any time. The value of the derivative is defined as the price of the replicating strategy. If the market is arbitrage-free, which is assumed in this model, then the price of the derivative must be equal to its value.

In this discrete-time modelling framework it is assumed that two assets are traded on the financial market, a stock and a money market fund. The stock prices are driven by a binomial process such that the next period price can only adopt two values. A more concrete specification of the process is given in the next section. The evolution of the prices of the money market fund is deterministic and henceforth known in advance.

A trading strategy ϕ is a sequence of portfolios $\phi_{t_i} = (\alpha_{t_i}, \beta_{t_i})$, $i = 0, \ldots, N-1$, indicating the number of stocks and money market funds, respectively, held in period $(t_i, t_{i+1}]$. The price of a portfolio ϕ_{t_i} that has to be paid immediately after t_i, $i = 0, \ldots, N-1$, is given by

$$\Pi^+_{t_i} := \alpha_{t_i} S_{t_i} + \beta_{t_i} B_{t_i} . \tag{1}$$

The price of the trading strategy ϕ in t_0 is defined as

$$\Pi_{t_0} := \alpha_{t_0} S_{t_0} + \beta_{t_0} B_{t_0} ,$$

such that $\Pi_{t_0} = \Pi^+_{t_0}$ is always satisfied.

The introduction of taxes requires a refinement of notation, the modification of some basic principles and a clear formulation of the payoff to be replicated. Let us start with the payoff structure and assume that we are to build a portfolio in the last but one period that replicates the payoff at maturity. We have to consider tax payments, so we cannot realize $\alpha_{t_{N-1}} S_{t_N} + \beta_{t_{N-1}} B_{t_N}$ at maturity. Furthermore, if we have to deduce the payment from a contract, say $\tilde{X}_{t_N}$, then the taxation of the derivative has to be taken into account. Finally, taxation will generally lead to a modified strategy, which we will denote $\tilde{\phi}$ in the following. Thus, the replicating strategy under taxation has to satisfy

$$\tilde{\alpha}_{t_{N-1}} \left(S_{t_N}(\omega) - \mathcal{T}^S_{t_N}(\omega) \right) + \tilde{\beta}_{t_{N-1}} \left(B_{t_N}(\omega) - \mathcal{T}^B_{t_N}(\omega) \right) = \tilde{X}_{t_N}(\omega) - \mathcal{T}^X_{t_N}(\omega) .$$

$T^P_{t_i}(\omega)$ is the absolute tax amount related to asset P and payable in t_i. The dependence of the state of the world is induced by the stochastic price behavior of the corresponding asset, which directly influences the tax base.

There still remains the question what payment should be replicated. We will analyze two cases or — as it is labelled in this paper — two views on the taxation of contingent claims. The first view is

directed to payoffs. The underlying question of this perspective is: What is the price of a portfolio exactly replicating a given payoff after a certain tax scheme has been introduced? The given payoff we are interested in is the payoff that would be realized in a world without taxes, so the complete impact of the tax is incorporated in the modified value. If this case is analyzed, then the after-tax payoff is equal to the given pre-tax value, i.e.

$$\tilde{X}_{t_N}(\omega) - \mathcal{T}^X_{t_N}(\omega) = X_{t_N}(\omega)\,.$$

This question will always be the decisive one if the economic consequences of a contract are relevant — not its formal specification. This is true for example if someone wants an uncertain payoff to be hedged.

The second view is aimed at the economic consequences of the introduction of a certain tax scheme on a certain contract. This perspective focuses on the tax system and its impact on investments. Another case where this question becomes relevant is the situation in which an investor already owns a contract and the tax system is supposed to be changed. Hence, the investor is interested in the economic consequences on the contract he owns and not in the payoffs he cannot realize anymore. If these aspects are analyzed, then we set

$$\tilde{X}_{t_N}(\omega) - \mathcal{T}^X_{t_N}(\omega) = X_{t_N}(\omega) - \mathcal{T}^X_{t_N}(\omega) \quad \Leftrightarrow \quad \tilde{X}_{t_N}(\omega) = X_{t_N}(\omega)\,,$$

i.e. the payoff before taxes corresponds to the contractual payoff. The problem with this approach is that we have to replicate an after-tax payoff we do not know in advance. The way to solve this problem is explained in detail below.

Let us shortly review the cyclic actions that take place in each trading date in $\mathbb{T} \setminus \{t_0\}$. To lighten the notation, the state of the world is omitted if the realization of the corresponding period is known or of minor relevance. Furthermore, it is assumed that the price evolution of the money market fund is deterministic. If at any date t_i new prices are observed, the portfolio built in t_i is dissolved yielding the pre-tax price (or value)

$$\tilde{\alpha}_{t_{i-1}} S_{t_i}(\omega_i) + \tilde{\beta}_{t_{i-1}} B_{t_i} = \tilde{\Pi}_{t_i}(\omega_i)\,.$$

Then, taxes are raised immediately resulting in an after-tax portfolio value

$$\tilde{\alpha}_{t_{i-1}} S^{\tau}_{t_i}(\omega_i) + \tilde{\beta}_{t_{i-1}} B^{\tau}_{t_i} = \tilde{\Pi}^{\tau}_{t_i}(\omega_i)\,, \tag{2}$$

which is the economically relevant variable. Unless the final period is reached, the after-tax value is

reinvested in portfolio $\phi_{t_{i+1}}$ at the price

$$\tilde{\Pi}^{+}_{t_i}(\omega_i) = \tilde{\alpha}_{t_i} S_{t_i}(\omega_i) + \tilde{\beta}_{t_i} B_{t_i}. \tag{3}$$

Since we assumed that only two states can occur after a subperiod has passed, say u and d, the number of stocks must satisfy

$$\tilde{\alpha}_{t_i} = \frac{\tilde{\Pi}^{\tau}_{t_{i+1}}(u) - \tilde{\Pi}^{\tau}_{t_{i+1}}(d)}{S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)} \tag{4}$$

and the number of money market funds

$$\tilde{\beta}_{t_i} = \frac{S^{\tau}_{t_{i+1}}(u)\,\tilde{\Pi}^{\tau}_{t_{i+1}}(d) - S^{\tau}_{t_{i+1}}(d)\,\tilde{\Pi}^{\tau}_{t_{i+1}}(u)}{B^{\tau}_{t_{i+1}}\left(S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)\right)} \tag{5}$$

in order to be a replicating strategy. However, in general

$$\tilde{\Pi}_{t_i}(\omega_i) \neq \tilde{\Pi}^{+}_{t_i}(\omega_i),$$

so this sequence of portfolios is not self-financing in the classical sense. Since the government obtains or pays a certain cash flow after each period, the concept of self-financing strategies has to be reformulated if portfolios are analyzed in a world with taxes. This is done in the following definition:

Definition 2.1 (self-financing trading strategy under a tax regime $\mathcal{T}$) *A trading strategy $\tilde{\phi}$ is called self-financing under a tax regime $\mathcal{T}$ if the after tax value of $\tilde{\phi}_{t_{i-1}}$ in t_i equals the price of the newly formed portfolio $\tilde{\phi}_{t_i}$ held in period $(t_i, t_{i+1}]$, i.e.*

$$\tilde{\alpha}_{t_{i-1}} S^{\tau}_{t_i} + \tilde{\beta}_{t_{i-1}} B^{\tau}_{t_i} = \tilde{\alpha}_{t_i} S_{t_i} + \tilde{\beta}_{t_i} B_{t_i} \tag{6}$$

or

$$\left(\tilde{\alpha}_{t_i} - \tilde{\alpha}_{t_{i-1}}\right) S_{t_i} + \left(\tilde{\beta}_{t_i} - \tilde{\beta}_{t_{i-1}}\right) B_{t_i} = -\left(\tilde{\alpha}_{t_{i-1}} \mathcal{T}^{S}_{t_i} + \tilde{\beta}_{t_{i-1}} \mathcal{T}^{B}_{t_i}\right). \tag{7}$$

The second formulation in equation (7) underscores the constraint that the taxes to be paid (or received) have to be completely financed by or absorbed by the change of the portfolio value.

As mentioned above, only the classical capital gains tax is reviewed, i.e. if prices $P_{t_{i-1}}$ and P_{t_i} are observed in t_{i-1} and t_i, respectively, then the tax payable in t_i is determined according to

$$\mathcal{T}^{P}_{t_i} = \tau_P \left(P_{t_{i-1}} - P_{t_i}\right).$$

The range of values the tax rate τ_P can adopt will be important in economical interpretations. For that reason, we assume that

$$0 \leq \tau_P < 1$$

will be satisfied for any tax rate.

2.2 A decision-neutral tax: a fallacy in discrete time

One might be tempted to replace the portfolio value in t_i by the value of its components $\tilde{\alpha}_{t_i} S_{t_i}$ and $\tilde{\beta}_{t_i} B_{t_i}$. Proceeding this way, one obtains an after tax portfolio value

$$\tilde{\Pi}^{\tau}_{t_{i+1}}(\omega_{i+1}) = (1-\tau_X)\,\tilde{\Pi}_{t_{i+1}}(\omega_{i+1}) + \tau_X\left(\tilde{\alpha}_{t_i} S_{t_i} + \tilde{\beta}_{t_i} B_{t_i}\right)$$

in period t_{i+1}. Inserting the possible prices, the portfolio structure can be resolved. Hence, the number of stock is given by

$$\tilde{\alpha}_{t_i} = \frac{1-\tau_X}{1-\tau_S} \cdot \frac{\tilde{\Pi}_{t_{i+1}}(u) - \tilde{\Pi}_{t_{i+1}}(d)}{S_{t_{i+1}}(u) - S_{t_{i+1}}(d)},$$

the number of money market funds by

$$\begin{aligned}\tilde{\beta}_{t_i} &= (1-\tau_X)\,\frac{\left(S^{\tau}_{t_{i+1}}(u) - \tau_X S_{t_i}\right)\tilde{\Pi}_{t_{i+1}}(d) - \left(S^{\tau}_{t_{i+1}}(d) - \tau_X S_{t_i}\right)\tilde{\Pi}_{t_{i+1}}(u)}{\left(B^{\tau}_{t_{i+1}} - \tau_X B_{t_i}\right)\left(S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)\right)} \\ &= \frac{1-\tau_X}{1-\tau_B} \cdot \frac{\left(S_{t_{i+1}}(u) - \frac{\tau_X-\tau_S}{1-\tau_S} S_{t_i}\right)\tilde{\Pi}_{t_{i+1}}(d) - \left(S_{t_{i+1}}(d) - \frac{\tau_X-\tau_S}{1-\tau_S} S_{t_i}\right)\tilde{\Pi}_{t_{i+1}}(u)}{\left(B_{t_{i+1}} - \frac{\tau_X-\tau_B}{1-\tau_B} B_{t_i}\right)\left(S_{t_{i+1}}(u) - S_{t_{i+1}}(d)\right)}.\end{aligned}$$

One obtains the famous result that the decision variable, i.e. the complete portfolio process, remains unchanged if the tax rate of interest-bearing assets τ_B equals the tax rate of dividend paying assets τ_S.

Unfortunately, the trading strategy that has just been determined does not have the properties we required. Since

$$\tilde{\alpha}_{t_i} S_{t_i} + \tilde{\beta}_{t_i} B_{t_i}$$

is the price one has to pay for a portfolio held from $(t_i, t_{i+1}]$, it corresponds to the after tax portfolio value in t_i which does not determine the base for taxation. Instead, the pre-tax price $\tilde{\alpha}_{t_{i-1}} S_{t_i} + \tilde{\beta}_{t_{i-1}} B_{t_i}$ has to be inserted. To put it the other way round, in general the trading strategy cannot be self-financing under a tax regime.

2.3 The payoff view of replication under taxation

As mentioned in the introduction, the impact of taxation on derivative values is analyzed from two perspectives. We start the discussion concentrating on a given payoff. In other words, in this section we try answer the question: If we want to have a certain payoff structure (e.g. the payoff structure of a certain derivative in a world without taxes), what is the value of a derivative that offers the same payoff after taxes have been paid?

The basic principles are still valid. First, a trading strategy has to be found such that the payoff is replicated by the payoffs of the market traded assets. The only difference to standard derivative pricing is the taxation of the assets contained in the portfolio strategy. As before, the value of the derivative is defined as the price one has to pay for initializing the portfolio strategy.

Since the value at maturity is known and equal to the given payoff, i.e.

$$\tilde{\Pi}^{\tau}_{t_N} = X_{t_N},$$

equation (4), the number of stocks, and equation (5), the number of money market funds, can directly be applied to determine the replicating portfolio structure in each period.

Inserting the portfolio structure of (4) and (5) into equation (3), the price of a portfolio right after taxation, yields

$$\tilde{\Pi}^{+}_{t_i} = \frac{\tilde{\Pi}^{\tau}_{t_{i+1}}(u) - \tilde{\Pi}^{\tau}_{t_{i+1}}(d)}{S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)} S_{t_i} + \frac{S^{\tau}_{t_{i+1}}(u)\tilde{\Pi}^{\tau}_{t_{i+1}}(d) - S^{\tau}_{t_{i+1}}(d)\tilde{\Pi}^{\tau}_{t_{i+1}}(u)}{B^{\tau}_{t_{i+1}}\left(S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)\right)} B_{t_i}. \tag{8}$$

It is well known since Harrison and Kreps [12] and Harrison and Pliska [13] that the value of a derivative can be represented as an expected value under an equivalent martingale measure. Since this method allows for pricing derivatives in a very elegant way, we analyze the consequences of the tax system on the martingale method and its characteristics.

Following the idea of Harrison and Kreps [12], equation (8) can be rewritten as

$$\tilde{\Pi}^{\tau}_{t_i} = \frac{S_{t_i}B^{\tau}_{t_{i+1}} - S^{\tau}_{t_{i+1}}(d)B_{t_i}}{S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)} \left(B^{\tau}_{t_{i+1}}\right)^{-1} \tilde{\Pi}^{\tau}_{t_{i+1}}(u) + \frac{S^{\tau}_{t_{i+1}}(u)B_{t_i} - S_{t_i}B^{\tau}_{t_{i+1}}}{S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)} \left(B^{\tau}_{t_{i+1}}\right)^{-1} \tilde{\Pi}^{\tau}_{t_{i+1}}(d)$$

or equivalently as

$$\begin{aligned} B^{-1}_{t_i}\tilde{\Pi}^{\tau}_{t_i} &= \frac{S_{t_i}B^{-1}_{t_i}B^{\tau}_{t_{i+1}} - S^{\tau}_{t_{i+1}}(d)}{S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)} \left(B^{\tau}_{t_{i+1}}\right)^{-1} \tilde{\Pi}^{\tau}_{t_{i+1}}(u) + \frac{S^{\tau}_{t_{i+1}}(u) - S_{t_i}B^{-1}_{t_i}B^{\tau}_{t_{i+1}}}{S^{\tau}_{t_{i+1}}(u) - S^{\tau}_{t_{i+1}}(d)} \left(B^{\tau}_{t_{i+1}}\right)^{-1} \tilde{\Pi}^{\tau}_{t_{i+1}}(d) \\ &= q^{\tau}_{t_i} \left(B^{\tau}_{t_{i+1}}\right)^{-1} \tilde{\Pi}^{\tau}_{t_{i+1}}(u) + \left(1 - q^{\tau}_{t_i}\right) \left(B^{\tau}_{t_{i+1}}\right)^{-1} \tilde{\Pi}^{\tau}_{t_{i+1}}(d) \end{aligned}$$

where the factors

$$q_{t_i}^{\tau} := \frac{S_{t_i} B_{t_i}^{-1} B_{t_{i+1}}^{\tau} - S_{t_{i+1}}^{\tau}(d)}{S_{t_{i+1}}^{\tau}(u) - S_{t_{i+1}}^{\tau}(d)} \tag{9}$$

satisfy the properties of (transition) probabilities under suitable restrictions on the growth factors of the price processes. If we consider $B_{t_{i+1}}^{\tau}$ to be a numeraire, then the portfolio process expressed in units of the numeraire is obviously not a martingale. However, we can find a numeraire such that all price processes in units of the numeraire are martingales.

Define for all $t_i \in \mathbb{T}$ the numeraire process by

$$\mathcal{N}_{t_i} := \begin{cases} B_{t_0}, & t_i = t_0; \\ B_{t_0} \prod_{s=1}^{i} \frac{B_{t_s}^{\tau}}{B_{t_{s-1}}} & t_i > t_0; \end{cases} \tag{10}$$

$B_{t_0} > 0$, then the process

$$\hat{\Pi}_{t_i}^{\tau} := \mathcal{N}_{t_i}^{-1} \tilde{\Pi}_{t_i}^{\tau}$$

is a martingale. The following proposition summarizes the main result of this section.

Proposition 2.1 *Let $(\mathcal{N}_{t_i})_{i=0}^{N}$ be the price process of a numeraire. Let the price process of a tradeable asset under taxation be given by*

$$Z_{t_i} := \begin{cases} S_{t_0}, & t_i = t_0; \\ S_{t_0} \prod_{s=1}^{i} \frac{S_{t_s}^{\tau}}{S_{t_{s-1}}} & t_i > t_0. \end{cases} \tag{11}$$

Assume that the price processes behave in a way that the two state market is complete at any time. Define a measure Q^{τ} such that the transition probabilities are given by (9) and the price process of the tradeable asset in units of the numeraire by

$$\hat{Z}_{t_i} = \hat{Z}_{t_0} \prod_{s=1}^{i} \frac{\left(B_{t_s}^{\tau}\right)^{-1} S_{t_s}^{\tau}}{B_{t_{s-1}}^{-1} S_{t_{s-1}}}.$$

Finally, assume that $\mathrm{E}\left[|\hat{Z}_{t_i}|\right] < \infty$ is satisfied for all $t_i \in \mathbb{T}$.

Then the process $\left(\hat{Z}_{t_i}\right)_{i=1}^{N}$ is a martingale and the value of a derivative in t_0 — expressed as units of a numeraire — generating a payoff X_{t_N} under a tax regime $\mathcal{T}$ is given by

$$\hat{V}_{t_0}^{\tau} = \mathrm{E}_{Q^{\tau}}\left[\hat{X}_{t_N} \middle| \mathcal{F}_{t_0}\right].$$

2.4 The contract-centric view

In this section the view changes from payoffs to fixed contract specifications. The question that is to be answered in this section is: What is the value of a contract with given specifications after a tax system has been introduced?

The main problem that is to be solved first is the fact that the final after-tax value is not known. We just know the pre-tax value from the contract but the value that investors are actually interested in is the after-tax value. However, the final value has to be known if the basic principle of replication is to be applied.

The strategy is to formulate an arbitrary pre-tax portfolio value as a function of the current after-tax value, all preceding after-tax values and the value of the initial portfolio at time t_0. Hence, this pre-tax portfolio value can be replaced by values that, in turn, can be substituted by the corresponding after-tax portfolio values. Thus, we first show that the relation in the following lemma is true.

Lemma 2.1 *Let V_{t_0} be a start value and consider the scheme*

$$V_{t_s}^{\tau} = V_{t_s} - \tau_X \left(V_{t_s} - V_{t_{s-1}} \right) ,$$

then for any $t_i \in \mathbb{T}$, the pre-tax value can be written explicitly as

$$V_{t_i} = \rho_{i,0} V_{t_0} + \sum_{s=1}^{i} \rho_{i,s} \frac{V_{t_s}^{\tau}}{1-\tau_X} , \tag{12}$$

where

$$\rho_{i,s} := \rho_{i-s} = \left(\frac{-\tau_X}{1-\tau_X} \right)^{i-s} . \tag{13}$$

The straightforward proof is given in the appendix.

The strategy we mentioned above is applied to a European style contract. The value of the derivative at maturity, V_{t_N}, equals the payoff generated by the contract, which is denoted X_{t_N}. Consequently, the replicating portfolio must have an after-tax value

$$\tilde{\Pi}_{t_N}^{\tau} = V_{t_N}^{\tau} = X_{t_N} - \mathfrak{T}_{t_N}^{X}$$

at the end of the contract's lifetime. The standard technique, backwards induction, cannot be applied directly to the problem since the tax amount depends on the price of the previous period which depends implicitly on the tax amount of the previous period and so on.

Since a gains tax is assumed, the after-tax value can be written as

$$\begin{aligned}\tilde{\Pi}^{\tau}_{t_N} = V^{\tau}_{t_N} &= X_{t_n} - \tau_X \left(X_{t_N} - V_{t_{N-1}}\right) \\ &= (1-\tau_X)X_{t_N} + \tau_X V_{t_{N-1}}\,,\end{aligned}$$

where $V_{t_{N-1}}$ is the pre-tax value of the derivative implicitly determined such that the after-tax value of the derivative is equal to the after-tax value of the replicating portfolio. Whereas

$$\tilde{\Pi}^{\tau}_{t_i} = V^{\tau}_{t_i}$$

must be satisfied in all points of time to avoid arbitrage,

$$\tilde{\Pi}_{t_i} = V_{t_i}$$

will not be true in general.

The idea is to express the pre-tax value V_{t_i} by those components that are of economic relevance, i.e. by the after-tax values and the initial price. Though it is possible to substitute the variable in a single step according to (12), we do it iteratively using the relation

$$V_{t_{N-1}} = \frac{1}{1-\tau_X} V^{\tau}_{t_{N-1}} - \frac{\tau_X}{1-\tau_X} V_{t_{N-2}}\,.$$

To build a replicating portfolio held from t_{N-1} to t_N the following equation has to be satisfied,

$$\tilde{\alpha}_{t_{N-1}} S^{\tau}_{t_N} + \tilde{\beta}_{t_{N-1}} B^{\tau}_{t_N} = (1-\tau_X)X_{t_N} + \frac{\tau_X}{1-\tau_X}\left(\tilde{\alpha}_{t_{N-1}} S_{t_{N-1}} + \tilde{\beta}_{t_{N-1}} B_{t_{N-1}}\right) - \frac{\tau_X^2}{1-\tau_X} V_{t_{N-2}}\,,$$

where the after-tax value $V^{\tau}_{t_{N-1}}$ on the right hand side has been replaced by the after-tax value of the replicating portfolio. If we define

$$\bar{\Pi}^{\tau}_{t_N} := (1-\tau_X)\, X_{t_N}$$

and

$$\delta_{t_N} := \tau_X\,,$$

then this condition is equivalent to

$$\tilde{\alpha}_{t_{N-1}}\left(S^{\tau}_{t_N} - \frac{\delta_{t_N}}{1-\tau_X} S_{t_{N-1}}\right) + \tilde{\beta}_{t_{N-1}}\left(B^{\tau}_{t_N} - \frac{\delta_{t_N}}{1-\tau_X} B_{t_{N-1}}\right) = \bar{\Pi}^{\tau}_{t_N} + \delta_{t_N}\rho_1 V_{t_{N-2}}\,.$$

$\bar{\Pi}^{\tau}_{t_N}$ and δ_{t_N} have been defined in a way to outline a general structure occurring in each iteration. This should become clear in the course of this section.

Solving for the portfolio components yields

$$\begin{aligned}\tilde{\alpha}_{t_{N-1}} &= \frac{\bar{\Pi}^{\tau}_{t_N}(u) - \bar{\Pi}^{\tau}_{t_N}(d)}{S^{\tau}_{t_N}(u) - S^{\tau}_{t_N}(d)} \\ &= \frac{1-\tau_X}{1-\tau_S} \cdot \frac{X_{t_N}(u) - X_{t_N}(d)}{S_{t_N}(u) - S_{t_N}(d)}\end{aligned}$$

and

$$\begin{aligned}\tilde{\beta}_{t_{N-1}} &= \frac{\left(S^{\tau}_{t_N}(u) - \frac{\delta_{t_N}}{1-\tau_X} S_{t_{N-1}}\right)\bar{\Pi}^{\tau}_{t_N}(d) - \left(S^{\tau}_{t_N}(d) - \frac{\delta_{t_N}}{1-\tau_X} S_{t_{N-1}}\right)\bar{\Pi}^{\tau}_{t_N}(u)}{\left(B^{\tau}_{t_N} - \frac{\delta_{t_N}}{1-\tau_X} B_{t_{N-1}}\right)\left(S^{\tau}_{t_N}(u) - S^{\tau}_{t_N}(d)\right)} \\ &\quad + \delta_{t_N}\rho_1 \frac{B_{t_{N-1}}}{B^{\tau}_{t_N} - \frac{\delta_{t_N}}{1-\tau_X} B_{t_{N-1}}} \frac{V_{t_{N-2}}}{B_{t_{N-1}}} \\ &= \gamma_{t_{N-1}} + \delta_{t_{N-1}} \frac{V_{t_{N-2}}}{B_{t_{N-1}}},\end{aligned}$$

where

$$\delta_{t_{N-1}} = \delta_{t_N}\rho_1 \frac{B_{t_{N-1}}}{B^{\tau}_{t_N} - \frac{\delta_{t_N}}{1-\tau_X} B_{t_{N-1}}}.$$

Some characteristics are worth being mentioned at this step. The number of stocks can be computed directly, though the structure will not be that simple in the next iterations. The formula for the number of money market funds is more involved. However, $\tilde{\beta}_{t_i}$ always can be separated into a part independent of the past and a part that directly depends on the previous value. The same statement hold for the after-tax value that can be split up into a part adapted to $\mathcal{F}_{t_{N-1}}$,

$$\bar{\Pi}^{\tau}_{t_{N-1}} := \tilde{\alpha}_{t_{N-1}} S_{t_{N-1}} + \gamma_{t_{N-1}} B_{t_{N-1}},$$

and a part depending on the portfolio values of the past and hence unknown at this step in the recursive algorithm. So we get the decomposition

$$\begin{aligned}\tilde{\Pi}^{\tau}_{t_{N-1}} &= \tilde{\alpha}_{t_{N-1}} S_{t_{N-1}} + \gamma_{t_{N-1}} B_{t_{N-1}} + \delta_{t_{N-1}} V_{t_{N-2}} \\ &= \bar{\Pi}^{\tau}_{t_{N-1}} + \delta_{t_{N-1}} V_{t_{N-2}},\end{aligned}$$

which is useful since the unknown part can be rolled back until the portfolio value does no longer depend on the past.

We can now proceed and obtain

$$\begin{aligned}\tilde{\alpha}_{t_{N-2}} S^{\tau}_{t_{N-1}} + \tilde{\beta}_{t_{N-2}} B^{\tau}_{t_{N-1}} &= \bar{\Pi}^{\tau}_{t_{N-1}} + \delta_{t_{N-1}} V_{t_{N-2}} \\ &= \bar{\Pi}^{\tau}_{t_{N-1}} + \delta_{t_{N-1}} \left(\frac{1}{1-\tau_X} (\tilde{\alpha}_{t_{N-2}} S_{t_{N-2}} + \tilde{\beta}_{t_{N-2}} B_{t_{N-2}}) - \frac{\tau_X}{1-\tau_X} V_{t_{N-3}} \right)\end{aligned}$$

or

$$\begin{aligned}\tilde{\alpha}_{t_{N-2}} \left(S^{\tau}_{t_{N-1}} - \frac{\delta_{t_{N-1}}}{1-\tau_X} S_{t_{N-2}} \right) + \tilde{\beta}_{t_{N-2}} \left(B^{\tau}_{t_{N-1}} - \frac{\delta_{t_{N-1}}}{1-\tau_X} B_{t_{N-2}} \right) \\ = \bar{\Pi}^{\tau}_{t_{N-1}} - \frac{\tau_X}{1-\tau_X} \delta_{t_{N-1}} V_{t_{N-3}} \,.\end{aligned}$$

If we go on in the same manner, the algorithm ends up with

$$\tilde{\Pi}^{\tau}_{t_1} = \bar{\Pi}^{\tau}_{t_1} + \delta_{t_1} V_{t_0} \,,$$

so we have

$$\tilde{\alpha}_{t_0} \left(S^{\tau}_{t_1} - \delta_{t_1} S_{t_0} \right) + \tilde{\beta}_{t_0} \left(B^{\tau}_{t_1} - \delta_{t_1} B_{t_0} \right) = \bar{\Pi}^{\tau}_{t_1} \,,$$

which is true for

$$\tilde{\alpha}_{t_0} = \frac{\bar{\Pi}^{\tau}_{t_1}(u) - \bar{\Pi}^{\tau}_{t_1}(d)}{S^{\tau}_{t_1}(u) - S^{\tau}_{t_1}(d)}$$

and

$$\tilde{\beta}_{t_0} = \frac{\left(S^{\tau}_{t_1}(u) - \delta_{t_1} S_{t_0} \right) \bar{\Pi}^{\tau}_{t_1}(d) - \left(S^{\tau}_{t_1}(d) - \delta_{t_1} S_{t_0} \right) \bar{\Pi}^{\tau}_{t_1}(u)}{\left(B^{\tau}_{t_1} - \delta_{t_1} B_{t_0} \right) \left(S^{\tau}_{t_1}(u) - S^{\tau}_{t_1}(d) \right)} \,.$$

The following proposition formulates the general result:

Proposition 2.2 *Define*

$$\bar{S}^{\tau}_{t_i} := S^{\tau}_{t_i} - \tilde{\delta}_{t_i} S_{t_{i-1}}$$

and

$$\bar{B}^{\tau}_{t_i} := B^{\tau}_{t_i} - \tilde{\delta}_{t_i} B_{t_{i-1}} \,,$$

where

$$\tilde{\delta}_{t_i} := \begin{cases} \delta_{t_1}, & t_i = t_1; \\ \frac{\delta_{t_i}}{1-\tau_X}, & t_i > t_1; \end{cases}$$

and

$$\delta_{t_i} = \tau_X \rho_{N-i} \prod_{s=1}^{N-i} \frac{B_{t_{N-s}}}{\bar{B}^{\tau}_{t_{N-s+1}}}.$$

Then the portfolio process with

$$\tilde{\alpha}_{t_i} = \frac{\bar{\Pi}^{\tau}_{t_{i+1}}(u) - \bar{\Pi}^{\tau}_{t_{i+1}}(d)}{\bar{S}^{\tau}_{t_{i+1}}(u) - \bar{S}^{\tau}_{t_{i+1}}(d)} \tag{14}$$

and

$$\tilde{\beta}_{t_i} = \begin{cases} \gamma_{t_0}, & t_i = t_0; \\ \gamma_{t_i} - \delta_{t_i} \frac{V_{t_{i-1}}}{B_{t_i}}, & t_i > t_0; \end{cases} \tag{15}$$

where γ_{t_i} is given by

$$\gamma_{t_i} = \frac{\bar{S}^{\tau}_{t_{i+1}}(u)\bar{\Pi}^{\tau}_{t_{i+1}}(d) - \bar{S}^{\tau}_{t_{i+1}}(d)\bar{\Pi}^{\tau}_{t_{i+1}}(u)}{\bar{B}^{\tau}_{t_{i+1}}\left(\bar{S}^{\tau}_{t_{i+1}}(u) - \bar{S}^{\tau}_{t_{i+1}}(d)\right)}, \tag{16}$$

defines a strategy replicating the pre-tax value of a given contract with payoff X_{t_N}.

It is important to note that we do not need to specify the complete portfolio price process. To determine the value of the taxed option one can concentrate on the part

$$\bar{\Pi}^{\tau}_{t_i} = \tilde{\alpha}_{t_i} S_{t_i} + \gamma_{t_i} B_{t_i}.$$

Inserting (14) and (16) into the equation yields

$$\bar{\Pi}^{\tau}_{t_i} = \frac{\bar{\Pi}^{\tau}_{t_{i+1}}(u) - \bar{\Pi}^{\tau}_{t_{i+1}}(d)}{\bar{S}^{\tau}_{t_{i+1}}(u) - \bar{S}^{\tau}_{t_{i+1}}(d)} S_{t_i} + \frac{\bar{S}^{\tau}_{t_{i+1}}(u)\bar{\Pi}^{\tau}_{t_{i+1}}(d) - \bar{S}^{\tau}_{t_{i+1}}(d)\bar{\Pi}^{\tau}_{t_{i+1}}(u)}{\bar{B}^{\tau}_{t_{i+1}}\left(\bar{S}^{\tau}_{t_{i+1}}(u) - \bar{S}^{\tau}_{t_{i+1}}(d)\right)} B_{t_i}.$$

This can be rewritten as

$$B^{-1}_{t_i}\bar{\Pi}^{\tau}_{t_i} = \frac{B^{-1}_{t_i}\bar{B}^{\tau}_{t_{i+1}}S_{t_i} - \bar{S}^{\tau}_{t_{i+1}}(d)}{\bar{S}^{\tau}_{t_{i+1}}(u) - \bar{S}^{\tau}_{t_{i+1}}(d)} \left(\bar{B}^{\tau}_{t_{i+1}}\right)^{-1} \bar{\Pi}^{\tau}_{t_{i+1}}(u) + \frac{\bar{S}^{\tau}_{t_{i+1}}(u) - B^{-1}_{t_i}\bar{B}^{\tau}_{t_{i+1}}S_{t_i}}{\bar{S}^{\tau}_{t_{i+1}}(u) - \bar{S}^{\tau}_{t_{i+1}}(d)} \left(\bar{B}^{\tau}_{t_{i+1}}\right)^{-1} \bar{\Pi}^{\tau}_{t_{i+1}}(d),$$

where the factors

$$q_{t_i}^{\tau} = \frac{B_{t_i}^{-1}\bar{B}_{t_{i+1}}^{\tau} S_{t_i} - \bar{S}_{t_{i+1}}^{\tau}(d)}{\bar{S}_{t_{i+1}}^{\tau}(u) - \bar{S}_{t_{i+1}}^{\tau}(d)} \tag{17}$$

satisfy the properties of (transition) probabilities under suitable restrictions on the growth factors of the price processes.

Define for all $t_i \in \mathbb{T}$ the numeraire process as

$$\mathcal{N}_{t_i} := \begin{cases} B_{t_0}, & t_i = t_0; \\ B_{t_0} \prod_{s=1}^{i} \frac{\bar{B}_{t_s}^{\tau}}{B_{t_{s-1}}}, & t_i > t_0; \end{cases}$$

then the process

$$\hat{\Pi}_{t_i}^{\tau} := \mathcal{N}_{t_i}^{-1} \bar{\Pi}_{t_i}^{\tau}$$

is a martingale. We can formulate a result similar to proposition 2.1, only the numeraire process has changed.

3 Binomial model implementation and limit results

3.1 The structure in binomial models

We get more concrete now and assume that trading only takes place at equidistant points in time such that the set $\mathbb{T}$ can be written as

$$\mathbb{T} = \{t_0, t_0 + \Delta, \ldots, t_0 + N \cdot \Delta\} \tag{18}$$

with the overall time interval between 0 and T being fixed. Suppose that the price of the money market fund is determined by the non-stochastic one-period interest rate $r \geq 0$, such that its price evolution can be described by

$$B_{t_n} = \begin{cases} B_{t_0}, & \text{if } t_n = t_0, \\ B_{t_{n-1}} \exp(r\Delta), & \text{if } t_0 < t_n \leq t_N, \end{cases}$$

where $B_{t_0} > 0$ for economical reasons. The stochastic process that governs the evolution of the stock price is given by

$$S_{t_n} = \begin{cases} S_{t_0}, & \text{if } t_n = t_0; \\ S_{t_{n-1}} \exp(\mu\Delta + \sigma\sqrt{\Delta} X_{t_n}), & \text{if } t_0 < t_n \leq t_N; \end{cases} \tag{19}$$

where $S_{t_0} > 0$ and X_{t_n} is a sequence of independently identically distributed (i.i.d.) Bernoulli random variables

$$X_{t_n} : (\Omega_{t_n}, \mathcal{F}_{t_n}) \to (\mathcal{X}_{t_n}, \mathcal{B}_{t_n})$$

with outcomes in the state space $\mathcal{X}_{t_n} = \{-1, 1\}$. Given the information at t_n, the probability that $X_{t_{n+1}} = 1$ is p and that $X_{t_{n+1}} = -1$ is $(1-p)$. The parameter $\mu \in \mathbb{R}$ is referred to as the drift coefficient and the parameter $\sigma > 0$ as the diffusion coefficient of the process.

First, the transition probabilities induced by the martingale measure Q^τ are analyzed under the additional assumptions introduced in this section. Though the structure of the transition probabilities seems to be quite different, it turns out that they are actually equal. We state this result as a proposition. The applied technique is similar to the one in Amin [1]. The transition probabilities are chosen such that a stock process with a given drift coefficient, say α, is a martingale.[1] Though this drift parameter is of no significance in the limit, the additional degree of freedom is used to improve numerical properties and to guarantee value of q^τ in the interval $(0;1)$. This idea is resumed in section 4.

Proposition 3.1 *Let α be the drift term of the stock price process in a binomial model such that the process is a martingale. The transition probabilities without taxation are given by*

$$q(\alpha) = \frac{e^{r\Delta} - e^{\alpha\Delta - \sigma\sqrt{\Delta}}}{e^{\alpha\Delta + \sigma\sqrt{\Delta}} - e^{\alpha\Delta - \sigma\sqrt{\Delta}}},$$

accordingly. Define $q_p^\tau(\alpha)$ as in equation (9) and $q_c^\tau(\alpha)$ as in equation (17).

Then the transition probabilities satisfy

$$q_p^\tau(\alpha) = q_c^\tau(\alpha) =: q^\tau(\alpha).$$

[1] The binomial model by Cox, Ross, and Rubinstein [7] is contained as a special case in this model class. We obtain this specification if we set α equal to 0.

Moreover, the equality

$$q(\alpha) = q^{\tau}(\alpha)$$

is given if and only if $\tau_B = \tau_S$.

Proof. For the transition probability in (17) we obtain

$$\begin{aligned} q_c^{\tau}(\alpha) &= \frac{B_{t_i}^{-1} B_{t_{i+1}}^{\tau} - \tilde{\delta}_{t_{i+1}} - S_{t_i}^{-1} S_{t_{i+1}}^{\tau}(d) + \tilde{\delta}_{t_{i+1}}}{S_{t_i}^{-1} S_{t_{i+1}}^{\tau}(u) - S_{t_i}^{-1} S_{t_{i+1}}^{\tau}(d)} \\ &= \frac{(1-\tau_B)e^{r\Delta} + \tau_B - (1-\tau_S)e^{\alpha\Delta - \sigma\sqrt{\Delta}} - \tau_S}{(1-\tau_S)\left(e^{\alpha\Delta+\sigma\sqrt{\Delta}} - e^{\alpha\Delta-\sigma\sqrt{\Delta}}\right)} \\ &= \frac{\frac{1-\tau_B}{1-\tau_S}e^{r\Delta} + \frac{\tau_B-\tau_S}{1-\tau_S} - e^{\alpha\Delta-\sigma\sqrt{\Delta}}}{e^{\alpha\Delta+\sigma\sqrt{\Delta}} - e^{\alpha\Delta-\sigma\sqrt{\Delta}}} \end{aligned} \tag{20}$$

and in (9) immediately

$$\begin{aligned} q_p^{\tau}(\alpha) &= \frac{B_{t_i}^{-1} B_{t_{i+1}}^{\tau} - S_{t_i}^{-1} S_{t_{i+1}}^{\tau}(d)}{S_{t_i}^{-1} S_{t_{i+1}}^{\tau}(u) - S_{t_i}^{-1} S_{t_{i+1}}^{\tau}(d)} \\ &= \frac{\frac{1-\tau_B}{1-\tau_S}e^{r\Delta} + \frac{\tau_B-\tau_S}{1-\tau_S} - e^{\alpha\Delta-\sigma\sqrt{\Delta}}}{e^{\alpha\Delta+\sigma\sqrt{\Delta}} - e^{\alpha\Delta-\sigma\sqrt{\Delta}}}, \end{aligned} \tag{21}$$

so the equality $q_p^{\tau}(\alpha) = q_c^{\tau}(\alpha)$ is always satisfied. The equality

$$q(\alpha) = q^{\tau}(\alpha)$$

holds iff

$$\frac{\tau_B - \tau_S}{1-\tau_S} = \left(1 - \frac{1-\tau_B}{1-\tau_S}\right) e^{r\Delta},$$

which is satisfied if and only if $\tau_B = \tau_S$. ■

3.2 An explicit pricing formula

Now that we have a deep economic insight into the payoffs and the replicating strategies, we can derive the limit results. First, the limit of the numeraire, i.e. the taxed money market fund, is analyzed. We will concentrate on the payoff view modelling and only consider the process in equation (10). The differences between the payoff and contract view in the limit are discussed in section 4 as part of the numerical analysis.

Proposition 3.2 *Let $\mathbb{T}$ be a discretization of the real line as in (18) and set $t_0 = 0$ and $t_N = T$. Let $(\mathcal{N}_{t_i})_{i=0}^N$ be a sequence of real numbers given by*

$$\mathcal{N}_{t_i} = B_{t_0} \prod_{s=1}^{i} \frac{B_{t_s}^{\tau}}{B_{t_{s-1}}} .$$

Then the sequence converges for a fixed T and $N \to \infty$ to

$$\mathcal{N}_T = \mathcal{N}_0 e^{(1-\tau_B) r T} . \tag{22}$$

The next step is to determine the limit distribution of the stock price at an arbitrary but fixed date T. We obtain the following result:

Proposition 3.3 *Let $\mathbb{T}$ be a discretization of the real line as in (18) and set $t_0 = 0$ and $t_N = T$. Let the process $(S_{t_i})_{i=0}^N$ be given by (19) with transition probabilities defined by (20) or (21).*

Then the sequence of distribution functions $F_N(x)$ of random variables

$$R_N := \sum_{n=1}^{N} \ln \left(\frac{S_{t_n}}{S_{t_{n-1}}} \right)$$

converges for a fixed T and $N \to \infty$ to the distribution function $\Phi\left(x; \left(\frac{1-\tau_B}{1-\tau_S} r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right)$ of a normal distributed variable R_T with mean $\left(\left(\frac{1-\tau_B}{1-\tau_S} r - \frac{1}{2}\sigma^2\right) T\right)$ and variance $\sigma^2 T$, i.e.

$$F_N(x) \to \Phi\left(x; \left(\frac{1-\tau_B}{1-\tau_S} r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right) \tag{23}$$

or symbolically,

$$R_N \xrightarrow{d} R .$$

The proofs of both propositions can be found in the appendix.

We are now able to determine an explicit pricing formula for a European call option under the capital gains tax regime.

Theorem 3.1 (Black-Scholes pricing formula under a capital gains taxation) *Let the price process of a numeraire be given by (22). Assume that the price process of the underlying under a martingale measure Q^{τ} can be described by the stochastic process*

$$S_T = S_{t_0} e^{\left(\frac{1-\tau_B}{1-\tau_S} r - \frac{1}{2}\sigma^2\right)(T-t_0) + \sigma\left(W_T - W_{t_0}\right)} ,$$

such that the distribution of the logarithmic return is given by (23).

Then the money value of a European call option is given by

$$V_{t_0}^C = S_{t_0} \exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r\,(T-t_0)\right) \Phi\left(d_1^\tau\right) - K \exp\left(-\left(1-\tau_B\right) r\,(T-t_0)\right) \Phi\left(d_2^\tau\right), \tag{24}$$

where

$$d_1^\tau := \frac{\ln\left(\frac{S_{t_0}}{K}\right) + \left(\frac{1-\tau_B}{1-\tau_S} r + \frac{1}{2}\sigma^2\right)(T-t_0)}{\sigma\sqrt{T-t_0}}$$

and

$$d_2^\tau = d_1^\tau - \sigma\sqrt{T-t_0}\,.$$

Proof. Since the distributional structure of the Black-Scholes world is valid, we can use the general pricing formula

$$\begin{aligned}\hat{V}_{t_0}^C &= \mathrm{E}_{Q^\tau}\left[\left(\hat{S}_T - \hat{K}\right)^+ \middle| \mathcal{F}_{t_0}\right] \\ &= \mathrm{E}_{Q^\tau}\left[\hat{S}_T \middle| \mathcal{F}_{t_0}\right] \Phi\left(\frac{\ln\left(\frac{\mathrm{E}_{Q^\tau}\left[\hat{S}_T \middle| \mathcal{F}_{t_0}\right]}{\hat{K}} + \sigma^2\,(T-t_0)\right)}{\sigma\sqrt{T-t_0}}\right) - \hat{K}\Phi\left(\frac{\ln\left(\frac{\mathrm{E}_{Q^\tau}\left[\hat{S}_T \middle| \mathcal{F}_{t_0}\right]}{\hat{K}} - \sigma^2\,(T-t_0)\right)}{\sigma\sqrt{T-t_0}}\right).\end{aligned}$$

We know that

$$\mathrm{E}_{Q^\tau}\left[\hat{S}_T \middle| \mathcal{F}_{t_0}\right] = \hat{S}_{t_0} \exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r\,(T-t_0)\right)$$

is valid under the martingale measure, so the result of the theorem follows immediately. ■

We can now review the discussion if the value of an option remains unchanged if the tax rates τ_B and τ_S are equal. Thus, let us assume that the tax rates satisfy

$$\tau_B = \tau_S = \tau\,.$$

Then the pricing formula reduces to

$$V_{t_0}^C = e^{\tau r(T-t_0)} V_{t_0}^{C,BS},$$

where $V_{t_0}^{C,BS}$ is the value of a European call due to the standard model by Black and Scholes.

The value of a European put can be determined using the put-call parity (cf. Stoll [20]). Since

$$\mathrm{E}_{Q^\tau}\left[\left(\hat{K}-\hat{S}_T\right)^+\middle|\mathcal{F}_{t_0}\right]=\mathrm{E}_{Q^\tau}\left[\left(\hat{S}_T-\hat{K}\right)^+-\hat{S}_T+\hat{K}\middle|\mathcal{F}_{t_0}\right]$$

the value is given by

$$\begin{aligned}V_{t_0}^P&=V_{t_0}^C-S_{t_0}\exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S}r\left(T-t_0\right)\right)+K\exp\left(-(1+\tau_B)r\left(T-t_0\right)\right)\\&=S_{t_0}\exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S}r\left(T-t_0\right)\right)\left[\Phi\left(d_1^\tau\right)-1\right]-K\exp\left(-\left(1-\tau_B\right)r\left(T-t_0\right)\right)\left[\Phi\left(d_2^\tau\right)-1\right],\end{aligned}$$

so the well-known structure of the put pricing formula — adapted to taxes — is obtained.

3.3 Hedging and Sensitivity Analysis

In this section the economic aspects are analyzed, i.e. we investigate those relations that leads to a riskless portfolio. Consider the value of a derivative in units of a numeraire to be a function of time and stock price level, thus

$$\hat{V}_t=\hat{V}_t\left(t,S_t\right).$$

Applying Itos Lemma yields

$$d\hat{V}_t=\left(\frac{\partial\hat{V}_t}{\partial t}+\frac{1-\tau_B}{1-\tau_S}rS_t\frac{\partial\hat{V}_t}{\partial S_t}+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2\hat{V}_t}{\partial S_t^2}\right)dt+\sigma S_t\frac{\partial\hat{V}_t}{\partial S_t}.$$

Since on an arbitrage-free market this process must also be a martingale, the drift must vanish. In conjunction with

$$\frac{\partial\hat{V}_t}{\partial t}=\frac{\partial\mathcal{N}_t^{-1}V_t}{\partial t}=(1-\tau_B)r\mathcal{N}_t^{-1}V_t+\mathcal{N}_t^{-1}\frac{\partial V_t}{\partial t}$$

one obtains the partial differential equation

$$\frac{\partial V_t}{\partial t}+\frac{1-\tau_B}{1-\tau_S}rS_t\frac{\partial V_t}{\partial S_t}+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2V_t}{\partial S_t^2}=\left(1-\tau_B\right)rV_t. \tag{25}$$

A comparison with equation (15) in Scholes [18] points out the modification we made. Scholes formed a portfolio consisting of a stock and a European option such that any risk is eliminated. This procedure leads to the left hand side of equation (25). The term was equated to $r_{\tau_B}V_t$, the return of a riskless portfolio. However, gains from interest-bearing investments are taxed with rate τ_B, so it seems consequent to equate the riskless portfolio return to $\left(1-\tau_B\right)rV_t$. This finally leads to the formulas derived in this paper.

The partial derivatives in the differential equation correspond to the sensitivity functions Θ_C^τ, Δ_C^τ and Γ_C^τ, also known as the Greeks. Δ is the first derivative of the pricing formula (24) with respect to the stock price. Calculating this derivative yields

$$\begin{aligned}\Delta_C^\tau &= \exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r(T-t_0)\right)\Phi\left(d_1^\tau\right) \\ &\quad + \exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r(T-t_0)\right)\frac{\phi\left(d_1^\tau\right)}{\sigma\sqrt{T-t_0}} - \frac{K}{S_{t_0}}\exp\left(-\left(1-\tau_B\right)r\left(T-t_0\right)\right)\frac{\phi\left(d_2^\tau\right)}{\sigma\sqrt{T-t_0}}.\end{aligned}$$

Since

$$\phi\left(d_2^\tau\right) = \frac{S_{t_0}}{K}\exp\left(\tfrac{1-\tau_B}{1-\tau_S} r(T-t_0)\right)\phi\left(d_1^\tau\right) \tag{26}$$

the expression reduces to

$$\Delta_C^\tau = \exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r(T-t_0)\right)\Phi\left(d_1^\tau\right),$$

which corresponds to the number of stocks held in the replicating portfolio. If $\tau_B \geq \tau_S;\ \tau_B, \tau_S > 0$, the portfolio contains more stocks compared to the case without taxation. If the opposite relation holds, no general statement can be made.

The second derivative is known as Γ. Applied to the pricing formula, we obtain

$$\begin{aligned}\Gamma_C^\tau &= \exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r(T-t_0)\right)\frac{\phi\left(d_1^\tau\right)}{\sigma S_{t_0}\sqrt{T-t_0}} \\ &= \frac{\Delta_C^\tau}{\sigma S_{t_0}\sqrt{T-t_0}}\frac{\phi\left(d_1^\tau\right)}{\Phi\left(d_1^\tau\right)}.\end{aligned}$$

Finally, we take the derivative with respect to *time to maturity*, Θ. Note that the differential equation contains the derivative with respect to time, so the relation is

$$\Theta_C^\tau = -\frac{\partial V_t}{\partial t}.$$

We obtain

$$\begin{aligned}\Theta_C^\tau &= S_{t_0}\exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r\left(T-t_0\right)\right)\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S} r\Phi\left(d_1^\tau\right) + \phi\left(d_1^\tau\right)\left(\frac{\partial d_2^\tau}{\partial(T-t_0)} + \frac{\sigma}{2\sqrt{T-t_0}}\right)\right) \\ &\quad + \exp\left(-\left(1-\tau_B\right)r(T-t_0)\right)K\left(\left(1-\tau_B\right)r\Phi\left(d_2^\tau\right) - \phi\left(d_2^\tau\right)\frac{\partial d_2^\tau}{\partial(T-t_0)}\right)\end{aligned}$$

or equivalently using (26),

$$\Theta_C^\tau = (1-\tau_B)r\left(\frac{\tau_S}{1-\tau_S}S_{t_0}\exp\left(\tfrac{\tau_S(1-\tau_B)r(T-t_0)}{1-\tau_S}\right)\Phi\left(d_1^\tau\right)+K\exp\left(-(1-\tau_B)r(T-t_0)\right)\Phi\left(d_2^\tau\right)\right)$$
$$+\frac{\sigma S_{t_0}\phi\left(d_1^\tau\right)}{2\sqrt{T-t_0}}\exp\left(\tfrac{\tau_S(1-\tau_B)}{1-\tau_S}r\left(T-t_0\right)\right).$$

Inserting these functions into the partial differential equation (25) yields

$$-\Theta_C^\tau+\frac{1-\tau_B}{1-\tau_S}rS_t\Delta_C^\tau+\frac{1}{2}\sigma^2S_t^2\Gamma_C^\tau=(1-\tau_B)rV_t$$

and henceforth confirms the previous result.

4 Numerical Analysis

In this section the economic consequences of taxation on the value of a European option are exemplified. We mainly focus on two topics. First, the difference between the payoff view and the contract view is investigated. Secondly, the limit results of both discrete-time approaches are compared to the closed form solution derived in a continuous-time setting.

Consider a stock price trading at 100 and a money market fund at 1. Assume that the stock's log volatility is .4 and the riskless interest .05. We demonstrate the consequences of taxation on a European option written on the stock with a strike price of 90 and a time to maturity of .5 years. The option value due to the Black Scholes formula is $V^{C,BS}=17.7629$. The results in column *payoff centric* and

tax rates	payoff centric	contract centric	BS with taxes
$\tau_B=.00\quad\tau_S=.00$	17.7628	17.7628	17.7629
$\tau_B=.00\quad\tau_S=.25$	18.3752	18.3751	18.3749
$\tau_B=.00\quad\tau_S=.50$	19.6385	19.6385	19.6388
$\tau_B=.25\quad\tau_S=.00$	17.4220	17.4219	17.4211
$\tau_B=.25\quad\tau_S=.25$	17.8742	17.8741	17.8742
$\tau_B=.25\quad\tau_S=.50$	18.8029	18.8029	18.8031
$\tau_B=.50\quad\tau_S=.00$	17.0818	17.0818	17.0818
$\tau_B=.50\quad\tau_S=.25$	17.3811	17.3811	17.3799
$\tau_B=.50\quad\tau_S=.50$	17.9862	17.9862	17.9863

Table 1: Taxation: algorithmic and closed form solutions

contract centric were determined according to the numerical algorithms in sections 2.3 and 2.4 under the specifications introduced in 3.1. The drift of the stock price was chosen such that the transition

probabilities implied by the martingale measure were $q = \frac{1}{2}$, i.e. α was set to

$$\alpha = -\frac{1}{\Delta} \ln \left(\frac{(1-\tau_S)\cosh\left(\sigma\sqrt{\Delta}\right)}{(1-\tau_B)\,e^{r\Delta} + \tau_B - \tau_S} \right),$$

since in general this specification yields good results. The routine was written in Java and performed on base of 2,000 periods. The average computing time[2] to determine the option value under taxation was .239 seconds (payoff view) and .245 seconds (contract view), respectively, so the procedure could even be applied in an option trading environment.

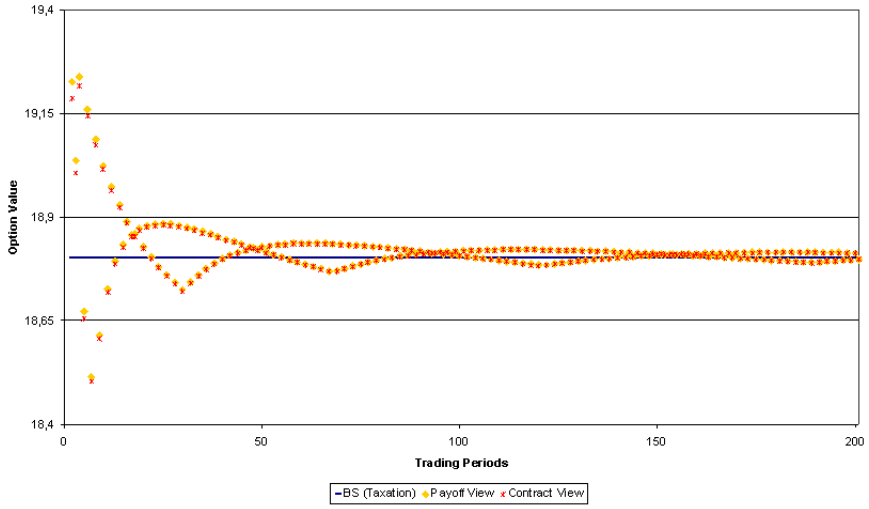

Figure 1: Convergence of discrete-time models to the continuous-time model

The data in table 1 indicate that all values are very close. The convergence process is illustrated in figure 1. It can be observed that the option value from the payoff perspective is always higher than the value from the contract view if there are only few trading periods. However, both values converge to the value determined by the continuous-time model (Black Scholes under taxation). The convergence is not monotone but shows the typical pattern of a sequence of binomial model values (cf. Reimer [15] for a detailed analysis of the convergence of binomial models).

Figure 2 summarizes and visualizes the main characteristics of the Black Scholes model under taxation. The plane parallel to the τ_B-τ_S-plane represents the value of the standard Black Scholes formula. The inclined plane reflects the values of the option pricing formula that incorporates taxation. One can clearly recognize that the intersection of the planes is not a straight line through the points $\left(0,0,V^{C,BS}\right)$ and $\left(.5,.5,V^{C,BS}\right)$. This would have been the case if the taxation was neutral under equal tax rates.

We can also observe that for any tax rate τ_B the option value increases when τ_S increases. On the other hand, for any τ_S the option value decreases when τ_B increases. This is true for any input parameters

[2] Computation was performed on a PC with an AMD Athlon^{TM} XP 1700+, 1.48 GHz processor. The average computation time was calculated from 100 samples with 2,000 iterations each.

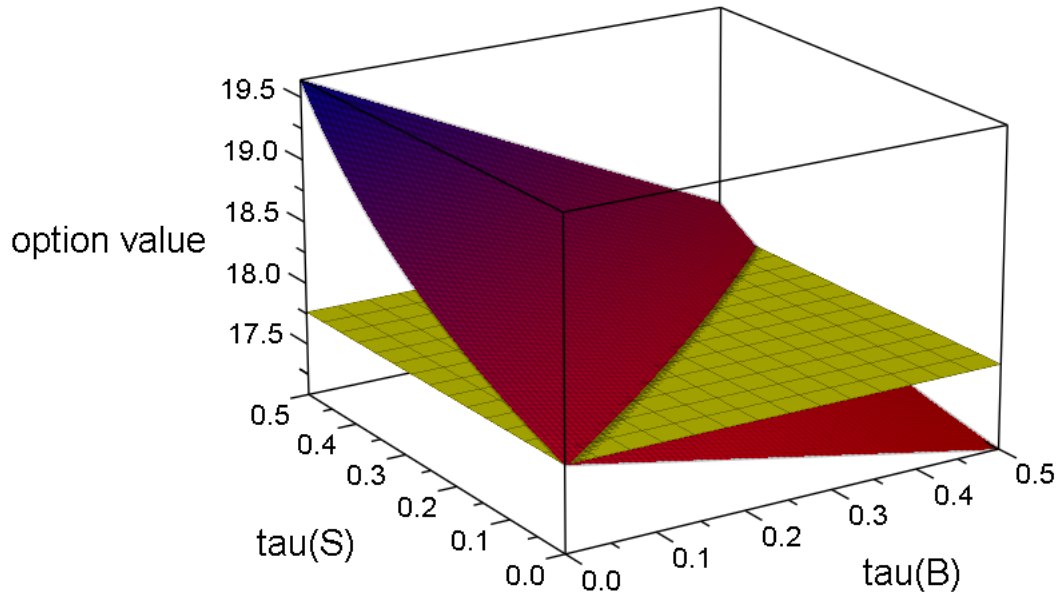

Figure 2: The impact of taxation on the option value

and can be shown as follows. Let us express the ratio of the tax rates as

$$\gamma := \frac{\tau_B}{\tau_S}$$

and substitute τ_B in equation (24) by $\gamma\,\tau_S$. Calculating the derivative of the option value with respect to the tax rates ratio yields

$$\begin{aligned}\frac{\partial V_{t_0}}{\partial \gamma} = &-\left(\frac{\tau_S^2 r(T-t_0)}{1-\tau_S}\right) S_{t_0} \exp\left(\frac{\tau_S(1-\gamma\tau_S)}{1-\tau_S} r(T-t_0)\right) \Phi\left(d_1^{\tau}\right) \\ &-(\tau_S r(T-t_0)) K \exp\left(-(1-\gamma\tau_S) r(T-t_0)\right) \Phi\left(d_2^{\tau}\right)\end{aligned}$$

Since the derivative is negative, the above mentioned statements are already shown.

5 Conclusion

We have analyzed the impact of a gains tax regime on the arbitrage-free valuation of options on a sound basis and in great detail. The discrete-time approaches have been motivated by consequently applying the replication principle to the valuation problem under taxation. The limiting behavior of market prices under the stated assumptions could be determined for the payoff view. Numerical results gave a strong

hint that the difference with respect to the contract view is negligible in the limit.

The limit results could be used to determine a closed form solution for the value of a European option. The formula allows for the analysis of tax rates that differentiate between gains from interest-bearing instruments and capital gains, respectively. If both rates are equal, the tax system is not neutral. We have shown that the trading strategy and henceforth the value of the derivative is influenced under this assumption.

Both approaches, the binomial framework and the partial differential equation provide an access point for those modifications that can only be managed using numerical schemes. This might be necessary when more realistic tax scheme are to be modelled.

Appendix

Proofs

Proof of lemma 2.1. We will prove the lemma by induction. First, note that the assertion is true for $i = 1$ since we get

$$V_{t_1} = \frac{-\tau_X}{1-\tau_X} V_{t_0} + \frac{1}{1-\tau_X} V_{t_1}\,,$$

which is true according to the prerequisites. Furthermore, note that $V_{t_{s+1}}$ satisfies

$$V_{t_{s+1}} = \frac{V^{\tau}_{t_{s+1}}}{1-\tau_X} - \frac{\tau_X}{1-\tau_X} V_{t_s}\,.$$

Let us assume that equation (12) is valid for an arbitrary i. Then

$$\begin{aligned}
V_{t_{i+1}} &= \frac{(-\tau_X)^{i+1}}{(1-\tau_X)^{i+1}} V_{t_0} + \sum_{s=1}^{i+1} \frac{(-\tau_X)^{(i+1)-s}}{(1-\tau_X)^{(i+1)-s}} \frac{V^{\tau}_{t_s}}{1-\tau_X} \\
&= \frac{(-\tau_X)^{i+1}}{(1-\tau_X)^{i+1}} V_{t_0} + \sum_{s=1}^{i} \frac{(-\tau_X)^{(i+1)-s}}{(1-\tau_X)^{(i+1)-s}} \frac{V^{\tau}_{t_s}}{1-\tau_X} + \frac{V^{\tau}_{t_{i+1}}}{1-\tau_X} \\
&= -\frac{\tau_X}{1-\tau_X} \left(\frac{(-\tau_X)^{i}}{(1-\tau_X)^{i}} V_{t_0} + \sum_{s=1}^{i} \frac{(-\tau_X)^{i-s}}{(1-\tau_X)^{i-s}} \frac{V^{\tau}_{t_s}}{1-\tau_X} \right) + \frac{V^{\tau}_{t_{i+1}}}{1-\tau_X}\,,
\end{aligned}$$

which completes the proof. ■

Proof of proposition 2.1. We have to show that the process

$$\hat{Z}_{t_i} := \aleph^{-1}_{t_i} Z_{t_i} = \hat{Z}_{t_0} \prod_{s=1}^{i} \frac{\left(B^{\tau}_{t_s}\right)^{-1} S^{\tau}_{t_s}}{B^{-1}_{t_{s-1}} S_{t_{s-1}}}$$

satisfy the condition

$$\hat{Z}_{t_{i-1}} = \mathrm{E}\left[\hat{Z}_{t_i} | \mathcal{F}_{t_{i-1}}\right]$$

for all $t_i \in \mathbb{T} \setminus \{t_0\}$. Calculating the conditional expectation of $\frac{\hat{Z}_{t_i}}{\hat{Z}_{t_{i-1}}}$ given the information in t_{i-1} yields

$$\frac{S_{t_{i-1}} B^{-1}_{t_{i-1}} B^{\tau}_{t_i} - S^{\tau}_{t_i}(d)}{S^{\tau}_{t_i}(u) - S^{\tau}_{t_i}(d)} \frac{\left(B^{\tau}_{t_i}\right)^{-1} S^{\tau}_{t_i}(u)}{B^{-1}_{t_{i-1}} S_{t_{i-1}}} + \frac{S^{\tau}_{t_i}(u) - S_{t_{i-1}} B^{-1}_{t_{i-1}} B^{\tau}_{t_i}}{S^{\tau}_{t_i}(u) - S^{\tau}_{t_i}(d)} \frac{\left(B^{\tau}_{t_i}\right)^{-1} S^{\tau}_{t_i}(d)}{B^{-1}_{t_{i-1}} S_{t_{i-1}}} = 1\,.$$

Since $\mathrm{E}\left[\left|\hat{Z}_{t_i}\right|\right] < \infty$ is satisfied by assumption, this implies that $\hat{Z}_{t_i}$ is a Q^{τ}-martingale.

The probability measure is unique, since the market is complete. It follows from the fundamental

theorem of asset pricing (cf. Bingham and Kiesel [2], pp. 96-102) that the value of a derivative in units of the numeraire generating payoff X_{t_N} is given by

$$\hat{V}_{t_0}^{\tau} = \mathrm{E}_{Q^{\tau}}\left[\hat{X}_{t_N} \middle| \mathcal{F}_{t_0}\right] .$$

■

Proof of proposition 3.2. Fix the point $T = t_N$ and define

$$\mathcal{N}_{t_N} := B_{t_0} \prod_{s=1}^{N} \frac{B_{t_s}^{\tau}}{B_{t_{s-1}}}$$

the value of the numeraire. Define the logarithmic return of the numeraire over $[0,T]$ as

$$\begin{aligned} Y_{t_N} := \ln\left(\frac{\mathcal{N}_{t_N}}{\mathcal{N}_{t_0}}\right) &= \sum_{s=1}^{N} \ln\left((1-\tau_B)e^{r\Delta} + \tau_B\right) \\ &= T\frac{1}{\Delta}\ln\left((1-\tau_B)e^{r\Delta} + \tau_B\right) . \end{aligned}$$

The limit of this sequence as $N \to \infty$ is

$$\lim_{\Delta \to 0} Y_{t_N} = \frac{r(1-\tau_B)e^{r\Delta}}{(1-\tau_B)e^{r\Delta} + \tau_B} T = r(1-\tau_B)T .$$

Thus, the price of the numeraire converges to

$$\mathcal{N}_T = \mathcal{N}_0 e^{r(1-\tau_B)T} .$$

■

Proof of proposition 3.3. For $t_n \in \mathbb{T} \setminus \{t_0\}$, let

$$R_{t_n} := \ln\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)$$

be the one-period logarithmic return of stock price, then $R_{t_1}, R_{t_2}, \ldots, R_{t_N}$ is a sequence of i.i.d. random variables with characteristic function

$$\begin{aligned} \chi_{R_{t_n}}(\theta) &:= \mathrm{E}_Q\left[e^{i\theta R_{t_n}}\right] \\ &= q^{\tau}(\alpha) e^{i\theta(\alpha\Delta + \sigma\sqrt{\Delta})} + (1 - q^{\tau}(\alpha))\, e^{i\theta(\alpha\Delta - \sigma\sqrt{\Delta})} . \end{aligned}$$

Defining

$$Y_N := \sum_{n=1}^{N} R_{t_n} = \ln\left(\frac{S_T}{S_0}\right)$$

yields the overall logarithmic stock return whose distribution under the martingale measure is to be determined. Since the random variables of the sequence (R_{t_n}) are independent, the characteristic function of the sum can be expressed as the product of the components' characteristic functions. We obtain

$$\chi_{Y_N}(\theta) = \mathrm{E}_Q\left[e^{i\theta Y_N}\right] = \prod_{n=1}^{N} \chi_{R_{t_n}}(\theta)\,,$$

the characteristic function of the overall return. Its logarithm, which we use for convenience, is given by

$$\begin{aligned}\ln\left(\chi_{Y_N}(\theta)\right) &= \frac{1}{\Delta}\ln\left(q^{\tau}(\alpha)e^{i\theta(\alpha\Delta+\sigma\sqrt{\Delta})} + (1-q^{\tau}(\alpha))\,e^{i\theta(\alpha\Delta-\sigma\sqrt{\Delta})}\right)T \\ &= \alpha i\theta T + \frac{1}{\Delta}\ln\left(q^{\tau}(\alpha)e^{i\theta\sigma\sqrt{\Delta})} + (1-q^{\tau}(\alpha))\,e^{-i\theta\sigma\sqrt{\Delta})}\right)T\,. \qquad (27)\end{aligned}$$

Now, the second term of (27) can be expressed by a Taylor series at $\theta = 0$ according to

$$\begin{aligned}g_{\Delta}(\theta) &:= \frac{1}{\Delta}\ln\left(q^{\tau}(\alpha)e^{i\theta\sigma\sqrt{\Delta})} + (1-q^{\tau}(\alpha))\,e^{-i\theta\sigma\sqrt{\Delta})}\right)T \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}g_{\Delta}^{(n)}(0)\theta^n\,.\end{aligned}$$

In the following, the limit of each term of the sum is analyzed as Δ tends to 0. Starting with

$$\lim_{\Delta\to 0} g_{\Delta}(\theta) = \lim_{\Delta\to 0}\frac{T}{\Delta}\ln(1) = 0$$

reveals that the first term can be neglected. For the second term, one obtains

$$g'_{\Delta}(\theta) = \frac{1}{\sqrt{\Delta}}\frac{q^{\tau}(\alpha)e^{i\theta\sigma\sqrt{\Delta}} - (1-q^{\tau}(\alpha))e^{-i\theta\sigma\sqrt{\Delta}}}{q^{\tau}(\alpha)e^{i\theta\sigma\sqrt{\Delta}} + (1-q^{\tau}(\alpha))e^{-i\theta\sigma\sqrt{\Delta}}}i\sigma T$$

immediately. Using the definitions

$$\cosh(x) := \frac{\exp(x)+\exp(-x)}{2}$$

and

$$\sinh(x) := \frac{\exp(x)-\exp(-x)}{2}$$

the transition probabilities can be substituted by the decomposition

$$\begin{aligned} q^{\tau}(\alpha) &= \frac{1}{2} + \frac{1}{2} \frac{\frac{1-\tau_B}{1-\tau_S} e^{-(\alpha-r)\Delta} + \frac{\tau_B-\tau_S}{1-\tau_S} e^{-\alpha\Delta} - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sinh\left(\sigma\sqrt{\Delta}\right)} \\ &=: \frac{1}{2} + \frac{1}{2}\eta^{\tau}(\alpha) \end{aligned}$$

and henceforth the first derivative by

$$g'_{\Delta}(\theta) = \frac{1}{\sqrt{\Delta}} \frac{\sinh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta^{\tau}(\alpha)\cosh\left(i\theta\sigma\sqrt{\Delta}\right)}{\cosh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta^{\tau}(\alpha)\sinh\left(i\theta\sigma\sqrt{\Delta}\right)}.$$

Evaluated at point $\theta = 0$ the expression reduces to

$$g'_{\Delta}(0) = \frac{1}{\sqrt{\Delta}}\eta^{\tau}(\alpha) i\sigma T\,. \tag{28}$$

The limit of (28) as Δ tends to 0 is given by

$$\begin{aligned} \lim_{\Delta\to 0} g'_{\Delta}(0) &= \lim_{\Delta\to 0} \frac{\frac{1-\tau_B}{1-\tau_S} e^{-(\alpha-r)\Delta} + \frac{\tau_B-\tau_S}{1-\tau_S} e^{-\alpha\Delta} - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sqrt{\Delta}\sinh\left(\sigma\sqrt{\Delta}\right)} i\sigma T \\ &= \lim_{\Delta\to 0} -\frac{\frac{1-\tau_B}{1-\tau_S}(\alpha-r) e^{-(\alpha-r)\Delta} + \frac{\tau_B-\tau_S}{1-\tau_S}\alpha e^{-\alpha\Delta} - \sinh\left(\sigma\sqrt{\Delta}\frac{\sigma}{2\sqrt{\Delta}}\right)}{\frac{1}{2\sqrt{\Delta}}\sinh\left(\sigma\sqrt{\Delta}\right) + \cosh\left(\sigma\sqrt{\Delta}\right)\frac{\sigma}{2}} i\sigma T \\ &= \left(\frac{1-\tau_B}{1-\tau_S} r - \frac{1}{2}\sigma^2\right) iT\,, \end{aligned}$$

where the theorem of de l'Hospital and the relation

$$\lim_{\Delta\to 0} \frac{1}{\sqrt{\Delta}} \sinh\left(\sigma\sqrt{\Delta}\right) = \sigma$$

has been used.

The second derivative is given by

$$g''_{\Delta}(\theta) = -\left(1 - \frac{\left(\sinh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta^{\tau}(\alpha)\cosh\left(i\theta\sigma\sqrt{\Delta}\right)\right)^2}{\left(\cosh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta^{\tau}(\alpha)\sinh\left(i\theta\sigma\sqrt{\Delta}\right)\right)^2}\right)\sigma^2 T$$

Evaluating at $\theta = 0$ and taking the limit leads to

$$\begin{aligned}\lim_{\Delta\to 0} g''_\Delta(0) &= \lim_{\Delta\to 0} -\left(1-\left(\eta^\tau(\alpha)\right)^2\right)\sigma^2 T \\ &= -\sigma^2 T\end{aligned}$$

since

$$\lim_{\Delta\to 0}\eta^\tau(\alpha) = 0\,.$$

It is not difficult to see that all higher derivatives vanish as Δ tends to 0. Therefore the limit of the characteristic function is

$$\chi_Y(\theta) := \lim_{N\to\infty}\chi_{Y_N}(\theta) = \exp\left(\left(\tfrac{1-\tau_B}{1-\tau_S}r - \tfrac{1}{2}\sigma^2\right)Ti\theta - \tfrac{1}{2}\sigma^2 T\theta^2\right),$$

which is the characteristic function of a normal distributed random variable Z with expected value $\frac{1-\tau_B}{1-\tau_S}rT - \frac{1}{2}\sigma^2 T$ and a variance $\sigma^2 T$. From the uniqueness theorem for characteristic functions we know that a distribution function is uniquely determined by its characteristic function. ■

References

[1] Amin, Kaushik I., 1991, On the computation of continuous time option prices using discrete approximations, *Journal of Financial and Quantitative Analysis* 26, 477–495.

[2] Bingham, Nicholas H., and Rüdiger Kiesel, 1998, *Risk-Neutral Valuation. Pricing and Hedging of Financial Derivatives* (Springer-Verlag: London, Berlin, Heidelberg et al.).

[3] Black, Fischer, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–654.

[4] Constantinides, George M., 1983, Capital market equilibrium with personal tax, *Econometrica* 51, 611–636.

[5] ——— , 1984, Optimal stock trading with personal taxes: Implications for prices and the abnormal january returns, *Journal of Financial Economics* 13, 299–335.

[6] ——— , and Jonathan E. Ingersoll, 1984, Optimal bond trading with personal taxes, *Journal of Financial Economics* 13, 299–335.

[7] Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option pricing: A simplified approach, *Journal of Financial Economics* 7, 229–263.

[8] Dammon, Robert M., and Chester S. Spatt, 1996, The optimal trading and pricing of securities with asymmetrix capital gains taxes and transaction costs, *Review of Financial Studies* 9, 921–952.

[9] ——— , and Harold H. Zhang, 2001, Optimal consumption and investment with capital gains taxes, *Review of Financial Studies* 14, 583–616.

[10] Gallmeyer, Michael F., Ron Kaniel, and Stathis Tompaidis, 2001, Tax management strategies with multiple risky assets, Working Paper.

[11] Garlappi, Lorenzo, Vasant Naik, and Joshua Slive, 2001, Portfolio selection with multiple assets and capital gains taxes, Working Paper, University of British Columbia.

[12] Harrison, J. Michael, and David M. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* 20, 381–408.

[13] Harrison, J. Michael, and Stanley R. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* 11, 251–260.

[14] Miller, Merton H., and Myron S. Scholes, 1978, Dividends and taxes, *Journal of Financial Economics* 6, 333–364.

[15] Reimer, Matthias, 1997, Examing binomial option price approximations, Ph.D. thesis University of Bonn.

[16] Ross, Stephen A., 1985, Debt and taxes and uncertainty, *Journal of Finance* 40, 637–657.

[17] ———, 1987, Arbitrage and martingales with taxation, *Journal of Political Economy* 95, 371–393.

[18] Scholes, Myron, 1976, Taxes and the pricing of options, *Journal of Finance* 31, 319–332.

[19] Schultz, Paul, 1988, Tax loss selling in common stocks, Ph.D. thesis University of Chicago.

[20] Stoll, Hans R., 1969, The relationship between put and call option prices, *Journal of Finance* 31, 802–824.

[21] Williams, Joseph T., 1985, Trading and valuing depreciable assets, *Journal of Financial Economics* 14, 283–308.

Diskussionspapiere der Fakultät für Wirtschaftswissenschaften ab Nr. 441/2000

441 Anne Chwolka, Dirk Simons — Impacts of Revenue Sharing, Profit Sharing, and Transfer Pricing on Quality-Improving Investments, Januar 2000

442. Carsten Köper, Peter Flaschel — Real-Financial Interaction: A Keynes-Metzler-Goodwin Portfolio Approach, Januar 2000

443. Th. Spitta, R. Decker, A. Sigge, P. Wolf, V. Tiemann — Erste Bilanz des Kreditpunktesystems der Fakultät für Wirtschaftswissenschaften, Januar 2000

444. Imre Dobos — A dynamic theory of production: flow or stock-flow production Functions, Februar 2000

445. Carl Chiarella, Peter Flaschel — Applying Disequilibrium Growth Theory: I. Investment, Debt and Debt Deflation, January 2000

446. Imre Dobos — A Dynamic Environmental Theory of Production, Maerz 2000

447. Anne Chwolka — "Marktorientierte Zielkostenvorgaben als Instrument der Verhaltenssteuerung im Kostenmanagement", März 2000

448. Volker Böhm, Carl Chiarella — Mean Variance Preferences, Expectations Formation, and the Dynamics of Random Asset Prices, April 2000

449. Beate Pilgrim — Non-equivalence of uniqueness of equilibria in complete and in incomplete market models, March 2000

450. Beate Pilgrim — A Brief Note on Mas-Colell's First Observation on Sunspots, March 2000

451. Thorsten Temme — An Integrated Approach for the Use of CHAID in Applied Marketing Research, May 2000

452. Reinhold Decker, Claudia Bornemeyer — Ausgewählte Ansätze zur Entscheidungsunterstützung im Rahmen der Produktliniengestaltung, Mai 2000

453. Martin Feldmann — Threshold Accepting with a Back Step. Excellent results with a hybrid variant of Threshold Accepting, Mai 2000

454. Willi Semmler, Malte Sieveking — Credit Risk and Sustainable Debt: A Model and Estimations for Euroland, November 1999

455. Alexander Krüger, Ralf-Michael Marquardt — Der Euro - eine schwache Währung? Mai 2000

456. Veith Tiemann — Symmetrische/klassische Kryptographie - Ein interaktiver Überblick, Mai 2000

457. Imre Dobos — Umweltbewusste Produktionsplanung auf Grundlage einer dynamischen umweltorientierten Produktionstheorie: Eine Projektbeschreibung, Juni 2000

458. Imre Dobos — Optimal production-inventory strategies for a HMMS-type reverse logistics system, Juli 2000

459. Joachim Frohn — Ein Marktmodell zur Erfassung von Wanderungen (revidierte Fassung) Juli 2000

460. Klaus-Peter Kistner, Imre Dobos — Ansaetze einer umweltorientierten Produktionsplanung: Ergebnisse eines Seminars, Juli 2000

461. Reinhold Decker — Instrumentelle Entscheidungsunterstützung im Marketing am Beispiel der Verbundproblematik, September 2000

462. Caren Sureth — The influence of taxation on partially irreversible investment decisions - A real option approach, April 2000

463. Veith Tiemann — Asymmetrische/moderne Kryptographie - Ein interaktiver Überblick Oktober 2000

464. Carsten Köper — Stability Analysis of an Extended KMG Growth Dynamics, December 2000

465. Stefan Kardekewitz — Analyse der unilateralen Maßnahmen zur Vermeidung der Doppelbesteue-rung im deutschen Erbschaftsteuer-recht, Februar 2001

466. Werner Glastetter — Zur Kontroverse über das angemessene wirtschafts- und konjunkturpolitische Paradigma – Einige Akzente der gesamtwirtschaftlichen Entwicklung Westdeutschlands von 1950 bis 1993, März 2001

467. Thomas Braun, Ariane Reiss — Benchmarkorientierte Portfolio-Strategien, Mai 2001

468. Martin Feldmann, Stephanie Müller — An incentive scheme for true information providing in SUPPLY CHAINS, Juni 2001

469. Wolf-Jürgen Beyn, Thorsten Pampel, Willi Semmler — Dynamic optimization and Skiba sets in economic examples, August 2001

470. Werner Glastetter — Zur Kontroverse über das angemessene wirtschafts- und konjunkturpolitische Paradigma (II) – Einige Akzente der gesamtwirtschaftlichen Entwicklung der Bundesrepublik

Diskussionspapiere der Fakultät für Wirtschaftswissenschaften ab Nr. 441/2000

		Deutschland von 1991-1999, September 2001
471.	Hermann Jahnke, Anne Chwolka, Dirk Simons	Coordinating demand and capacity by adaptive decision making September 2001
472.	Thorsten Pampel	Approximation of generalized connecting orbits with asymptotic rate, September 2001
473.	Reinhold Decker Heiko Schimmelpfennig	Assoziationskoeffizienten und Assoziationsregeln als Instrumente der Verbundmessung - Eine vergleichende Betrachtung, September 2001
474.	Peter Naeve	Virtuelle Tabellensammlung, September 2001
475.	Heinz-J. Bontrup Ralf-Michael Marquardt	Germany's Reform of the Pension System: Choice between „Scylla and Charybdis" Oktober 2001
476.	Alexander M. Krüger	Wechselkurszielzonen zwischen Euro, Dollar und Yen -- nur eine Illusion? Oktober 2001
477.	Jan Wenzelburger	Learning to predict rationally when beliefs are heterogeneous. October 2001
478.	Jan Wenzelburger	Learning in linear models with expectational leads October 2001
479.	Claudia Bornemeyer, Reinhold Decker	Key Success Factors in City Marketing – Some Empirical Evidence - October 2001
	Fred Becker Michael Tölle	Personalentwicklung für Nachwuchswissenschaftler an der Universität Bielefeld: Eine explorative Studie zur Erhebung des Ist-Zustands und zur Begründung von Gestal-tungsvorschlägen Oktober 2001
480.	Dirk Biskup, Martin Feldmann	On scheduling around large restrictive common due windows December 2001
481.	Dirk Biskup	A mixed-integer programming formulation for the ELSP with sequence-dependent setup-costs and setup-times December 2001
482.	Lars Grüne, Willi Semmler, Malte Sieveking	Thresholds in a Credit Market Model with Multiple Equilibria August 2001
483.	Toichiro Asada	Price Flexibility and Instability in a Macrodynamic Model with Debt Effect, February 2002
484.	Rolf König, Caren Sureth	Die ökonomische Analyse der Auswirkungen der Unternehmenssteuerreform auf Sachinvestitionsentscheidungen vor dem Hintergrund von Vorteilhaftigkeits- und Neutralitätsüberlegungen - diskreter und stetiger Fall - März 2002
485.	Fred G. Becker, Helge Probst	Personaleinführung für Universitätsprofessoren: Eine explorative Studie an den Universitäten in Nordrhein-Westfalen zum Angebot und an der Universität Bielefeld zum Bedarf März 2002
486.	Volker Böhm, Tomoo Kikuchi	Dynamics of Endogenous Business Cycles and Exchange Rate Volatility April 2002
487.	Caren Sureth	Die Besteuerun g von Beteiligungsveräußerungen - eine ökonomische Analyse der Interdependenzen von laufender und einmaliger Besteuerung vor dem Hintergrund der Forderung nach Rechtsformneutralität - Juli 2002
488.	Reinhold Decker	Data Mining und Datenexploration in der Betriebswirtschaft Juli 2002
489.	Ralf Wagner, Kai-Stefan Beinke, Michael Wendling	Good Odd Prices and Better Odd Prices - An Empirical Investigation September 2002
490.	Hans Gersbach, Jan Wenzelburger	The Workout of Banking Crises: A Macroeconomic Perspective September 2002
491.	Dirk Biskup, Dirk Simons	Common due date scheduling with autonomous and induced learning September 2002
492.	Martin Feldmann, Ralf Wagner	Navigation in Hypermedia: Neue Wege für Kunden und Mitarbeiter September 2002
493.	Volker Böhm, Jan Wenzelburger	On the Performance of Efficient Portfolios November 2002
494.	J. Frohn, P. Chen, W. Lemke, Th. Archontakis, Th. Domeratzki, C. Flöttmann, M. Hillebrand, J. Kitanovic, R. Rucha, M. Pullen	Empirische Analysen von Finanzmarktdaten November 2002
495.	Volker Böhm	CAPM Basics December 2002
496.	Susanne Kalinowski , Stefan Kardekewitz	Betriebstätte vs. Kapitalgesellschaft im Ausland - eine ökonomische Analyse März 2003
497.	Jochen Jungeilges	On Chaotic Consistent Expectations Equilibria March 2003

Diskussionspapiere der Fakultät für Wirtschaftswissenschaften ab Nr. 441/2000

498.	Volker Böhm	MACRODYN - The Handbook - March 2003
499.	Jochen A. Jungeilges	Sequential Computation of Sample Moments and Sample Quantiles – A Tool for MACRODYN - April 2003
500.	Fred G. Becker, Vera Brenner	Personalfreisetzung in Familien-unternehmen: Eine explorative Studie zur Proble-matik Juni 2003
501.	Michael J. Fallgatter, Dirk Simons.	"Zum Überwachungsgefüge deutscher Kapitalgesellschaften - Eine anreiz-theoretische Analyse der Vergütung, Haftung und Selbstverpflichtung des Aufsichtsrates" Juni 2003
502.	Pu Chen	Weak exogeneity in simultaneous equations models Juli 2003
503.	Pu Chen	Testing weak exogeneity in VECM Juli 2003
504.	Fred G. Becker, Carmen Schröder	Personalentwicklung von Nachwuchs-wissenschaftlern: Eine empirische Studie bei Habilitanden des Fachs "Betriebswirtschaftslehre" Juli 2003
505.	Caren Sureth	Die Wirkungen gesetzlicher und theo-retischer Übergangsregelungen bei Steuerreformen – eine ökonomische Analyse steuerinduzierter Verzerrun-gen am Beispiel der Reform der Be-steuerung von Beteiligungserträgen - August 2003
506.	Jan Wenzelburger	Learning to play best response in duopoly games" August 2003
507.	Dirk Simons	Quasirentenansätze und Lerneffekte September 2003
508.	Dirk Simons Dirk Biskup	Besteht ein Bedarf nach Dritthaftung des gesetzlichen Jahresabschluss-prüfers? September 2003
509.	Tomoo Kikuchi	A Note on Symmetry Breaking in a World Economy with an International Financial Market., October 2003
510.	Fred G. Becker Oliver Krah	Explorative Studie zur Personalein-führung bei Unternehmen in OWL: Ergebnisübersicht Oktober 2003
511.	Martin Feldmann Stephanie Müller	Simulation von Reentrant Lines mit ARENA: Ergebnisse eines Projektes zur Betriebsinformatik Januar 2004
512.	Xuemin Zhao Reinhold Decker	Choice of Foreign Market Entry Mode Cognitions from Empirical and Theoretical Studies January 2004
513.	Volker Böhm Jochen Jungeilges	Estimating Affine Economic Models With Discrete Random Perturbations January 2004
514.	Ralf Wagner	Mining Promising Qualification Patterns February 2004
515.	Ralf Wagner	Contemporary Marketing Practices in Russia February 2004
516.	Reinhold Decker Ralf Wagner Sören Scholz	Environmental Scanning in Marketing Planning – An Internet-Based Approach –
517.	Dirk Biskup Martin Feldmann	Lot streaming with variable sublots: an integer programming formulation April 2004
518.	Andreas Scholze	Folgebewertung des Geschäfts- oder Firmenswerts aus Sicht der Meß- bzw. Informationsgehaltsperspektive April 2004
519.	Hans Gersbach Jan Wenzelburger	Do risk premia protect from banking crises? May 2004
520.	Marten Hillebrand Jan Wenzelburger	The impact of multiperiod planning horizons on portfolios and asset prices in a dynamic CAPM May 2004
521.	Stefan Wielenberg	Bedingte Zahlungsversprechen in der Unternehmenssanierung Juni 2004
522.	Sören Scholz, Ralf Wagner	The Quality of Prior Information Structure in Business Planning - An Experiment in Environmental Scanning - August 2004
523.	Jan Thomas Martini Claus-Jochen Haake	Negotiated Transfer Pricing in a Team-Investment Setting October 2004
524.	Reinhold Decker	Market basket analysis by means of a growing neural network November 2004
525.	Reinhold Decker Sören Scholz	Wie viel darf guter Service kosten? Einkaufsstättenbedingte Preiswahr-nehmung im Selbstmedikationsmarkt November 2004
526.	Fred G. Becker Roman Bobrichtchev Natascha Henseler	Ältere Arbeitnehmer und alternde Belegschaften: Eine empirische Studie bei den 100 größten deutschen Unternehmen Dezember 2004
527.	Jan Wenzelburger Hans Gersbach	Risk Premia in Banking and the Macroeconomy" December 2004

Diskussionspapiere der Fakultät für Wirtschaftswissenschaften ab Nr. 441/2000

528.	Joachim Frohn, Chen Pu	Alternative ökonometrische Zeitverwendungsmodelle Dezember 2004
529.	Stefan Niermann Joachim Frohn	Standortfaktoren und ihre Bedeutung für das Abwandern von Unternehmen
530.	Christoph Wöster	Constructing Arbitrage-free Binomial Models December 2004
531.	Fred G. Becker, Natascha Henseler u.a.	Fremdmanagement in Familienunternehmen Januar 2005
532.	Andreas Scholze	Die Bestimmung des Fortführungswerts in der Unternehmensbewertung mithilfe des Residualgewinnmodells Februar 2005
533.	Marten Hillebrand Jan Wenzelburger	On the Dynamics of Asset Prices and Portfolios in a Multiperiod CAPM" February 2005
534.	Jan Thomas Martini	Transfer Pricing for Coordination and Profit Determination: An Analysis of Alternative Schemes February 2005
535.	Klaus Wersching	Innovation and Knowledge Spillover with Geographical and Technological Distance in an Agentbased Simulation Model May 2005
536.	Anne Chwolka Jan Thomas Martini Dirk Simons	Accounting-Data-Based Transfer Prices in a Team-Investment Setting May 2005
537.	Sören W. Scholz Ralf Wagner	Autonomous Environmental Scanning on the World Wide Web June 2005
538.	Thorsten Pampel	On the convergence of balanced growth in continous time July 2005
539.	Fred G. Becker Michael K. Ruppel	Karrierestau - Ein Problem von Führungskräften wie Organisationen Juli 2005
540.	Li Xihao Jan Wenzelburger	Auction Prices and Asset Allocations of the Electronic Equity Trading System *Xetra* August 2005
541.	Volker Böhm Luca Colombo	Technology Choice with Externalities - A General Equilibrium Approach August 2005
542.	Martin Feldmann Dirk Biskup	On lot streaming with multiple products August 2005
543.	Christoph Wöster	Die Ermittlung des Conversion Factors im Futures-Handel September 2005
544.	Thomas Braun	The impact of taxation on upper and lower bounds of enterprise value October 2005
545.	Christoph Wöster	Replication in Consistent Binomial Models November 2005
546.	Thomas Braun	Asymmetrische Information, Beteiligungsfinanzierung und drohende Überschuldung Dezember 2005
547.	Volker Böhm, Tomoo Kikuchi, George Vachadze	Welfare and the Role of Equity in an Economy with Capital Accumulation. December 2005
548.	Volker Böhm, Thorsten Pampel, Jan Wenzelburger	On the stability of balanced growth December 2005
549.	Jan Wenzelburger Hans Gersbach	Sophistication in Risk Management and Banking Stability: The Long Term February 2006
550.	Jan Wenzelburger Hans Gersbach	Sophistication in Risk Management and Banking Stability: The Short Term February 2006
551.	Volker Böhm, Tomoo Kikuchi, George Vachadze	On the Role of Equity for the Dynamics of Capital Accumulation May 2006
552.	Andreas Scholze	Buchwertorientierte Finanzierungspolitik in der Unternehmensbewertung Juni 2006
553.	Volker Böhm George Vachadze	Endogenous Inequality of Nations through Asset Market Integration August 2006
554.	Tomoo Kikuchi	Inequality of Nations and Endogenous Fluctuations in a Two Country Model August 2006
555.	Volker Böhm, George Vachadze	Credit Risk and Symmetry Breaking Through Financial Market Integration August 2006
556.	Tomoo Kikuchi	"International Asset Market, Nonconvergence, and Endogenous Fluctuations" September 2006
557.	Hermann Jahnke, Stefan Wielenberg, Heinrich Schumacher.	Integration von in- und externem Rechnungswesen in mittelständisch strukturierten Unternehmen: Aktueller Stand und Perspektiven unter dem Eindruck der Einführung der Internationalen Rechnungslegungsstandards September 2006
558.	Volker Böhm, Tomoo Kikuchi George Vachadze	Asset Pricing with Markovian Productivity Growth October 2006

Diskussionspapiere der Fakultät für Wirtschaftswissenschaften ab Nr. 441/2000

559. Christoph Wöster — Pricing Derivatives Under a Gains Tax Regime: New Impacts
October 2006